THE GLOBAL STRUCTURE OF ITERATED FUNCTION SYSTEMS

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I study sets of attractors and non-attractors of finite iterated function systems. I provide examples of compact sets which are attractors of iterated function systems as well as compact sets which are not attractors of any iterated function system. I show that the set of all attractors is a dense F_σ set and the space of all non-attractors is a dense G_δ set it the space of all non-empty compact subsets of a space X. I also investigate the small trans-finite inductive dimension of the space of all attractors of iterated function systems generated by similarity maps on [0,1].

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CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

In [12], Karl Weierstrass defined the function

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

where $0 < a < 1$, b is a positive integer, and

$$ab > 1 + \frac{3}{2}\pi.$$

This function is everywhere continuous but nowhere differentiable. The graph of such a function would today be considered a fractal. Later, in [5], Helge von Koch gave a definition for a similar curve, this curve later became known as the von Koch Snowflake. Later, in 1915, Wacław Sierpiński gave a definition for his triangle and

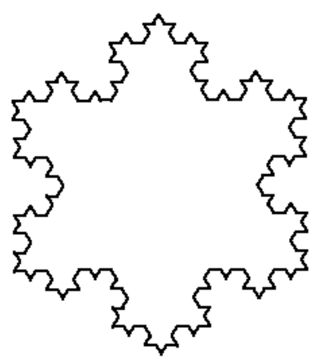

FIGURE 1.1. The von Koch Snowflake

one year later produced his carpet. George Cantor also constructed several subsets of the real line with peculiar properties, today these Cantor Sets are recognized as fractals as well.

All of these structures share a common attribute, they are all self-similar, in other words they are similar or approximately similar to parts of themselves.

The main topic of this dissertation is the iterated function system. The background for this concept was introduced in 1981 in [4] by John Hutchinson, where it is shown that there is a compact set which is invariant with respect to a finite collection of special functions called contractions. As it turns out, these invariant sets, called attractors, are often self-similar.

The contents of ths dissertation are summarized as follows:

In this chapter, we gather together some theorems and definitions from topology which are important to the theory of iterated function systems. We define three different notions of dimension. We also define the notion of an iterated function system, and we give a few examples of attractors. All theorems and definitions in the chapter are generally known, and proofs are only given to those results that are either essential to the theory of iterated function systems or are not known to the general mathematical community.

In chapter 2 we give further examples of attractors of iterated function systems. We show that every geometric sequence with positive common ratio less than 1 is an attractor of an iterated function. We also show that for every $p > 0$, the set $F_p = \{1/n^p\}_{n\geq 1} \cup \{0\}$ is also an attractor. We also give examples of sets which are not attractors of any iterated function system, one of which is countable.

In chapter 3, we investigate the topological properties of sets of attractors and non-attractors of iterated function systems. We show that the set of all attractors forms a dense F_σ set in the space of all compact subspaces in some topological space X. We also show that the set of all non-attractors forms a dense G_δ set in the space of all compact subspaces of X.

Finally, in chapter 4, we investigate the dimensional properties of the set of all attractors of iterated function systems whose contractions satisfy certain properties. Some of these properties include similarity maps in $C([0, 1])$ and collection of contractions which are closed under multiplication by elements of $[0, 1]$.

For the purposes of this paper, let (X, d) be a compact metric space, and let A be a finite alphabet. We will also primarily consider X to be a real Euclidean space. Infinite iterated function systems have been studied by Mauldin and Urbanski in [8] and [9].

1.1. Topological Background

One of the most important concepts in the theory of iterated function systems is the concept of a contraction. A *contraction* is a function $f : X \to X$ such that there exists a real number $0 \leq r < 1$ so that for any $x, y \in X$, $d(f(x), f(y)) \leq r \cdot d(x, y)$. Such an r will be called a *contraction factor* of the function f. Note that f is a special case of a Lipschitz function, therefore it follows that f is uniformly continuous. Also note that if f is a contraction, then there is a least r which satisfies the contraction factor condition. For the rest of this paper, when we refer to the contraction factor of a contraction, we are refering to this least r.

The following theorem is essential to to theory of iterated function systems, and so a proof is given of this result. Note, however, that this is a widely known result and can be found in any introductory level topology text book.

THEOREM 1.1 (Banach Fixed Point Theorem). *If $f : X \to X$ is a contraction map with contraction factor r, then there exists a unique point $x_0 \in X$ with $f(x_0) = x_0$. Furthermore, for any $x \in X$ and any $n \in \mathbb{N}$, we have $d(f^{(n)}(x), x_0) \leq r^n \cdot d(x, x_0)$ where*

$$f^{(n)}(x) = \left(\underbrace{f \circ f \circ \ldots \circ f}_{\text{n times}} \right) (x).$$

PROOF. For $x \in X$, consider the sequence $\{x_n\}_{n \geq 1}$ where and $x_n = f^{(n)}(x_{n-1})$ for $n \geq 1$. It is clear that $\{x_n\}_{n \geq 1}$ is a Cauchy sequence.

Since (X, d) is a complete metric space and the sequence $\{x_n\}_{n \geq 1}$ is Cauchy, we may let $x_0 \in X$ so that $x_n \to x_0$ as $n \to \infty$. We now show that x_0 is the unique fixed point of the contraction $f : X \to X$.

First note that as $x_n \to x_0$ and $f : X \to X$ is continuous, we conclude that $f(x_n) \to f(x_0)$. But the sequence $\{f(x_n)\}_{n\geq 0}$ is the same as the sequence $\{x_n\}_{n\geq 1}$, and is therefore a subsequence of $\{x_n\}_{n\geq 0}$. Hence both sequences must converge to the same point, i.e, $f(x_0) = x_0$.

Now suppose $y \in X$ with $f(y) = y$ and $x \neq y$. Then

$$0 < d(x, y) = d(f(x), f(y)) < r \cdot d(x, y).$$

Which is impossible as $d(x, y) > 0$ and $0 < r < 1$. Therefore $x = y$, and x is the unique fixed point of the contraction $f : X \to X$.

Finally, a simple induction argument shows that for every $n \in \mathbb{N}$ and for every $x \in X$, $d(f^{(n)}(x), x_0) \leq r^n \cdot d(x, x_0)$. $\square$

Now, let $f, g : X \to X$ be contraction maps with contraction factors r and s respectively, and observe that

$$d(f(g(x)), f(g(y)) \leq r \cdot d(g(x), g(y)) \leq rs \cdot d(x, y),$$

for every $x, y \in X$. Also, since $0 < r, s < 1$ we have $0 < rs < \max\{r, s\} < 1$, and therefore $f \circ g : X \to X$ is a contraction map with contraction factor at most $\max\{r, s\}$. Using mathematical induction, this result can be extended to finite compositions of contraction maps. That is, if $f_1, f_2, \ldots f_n : X \to X$ are contractions with contraction factors $r_1, r_2, \ldots r_n$ respectively, then $f_1 \circ f_2 \circ \ldots \circ f_n : X \to X$ is a contraction map with contraction factor at most $r = \max\{r_i\}_{i=1}^n$.

As attractors of iterated function systems are compact subsets of X, the natural topological space to consider in studying attractors is the metric space $(\mathcal{K}(X), d_H)$ where

$$\mathcal{K}(X) := \{K \subseteq X : K \neq \emptyset \text{ is compact}\}$$

and $d_H : \mathcal{K}(X) \times \mathcal{K}(X) \to [0, +\infty)$ is defined by

$$d_H(K_1, K_2) = \max\{d(K_1, K_2), d(K_2, K_1)\},$$

where

$$d(K_1, K_2) = \sup_{k_1 \in K_1} \inf_{k_2 \in K_2} d(k_1, k_2).$$

The topology on $\mathcal{K}(X)$ generated by the metric d_H will be called the Vietoris topology on $\mathcal{K}(X)$. Note that since (X, d) is a compact metric space, then $(\mathcal{K}(X), d_H)$ is also a compact metric space. Also note that the set $F = \{K \in \mathcal{K}(X) : |K| \text{ is finite}\}$ is dense in $\mathcal{K}(X)$. When considering any collection of attractors as a topological space, we will give the collection the subspace topology inherited from the Vietoris topology on $\mathcal{K}(X)$.

Now consider a collection of contractions $f_1, f_2, \ldots, f_n : X \to X$ with contraction factors r_i. Since each f_i is continuous, for every $A \in \mathcal{K}(X)$, the set $f_i(A) = \{f_i(a) : a \in A\} \in \mathcal{K}(X)$. Therefore, we may think of each f_i as a function from $\mathcal{K}(X)$ to $\mathcal{K}(X)$. Next, define the Hutchinson operator as $F : \mathcal{K}(X) \to \mathcal{K}(X)$ by

$$F(A) = \bigcup_{i=1}^{n} f_i(A).$$

In [4], it is shown that there is a unique set $A \in \mathcal{K}(X)$ so that $F(A) = A$, where F is the Hutchinson operator. What follows is an alternate proof of this fact, however, we show that the Hutchinson operator is actually a contraction on the space $\mathcal{K}(X)$.

THEOREM 1.2. *The function $F : \mathcal{K}(X) \to \mathcal{K}(X)$ as defined is a contraction.*

PROOF. Let $K_1, K_2 \in \mathcal{K}(X)$. Then $F(K_1) = \bigcup_{i \in A} f_i(K_1)$ and $F(K_2) = \bigcup_{i \in A} f_i(K_2)$. Thus for any point $x \in F(K_1)$,

$$\begin{aligned} d(x, F(K_2)) &= \min_{y \in F(K_2)} d(x, y) \\ &= \min_{i \in A}\{d(x, f_i(K_2))\} \\ &\leq d(x, f_j(K_2)) \end{aligned}$$

for any $j \in A$. Next we compute $d(x, F(K_2))$ over all $x \in F(K_1)$. Since $x \in F(K_1)$, there exists $j \in A$ and $y \in K_1$ so that $x = f_j(y)$. Then, by the previous inequality, we have

$$d(x, F(K_2)) = d(f_j(y), F(K_2)) \leq d(f_j(y), f_j(K_2)).$$

Now, for any $z \in X$, $d(f_j(y), f_j(z)) \leq r_j \cdot d(y, z)$. Thus,

$$d(x, F(K_2)) \leq r_j \cdot d(y, K_2) \leq r \cdot d(y, K_2).$$

Therefore,

$$\max_{x \in f_j(K_1)} d(x, F(K_2)) \leq r \cdot \max_{y \in K_1} d(y, K_2).$$

Taking the maximum over all j, we conclude

$$\max_{x \in F(K_1)} d(x, F(K_2)) \leq r \cdot \max_{y \in K_1} d(y, K_2).$$

By a symmetric argument, one can show that

$$\max_{x \in F(K_2)} d(x, F(K_1)) \leq r \cdot \max_{y \in K_2} d(y, K_1).$$

The last two inequalities imply that $d_H(F(K_1), F(K_2)) \leq r \cdot d_H(K_1, K_2)$, thus showing that $F : \mathcal{K}(X) \to \mathcal{K}(X)$ as defined above is a contraction. □

Since $(\mathcal{K}(X), d_H)$ is a complete metric space, the Banach Fixed Point Theorem yields a unique fixed point for F, label this set by $\mathcal{J}_F \in \mathcal{K}(X)$.

As this dissertation does make use of some dimension theory, a few definitions and basic theorems about these dimensions are in order.

The first dimension we will define will be the *Hausdorff dimension*, these definitions and the following propositions can all be found in [3].

Suppose that X is a subset of $\mathbb{R}^n$ for some n and that $\{U_i\}$ is a countable collection of sets each with diameter at most δ so that

$$F \subset \bigcup_{i=1}^{\infty} U_i,$$

then the collection $\{U_i\}$ is called a δ-cover of F. Now suppose $s \in [0, +\infty)$, for any $\delta > 0$, we may define

$$\mathcal{H}^s_\delta(X) := \inf \left\{ \sum_{i=1}^{\infty} \text{diam}\,(U_i)^s : \{U_i\} \text{ is a } \delta\text{-cover of } X \right\}.$$

Now we write

$$\mathcal{H}^s(X) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(X).$$

Note that this limit may be 0 or even $+\infty$. $\mathcal{H}^s(X)$ is called the *s-dimensional Hausdorff measure* of F.

PROPOSITION 1.3. *Let $X \subset \mathbb{R}^n$ and let $f : X \to X$ be a Hölder continuous function with exponent α, i.e.,*

$$|f(x) - f(y)| \leq r \cdot |x - y|^\alpha$$

for all $x, y \in X$ and some constant $r > 0$. Then for each $s \in [0, \infty)$

$$\mathcal{H}^{s/\alpha}(f(X)) \leq r^{s/\alpha} \mathcal{H}^s(X).$$

PROOF. Let $\{U_i\}$ be a δ-cover of X. Then, since $\text{diam}\,(f(X \cap U_i)) \leq r \cdot \text{diam}\,(U_i)^\alpha$, it follows that $\{f(X \cap U_i)\}$ is an ϵ-cover of $f(X)$ where $\epsilon = r\delta^\alpha$. Thus

$$\sum_{i=1}^{\infty} \text{diam}\,(f(X \cap U_i))^{s/\alpha} \leq r^{s/\alpha} \sum_{i=1}^{\infty} \text{diam}\,(U_i)^s.$$

Therefore, $\mathcal{H}^{s/\alpha}_\epsilon(f(X)) \leq r^{s/\alpha} \mathcal{H}^s_\delta(X)$. But note that as $\delta \to 0$, $\epsilon \to 0$, and therefore $\mathcal{H}^{s/\alpha}(f(X)) \leq r^{s/\alpha} \mathcal{H}^s(X)$. □

Note that if $f : X \to \mathbb{R}^n$ is Hölder continuous with exponent $\alpha = 1$, then f is Lipschitz continuous, and the above result reduces to

$$\mathcal{H}^s(f(X)) \leq r^s \mathcal{H}^s(X).$$

Now suppose $t > s$ and that $\{U_i\}$ is a δ-cover of X, then

$$\sum_i \text{diam}\,(U_i)^t \leq \sum_i \text{diam}\,(U_i)^{t-s} \cdot \text{diam}\,(U_i)^s \leq \delta^{t-s} \sum_i \text{diam}\,(U_i)^s.$$

Thus, by taking infima, we see that

$$\mathcal{H}^t_\delta(X) \leq \delta^{r-s}\mathcal{H}^s_\delta(X).$$

Therefore, by taking $\delta \to 0$, we see that if $\mathcal{H}^s(X) < \infty$, then $\mathcal{H}^t(X) = 0$ for any $t > s$. Thus there is a critical value of s where $\mathcal{H}^s(X)$ jumps from $+\infty$ to 0, this critical value is called the *Hausdorff dimension* of X.

Definition 1.4. The *Hausdorff dimension* of a set $X \subset \mathbb{R}^n$ is given by

$$\dim_H(X) := \inf\{s \in [0, +\infty) : \mathcal{H}^s(X) = 0\} = \sup\{s \in [0, +\infty) : \mathcal{H}^s(X) = +\infty\}.$$

Note that if $f : X \to \mathbb{R}^n$ is Lipschitz continuous, we have

$$\begin{aligned}
\dim_H(f(X)) &= \inf\{s \in [0, +\infty) : \mathcal{H}^s(f(X)) = 0\} \\
&\leq \inf\{s \in [0, +\infty) : r^s\mathcal{H}^s(X) = 0\} \\
&= \inf\{s \in [0, +\infty) : \mathcal{H}^s(X) = 0\} \\
&= \dim_H(X).
\end{aligned}$$

The next dimension we will define is the small transfinite inductive dimension, this definition can be found in [2].

Definition 1.5. Let X be a regular space and define the *small transfinite inductive dimension* of X, denoted by $\operatorname{trind} X$, in the following way:

- $\operatorname{trind} X = -1$ if and only if $X = \emptyset$;
- $\operatorname{trind} X \leq \alpha$, where α is an ordinal number, if for every point $x \in X$ and each open neighborhood V of x there is an open set $U \subset X$ such that

  $$x \in U \subset V \text{ and } \operatorname{trind} \operatorname{Fr} U < \alpha;$$

- $\operatorname{trind} X = \alpha$ if $\operatorname{trind} X \leq \alpha$ and the inequality $\operatorname{trind} X \leq \beta$ does not hold for any ordinal $\beta < \alpha$;
- $\operatorname{trind} X = \infty$ if the inequality $\operatorname{trind} X \leq \alpha$ does not hold for any ordinal α.

It is easy to see that the small transfinite inductive dimension is a tological invariant. Also if $U \subset X$, then $\operatorname{trind} U \leq \operatorname{trind} X$.

Now for the definition of the *small inductive dimension* of X, denoted by $\operatorname{ind} X$, simply take the definition of $\operatorname{trind} X$ and replace all ordinals with natural numbers.

Obviously, for $n \geq 0$, a regular space X satisfies $\operatorname{ind} X \leq n$ if and only if X has a base $\mathcal{B}$ such that $\operatorname{ind} \operatorname{Fr} U \leq n-1$ for every $U \in \mathcal{B}$. However, since every base $\mathcal{B}$ for a separable space X contains a countable family $\mathcal{B}'$ which is still a base for X, we have

THEOREM 1.6. *For $n \geq 0$, a separable metric space X satisfies* $\operatorname{ind} X \leq n$ *if and only if X has a countable base $\mathcal{B}$ such that* $\operatorname{ind} \operatorname{Fr} U \leq n-1$ *for every $U \in \mathcal{B}$.*

A very useful theorem that relates the Hausdorff dimension and the small inductive dimension can be found in [11] the proof of which is due to Eilenberg, and is stated as follows:

THEOREM 1.7. *If a space X satisfies* $\operatorname{ind} X \leq n$ *then there is a homeomorphism h of X into I^{2n+1} such that for every real number $r > n$*

$$\mathcal{H}^r(\overline{h(X)}) = 0.$$

Moreover, the space $(I^{2n+1})^X$ contains a dense G_δ set of homeomorphisms satisfying the above condition.

Thus we may conclude that $\dim_H(X) \geq \operatorname{ind} X$.

Recall from topology that a separable metric space X is said to be 0-dimensional if and only if X is non-empty and has a countable base consisting of clopen sets. Also note that a separable metric space X is 0-dimensional if and only if $\operatorname{ind} X = 0$, as can be seen from the previous theorem.

The last dimension that we will define is the box-counting dimension, this definition can also be found in [3].

DEFINITION 1.8. Let X be any bounded set and let $N_\delta(X)$ be the smallest number of sets of diameter less than or equal to δ which can cover X. The *lower box-counting dimension* and the *upper box-counting dimension* are defined respectively as follows:

$$\underline{\dim}_B X = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(X)}{-\log \delta}$$

$$\overline{\dim}_B X = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(X)}{-\log \delta}.$$

If the lower box-counting dimension agrees with the upper box-counting dimension, then this common value is called the *box-counting dimension* of X, and

$$\dim_B X = \lim_{\delta \to 0} \frac{\log N_\delta(X)}{-\log \delta}.$$

When calculating the box-counting dimension of a set, it is enough to consider limits as δ tends to 0 through any decreasing sequence δ_k such that $\delta_{k+1} \geq c\delta_k$ for some constant $0 < c < 1$. To see this, note that if $\delta_{k+1} \leq \delta < \delta_k$, then

$$\begin{aligned} \frac{\log N_\delta(F)}{-\log_\delta} &\leq \frac{\log N_{\delta_{k+1}}(F)}{-\log \delta_k} \\ &= \frac{\log N_{\delta_{k+1}}(F)}{-\log \delta_{k+1} + \log(\delta_{k+1}/\delta_k)} \\ &\leq \frac{\log N_{\delta_{k+1}}(F)}{-\log \delta_{k+1} + \log c}, \end{aligned}$$

so that

$$\overline{\lim}_{\delta \to 0} \frac{\log N_\delta(X)}{-\log \delta} \leq \overline{\lim}_{k \to \infty} \frac{\log N_{\delta_k}(F)}{-\log \delta_k}.$$

The opposite inequality is trivial; the case of lower limits is handled in a similar way.

PROPOSITION 1.9. *For any $p > 0$ let $F_p = \{1/n^p\}_{n \geq 1} \cup \{0\}$. Then*

$$\dim_B F_p = \frac{1}{p+1}.$$

PROOF. First note that by letting $f(x) = x^{-p}$ for $x > 0$, we have $f'(x) = -px^{-(p+1)}$, and therefore, by the mean value theorem, $\exists c \in [k, k+1]$ so that $f(k) - f(k+1) = pc^{-(p+1)}$. In other words, $\exists c \in [k, k+1]$ so that

$$\frac{1}{k^p} - \frac{1}{(k+1)^p} = \frac{p}{c^{p+1}},$$

and also note that

$$\frac{p}{(k+1)^{p+1}} \le \frac{p}{c^{p+1}} \le \frac{p}{k^{p+1}}.$$

First we give a lower estimate of the lower box-counting dimension. Let

$$\delta_k = p/(k+1)^{(p+1)},$$

and note that an interval of length δ_k can contain at most one point of the set

$$\{1, 1/2^p, 1/3^p, \ldots, 1/k^p\}.$$

Therefore, at least k intervals of length δ_k are required to cover F_p. Therefore

$$\begin{aligned} \frac{\log N_{\delta_k}(F_p)}{-\log \delta_k} &\ge \frac{\log k}{\log \frac{(k+1)^{p+1}}{p}} \\ &= \frac{\log k}{(p+1)\log(k+1) - \log p}. \end{aligned}$$

Letting $k \to \infty$, so that $\delta_k \to 0$, yields

$$\underline{\dim}_B F_p \ge \frac{1}{p+1}.$$

Now, we give an upper estimate of the upper box-counting dimension. Let $\delta_k = p/k^{(p+1)}$, and note that k/p intervals of length δ_k are required to cover $[0, 1/k^p]$, leaving another $k-1$ points in F_p not yet covered. Also note that an interval of length δ_k can contain at most one point of the set $\{1, 1/2^p, 1/3^p, \ldots, 1/(k-1)^p\}$. Thus we have

$$\begin{aligned} \frac{\log N_{\delta_k}(F_p)}{-\log \delta_k} &\le \frac{\log(\frac{k}{p} + k - 1)}{\log \frac{k^{p+1}}{p}} \\ &= \frac{\log(\frac{k}{p} + k - 1)}{(p+1)\log k - \log p}. \end{aligned}$$

Again, letting $k \to \infty$, so that $\delta_k \to 0$, yields

$$\overline{\dim}_B F_p \leq \frac{1}{p+1}.$$

Therefore, we conclude that

$$\dim_B F_p = \frac{1}{p+1}.$$

□

We also have

PROPOSITION 1.10. *Let* $0 < r < 1$, *and let* $G_r = \{r^n\}_{n \geq 0} \cup \{0\}$. *Then* $\dim_B G_r = 0$.

1.2. Basic Definitions and Examples of Iterated Function Systems

In this section we will give a brief introduction to the theory of iterated function systems and we will give 2 common examples of attractors.

DEFINITION 1.11. Let A be a finite set and for each $i \in A$, let $\varphi_i : X \to X$ be a contraction with contraction factor r_i. The space X together with the collection $\{\varphi_i\}_{i \in A}$ is called an *Iterated Function System*. The notation for such a system will be $\{X; \varphi_i : i \in A\}$. The number $r = \max\{r_i : i \in A\}$ is the *contraction factor* of the iterated function system.

For the alphabet A, we define A^∞ to be the set of all infinite words on A, A^* is the set of all finite words on A, and A^n is the set of all words on A of length n. For any $\omega \in A^*$, let $|\omega| = n$ be the unique natural number so that $\omega \in A^n$. Also, for $\omega \in A^\infty$, let $\omega|_n = \omega_1\omega_2 \ldots \omega_n$.

For each $\omega \in A^*$, say $\omega = \omega_1\omega_2 \ldots \omega_n$, define the map coded by ω as the function

$$\varphi_\omega = \varphi_{\omega_1} \circ \varphi_{\omega_2} \circ \ldots \circ \varphi_{\omega_n} : X \to X.$$

Note that the map coded by ω is a finite composition of contractions and is therefore a contraction with contraction factor r.

Also note that for $\omega \in A^\infty$, the collection of sets $\{\varphi_{\omega|_n}(X)\}_{n\in\mathbb{N}}$ forms a nested collection of closed subsets of X and also note that $\operatorname{diam}(\varphi_{\omega|_n}(X)) \leq r^n \cdot \operatorname{diam}(X) \to 0$ as $n \to \infty$. Thus,

$$\bigcap_{n\in\mathbb{N}} \varphi_{\omega|_n}(X) = \{x\}$$

for some $x \in X$. Thus we can define a map $\pi : A^\infty \to X$ by letting $\pi(\omega)$ be the unique element in this intersection. Now let

$$\mathcal{J} := \pi(A^\infty) = \bigcup_{\omega\in A^\infty} \bigcap_{n\in\mathbb{N}} \varphi_{\omega|_n}(X).$$

The set $\mathcal{J}$ is called the *attractor* of the iterated function system $\{X; \varphi_i : i \in A\}$.

For an alternative formulation for the attractor of an iterated function system, we must first state a lemma due to Dénes König and appears in [6].

LEMMA 1.12 (König's Lemma). *If G is a connected graph with infinitely many vertices such that every vertex has finite degree, then every vertex of G is part of an infinitely long simple path, that is, a path with no repeated vertices.*

A special case of this lemma states any finitely branching tree with infinitely many vertices has an infinitely long branch.

THEOREM 1.13. *If $\{X; \varphi_1, \varphi_2, \ldots, \varphi_n\}$ is an iterated function system with attractor $\mathcal{J}$, then*

$$\mathcal{J} = \bigcap_{n\in\mathbb{N}} \bigcup_{\omega\in A^n} \varphi_\omega(X).$$

PROOF. First, suppose $x \in \mathcal{J}$. Therefore, $\pi(\omega) = x$ for some $\omega \in A^\infty$. Thus, for every $n \in \mathbb{N}$, there exists $\omega' \in A^n$ with $x \in \varphi_{\omega'}(X)$, i.e.,

$$x \in \bigcap_{n\in\mathbb{N}} \bigcup_{\omega\in A^n} \varphi_\omega(X).$$

Now, let

$$x \in \bigcap_{n\in\mathbb{N}} \bigcup_{\omega\in A^n} \varphi_\omega(X).$$

Thus, for every $n \in \mathbb{N}$ there exists $\omega' \in A^n$ with $x \in \varphi_{\omega'}(X)$. Therefore, by the note following König's Lemma, there exists an $\omega \in A^\infty$ with $\pi(\omega) = x$, and thus $x \in \mathcal{J}$. □

Now we present a method for determining if a given set is the attractor of a given iterated function system.

LEMMA 1.14. *Consider the iterated function system $\{X; \varphi_i : i \in A\}$ with attractor $\mathcal{J}$. Then*

$$\mathcal{J} = \bigcup_{i \in A} \varphi_i(\mathcal{J}).$$

PROOF. Let $x_0 \in \bigcup_{i \in A} \varphi_i(\mathcal{J})$. Then there exists $i(0) \in A$ and $x_1 \in \mathcal{J}$ such that $x_0 = \varphi_{i(0)}(x_1)$. Since $x_1 \in \mathcal{J}$, there exists $\omega \in A^\infty$ with $x_1 = \pi(\omega)$. But then $\omega' = (i(0)\omega) \in A^\infty$ and clearly $\pi(\omega') = x_0$. Therefore $x_0 \in \mathcal{J}$.

Now suppose $x \in \mathcal{J}$, then $x = \pi(\omega) = \bigcap_{n \in \mathbb{N}} \varphi_{\omega|_n}(X)$ for some $\omega \in A^\infty$. But note that for any $\omega \in A^\infty$ we have

$$\begin{aligned}
\pi(\omega) &= \bigcap_{n=1}^{\infty} \varphi_{\omega|_n}(X) \\
&\subset \varphi_{\omega_1}\left(\bigcap_{k=1}^{\infty} \varphi_{\sigma\omega|_k}(X)\right) \\
&= \varphi_{\omega_1}(\pi(\sigma\omega)) \\
&\in \varphi_{\omega_1}(\mathcal{J})
\end{aligned}$$

Thus $x \in \varphi_{\omega_1}(\mathcal{J})$, which concludes the proof. □

Recall that the function $\hat{\Phi} : \mathcal{K}(X) \to \mathcal{K}(X)$ defined by $\hat{\Phi}(K) = \bigcup_{i \in A} \varphi_i(K)$ is a contraction, and therefore has a unique fixed point, but the previous lemma shows that $\mathcal{J}$ is a fixed point of this map. Thus we have

THEOREM 1.15. *For any iterated function system $\{X; \varphi_i : i \in A\}$ there is a unique attractor $\mathcal{J} \in \mathcal{K}(X)$. Furthermore,*

$$\mathcal{J} \in \mathcal{K}(X)$$

is the attractor of the iterated function system $\{X; \varphi_i : i \in A\}$ if and only if

$$\mathcal{J} = \bigcup_{i \in A} \varphi_i(\mathcal{J}).$$

Now we present two examples of iterated function systems, and we also present their attractors.

EXAMPLE 1.16 (Cantor's Middle Third Set). *Consider the iterated function system*

$$\{[0,1]; \varphi_1, \varphi_2\}$$

where

$$\begin{cases} \varphi_1(x) = \frac{1}{3}x \\ \varphi_2(x) = \frac{1}{3}x + \frac{2}{3} \end{cases}$$

According to the Banach Fixed Point Theorem, $\lim_{n\to\infty} \hat{\Phi}^{(n)}(K) = \mathcal{J}$ *for any compact set* $K \subset [0,1]$.

Take as our K*, the compact set* $[0,1]$*. Then*

$$\hat{\Phi}(K) = \varphi_1([0,1]) \cup \varphi_2([0,1]) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Now

$$\begin{aligned} \hat{\Phi}^{(2)}(K) &= \hat{\Phi}(\hat{\Phi}([0,1])) \\ &= \hat{\Phi}\left(\left[0, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, 1\right]\right) \\ &= \hat{\Phi}\left(\left[0, \tfrac{1}{3}\right]\right) \cup \hat{\Phi}\left(\left[\tfrac{2}{3}, 1\right]\right) \\ &= \left[0, \tfrac{1}{9}\right] \cup \left[\tfrac{2}{9}, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, \tfrac{7}{9}\right] \cup \left[\tfrac{8}{9}, 1\right] \end{aligned}$$

At each step, we are removing the middle third of the remaining intervals from the previous step. This naturally leads to Cantor's Middle Third Set.

For the next example, we will consider an iterated function system whose attractor is a compact subset of $\mathbb{C}$.

Example 1.17 (The Sierpinski Gasket). *Consider the iterated function system*

$$\{T; \varphi_1, \varphi_2, \varphi_3\}$$

where T *is the closed triangle in the Complex plane with vertices given by the complex numbers* $0 + 0i, 2 + 0i,$ and $1 + \sqrt{3}i$ *and*

$$\begin{cases} \varphi_1(t) = \frac{1}{2}t \\ \varphi_2(t) = \frac{1}{2}t + 1 \\ \varphi_3(t) = \frac{1}{2}t + \frac{1+\sqrt{3}i}{2} \end{cases}$$

Again, the function $\hat{\Phi}(K) = \varphi_1(K) \cup \varphi_2(K) \cup \varphi_3(K)$ *has a fixed point, and this fixed point can be found by iterating the function* $\hat{\Phi}$ *with* T *as the initial set, as shown in fig. 1.2.*

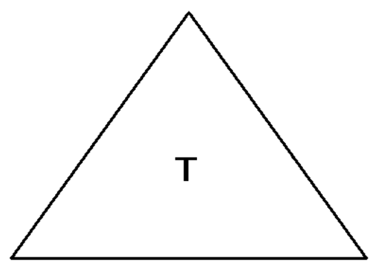

Figure 1.2. The initial set T

After one iteration, we arrive at the set $\hat{\Phi}(T)$*, which is shown in fig. 1.3. Note that in this figure, the triangle labeled as* 1 *is the image of the contraction* φ_1*.*

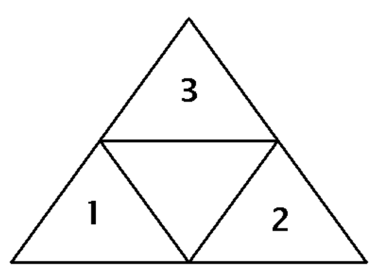

Figure 1.3. The set $\hat{\Phi}(T)$

For the n^{th} iterate, $\hat{\Phi}^{(n)}(T)$, we would remove the middle upside down triangle from each of the remaining triangles from the $(n-1)^{st}$ iterate. This process will approach the set, in the Hausdorff distance, to the set shown in fig. 1.4.

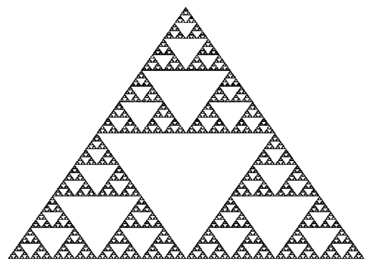

FIGURE 1.4. The attractor of the IFS $\{T; \varphi_1, \varphi_2, \varphi_3\}$

1.3. Applications of Iterated Function Systems

In this section we give a brief discussion of one of the possible applications of iterated functions systems. This application is to the area of image compression. The following theorem, the Collage Theorem, which was originally proven by Michael Barnsley, is an important theorem to theory of iterated function systems, and so its proof is given here. The statement of the following theorem and corollary along with there proofs can be found in [3].

THEOREM 1.18 (Collage Theorem). *Let $\{X; \varphi_1, \varphi_2, \ldots, \varphi_n\}$ be any iterated function system with contraction factor r and attractor $\mathcal{J}$. For any non-empty set $E \in \mathcal{K}(X)$*

$$d_H(E, \mathcal{J}) \leq d_H\left(E, \bigcup_{i=1}^{n} \varphi_i(E)\right) \frac{1}{1-r}.$$

PROOF. By using the triangle inequality for the Hausdorff metric and the definition of the attractor of an iterated function system, we have

$$\begin{aligned} d_H(E, \mathcal{J}) &\leq d_H\left(E, \bigcup_{i=1}^{n} \varphi_i(E)\right) + d_H\left(\bigcup_{i=1}^{n} \varphi_i(E), \mathcal{J}\right) \\ &= d_H\left(E, \bigcup_{i=1}^{n} \varphi_i(E)\right) + d_H\left(\bigcup_{i=1}^{n} \varphi_i(E), \bigcup_{i=1}^{n} \varphi_i(\mathcal{J})\right) \end{aligned}$$

$$\leq \; d_H\left(E, \bigcup_{i=1}^{n} \varphi_i(E)\right) + r d_H(E, \mathcal{J}),$$

which finishes the proof. □

A corollary to the collage theorem is

Corollary 1.19. *If $E \neq \emptyset$ is a compact subset of X, then for every $\delta > 0$ there exists an iterated function system $\{X; \varphi_1, \varphi_2, \ldots, \varphi_n\}$ with attractor $\mathcal{J}$ satisfying $d_H(\mathcal{J}, E) < \delta$. In other words, the set of all attractors is a dense subset of $\mathcal{K}(X)$.*

Proof. Let $B_1, B_2, \ldots, B_n$ be a collection of open balls that cover E which have centers in E and whose radii are at most $\frac{1}{4}\delta$. Such a collection exists since E is compact. Therefore we have $E \subset \bigcup_{i=1}^{n} B_i \subset E_{\delta/4}$, where $E_{\delta/4}$ is the $\delta/4$-neighborhood of E. Now for each i, let φ_i be a contraction map so that $\varphi_i(E) \subset B_i$ and whose contraction factor is less than $\frac{1}{2}$. But then

$$\varphi_i(E) \subset B_i \subset (\varphi_i(E))_{\delta/2},$$

so that

$$\left(\bigcup_{i=1}^{n} \varphi_i(E)\right) \subset E_{\delta/4} \text{ and } E \subset \bigcup_{i=1}^{n} (\varphi_i(E))_{\delta/2}.$$

But then we have

$$d_H\left(E, \bigcup_{i=1}^{n} \varphi_i(E)\right) \leq \frac{1}{2}\delta.$$

It follows from the collage theorem that

$$\begin{aligned} d_H(\mathcal{J}, E) &\leq d_H\left(E, \bigcup_{i=1}^{n} \varphi_i(E)\right) \frac{1}{1-r} \\ &< \frac{1}{2}\delta\left(\frac{1}{1-1/2}\right) \\ &= \delta. \end{aligned}$$

□

Therefore, we may approximate any compact subset of $\mathbb{R}^n$ with an attractor of some iterated function system. Unfortunately, the number of contractions needed to approximate a compact set may be very large. For some fractals, we can try to calculate the box-counting dimension of the the fractal, thus we may restrict ourselves to sets of contractions which yield an attractor with the desired box-counting dimension. For example, the Sierpinski triangle has a box-counting dimension of $\log 3/\log 2$, and we have seen that we need three contractions each with contractive factor $1/2$ to generate the Sierpinski triangle.

However, for other fractals, such as the fern, we may draw a rough outline of the image, then cover this image by smaller similar or affine copies of itself. These similarities and affine maps may then be used to generate an iterated function system whose attractor approximates the original image.

FIGURE 1.5. The Barnsley Fern

CHAPTER 2

ATTRACTORS AND NON-ATTRACTORS OF ITERATED FUNCTION SYSTEMS

In this chapter we will present sets which are attractors of iterated function systems and sets which are not attractors of any iterated function system. For any $x \in [0, 1)$ we will construct a subset of $[0, 1]$ which is an attractor of some iterated function system and whose box-counting dimension is x. We will give an example of a countable subset of $[0, 1]$ which has box-counting dimension 1 and which is not an attractor of any iterated function system. Finally, for any $n \in \mathbb{N}$, we will give an example of a set which has exactly n points where local connectivity fails, these n points are also the only degenerate connected components of the set, and which is an attractor of some iterated function. These sets will become more important in Chapter 3.

2.1. Preliminaries

Let $I = [0, 1]$ be the closed unit interval in $\mathbb{R}$, and let $f : I \to \mathbb{R}^n$ be an embedding. We define the variation of f as follows: Let $P = \{x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n\}$ where $x_0 = 0$ and $x_n = 1$ be a partition of I. Let

$$V_f(P) := \sum_{k=1}^{n} |f(x_i) - f(x_{i-1})|$$

where $|f(x_i) - f(x_{i-1})|$ is the standard Euclidean distance from $f(x_i)$ and $f(x_{i-1})$. Then we take $V_f = \sup\{V_f(P) : P \text{ is a partition of } I\}$. The set $\mathrm{Im}(f) = \{f(x) : x \in I\}$ is called an *arc* and V_f is called the *arc length* of f. Note that if $f : I \to \mathbb{R}^n$ and $g : I \to \mathbb{R}^n$ are two embeddings and $\mathrm{Im}(f) = \mathrm{Im}(g)$, then $V_f = V_g$. Thus it makes sense to talk about the arc length of an arc without relying on any specific embedding. If β is an arc in $\mathbb{R}^n$, then the arc length of β will be denoted by $V(\beta)$. Note that if β is an arc, then $\mathcal{H}^1(\beta) = V(\beta)$.

Since β is an arc in $\mathbb{R}^n$, there exist two distinct non-cutpoints of β, call these points $a, b \in \beta$. We will refer to these non-cutpoints as endpoints of the arc β. We can place a natural order on β so that $a < b$. Now, for $c, d \in \beta$ with $a < c < d < b$, we write β_c^d to denote the sub-arc of β with endpoints $c < d$.

Let β be an arc in $\mathbb{R}^n$ with endpoints $a < b$. Then

- $V(\beta) > 0$.
- If $c \in \beta$ with $a < c < b$, then $V(\beta_a^c) + V(\beta_c^b) = V(\beta)$.
- If $V(\beta) < \infty$, then the function $v : \beta \to [0, \infty)$ defined by $v(x) = V(\beta_a^x)$ is strictly increasing.
- If $V(\beta) < \infty$, then $v : \beta \to [0, \infty)$ as defined above is also continuous.

2.2. Examples of Attractors and Non-Attractors

In [10], it is shown that if β is an arc in $\mathbb{R}^n$ with $V(\beta) < \infty$, then β is an attractor of some iterated function system in $\mathbb{R}^n$. It is also shown that if $\{\beta_1, \beta_2, \ldots, \beta_n\}$ is a collection of arcs in $\mathbb{R}^n$ so that $V(\beta_i) < \infty$ for each $i = 1, 2, \ldots, n$, then

$$\bigcup_{i=1}^{n} \beta_i$$

is also an attractor of some iterated function system in $\mathbb{R}^n$.

However, it is also shown in [10], that if β is an arc in $\mathbb{R}^n$ with endpoints $a < b$ satisfying

(i) $V(\beta_x^y) < \infty$ whenever $x, y \in \beta$ with $x, y \neq b$, and

(ii) $V(\beta_x^b) = \infty$ whenever $x \in \beta$ with $x \neq b$,

then β is not an attractor of any iterated function system in $\mathbb{R}^n$.

We would now like to generalize this statement. To this end, let β be an arc in $\mathbb{R}^k$ satisfying (i) and (ii) above, also assume that $\vec{0} = (0, 0, \ldots, 0)$ and $\vec{1} = (1, 0, \ldots, 0)$ are the endpoints of β and order β so that $\vec{1} < \vec{0}$. For simplicity, let $\mathcal{J} = \beta \times I^n$.

LEMMA 2.1. *Suppose X is a compact subset of $\mathbb{R}^{k+n}$ containing $\mathcal{J}$ and $\mathcal{J}$ is the attractor of the iterated function system $\{X; \varphi_1, \varphi_2, \ldots, \varphi_m\}$. Let $(x_i, y_i) \in \mathcal{J}$ be the unique fixed point of φ_i. Then $x_i = \vec{0}$ or for every $y \in I^n$, $(\vec{0}, y) \notin \varphi_i(\mathcal{J} \setminus (\{\vec{0}\} \times I^n))$.*

PROOF. Suppose, by way of contradiction, $x_i \neq \vec{0}$ and there exists $y \in I^n$ so that $(\vec{0}, y) \in \varphi_i(\mathcal{J} \setminus (\{\vec{0}\} \times I^n))$. Let $(c, d) \in \mathcal{J} \setminus (\{\vec{0}\} \times I^n)$ so that $(\vec{0}, y) = \varphi_i((c, d))$. Now take γ to be a rectifiable arc joining (c, d) and (x_i, y_i); this can be done as $x_i, c \neq \vec{0}$. But then $\varphi_i(\gamma)$ is a rectifiable arc joining $\varphi_i((c, d)) = (\vec{0}, y)$ and $\varphi_i((x_i, y_i)) = (x_i, y_i)$, which is impossible. Therefore, either $x_i = \vec{0}$ or for every $y \in I^n$, $(\vec{0}, y) \notin \varphi_i(\mathcal{J} \setminus (\{\vec{0}\} \times I^n))$. $\square$

LEMMA 2.2. *Suppose X is a compact subset of $\mathbb{R}^{k+n}$ containing $\mathcal{J}$ and $\mathcal{J}$ is the attractor of the iterated function system $\{X; \varphi_1, \varphi_2, \ldots, \varphi_m\}$. Then there exists $1 \leq i \leq m$ so that $(\{\vec{0}\} \times I^n) \cap \varphi_i(\mathcal{J}) \neq \emptyset$ and $\varphi_i(\mathcal{J}) \not\subseteq \{\vec{0}\} \times I^n$.*

PROOF. Suppose, by way of contradiction, that for all i, $(\{\vec{0}\} \times I^n) \cap \varphi_i(\mathcal{J}) = \emptyset$ or $\varphi_i(\mathcal{J}) \subseteq \{\vec{0}\} \times I^n$, and for convenience set $A = \{1, 2, \ldots, m\}$. Let $A_1 := \{i \in A : (\{\vec{0}\} \times I^n) \cap \varphi_i(\mathcal{J}) \neq \emptyset\}$ and note that $A_1 \neq \emptyset$. Also note that if $i \in A_1$, then $\varphi_i(\mathcal{J}) \subseteq \{\vec{0}\} \times I^n$. Therefore, we may conclude that

$$\mathcal{J} = \left(\bigcup_{i \in A_1} \varphi_i(\mathcal{J}) \right) \cup \left(\bigcup_{i \in A \setminus A_1} \varphi_i(\mathcal{J}) \right).$$

By hypothesis,

$$\left(\bigcup_{i \in A_1} \varphi_i(\mathcal{J}) \right) \cap \left(\bigcup_{i \in A \setminus A_1} \varphi_i(\mathcal{J}) \right) = \emptyset.$$

Since both A_1 and $A \setminus A_1$ are finite, both unions are finite unions of closed sets and are therefore closed. Thus we have written $\mathcal{J}$ as a finite disjoint union of closed sets. This violates the connectedness of $\mathcal{J}$. Therefore, there exists $1 \leq i \leq m$ so that $(\{0\} \times I^n) \cap \varphi_i(\mathcal{J}) \neq \emptyset$ and $\varphi_i(\mathcal{J}) \not\subseteq \{\vec{0}\} \times I^n$. $\square$

Now we present the generalization of the result from [10].

THEOREM 2.3. *$\mathcal{J}$ is not an attractor of any iterated function system on $\mathbb{R}^{k+n}$ for any $k \in \mathbb{N}$.*

PROOF. Suppose, by way of contradiction, that there exists a compact subset, X, of $\mathbb{R}^{k+n}$ containing $\mathcal{J}$ and that $\mathcal{J}$ is the attractor of the iterated function system $\{X, \varphi_1, \varphi_2, \dots, \varphi_m\}$. By lemma 2.2, let $i \in \{1, 2, \dots, m\}$ be so that $(\{\vec{0}\} \times I^n) \cap \varphi_i(\mathcal{J}) \neq \emptyset$ and $\varphi_i(\mathcal{J}) \not\subset \{\vec{0}\} \times I^n$, and let $(x_i, y_i) \in \mathcal{J}$ be the unique fixed point of φ_i. Thus, by lemma 2.1, $x_i = \vec{0}$ of for every $y \in I^n$, $(\vec{0}, y) \notin \varphi_i(\mathcal{J} \setminus (\{\vec{0}\} \times I^n))$.

First, suppose $x_i = \vec{0}$. By lemma 2.2, there exists $z \in (\beta \setminus \{\vec{0}\}) \times I^n$ so that $\varphi_i(z) \notin \{\vec{0}\} \times I^n$. Thus we may join z and $\varphi_i(z)$ with a rectifiable arc γ. Therefore, by letting

$$\gamma_\infty = \left(\bigcup_{k=0}^{\infty} \varphi_i^{(k)}(\gamma) \right) \cup \{(x_i, y_i)\},$$

we get a rectifiable curve joining z and (x_i, y_i) which is impossible.

Therefore, for every $y \in I^n$, $(\vec{0}, y) \notin \varphi_i(\mathcal{J} \setminus (\{\vec{0}\} \times I^n))$. But then

$$\{\vec{0}\} \times I^n \subseteq \varphi_i^{(k)}(\{\vec{0}\} \times I^n) \to (x_i, y_i)$$

in the Hausdorff metric as $k \to \infty$, but this is also impossible.

Therefore, $\mathcal{J}$ is not the attractor of the iterated function system $\{X, \varphi_1, \varphi_2, \dots, \varphi_m\}$.

□

For another example of a non-attractor, we refer to [1], where it is shown that there exists a set $C \subset [0, 1]$ which is homeomorphic the the Cantor Middle Third set and is not the attractor of any iterated function system $\{[0, 1]; \varphi_1, \varphi_2, \dots, \varphi_n\}$. This implies that being an attractor of an iterated function system is not topologically invariant.

The following will become important in chapter 3 of this dissertation, but we define these sets now for purposes of continuity. We set

$$\mathcal{J}^{(1)} := \{0\} \cup \left(\bigcup_{i=0}^{\infty} \left[\frac{1}{2^{2i+1}}, \frac{1}{2^{2i}} \right] \right),$$

and, for $n \geq 2$, we set

$$\mathcal{J}^{(n)} := \mathcal{J} \cup \bigcup_{m=1}^{n-1} \mathcal{J}_m,$$

where

$$\mathcal{J}_m := \{2^{2m-1}\} \cup \bigcup_{i=1}^{\infty} \left[\frac{1+2^{2i+1}}{2^{2(i-m)+2}}, \frac{1+2^{2i}}{2^{2(i-m)+1}} \right].$$

THEOREM 2.4. *$\mathcal{J}^{(1)}$ is the attractor of an iterated function system.*

PROOF. Consider the iterated function system $\{[0,1], \varphi_1, \varphi_2, \varphi_3\}$ where

$$\begin{cases} \varphi_1(x) = \frac{1}{4}x \\ \varphi_2(x) = \frac{1}{2}x + \frac{1}{2} \\ \varphi_3(x) = \frac{1}{4}x + \frac{1}{2} \end{cases}$$

Now

$$\begin{aligned} \varphi_1(\mathcal{J}^{(1)}) &= \frac{1}{4}(\mathcal{J}^{(1)}) \\ &= \{0\} \cup \frac{1}{4}\left(\bigcup_{i=0}^{\infty} \left[\frac{1}{2^{2i+2}}, \frac{1}{2^{2i+1}} \right] \right) \\ &= \{0\} \cup \left(\bigcup_{i=1}^{\infty} \left[\frac{1}{2^{2i+2}}, \frac{1}{2^{2i+1}} \right] \right) \end{aligned}$$

and

$$\begin{aligned} \varphi_2(\mathcal{J}^{(1)}) - \frac{1}{2} &= \frac{1}{2}(\mathcal{J}^{(1)}) \\ &= \{0\} \cup \frac{1}{2}\left(\bigcup_{i=0}^{\infty} \left[\frac{1}{2^{2i+2}}, \frac{1}{2^{2i+1}} \right] \right) \\ &= \{0\} \cup \left(\bigcup_{i=1}^{\infty} \left[\frac{1}{2^{2i+1}}, \frac{1}{2^{2i}} \right] \right). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \varphi_1(\mathcal{J}^{(1)}) \cup \left(\varphi_2(\mathcal{J}^{(1)}) - \frac{1}{2} \right) &= \left(\varphi_3(\mathcal{J}^{(1)}) - \frac{1}{2} \right) \cup \left(\varphi_2(\mathcal{J}^{(1)}) - \frac{1}{2} \right) \\ &= \left(\varphi_3(\mathcal{J}^{(1)}) \cup \varphi_2(\mathcal{J}^{(1)}) \right) - \frac{1}{2} \end{aligned}$$

$$
\begin{aligned}
&= \{0\} \cup \left(\bigcup_{i=1}^{\infty} \left[\frac{1}{2^{2i+2}}, \frac{1}{2^{2i+1}} \right] \right) \cup \left(\bigcup_{j=1}^{\infty} \left[\frac{1}{2^{2j+1}}, \frac{1}{2^{2j}} \right] \right) \\
&= \left[0, \frac{1}{2} \right].
\end{aligned}
$$

Thus, we conclude that $\varphi_2(\mathcal{J}^{(1)}) \cup \varphi_3(\mathcal{J}^{(1)}) = [\frac{1}{2}, 1]$, and therefore

$$\hat{\Phi}(\mathcal{J}^{(1)}) = \mathcal{J}^{(1)}$$

and thus $\mathcal{J}^{(1)}$ is the attractor of the iterated function system $\{[0,1]; \varphi_1, \varphi_2, \varphi_3\}$. □

THEOREM 2.5. *For every $n \geq 2$, the set $\mathcal{J}^{(n)}$ is an attractor of some iterated function system.*

PROOF. Let

$$X = [0,1] \cup \bigcup_{i=1}^{n-1} [2^{2i-1}, 2^{2i}],$$

and consider the iterated function system $\{X; \varphi_1, \varphi_2, \ldots, \varphi_{2n}\}$ where $\varphi_1(x) = (4^{1-n})x$,

$$
\varphi_2(x) = \begin{cases}
(4^{1-n})x & x \in [0,1] \\
2^{1-2n}x + 4^{1-n} & x \in [2,4] \\
2^{1-2n}x + 4^{2-n} & x \in [8,16] \\
\vdots & \\
2^{1-2n}x + 4^{m-n} & x \in [2^{2m-1}, 2^{2m}] \\
\vdots & \\
2^{1-2n}x + 4^{-1} & x \in [2^{2n-3}, 2^{2n-2}]
\end{cases}
$$

and for $3 \leq k \leq 2n$,

$$
\varphi_k(x) = \begin{cases}
(2^{k-2}\varphi_1 + 2^{k-2})(x) & \text{for } k \text{ odd} \\
(2^{k-3}\varphi_2 + 2^{k-3})(x) & \text{for } k \text{ even}
\end{cases}
$$

We will first show that $\varphi_1(\mathcal{J}^{(n)}) \cup \varphi_2(\mathcal{J}^{(n)}) = \mathcal{J}^{(1)}$. To start, note that

$$\varphi_1(\mathcal{J}^{(1)}) = \frac{1}{2^{2n-2}} \left(\{0\} \cup \bigcup_{i=0}^{\infty} \left[\frac{1}{2^{2i+1}}, \frac{1}{2^{2i}} \right] \right)$$

$$= \{0\} \cup \bigcup_{i=0}^{\infty} \left[\frac{1}{2^{2(i+n)-1}}, \frac{1}{2^{2(i+1)-2}} \right]$$

$$= \{0\} \cup \bigcup_{i=n-1}^{\infty} \left[\frac{1}{2^{2i+1}}, \frac{1}{2^{2i}} \right]$$

Now

$$\begin{aligned}
\varphi_1(\mathcal{J}_m) &= \frac{1}{2^{2n-2}} \left(\{2^{2m-1}\} \cup \bigcup_{i=0}^{\infty} \left[\frac{1+2^{2i+1}}{2^{2(i-m)+2}}, \frac{1+2^{2i}}{2^{2(i-m)+1}} \right] \right) \\
&= \{2^{2(m-n)+1}\} \cup \bigcup_{i=0}^{\infty} \left[\frac{1+2^{2i+1}}{2^{2(i-m+n)}}, \frac{1+2^{2i}}{2^{2(i-m+n)-1}} \right]
\end{aligned}$$

Next, note that $\varphi_2(\mathcal{J}^{(1)}) = \varphi_1(\mathcal{J}^{(1)})$, and that

$$\begin{aligned}
\varphi_2(\mathcal{J}_m) &= \frac{1}{2^{2n-1}} \left(\{2^{2m-1}\} \cup \bigcup_{i=0}^{\infty} \left[\frac{1+2^{2i+1}}{2^{2(i-m)+2}}, \frac{1+2^{2i}}{2^{2(i-m)+1}} \right] \right) + \frac{1}{2^{2n-2m}} \\
&= \{2^{2m-2n}\} \cup \bigcup_{i=0}^{\infty} \left[\frac{1+2^{2i+1}}{2^{2(i-m+n)+1}}, \frac{1+2^{2i}}{2^{2(i-m+n)}} \right] + \frac{1}{2^{2n-2m}} \\
&= \{2^{2m-2n+1}\} \cup \bigcup_{i=0}^{\infty} \left[\frac{1+2^{2i+2}}{2^{2(i-m+n)+1}}, \frac{1+2^{2i+1}}{2^{2(i-m+n)}} \right]
\end{aligned}$$

Since the right endpoint of the intervals in $\varphi_2(\mathcal{J}_m)$ agree with the left endpoints of the intervals in $\varphi_1(\mathcal{J}_m)$, we see that

$$\varphi_1(\mathcal{J}_m) \cup \varphi_2(\mathcal{J}_m) = \left[\frac{1}{2^{2n-2m-1}}, \frac{1}{2^{2n-2m-2}} \right].$$

Thus, we get

$$\varphi_1(\mathcal{J}^{(n)}) \cup \varphi_2(\mathcal{J}^{(n)}) = \{0\} \cup \bigcup_{i=n-1}^{\infty} \left[\frac{1}{2^2 i+1}, \frac{1}{2^{2i}} \right] \cup \bigcup_{m=1}^{n-1} \left[\frac{1}{2^{2n-2m-1}}, \frac{1}{2^{2n-2m-2}} \right].$$

By letting $j = n - m - 1$, and re-indexing the above unions we get

$$\begin{aligned}
\varphi_1(\mathcal{J}^{(n)}) \cup \varphi_2(\mathcal{J}^{(n)}) &= \{0\} \cup \bigcup_{i=n-1}^{\infty} \left[\frac{1}{2^{2i+1}}, \frac{1}{2^{2i}} \right] \cup \bigcup_{j=0}^{n-2} \left[\frac{1}{2^{2j+1}}, \frac{1}{2^{2j}} \right] \\
&= \{0\} \cup \bigcup_{i=0}^{\infty} \left[\frac{1}{2^{2i+1}}, \frac{1}{2^{2i}} \right] \\
&= \mathcal{J}^{(1)}.
\end{aligned}$$

Now, to cover $\mathcal{J}_m$, note that $\operatorname{diam}(\mathcal{J}_m) = 2^{2m-1} = 2^{2m-1}\operatorname{diam}(\mathcal{J}^{(1)})$ and the left endpoint of $\mathcal{J}_m$ is $\{2^{2m-1}\}$. Therefore, in order to cover $\mathcal{J}_m$, we need to stretch φ_1 and φ_2 by a factor of 2^{2m-1} and shift them to the right 2^{2m-1} units. But this is exactly what the contractions φ_k and φ_{k+1} do for $k = 2m+1$. Therefore, for $k = 2m+1$, we have

$$\varphi_k(\mathcal{J}^{(n)}) \cup \varphi_{k+1}(\mathcal{J}^{(n)}) = \mathcal{J}_m.$$

Thus,

$$\begin{aligned}
\hat{\Phi}(\mathcal{J}^{(n)}) &= \bigcup_{i=1}^{2n} \varphi_i(\mathcal{J}^{(n)}) \\
&= (\varphi_1(\mathcal{J}^{(n)}) \cup \varphi_2(\mathcal{J}^{(n)})) \cup \ldots \cup (\varphi_{2n-1}(\mathcal{J}^{(n)}) \cup \varphi_{2n}(\mathcal{J}^{(n)})) \\
&= \mathcal{J}^{(1)} \cup \mathcal{J}_1 \cup \ldots \cup \mathcal{J}_{n-1} \\
&= \mathcal{J}^{(n)}.
\end{aligned}$$

Thus, we have shown that $\mathcal{J}^{(n)}$ is indeed the attractor of the iterated function system $\{X; \varphi_1, \varphi_2, \ldots, \varphi_{2n}\}$. □

Recall that for every $p > 0$, we have $F_p := \{1/n^p\}_{n\geq 1} \cup \{0\}$ and for every $0 < r < 1$ we defined $G_r := \{r^n\}_{n\geq 0} \cup \{0\}$.

Proposition 2.6. *For every $p > 0$, F_p is an attractor of an iterated function system.*

Proof. Consider the functions

$$\begin{cases}
\varphi_1(x) = \left(\frac{1}{2}\right)^p x \\
\varphi_2(x) = \frac{x}{(2+\sqrt[p]{x})^p} \\
\varphi_3(x) = 1
\end{cases}$$

Clearly, φ_1 and φ_3 are contractions. It remains to show that φ_2 is a contraction and that $\hat{\Phi}(\mathcal{J}) = \mathcal{J}$. We will first show that φ_2 is a contraction.

To that end, note that

$$\varphi_2'(x) = \frac{2}{(2+\sqrt[p]{x})^{p+1}}$$

and

$$\varphi_2''(x) = -\frac{2(p+1)x^{(p-1)/p}}{p(2+\sqrt[p]{x})^{p+2}} < 0.$$

Since $\varphi_2''(x) < 0$ for every $x \in [0,1]$, we conclude that $\varphi_2'(x)$ is a strictly decreasing function, and therefore it attains its maximum value at $x = 0$. But note that

$$\varphi_2'(0) = \frac{1}{2^p} < 1$$

and therefore φ_2 is indeed a contraction.

Now, it only remains to show that $\hat{\Phi}(\mathcal{J}) = \mathcal{J}$. Note that

$$\varphi_1\left(\left\{\frac{1}{n^p}\right\}_{n\geq 1} \cup \{0\}\right) = \left\{\frac{1}{(2n)^p}\right\} \cup \{0\},$$

$$\begin{aligned}
\varphi_2(x)\left(\left\{\frac{1}{n^p}\right\}_{n\geq 1} \cup \{0\}\right) &= \left\{\frac{\frac{1}{n^p}}{(2+\frac{1}{n})^p}\right\}_{n\geq 1} \cup \{0\} \\
&= \left\{\frac{1}{n^p(2+\frac{1}{n})^p}\right\}_{n\geq 1} \cup \{0\} \\
&= \left\{\frac{1}{(2n+1)^p}\right\}_{n\geq 1} \cup \{0\}.
\end{aligned}$$

Therefore, we have $\varphi_1(F_p) \cup \varphi_2(F_p) \cup \varphi_3(F_p) = F_p$ as desired. □

Proposition 2.7. *For every $0 < r < 1$, the set G_r is an attractor of an iterated function system.*

Proof. For $0 < r < 1$ consider the two contractions

$$\begin{cases} \varphi_1(x) = rx \\ \varphi_2(x) = 1 \end{cases}$$

Then G_r is obviously the attractor of $\{[0,1]; \varphi_1, \varphi_2\}$. □

Recall that

$$\dim_B F_p = \frac{1}{p+1},$$

and $\dim_B G_r = 0$. Since $1/(p+1)$ ranges over $(0,1)$ as p ranges over $(0,\infty)$, we have examples of attractors whose box dimensions range over $[0,1)$.

Now we wish to show that there exists a compact countable set which is not an attractor of any iterated function system.

To this end, consider the space $X = [0, 1]$, and for $n \in \mathbb{N}$ let

$$I_n = \left[\frac{1}{2^{2n-1}}, \frac{1}{2^{2n-2}}\right].$$

Now, for $n \in \mathbb{N}$, define

$$P_n := \left\{\frac{1}{2^{2n-2}} = p_{(n,1)} > p_{(n,2)} > \ldots > p_{(n,(n+1)^{(n+1)})} = \frac{1}{2^{2n-1}}\right\},$$

so that the points in P_n are equidistant from each other, and set

$$P := \{0\} \cup \bigcup_{n=1}^{\infty} P_n.$$

Since P is the countable union of finite sets, P is countable. Now we show that P is not the attractor of any iterated function system.

LEMMA 2.8. *For any $n \in \mathbb{N}$*

$$n \sum_{i=1}^{n} i^i + n < (n+1)^{(n+1)}.$$

PROOF. We will begin this proof by first proving that

$$\sum_{i=1}^{n} i^i \leq (n+1)^n - 1$$

for every $n \in \mathbb{N}$. To accomplish this, we will proceed by mathematical induction.

First note that for $n = 1$, the result is certainly true.

Now assume

$$\sum_{i=1}^{k} i^i \leq (k+1)^k - 1.$$

Note that

$$\begin{aligned}
\sum_{i=1}^{k+1} i^i &= \sum_{i=1}^{k} i^i + (k+1)^{(k+1)} \\
&\leq (k+1)^k + (k+1)^{(k+1)} - 1 \\
&\leq 2(k+1)^{(k+1)} - 1
\end{aligned}$$

$$\leq \ (k+2)^{(k+1)} - 1.$$

Thus, by mathematical induction, we can conclude that

$$\sum_{i=1}^{n} i^i \leq (n+1)^n - 1.$$

Therefore, we have

$$n\sum_{i=1}^{n} i^i + n \leq n(n+1)^n < (n+1)^{n+1}.$$

□

THEOREM 2.9. *P as defined above is not an attractor of any iterated function system.*

PROOF. Suppose $\{[0,1]; \varphi_1, \varphi_2, \ldots, \varphi_m\}$ is an iterated function system so that $\hat{\Phi}(P) = P$. Without loss of generality, assume $k \in \{1, 2, \ldots, m\}$ so that for $1 \leq i \leq k$, $\varphi_i(0) = 0$ and for $k+1 \leq i \leq m$, $|\varphi_i(P)| < \aleph_0$. Note that if $\varphi_i(0) = 0$, then $\varphi_i(x) < x$ for all $x \in (0,1]$. So that the points of P_{n+1} can only be mapped to, under φ_i, by points from

$$\bigcup_{j=i}^{n} P_j,$$

or from points in P_{n+1} itself. But if φ_i takes a point of P_{n+1} and maps it into P_{n+1}, then $\varphi_i|_{P_{n+1}}$ is constant. Thus the maximum number of points of P_{n+1} covered by

$$\bigcup_{i=1}^{k} \varphi_i(P)$$

is given by

$$k\sum_{j=1}^{n} j^j + k.$$

But note that for $n > k$, $k\sum_{j=1}^{n} j^j + k < n\sum_{j=1}^{n} j^j + n < (n+1)^{(n+1)} = |P_{n+1}|$. Therefore, for every $n > k$, there are points in P_{n+1} which are not covered by the above union. Thus the number of total points not covered by this union is countably infinite. Since

$$\bigcup_{i=k+1}^{m} \varphi_i(P)$$

is finite, we see that $\hat{\Phi}(P) \neq P$. □

Theorem 2.10. $\dim_B P = 1$

Proof. First, since $P \subset [0,1]$ and $\dim_B[0,1] = 1$, we immediately have $\overline{\dim}_B P \leq 1$. Thus we only need to show that $\underline{\dim}_B P \geq 1$. To see this note that $|P_n| = (n+1)^{n+1}$ and the distance bewtween points in P_n is given by

$$r_n = \frac{1}{(n+1)^{n+1}2^{2n-1}}.$$

Therefore we have

$$\frac{N_{r_n}(P)}{-\log r_n} \geq \frac{(n+1)\log(n+1)}{(2n-1)\log 2 + (n+1)\log(n+1)}.$$

The right hand side of the above inequality converges to 1 as $n \to \infty$. Hence $\underline{\dim}_B P \geq 1$. Therefore we conclude that $\dim_B P = 1$. □

Conjecture 2.11. If X is a compact countable subset of $\mathbb{R}$ with $|X| = \aleph_0$ and $\dim_B X = 1$, then X is not an attractor of any iterated function system.

CHAPTER 3

THE STRUCTURE OF THE SETS OF ALL ATTRACTORS AND NON-ATTRACTORS

3.1. Defining Sets of Iterated Function Systems and Attractors

In this dissertation, we are concerned about the topological structure and dimension of the sets of attractors and non-attractors of iterated function systems. To that end, a few definitions are in order.

Let $\mathcal{C}$ be a collection of contraction maps from X into X. By the set IFS$(X, \mathcal{C})$ we mean the set of all iterated function systems on X consisting of finitely many contractions all of which belong to $\mathcal{C}$. In this setting, if $\Phi \in \text{IFS}(X, \mathcal{C})$, then $\Phi = \{X; \varphi_i : i \in A\}$ for some finite alphabet A, and we can define the function $\hat{\Phi} : \mathcal{K}(X) \to \mathcal{K}(X)$ by

$$\hat{\Phi}(K) = \bigcup_{i \in A} \varphi_i(K),$$

that is $\hat{\Phi}$ is the Hutchinson operator for Φ. Now define

$$\text{ATT}(X, \mathcal{C}) := \{\mathcal{J} \in \mathcal{K}(X) : \exists \Phi \in \text{IFS}(X, \mathcal{C}) \text{ so that } \hat{\Phi}(\mathcal{J}) = \mathcal{J}.\}$$

If $n \in \mathbb{N}$, then we define IFS$(X, \mathcal{C}, n)$ as the set of all iterated function systems on X which consist of exaclty n contraction maps, all of which belong to $\mathcal{C}$. We define ATT$(X, \mathcal{C}, n)$ in a similar manner as before.

Finally, let $0 \leq \varepsilon < s < 1$, by the set IFS$(X, \mathcal{C}, n, \varepsilon, s)$ we mean the set of all iterated function systems on X which consist of exaclty n contraction maps all belonging to $\mathcal{C}$, and whose contraction factors are uniformly bounded below by ε and above by s. ATT$(X, \mathcal{C}, n, \varepsilon, s)$ is defined in a similar fashion as ATT$(X, \mathcal{C})$.

Note that for any collection of contraction maps $\mathcal{C}$, we have

$$\mathrm{IFS}(X,\mathcal{C},n,\varepsilon,s) \subset \mathrm{IFS}(X,\mathcal{C},n) \subset \mathrm{IFS}(X,\mathcal{C}),$$

and therefore

$$\mathrm{ATT}(X,\mathcal{C},n,\varepsilon,s) \subset \mathrm{ATT}(X,\mathcal{C},n) \subset \mathrm{ATT}(X,\mathcal{C}).$$

Also note that

$$\mathrm{IFS}(X,\mathcal{C},n) = \bigcup_{k\geq 3} \mathrm{IFS}\left(X,\mathcal{C},n,0,1-\frac{1}{k}\right),$$

and

$$\mathrm{IFS}(X,\mathcal{C}) = \bigcup_{n\geq 1} \mathrm{IFS}(X,\mathcal{C},n),$$

so that

$$\mathrm{IFS}(X,\mathcal{C}) = \bigcup_{n\geq 1}\left[\bigcup_{k\geq 3} \mathrm{IFS}\left(X,\mathcal{C},n,0,1-\frac{1}{k}\right)\right].$$

Similar equalities hold for $\mathrm{ATT}\left(X,\mathcal{C},n,0,1-\frac{1}{k}\right)$, $\mathrm{ATT}(X,\mathcal{C},n)$, and $\mathrm{ATT}(X,\mathcal{C})$.

3.2. Topological Properties of $\mathrm{IFS}(X,\mathcal{C})$ and $\mathrm{ATT}(X,\mathcal{C})$

In this section, we are going to demonstrate several topological properties of the sets $\mathrm{IFS}(X,\mathcal{C})$ and $\mathrm{ATT}(X,\mathcal{C})$. However, before we can continue on with this section, we must first state an important theorem from general topology. The proof of this theorem is omitted, and can be found in any topology book.

Theorem 3.1 (Arzela-Ascoli). *Let $X \subset \mathbb{R}^n$ be compact. If a sequence $\{f_n\}_{n\geq 1}$ in $C(X)$ is bounded and equicontinuous then it has a uniformly convergent subsequence.*

For the remained of this chapter let $\mathrm{IFS}(X)$ denote the set of all iterated function systems on X, and let

$$\mathrm{ATT}(X) := \{\mathcal{J} \in \mathcal{K}(X) : \exists \Phi \in \mathrm{IFS}(X) \text{ so that } \hat{\Phi}(\mathcal{J}) = \mathcal{J}\}.$$

We now wish to show that the set $\mathrm{ATT}(X)$ is a dense F_σ set while the set $\mathcal{K}(X)\setminus \mathrm{ATT}(X)$ is a dense G_δ set. This would say that the set $\mathcal{K}(X)\setminus\mathrm{ATT}(X)$ is topologically large in the space $\mathcal{K}(X)$. The next lemma states an important fact about convergent subsequences between two sequences, and will be used in the next theorem.

LEMMA 3.2. *Let X be a compact metric space and let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences of points from X. Then there is some sequence of natural numbers $n_1, n_2, n_3, \ldots$ so that both $\{X_{n_i}\}_{i=1}^{\infty}$ and $\{y_{n_1}\}_{i=1}^{\infty}$ converge.*

Note that the previous lemma can be extended, by mathematical induction, to any finite collection of sequences.

THEOREM 3.3. *Let X be a compact metric space. If $\mathcal{A}$ is the collection of all contractions on X, then the set $\mathrm{ATT}(X, \mathcal{A}, n, \varepsilon, s)$ is closed in $\mathcal{K}(X)$ for any $0 \leq \varepsilon < s < 1$.*

PROOF. Let $0 \leq \varepsilon < s < 1$ and take a sequence $\{\mathcal{J}_i\}_{i \geq 1}$ in $\mathrm{ATT}(X, \mathcal{A}, n, \varepsilon, s)$ which converges, in the Hausdorff metric, to the set $\mathcal{J}$, and let

$$\Phi_i = \{X; \varphi_{i,1}, \varphi_{i,2}, \ldots, \varphi_{i,n}\} \in \mathrm{IFS}(X, \mathcal{A}, n, \varepsilon, s)$$

so that $\hat{\Phi}_i(\mathcal{J}_i) = \mathcal{J}_i$. Now note that for each $j \in \{1, 2, \ldots, n\}$, the sequence $\{\varphi_{i,j}\}_{i \geq 1}$ is bounded and equicontinuous. Thus, by Theorem 3.1, this sequence has a uniformly convergent subsequence. Thus, by the previous lemma, there is a sequence of natural numbers $k_1, k_2, \ldots$ so that for every $j \in \{1, 2, \ldots, n\}$, the subsequence $\{\varphi_{k_i,j}\}_{i=1}^{\infty}$ converges uniformly to φ_j. Note that as φ_j is a uniform limit of uniformly bounded contractions, φ_j is also a contraction whose contraction factor is bounded below by ε and above by s. Thus we may consider the iterated function system

$$\Phi = \{X; \varphi_1, \varphi_2, \ldots, \varphi_n\} \in \mathrm{IFS}(X, \mathcal{A}, n, \varepsilon, s).$$

We now show that $\mathcal{J}$ is the attractor of the iterated function system Φ. To this end, note that

$$\begin{aligned} \mathcal{J} &= \lim_{i \to \infty} \mathcal{J}_{k_i} \\ &= \lim_{i \to \infty} \bigcup_{j=1}^{n} \varphi_{k_i,j}(\mathcal{J}_{k_i}) \end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{j=1}^{n} \lim_{i\to\infty} \varphi_{k_i,j}(\mathcal{J}_{k_i}) \\
&= \bigcup_{j=1}^{n} \varphi_j(\mathcal{J}) \\
&= \hat{\Phi}(\mathcal{J}).
\end{aligned}
$$

□

COROLLARY 3.4. *The set* ATT(X) *is an* F_σ *set in* $\mathcal{K}(X)$.

PROOF. Since

$$
\mathrm{ATT}(X) = \bigcup_{n\in\mathbb{N}} \left[\bigcup_{k\geq 3} \mathrm{ATT}\left(X, \mathcal{A}, n, 0, 1 - \frac{1}{K} \right) \right],
$$

it is a countable union of closed sets, and is therefore an F_σ set. □

COROLLARY 3.5. *The set* $\mathcal{K}(X) \setminus \mathrm{ATT}(X)$ *is a* G_δ *set.*

LEMMA 3.6. *Let* $J \in \mathrm{ATT}(X)$ *and suppose* $J = J_1 \cup J_2$, *where* $J_1 \cap J_2 = \emptyset$. *Suppose further that there exists a Lipschitz map* $g : X \to X$ *such that* $g(J_1) = J_2$. *Then* $J_1 \in \mathrm{ATT}(X)$.

PROOF. First, since $J \in \mathrm{ATT}(X)$ let $\Phi = \{X; \varphi_1, \varphi_2, \ldots, \varphi_m\} \in \mathrm{IFS}(X)$ so that $\hat{\Phi}(J) = J$ and set $A = \{1, 2, \ldots, m\}$. Now let $n \in \mathbb{N}$ be large enough so that for all ω in A^* with $|\omega| > n$, we have $\mathrm{diam}\,(\varphi_\omega(J)) < \mathrm{dist}\,(J_1, J_2)$. Set $A_1 := \{\omega \in A^n : \varphi_\omega(J) \cap J_1 \neq \emptyset\}$. Thus for any $\omega \in A_1$ we have $\varphi_\omega(J) \subset J_1$. Now consider the collection of functions

$$
\{\varphi_\omega : \omega \in A_1\} \cup \{\varphi_\omega \circ g : \omega \in A_1\}.
$$

Recall that the lipschitz constant of a composition of lipschitz maps is less than or equal to the product of the two lipschitz contants. Therefore, we may, if necessary, increase the size of our original n to ensure that this collection of functions is a collection of contractions.

It remains to show that J_1 is an attractor of some iterated function system. To this end, note that

$$\begin{aligned}\left(\bigcup_{\omega\in A_1}\varphi_\omega(J_1)\right)\cup\left(\bigcup_{\omega\in A_1}\varphi_\omega\circ g(J_1)\right) &= \left(\bigcup_{\omega\in A_1}\varphi_\omega(J_1)\right)\cup\left(\bigcup_{\omega\in A_1}\varphi_\omega(J_2)\right)\\ &= \bigcup_{\omega\in A_1}\varphi_\omega(J)\\ &= J_1.\end{aligned}$$

Thus J_1 is the attractor of an iterated function system, whose contractions are described as above. □

THEOREM 3.7. *The sets* $\mathrm{ATT}(X)$ *and* $\mathcal{K}(X)\setminus\mathrm{ATT}(X)$ *are both dense in* $\mathcal{K}(X)$.

PROOF. The collage theorem from chapter 1 shows that $\mathrm{ATT}(X)$ is dense in $\mathcal{K}(X)$.

To show that $\mathcal{K}(X)\setminus\mathrm{ATT}(X)$ is dense in $\mathcal{K}(X)$, we rely on the previous lemma. Let $A\in\mathcal{K}(X)$ and let $\varepsilon>0$. Since the collection of finite sets is dense in $\mathcal{K}(X)$, we may let $D\in\mathcal{K}(X)$ be finite so that $d_H(A,D)<\varepsilon/2$. Now let $d\in D$ and let β be an arc in $\mathbb{R}^m$ with endpoints $a<d$ which satisfies

(i) $V(\beta_x^y)<\infty$ whenever $x,y\in\beta$ with $x,y\neq d$,

(ii) $V(\beta_x^d)=\infty$ whenever $x\in\beta$ with $x\neq d$, and

(iii) $d_H(\beta,D)<\varepsilon/2$.

Thus $\beta\in\mathcal{K}(X)\setminus\mathrm{ATT}(X)$. Now, for every point $a\in D\setminus\{d\}$, let $f_a:X\to X$ be a Lipschitz map so that $f_a(d)=a$, $d_H(f_a(\beta),D)<\varepsilon/2$, and $f_a(\beta)\cap f_b(\beta)=\emptyset$ whenever $a\neq b$. Thus, by the previous lemma, we can conclude that $W=\beta\cup\bigcup_{a\in D\setminus\{d\}}f_a(\beta)$ is also in $\mathcal{K}(X)\setminus\mathrm{ATT}(X)$. Finally, the triangle inequality yields

$$\begin{aligned}d_H(W,A) &\leq d_H(W,D)+d_H(D,A)\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\ &= \varepsilon\end{aligned}$$

Therefore, $\mathcal{K}(X)\setminus\mathrm{ATT}(X)$ is dense in $\mathcal{K}(X)$. □

Now we turn our attention to attractors of iterated function systems whose contraction maps are all injective. Iterated function systems which are composed of only injective contractions will be called injective iterated function systems, and as we shall soon see, attractors of such systems must satisfy a very strong topological property.

THEOREM 3.8. *If $A \in \mathrm{ATT}(X, \mathcal{I})$, where $\mathcal{I}$ is the set of all injective contractions from X into X, with $|A| \geq 2$, then A is perfect.*

PROOF. Suppose A has an isolated point x, and suppose A is the attractor of

$$\{X; \varphi_1, \varphi_2, \dots, \varphi_n\} \in \mathrm{IFS}(X, \mathcal{I}).$$

Note that as x is an isolated point of A, the set $\{x\}$ is open in A.

Now let $\omega \in \{1, 2, \dots, n\}^\infty$ be so that $\pi(\omega) = x$, but then

$$\bigcap_{i=1}^{\infty} \varphi_{\omega|_i}(A) = \{x\}.$$

Thus, for some k large enough, $\varphi_{\omega|_k}(A) = \{x\}$. Therefore $\varphi_{\omega|_k}|_A$ is the constant map, and hence $\varphi_{\omega|_k}$ is not an injective map. But $\varphi_{\omega|_k}$ is a finite composition of injective maps, and is therefore injective, this is a contradiction. Therefore A can have no isolated points, i.e., it is perfect. $\square$

This theorem discounts a large class of sets which can be attractors of injective iterated function systems, namely any set which is not perfect must be an element of $\mathcal{K}(X) \setminus \mathrm{ATT}(X, \mathcal{I})$. Since every perfect subset if $\mathbb{R}^n$ is uncountable, we have the following corollary:

COROLLARY 3.9. *If J is a countable subset of $\mathbb{R}^n$ with $|J| \geq 2$, then J is not an attractor of any injective iterated function system.*

Now we shift over to finding sets which are homeomorphic to an attractor of some iterated function system. To this end, recall the sets

$$\mathcal{J}^{(1)} := \{0\} \cup \left(\bigcup_{i=0}^{\infty} \left[\frac{1}{2^{2i+1}}, \frac{1}{2^{2i}} \right] \right),$$

and, for $n \geq 2$, we set

$$\mathcal{J}^{(n)} := \mathcal{J} \cup \bigcup_{m=1}^{n-1} \mathcal{J}_m,$$

where

$$\mathcal{J}_m := \{2^{2m-1}\} \cup \bigcup_{i=1}^{\infty} \left[\frac{1+2^{2i+1}}{2^{2(i-m)+2}}, \frac{1+2^{2i}}{2^{2(i-m)+1}} \right].$$

from Chapter 2. Furthermore for any non-empty compact perfect set $X \subset \mathbb{R}$; denote by D the set of all points in X which are degenerate connected components of X, i.e., if $x \in D$, then the set $\{x\} \subset X$ is a connected component of X. Also, denote by L the set of all points in X where local connectedness fails, i.e., if $x \in L$, then there exists an open set V such that if U is an open set satisfying $x \in U \subset V$, then U is not connected. We now wish to show that if X is a compact subset of $\mathbb{R}$ for which $D = L$ and $0 \leq |D| < \infty$, then X is homeomorphic to an attractor of some iterated function system. In fact,

Theorem 3.10. *Let $X \subset \mathbb{R}$ be non-empty compact perfect set. Suppose further that $D = L$ and that $|D| = n$ where $n \in \mathbb{N}$. Then X is homeomorphic to $\mathcal{J}^{(n)}$. If $|D| = 0$, then X is a finite union of closed intervals.*

Proof. Clearly, if $D = L$ and $|D| = 0$, then X is a finite union of closed intervals, and is therefore an attractor of some iterated function system.

We will proceed by mathematical induction to prove the remainder of the theorem.

Suppose $|D| = 1$. First, map the single element $x \in D$ to 0. Now note that $X \setminus \{x\}$ must be a countable disjoint union of closed intervals. These intervals can be homeomorphically mapped to the closed intervals in $\mathcal{J}^{(1)}$ in such a way as to ensure that this map together with the map $x \mapsto 0$ is a homeomorphism. Therefore X is homeomorphic to $\mathcal{J}^{(1)}$.

Now assume that if $|D| = n-1$, then X is homeomorphic to $\mathcal{J}^{(n-1)}$. We now wish to show that if $|D| = n$, then X is homeomorphic to $\mathcal{J}^{(n)}$. To this end, let $x \in D$ and let $\{I_i\}_{i\in\mathbb{N}}$ be a collection of closed intervals which are all contained in some open interval centered at x which contains no other points of D. Then

$$X' := X \setminus \left(\{x\} \cup \bigcup_{i\in\mathbb{N}} I_i \right)$$

is a non-empty compact perfect subset of $\mathbb{R}$ with $|D| = n - 1$. Therefore, by our inductive hypothesis, X' is homeomorphic to $\mathcal{J}^{(n-1)}$. But we also have that the set

$$\left(\{x\} \cup \bigcup_{i\in\mathbb{N}} I_i \right)$$

is homeomorphic to $\mathcal{J}^{(1)}$ which is homeomorphic to $\mathcal{J}_{n-1}$. Thus we can combine the homeomorphisms so that we get a homeomorphism from X to $\mathcal{J}^{(n)}$. $\square$

CHAPTER 4

DIMENSIONAL PROPERTIES OF SETS OF ATTRACTORS

In this chapter we investigate dimensional properties of

$$\mathrm{ATT}([0,1],\mathcal{S},n) \text{ and } \mathrm{ATT}([0,1],\mathcal{S}),$$

where $\mathcal{S}$ is the collection of all contractive similarity maps mapping $[0,1]$ in itself, i.e., $\varphi \in \mathcal{S}$ if and only if

(i) $\varphi : [0,1] \to [0,1]$,

(ii) φ is a contraction, and

(iii) $\varphi(x) = mx + b$ for some $m, b \in \mathbb{R}$.

We will show that $\mathrm{ATT}([0,1],\mathcal{S})$ is a strongly countable-dimensional space and if it has small transfinite dimension, then its small transfinite dimension must be strictly greater than ω_0.

For this chapter, we will need to place a metric structure on the set $\mathrm{IFS}(X,\mathcal{C},n)$. We let $\rho : \mathrm{IFS}(X,\mathcal{C},n) \times \mathrm{IFS}(X,\mathcal{C},n) \to [0,+\infty)$ be defined as

$$\rho(\Phi,\Psi) = \max_{1\le i\le n}\left\{\sup_{x\in X} d(\varphi_i(x),\psi_i(x))\right\}.$$

Under this definition $(\mathrm{IFS}(X,\mathcal{C},n),\rho)$ is a metric space. We give $\mathrm{IFS}(X,\mathcal{C},n)$ the topology generated by this metric.

4.1. The Small Inductive Dimension of $\mathrm{ATT}([0,1],\mathcal{S},n)$

Before we can discuss the small inductive dimension of $\mathrm{ATT}([0,1],\mathcal{S},n)$ we must first evaluate $\operatorname{ind}\mathrm{ATT}([0,1],\mathcal{S},n,\varepsilon,s)$. Our first lemma tells that the map which takes a uniformly bounded iterated function system to its attractor is Lipschitz continuous. This allows us to use results about Hausdorff dimensions to easily calculate $\operatorname{ind}\mathrm{ATT}([0,1],\mathcal{S},n,\varepsilon,s)$.

LEMMA 4.1. *Let $0 \leq \varepsilon < s < 1$. The map $P : \mathrm{IFS}(X, \mathcal{C}, n, \varepsilon, s) \to \mathrm{ATT}(X, \mathcal{C}, n, \varepsilon, s)$ defined by $P(\Phi) = \mathcal{J}_\Phi$ is Lipschitz continuous with Lipschitz constant $(1-s)^{-1}$.*

PROOF. Let $\Phi, \Psi \in \mathrm{IFS}(X, \mathcal{C}, n, \varepsilon, s)$ and let $\omega \in \{1, 2, \ldots, n\}^\infty$. Define $D_n = ||\varphi_{\omega|_n} - \psi_{\omega|_n}||$. Then

$$\begin{aligned}
|\varphi_{\omega|_{n+1}}(x) - \psi_{\omega|_{n+1}}(x)| &= |\varphi_{\omega|_n}(\varphi_{\omega_{n+1}}(x)) - \psi_{\omega|_n}(\psi_{\omega_{n+1}}(x))| \\
&\leq |\varphi_{\omega|_n}(\varphi_{\omega_{n+1}}(x)) - \psi_{\omega|_n}(\varphi_{\omega_{n+1}}(x))| \\
&\qquad + |\psi_{\omega|_n}(\varphi_{\omega_{n+1}}(x)) - \psi_{\omega|_n}(\psi_{\omega_{n+1}}(x))| \\
&\leq D_n + |\psi_{\omega|_n}(\varphi_{\omega_{n+1}}(x)) - \psi_{\omega|_n}(\psi_{\omega_{n+1}}(x))| \\
&\leq D_n + s^n |\varphi_{\omega_{n+1}}(x) - \psi_{\omega_{n+1}}(x)| \\
&\leq D_n + s^n \rho(\Phi, \Psi).
\end{aligned}$$

Therefore we have $D_1 = \rho(\Phi, \Psi)$ and, by induction,

$$\begin{aligned}
D_k &\leq \rho(\Phi, \Psi) \sum_{j=0}^{k-1} s^j \\
&\leq (1-s)^{-1} \rho(\Phi, \Psi).
\end{aligned}$$

Therefore, $|\pi_\varphi(\omega) - \pi_\psi(\omega)| \leq (1-s)^{-1}\rho(\Phi, \Psi)$. Thus, $d_H(\mathcal{J}_\Phi, \mathcal{J}_\Psi) \leq (1-s)^{-1}\rho(\Phi, \Psi)$. □

Recall from Chapter 1 that the Hausdorff dimension of a space is greater than or equal to the Hausdorff dimension of any Lipschitz image of that space, also recall that the small inductive dimension of a space is less than or equal to its Hausdorff dimension. Using these two results, we are able to prove

LEMMA 4.2. *For any $0 \leq \varepsilon < s < 1$, we have $\operatorname{ind} \mathrm{ATT}([0,1], \mathcal{S}, n, \varepsilon, s) = 2n$.*

PROOF. First note that

$$\operatorname{ind} \mathrm{ATT}([0,1], \mathcal{S}, n, \varepsilon, s) \leq \dim_H \mathrm{ATT}([0,1], \mathcal{S}, n, \varepsilon, s)$$

$$\leq \dim_H \mathrm{IFS}([0,1], \mathcal{S}, n, \varepsilon, s)$$

$$\leq 2n,$$

where the last inequality is true because $\mathrm{IFS}([0,1], \mathcal{S}, n, \varepsilon, s)$ is homeomorphic to a subset of $\mathbb{R}^{2n}$.

Now let $\Delta_1, \Delta_2, \ldots, \Delta_n$ be pairwise disjoint closed subintervals of $[0,1]$ so that $\sup \Delta_k < \inf \Delta_{k+1}$ and let IFS^* be the subset of $\mathrm{IFS}([0,1], \mathcal{S}, n, \varepsilon, s)$ so that if

$$\Phi = \{[0,1]; \varphi_1, \varphi_2, \ldots, \varphi_n\} \in \mathrm{IFS}^*,$$

then $\varphi_i(x) = m_i x + b_i$ where

(i) $m_i > 0$ for each $1 \leq i \leq n$,

(ii) $b_i > 0$ for each $1 \leq i \leq n$, and

(iii) $\varphi_i(X) \subset \Delta_i$ for $1 \leq i \leq n$,

also let $\mathcal{J} \in \mathrm{ATT}^*$ if and only if there exists $\Phi \in \mathrm{IFS}^*$ so that $\hat{\Phi}(\mathcal{J}) = \mathcal{J}$.

We now show that $P|_{\mathrm{IFS}^*} : \mathrm{IFS}^* \to \mathrm{ATT}^*$ is a bijection, and therefore we may conclude that IFS^* and ATT^* are homeomorphic since IFS^* is compact. First of all, $P|_{\mathrm{IFS}^*}$ is clearly a surjection, thus it only remains to show that it is injective. To this end, let $\Phi = \{[0,1], \varphi_1, \ldots, \varphi_n\}, \Psi = \{[0,1]; \psi_1, \ldots, \psi_n\} \in \mathrm{IFS}^*$ with $\Phi \neq \Psi$. For each $1 \leq i \leq n$, consider the sets

$$\{\inf \varphi_i(\mathcal{J}_\Phi), \sup \varphi_i(\mathcal{J}_\Phi)\} \text{ and } \{\inf \psi_i(\mathcal{J}_\Psi), \sup \psi_i(\mathcal{J}_\Psi)\}.$$

It is clear that the fixed point of φ_1 is equal to $\inf \varphi_1(\mathcal{J}_\Phi)$ and the fixed point of φ_n is equal to $\sup \varphi_n(\mathcal{J}_\Phi)$. Similar conclusions can be made about about the fixed points for ψ_1 and ψ_n. Note that if $\inf \varphi_1(\mathcal{J}_\Phi) \neq \inf \psi_i(\mathcal{J}_\Psi)$ then $\mathcal{J}_\Phi \neq \mathcal{J}_\Psi$, also if $\sup \varphi_n(\mathcal{J}_\Phi) \neq \sup \psi_n(J_\Psi)$ then $\mathcal{J}_\Phi \neq \mathcal{J}_\Psi$.

Let x_1 be the fixed point of φ_1 and x_n be the fixed point of φ_n, and note that $\inf \varphi_i(\mathcal{J}_\Phi) = \varphi_i(x_1)$, $\sup \varphi_i(\mathcal{J}_\Phi) = \varphi_i(x_n)$. Letting y_1 be the fixed point of ψ_1 and y_n be the fixed point of ψ_n, similar conclusions can be drawn for ψ_i. Now suppose, by way of contradiction, that $\mathcal{J}_\Phi = \mathcal{J}_\Psi$. Therefore it must be the case that

$$\{x_1, \varphi_2(x_1), \ldots, \varphi_n(x_1)\} = \{y_1, \psi_2(y_1), \ldots, \psi_n(y_n)\},$$

and

$$\{\varphi_1(x_n), \varphi_2(x_n), \ldots, x_n\} = \{\psi_1(y_n), \psi_2(y_n), \ldots, y_n\}.$$

Therefore $x_1 = y_1$, $x_n = y_n$, $\varphi_i(x_1) = \psi_i(y_i)$, and $\varphi_i(x_n) = \varphi_i(x_n)$ for each $1 \leq i \leq n$. This can easily be seen from property (iii) from above. However, since each φ_i and ψ_i is of the form $mx + b$, we conclude that $\varphi_i = \psi_i$ for each $1 \leq i \leq n$, a contradiction as $\Phi \neq \Psi$. Therefore $P|_{\mathrm{ATT}^*}$ is injective.

Since we now have that IFS^* is homeomorphic with ATT^*, we have $\operatorname{ind}\mathrm{ATT}^* = 2n$. Thus we conclude that $2n \leq \operatorname{ind}\mathrm{ATT}([0,1], \mathcal{S}, n, \varepsilon, s) \leq 2n$. Therefore

$$\operatorname{ind}\mathrm{ATT}([0,1], \mathcal{S}, n, \varepsilon, s) = 2n.$$

□

Before we prove that $\operatorname{ind}\mathrm{ATT}([0,1], \mathcal{S}, n) = 2n$, we must state a short lemma and then a theorem from the theory of inductive dimensions. The proof of the lemma is omitted here, however the proof of the theorem will be given. The proof of the following lemma and theorem can be found in [2].

Lemma 4.3. *If a separable metric space X can be represented as the union of two subspaces Y and Z such that* $\operatorname{ind} Y \leq n - 1$ *and* $\operatorname{ind} Z \leq 0$, *then* $\operatorname{ind} X \leq n$.

Theorem 4.4 (The Sum Theorem). *If a separable metric space X can be written as the union of a sequence $F_1, F_2, F_3, \ldots$ of closed subspaces such that* $\operatorname{ind} F_i \leq n$ *for* $i = 1, 2, \ldots$, *then* $\operatorname{ind} X \leq n$.

Proof. We will prove this theorem by induction on n. The proof that this theorem is true when $n = 0$ is a result from general topology and can be found in [7]. Now assume that the theorem holds for spaces whose dimension is less than n, and consider a space $X = \bigcup_{i=1}^{\infty} F_i$, where each F_i is closed and $\operatorname{ind} F_i \leq n$ for $n \geq 1$. Now, by theorem 1.6, for $i = 1, 2, \ldots$ choose a countable base $\mathcal{B}_i$ for the space F_i such that $\operatorname{ind} \operatorname{Fr} U \leq n-1$ for every $U \in \mathcal{B}$, where Fr denotes the boundary operator in the space F_i. By the inductive hypothesis, the subspace

$$Y = \bigcup \left\{ \operatorname{Fr} U : U \in \bigcup_{i=1}^{\infty} \mathcal{B}_i \right\}$$

of X satisfies the inequality $\operatorname{ind} Y \leq n-1$. By the definition of a 0-dimensional space, we have that the space $Z_i = F_i \setminus Y$ satisfies the inequality $\operatorname{ind} Z_i \leq 0$; hence by the case of this theorem for $n = 0$, we have the subspace $Z = \bigcup_{i=1}^{\infty} Z_i = X \setminus Y$ of X also satisfies the inequality $\operatorname{ind} Z \leq 0$, since $Z_i = F_i \setminus Y = F_i \cap Z$ which is closed in Z. Thus by the previous lemma, we have $\operatorname{ind} X \leq n$. □

Theorem 4.5. *For any $n \in \mathbb{N}$, we have*

$$\operatorname{ind} \operatorname{ATT}([0,1], \mathcal{S}, n) = 2n.$$

Proof. Since

$$\operatorname{ATT}([0,1], \mathcal{S}, n) = \bigcup_{k=3}^{\infty} \operatorname{ATT}\left([0,1], \mathcal{S}, n, 0, 1 - \frac{1}{k}\right)$$

is a countable union of closed subsets each with small inductive dimension $2n$, we conclude that $\operatorname{ind} \operatorname{ATT}([0,1], \mathcal{S}, n) \leq 2n$ by the Sum Theorem. However, since

$$\operatorname{ATT}\left([0,1], \mathcal{S}, n, 0, 1 - \frac{1}{k}\right) \subset \operatorname{ATT}([0,1], \mathcal{S}, n),$$

we have that

$$\operatorname{ind} \operatorname{ATT}([0,1], \mathcal{S}, n) \geq \operatorname{ind} \operatorname{ATT}\left([0,1], \mathcal{S}, n, 0, 1 - \frac{1}{k}\right) = 2n$$

for all $k \geq 3$. Therefore $\operatorname{ind} \operatorname{ATT}([0,1], \mathcal{S}, n) = 2n$. □

4.2. The Small Transfinite Inductive Dimension of $\mathrm{ATT}([0,1],\mathcal{S})$

Since

$$\mathrm{ATT}([0,1],\mathcal{S}) = \bigcup_{n\geq 1} \mathrm{ATT}([0,1],\mathcal{S},n),$$

we see that $\mathrm{ATT}([0,1],\mathcal{S})$ is a countable union of closed sets each with finite small inductive dimension, and therefore $\mathrm{ATT}([0,1],\mathcal{S})$ is strongly countable-dimensional. However, this does not imply that $\mathrm{ATT}([0,1],\mathcal{S})$ has small transfinite dimension. We now wish to show that if $\mathrm{ATT}([0,1],\mathcal{S})$ has small transfinite dimension, then

$$\operatorname{trind} \mathrm{ATT}([0,1],\mathcal{S}) > \omega_0.$$

An interesting question that arises from this is: If $\mathrm{ATT}([0,1],\mathcal{S})$ has small transfinite dimension, then what is $\operatorname{trind} \mathrm{ATT}([0,1],\mathcal{S})$?

The Mazurkiewicz theorem for $\mathbb{R}^n$ states that any set $K \subset \mathbb{R}^n$ which satisfies the inequality $\operatorname{ind} K \leq n-2$ can not cut $\mathbb{R}^n$. This result is well known and can be found, for example, in [2]. The following lemma and Theorem generalize the Mazurkiewicz theorem to n-mainfolds.

Lemma 4.6. *Let M_n be an n-manifold. If $K \subset M_n$ satisfies the inequality $\operatorname{ind} K \leq n-1$, then K has empty interior.*

Proof. Suppose $\operatorname{Int} K \neq \emptyset$. Thus there exists $x \in K$ and an open set U so that $x \in U \subset K$. But then there exists an open set V so that V is homeomorphic to $\mathbb{R}^n$ and $x \in V \subset U \subset K$. Therefore $\operatorname{ind} K \geq \operatorname{ind} V = n$, and hence $\operatorname{ind} K = n$. □

Theorem 4.7 (Mazurkiewicz Theorem for n-manifolds). *Let M_n ba an n-manifold and let G be a region in M_n. If $K \subset G$ satisfies the inequality $\operatorname{ind} K \leq n-2$, then K does not cut G.*

Proof. Let $x, y \in G \setminus K$, where $K \subset M_n$ satisfies $\operatorname{ind} K \leq n-2$. Let $R_1, R_2, \ldots, R_\ell$ be a sequence of open sets in M_N satisfying

(i) R_i is homeomorphic to $\mathbb{R}^n$ for each i.

(ii) $x \in R_1$ and $y \in R_\ell$.

(iii) $R_i \cap R_{i+1} \neq \emptyset$ for $i = 1, 2, \ldots, \ell - 1$.

(iv) $R_i \subset G$ for each i.

Such a sequence is possible as G is connected and the open subsets of M_n which are homeomorphic to $\mathbb{R}^n$ form a basis for the topology on M_n.

By the previous lemma, K has an empty interior. Thus there exists a point

$$z_i \in (R_i \cap R_{i+1}) \setminus K,$$

for $i = 1, 2, \ldots, \ell - 1$. For convenience, let $z_0 = x$ and $z_\ell = y$. Now, by the Mazurkiewicz Theorem for $\mathbb{R}^n$, there is a continuum $C_i \subset R_i \setminus K$ which contains z_{i-1} and z_i. Thus the union

$$C = \bigcup_{i=1}^{\ell} C_i \subset G \setminus K$$

is a continuum which contains x and y. □

THEOREM 4.8. *If* ATT$([0,1], \mathcal{S})$ *has small transfinite dimension, then*

$$\operatorname{trind} \mathrm{ATT}([0,1], \mathcal{S}) > \omega_0.$$

PROOF. Suppose ATT$([0,1], \mathcal{S})$ has small transfinite dimension and let

$$\{[0,1]; \varphi_1, \varphi_2\} \in \mathrm{IFS}^*,$$

and note that the iterated function system $\{[0,1]; \varphi_\omega : \omega \in \{1,2\}^k\}$ is also in IFS* and these two iterated functions systems have the same attractor.

Now let $\omega \in \{1,2\}^k$ and let $t : \{1,2\}^k \to [0,1]$ and let $\Phi^t \in \mathrm{IFS}^*$ be defined as

$$\Phi^t = \{[0,1]; t(\omega)\varphi_\omega : \omega \in \{1,2\}^k\},$$

and note that the collection of iterated function systems $\{\Phi^t\}_{t \in \mathbb{R}^{\{1,2\}^k}}$ is a 2^k-manifold, and hence the collection of attractors of these iterated function systems would also be a 2^k-manifold. By letting $t_1 \equiv 1$ and $t_0 \equiv 0$, we see that our original iterated function system and the iterated function system whose contractions are all equivalently

0 belong to the above collection of iterated function systems. Hence, our original attractor and $\{0\}$ are among the attractors of this class of iterated function systems. Hence any open set containing our original attractor but not containing $\{0\}$ would cut the space of attractors and by the Mazurkiewicz Theorem for m-manifolds, the inductive dimension of this open set would have to be greater than $2^k - 1$. Therefore, for every k, we have found an open set whose boundary has small transfinite inductive dimension which is greater than $2^k - 1$, and therefore we may conclude that $\operatorname{trind} \mathrm{ATT}([0,1], \mathcal{S}) > \omega_0$. $\square$

4.3. Future Research

In this dissertation we discussed topological properties of sets of attractors and non-attractors. We also discussed dimensional properties of the sets

$$\mathrm{ATT}([0,1], \mathcal{S}, n)$$

and

$$\mathrm{ATT}([0,1], \mathcal{S}).$$

In the future we would like to investigate the same topological and dimensional properties for infinite iterated function systems.

Remaining in the field of finite iterated function systems, we would like to give a classification of which countable subsets of $\mathbb{R}^n$ are elements of $\mathrm{ATT}(X)$ and which are in $\mathcal{K}(X) \setminus \mathrm{ATT}(X)$, and we would like to solve contecture 2.11.

Another issue within finite iterated function systems, we would like address is

Conjecture 4.9. If $J \subset \mathbb{R}^n$ is countable and has infinite Cantor-Bendixson rank, then J is not an attractor of any iterated function system.

If this conjecture turns out to be true, then we would also like to solve the following problem

PROBLEM 4.10. What is the least ordinal α so that if $J \subset \mathbb{R}^n$ is countable and has Cantor-Bedixson rank greater than or equal to α, then J is not an attractor of any iterated function system.

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