THE ANALYTICAL DEVELOPMENT OF THE TRIGONOMETRIC FUNCTIONS

APPROVED:

E. H. Hanson
Major Professor

Carl T. York
Minor Professor

E. H. Hanson
Director of the Department of Mathematics

Jack Johnson
Dean of the Graduate School

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THESIS

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Pearl Cherrington Mackey, B. S.

Denton, Texas

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CHAPTER I

INTRODUCTION

Geometry is the source from which we first draw our knowledge of the trigonometric functions. The foundations of trigonometry are not quite as simple as a beginner might suppose, and the ordinary presentation of the theory rests on certain assumptions which need careful analysis. The most natural method is to follow as closely as we can the procedure of the ordinary textbooks, translating the geometrical language which they use into the language of analysis. Another method is to define the trigonometrical functions by infinite series. In this study, we will begin by defining the function A(x) as the integral $\int_0^x \frac{1}{1+u^2}$, and arrive at the trigonometric functions by inversion. By this process the definitions of the trigonometric functions are separated from geometry, and we will develop their properties also independently of geometry.

In this study, we will assume known the definitions of continuity, differentiability, limits, upper and lower bounds, and monotonic functions.

We will now define the Riemann integral. Let f(x) be a function defined and bounded on $[a,b]$. Let $\sigma: a = x_0 < x_1 < x_2 < \dots\dots < x_n = b$ be a subdivision of $[a,b]$. Then we define $\overline{\sum}\sigma$ to be $\sum_{i=1}^{m} u_i(x_i - x_{i-1})$ and we define $\underline{\sum}\sigma$ to be $\sum_{i=1}^{m} l_i(x_i - x_{i-1})$ where, for each i, u_i and l_i are respectively the least upper bound and the greatest lower bound of f(x) on $[x_{i-1}, x_i]$. Furthermore we define $\overline{\int} f(x)$ to be the greatest lower bound for all σ of $\overline{\sum}\sigma$, and we define $\underline{\int} f(x)$ to be the least upper bound for all σ of $\underline{\sum}\sigma$. If $\underline{\int} f(x) = \overline{\int} f(x)$, then f(x) is said to Riemann integrable on $[a,b]$ and their common value will be called the Riemann definite integral from a to b of f(x) and will be denoted by the symbol

$$\int_a^b f(x).$$

Remark: For any σ, $\overline{\sum}\sigma \geq \underline{\sum}\sigma$.

Extension of the Riemann integral: We will define $\int_a^a f(x)$ to be zero. If $a < b$, $\int_b^a f(x)$ is defined to be $-\int_a^b f(x)$.

We shall now state certain theorems without proof which will be referred to in the following chapters.

Theorem 1.1: A necessary and sufficient condition that a bounded function f(x) be integrable on $[a,b]$ is that for every $e > 0$ there exists a σ such that $\overline{\sum}\sigma - \underline{\sum}\sigma < e$.[1]

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[1] E. W. Hobson, The Theory of Functions of a Real Variable, Vol. I, p. 465.

<u>Theorem 1.2</u>: If f(x) is continuous on $[a,b]$ then f(x) is integrable on $[a,b]$.[2]

<u>Theorem 1.3</u>: If f(x) is bounded and integrable on $[a,b]$, then

$$\int_a^b k\, f(x) = k \int_a^b f(x),$$

where k is a constant.[3]

<u>Theorem 1.4</u>: If f(x) is bounded and integrable on $[a,b]$ and if α, β, γ are any three points in $[a,b]$, then

$$\int_\alpha^\gamma f(x) = \int_\alpha^\beta f(x) + \int_\beta^\gamma f(x).$$ [4]

<u>Theorem 1.5</u>: If f(x) and g(x) are both integrable in $[a,b]$ and $f(x) \geq g(x)$ at every point of $[a,b]$, then

$$\int_a^b f(x) \geq \int_a^b g(x).$$ [5]

<u>Theorem 1.6</u>: If f(x) is continuous on $[a,b]$, then at every point in $[a,b]$, $F(x) = \int_a^x f(x)$ possesses a derivative which is the function f(x).[6]

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[2] E. G. Phillips, <u>A Course of Analysis</u>, p. 173.

[3] <u>Ibid</u>., p. 176. [4] <u>Ibid</u>., p. 176.

[5] <u>Ibid</u>., p. 177. [6] <u>Ibid</u>., p. 180.

Theorem 1.7: If $f(x)$ is continuous on $[a,b]$, then $f(x)$ is bounded on $[a,b]$.[7]

Theorem 1.8: If $f(x)$ is continuous on $[a,b]$, and M and m are its least upper bound and its greatest lower bound respectively, then $f(x)$ assumes each of the values M and m at least once in the interval.[8]

Theorem 1.9: The sum, difference, and product of two continuous functions are themselves continuous. The quotient of two continuous functions is continuous, provided that the denominator remains different from zero.[9]

Theorem 1.10: If $f(x)$ is continuous on $[a,b]$ and k is any number between $f(a)$ and $f(b)$, then $f(x)$ assumes the value k at least once in the interval (a,b).[10]

Theorem 1.11: If the function $f(x)$ is single valued, continuous, and monotonic on $[a,b]$ and $f(a) = \alpha$, $f(b) = \beta$, then $f(x)$ has an inverse function $g(y)$ which is single valued, continuous, and monotonic on $[\alpha, \beta]$.[11]

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[7] G. H. Hardy, A Course in Pure Mathematics, sixth edition, p. 182.

[8] Ibid., p. 183.

[9] R. Courant, Differential and Integral Calculus, Vol. I, new revised edition, p. 54.

[10] Ibid., p. 67. [11] Ibid., p. 68.

<u>Theorem 1.12</u>: If at every point of $[a,b]$, the function $f(x)$ is differentiable and $f'(x) \neq 0$, then the inverse function $g(y)$ also has a deriative at every point of its interval of definition and $f'(x) \cdot g'(y) = 1$ for corresponding values of x and y.[12]

Theorem 1.13: If $f(x)$ is differentiable on $[a,b]$ and continuous at a and at b, then ~~their~~ There exists a point c of (a,b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.[13]

<u>Theorem 1.14</u>: If $f(x)$ is differentiable at $x = c$, then it is continuous at $x = c$.[14]

<u>Theorem 1.15</u>: If $f(x)$ and $g(x)$ are both differentiable, then

(1.) $F(x) = f(x) + g(x)$ is differentiable and $F'(x) = f'(x) + g'(x)$;

(2.) $F(x) = f(x) - g(x)$ is differentiable and $F'(x) = f'(x) - g'(x)$;

(3.) $F(x) = f(x) \cdot g(x)$ is differentiable and $F'(x) = f(x)g'(x) + g(x)f'(x)$;

(4.) $F(x) = \frac{f(x)}{g(x)}$ is differentiable, provided $g(x) \neq 0$, and $F'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{[g(x)]^2}$.[15]

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[12] <u>Ibid</u>., p. 145.

[13] <u>Ibid</u>., p. 103.

[14] <u>Ibid</u>., p. 97.

[15] <u>Ibid</u>., pp. 137-139.

Theorem 1.16: If $f'(x) = g'(x)$ on $[a,b]$, then there exists a constant k such that $f(x) = g(x) + k$ on $[a,b]$.[16]

Theorem 1.17: Every monotone function $f(x)$ on $[a,b]$ is bounded on $[a,b]$.[17]

Theorem 1.18: A necessary and sufficient condition that $f(x)$ be continuous at $x = c$ is that both $f(x)$ and $\lim_{x \to c} f(x)$ exist and the two are equal.[19]

The number p used in this paper may be evaluated to any desired degree of accuracy from its definition, and the number π, which occurs so frequently in mathematical analysis, may be defined to be 2p.

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[16] R. S. Burington and C. C. Torrance, Higher Mathematics, first edition, p. 279.

[17] Ibid., p. 279.

[18] Courant, op. cit., p. 61.

CHAPTER II

THE INTEGRAL OF $\frac{1}{1 + u^2}$

Definition 2.1: $A(x) = \int_0^x \frac{1}{1 + u^2}$.

Theorem 2.1: The function $\frac{1}{1 + u^2}$ is continuous for every real number u.

Since $1 + u^2 \neq 0$ for any real number u, $\frac{1}{1 + u^2}$ is continuous for every real number u from Theorem 1.9.

Theorem 2.2: The function $A(x) = \int_0^x \frac{1}{1 + u^2}$ is defined for every real number x; furthermore A(x) is differentiable and $A'(x) = \frac{1}{1 + x^2}$ for every real number x.

Since $\frac{1}{1 + u^2}$ is continuous on $[0,x]$ for $x > 0$, A(x) exists for all positive x. A(0) exists and is equal to 0. If $x < 0$, $\frac{1}{1 + u^2}$ is continuous on $[x,0]$ and the $\int_x^0 \frac{1}{1 + u^2}$ exists. But $\int_0^x \frac{1}{1 + u^2} = -\int_x^0 \frac{1}{1 + u^2}$.

Let a be any real number and $[r,s]$ an interval containing 0 and a. Since $\frac{1}{1 + u^2}$ is integrable on $[r,s]$ and continuous at a, A'(a) exists and equals $\frac{1}{1 + a^2}$. The theorem follows.

Theorem 2.3: A(x) is continuous and monotone increasing for every real number x.

Since A(x) is differentiable and every differentiable function is continuous, A(x) is continuous.

Let x_1 and x_2 be any two real numbers with $x_2 > x_1$. Then

$$\begin{aligned} A(x\ ) &= \int_0^{x_2} \frac{1}{1+u^2} \\ &= \int_0^{x_1} \frac{1}{1+u^2} + \int_{x_1}^{x_2} \frac{1}{1+u^2} \\ &= A(x_1) + \int_{x_1}^{x_2} \frac{1}{1+u^2}. \end{aligned}$$

Let m = maximum $\left[|x_1| , |x_2| \right]$. Then $\frac{1}{1+m^2} > 0$ and for every $x_1 \leq u \leq x_2$, $\frac{1}{1+u^2} \geq \frac{1}{1+m^2}$. Hence

$$\begin{aligned} \int_{x_1}^{x_2} \frac{1}{1+u^2} &\geq \int_{x_1}^{x_2} \frac{1}{1+m^2} \\ &= \frac{1}{1+m^2} \int_{x_1}^{x_2} 1 \\ &= \frac{x_2 - x_1}{1+m^2} > 0. \end{aligned}$$

Therefore $A(x_2) > A(x_1)$ and A(x) is monotone increasing.

<u>Theorem 2.4</u>: For every real number $x > 0$,

$$A(-x) = -A(x).$$

Let $[r,s]$ be any closed interval with $r > 0$. Since $\frac{1}{1+u^2} = \frac{1}{1+(-u)^2}$, the set of all heights assumed by $\frac{1}{1+u^2}$ in $[r,s]$ is the same set of numbers as the set of all heights assumed by $\frac{1}{1+u^2}$ in $[-s,-r]$. Hence the least upper bound of $\frac{1}{1+u^2}$ on $[r,s]$ equals the least upper bound of $\frac{1}{1+u^2}$ on $[-s,-r]$, and the greatest lower bound of $\frac{1}{1+u^2}$ on $[r,s]$ equals the greatest lower bound of $\frac{1}{1+u^2}$ on $[-s,-r]$. Let $\sigma : 0 = x_0 < x_1 < x_2 < \ldots\ldots\ldots\ldots < x_{n-1} < x_n = x'$ be any subdivision of $[0,x']$ such that $\overline{\sum}\sigma - \underline{\sum}\sigma < e$, $e > 0$ having been arbitrarily chosen in advance. Then

$$\underline{\sum}\sigma \leq \int_0^{x'} \frac{1}{1+u^2} \leq \overline{\sum}\sigma. \tag{1}$$

Corresponding to σ we have the subdivision $\sigma^*: -x' = -x_n < -x_{n-1} < \ldots\ldots\ldots < -x_1 < -x_0 = 0$ of $[-x',0]$. Also for every $0 < i \leq n$, the least upper bound of $\frac{1}{1+u^2}$ on $[x_{i-1}, x_i]$ equals the least upper bound of $\frac{1}{1+u^2}$ on $[-x_i, -x_{i-1}]$, and the greatest lower bound of $\frac{1}{1+u^2}$ on $[x_{i-1}, x_i]$ equals the greatest lower bound of $\frac{1}{1+u^2}$ on $[-x_i, -x_{i-1}]$. It follows that $\overline{\sum}\sigma^* = \overline{\sum}\sigma$, and $\underline{\sum}\sigma^* = \underline{\sum}\sigma$. Since

$$\underline{\sum}\sigma^* \leq \int_{-x'}^{0} \frac{1}{1+u^2} \leq \overline{\sum}\sigma^*,$$

then

$$\underline{\sum}\sigma \leq \int_{-x'}^{0} \frac{1}{1+u^2} \leq \overline{\sum}\sigma. \tag{2}$$

Subtracting (2) from (1), we will have

$$\left| \int_0^{x'} \frac{1}{1 + u^2} - \int_{-x'}^{0} \frac{1}{1 + u^2} \right| < e.$$

But e was arbitrarily chosen; therefore

$$\int_0^{x} \frac{1}{1 + u^2} = \int_{-x}^{0} \frac{1}{1 + u^2} .$$

The theorem follows.

<u>Theorem 2.5</u>: There exists a positive real number p such that p and -p are respectively the least upper bound and the greatest lower bound of A(x) on the continuum. Furthermore $\lim_{x \to +\infty} A(x) = p$, and $\lim_{x \to -\infty} A(x) = -p$.

Let $x > 0$ be arbitrarily chosen and choose an integer $n > x$. Let S be the sum of the convergent series, $\sum_{m=1}^{\infty} \frac{1}{m^2}$, and set $M = 1 + S$. Then

$$\begin{aligned} M &> 1 + \left(1 + \frac{1}{4} + \frac{1}{9} + \cdots\cdots\cdots\cdots + \frac{1}{(n-1)^2}\right) \\ &> 1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \cdots\cdots\cdots\cdots + \frac{1}{1 + (n-1)^2} \\ &= \overline{\sum} \sigma, \end{aligned}$$

where $\sigma : 0 < 1 < 2 < 3 < \cdots\cdots < n - 1 < n$ is a subdivision of $[0,n]$.

We have

$$\overline{\sum} \sigma \geq \int_0^{n} \frac{1}{1 + u^2} = A(n) > A(x).$$

But $A(0) = 0$ and for every $x \leq 0$, $A(x) \leq 0$. Thus M is an upper bound of $A(x)$ on the continuum. Let p equal the least upper bound of $A(x)$ on the continuum. Suppose, if possible, that $A(x_1) < -p$ for some value of x_1. Then $A(-x_1) = -A(x_1) > p$. This contradicts the fact that p is the least upper bound. Hence $-p$ is a lower bound. Suppose next that $-p$ is not the greatest lower bound. Then there exists a $d > 0$ so that $-p + d$ is a lower bound, i. e., for every x that is a real number $A(x) \geq -p + d$. Hence for every x, $A(-x) = -A(x) \leq p - d$ which would make $p - d$ a lower bound contrary to fact. Thus $-p$ is the greatest lower bound of $A(x)$ on the continuum.

Let $e > 0$ be arbitrarily chosen. Since p is the least upper bound of $A(x)$ on the continuum, there exists a real number a such that $A(a) > p - e$. We have for every x, $A(x) \leq p$. Also for every $x > a$, $A(x) > A(a)$. Thus, for every $x > a$, $p - e < A(x) \leq p < p + e$. Hence for every $x > a$, $|p - A(x)| < e$. Therefore $\lim_{x \to +\infty} A(x) = p$. Similarly, $\lim_{x \to -\infty} A(x) = -p$.

<u>Theorem 2.6</u>: For every $x > 0$, $A(1/x) = 2A(1) - A(x)$, and for every $x < 0$, $A(1/x) = -2A(1) - A(x)$.

Let $F(x) = -A(1/x)$. Then for every $c \neq 0$,

$$F'(c) = \frac{1}{1 + 1/c^2} \cdot \frac{1}{c^2} = \frac{1}{1 + c^2} .$$

Let $x_0 > 0$ be arbitrarily chosen. Choose $a > 0$ and $b > 0$ so that $a < x_0 < b$ and $a < 1 < b$. Since $F(x)$ and $A(x)$ have the same derivative at each point in $[a,b]$, there exists a real number k such that $A(x) + A(1/x) = k$ for every x in $[a,b]$. Therefore $A(1) + A(1) = k$ and $k = 2A(1)$. Then for every x in $[a,b]$, $A(x) + A(1/x) = 2A(1)$. Therefore, $A(x_0) + A(1/x_0) = 2A(1)$ and $A(1/x_0) = 2A(1) - A(x_0)$. Since $x_0 > 0$ was arbitrarily chosen, $A(1/x) = 2A(1) - A(x)$ for every $x > 0$.

Let $x_1 < 0$ be arbitrarily chosen. Choose $r < 0$ and $s < 0$ so that $r < x_1 < s$ and $r < -1 < s$. There exists a real number h such that $A(x) + A(1/x) = h$ for every x in $[r,s]$. Therefore $A(-1) + A(-1) = h$ and $h = 2A(-1) = -2A(1)$. Then for every x in $[r,s]$, $A(x) + A(1/x) = -2A(1)$. Therefore, $A(x_1) + A(1/x_1) = -2A(1)$ and $A(1/x_1) = -2A(1) - A(x_1)$. Since $x_1 < 0$ was arbitrarily chosen, $A(1/x) = -2A(1) - A(x)$ for every $x < 0$.

<u>Theorem 2.7</u>: $A(1) = p/2$.

Let $e > 0$ be arbitrarily chosen. Since $A(x)$ is continuous for $x = 0$, there exists a real number $d > 0$ such that, for every x where $|x - 0| < d$, $|A(x) - A(0)| < e$. But $A(0) = 0$; therefore there exists a $d > 0$ such that, for every x where $|x| < d$, $|A(x)| < e$. There exists a real number $M > 0$ such that, for every $x > M$, $|p - A(x)| < e$. Choose $0 < x <$ minimum $(d, 1/M)$. Then $0 < A(x) < e$. Also

$x < 1/M$, $Mx < 1$, $1/x > M$, and $|p - A(1/x)| < e$. Then $p - e < A(1/x) < p$. For every $x > 0$, $A(1/x) = 2A(1) - A(x)$. Thus $2\,A(1) > A(1/x)$ and $p - e < 2A(1) = A(x) + A(1/x) < p + e$. Therefore, for every $e > 0$, $|2A(1) - p| < e$. Hence $2A(1) = p$ and $A(1) = p/2$.

<u>Theorem 2.8</u>: If $x_1 > 0$ and $x_2 < 1/x_1$, or if $x_1 < 0$ and $x_2 > 1/x_1$, $A\left(\frac{x_1 + x_2}{1 - x_1 x_2}\right) = A(x_1) + A(x_2)$; if $x_1 > 0$ and $x_2 > 1/x_1$, $A\left(\frac{x_1 + x_2}{1 - x_1 x_2}\right) = A(x_1) + A(x_2) - 2p$; if $x_1 < 0$ and $x_2 < 1/x_1$, $A\left(\frac{x_1 + x_2}{1 - x_1 x_2}\right) = A(x_1) + A(x_2) + 2p$.

Let $G(x) = A\left(\frac{x_1 + x}{1 - x_1 x}\right)$. $G(x)$ is not defined and hence not differentiable at $x = 1/x_1$. For every $c \neq 1/x_1$,

$$G'(c) = \frac{1}{1 + \left(\frac{x_1 + c}{1 - x_1 c}\right)^2} \cdot \frac{1 + x_1^2}{(1 - x_1 c)^2}$$

$$= \frac{1 + x_1^2}{(1 - x_1 c)^2 + (x_1 + c)^2}$$

$$= \frac{1 + x_1^2}{(1 + x_1^2)\,(1 + c^2)} = \frac{1}{1 + c^2}\,.$$

For every $c \neq 1/x_1$, $G'(c) = A'(c)$. Suppose $a < 0$ and $b > 0$ are chosen in such a way that $[a,b]$ does not include $1/x_1$. Then there exists a real number k such that, for every x in $[a,b]$,

$$A\left(\frac{x_1 + x}{1 - x_1 x}\right) - A(x) = k.$$

Therefore when $x = 0$, $A(x_1) - 0 = k$. Then, for every x

in $[a,b]$,

$$A\left(\frac{x_1 + x}{1 - x_1 x}\right) - A(x) = A(x_1).$$

Case I. Let $x_1 > 0$ and $x_2 < 1/x_1$. We may choose $a < 0$ and $b > 0$ such that $a < x_2 < b < 1/x_1$. Then, for $x = x_2$,

$$A\left(\frac{x_1 + x_2}{1 - x_1 x_2}\right) - A(x_2) = A(x_1)$$

and

$$A\left(\frac{x_1 + x_2}{1 - x_1 x_2}\right) = A(x_1) + A(x_2).$$

Case II. Let $x_1 < 0$ and $x_2 > 1/x_1$. Now choose $a < 0$ and $b > 0$ such that $1/x_1 < a < x_2 < b$. Then, for $x = x_2$,

$$A\left(\frac{x_1 + x_2}{1 - x_1 x_2}\right) - A(x_2) = A\ (x_1)$$

and

$$A\left(\frac{x_1 + x_2}{1 - x_1 x_2}\right) = A(x_1) + A(x_2).$$

Case III. Let $x_1 > 0$ and $x_2 > 1/x_1$. Choose $a < 0$ and $b > 0$ such that $a < -1/x_2 < b < 1/x_1$. Then, using $x = -1/x_2$,

$$A\left(\frac{x_1 x_2 - 1}{x_1 + x_2}\right) + A(1/x_2) = A(x_1).$$

But

$$A\left(\frac{x_1 x_2 - 1}{x + x_2}\right) = p - A\left(\frac{x_1 + x_2}{x_1 x_2 - 1}\right)$$

and

$$-A\left(\frac{x_1 + x_2}{x_1x_2 - 1}\right) = A\left(\frac{x_1 + x_2}{1 - x_1x_2}\right).$$

Also $A(1/x_2) = p - A(x_2)$. Hence

$$p + A\left(\frac{x_1 + x_2}{1 - x_1x_2}\right) + p - A(x_2) = A(x_1)$$

and

$$A\left(\frac{x_1 + x_2}{1 - x_1x_2}\right) = A(x_1) + A(x_2) - 2p.$$

Case IV. Let $x_1 < 0$ and $x_2 < 1/x_1$. Choose $a < 0$ and $b > 0$ such that $1/x_1 < a < -1/x_2 < b$. Then, for $x = -1/x_2$,

$$A\left(\frac{x_1x_2 - 1}{x_1 + x_2}\right) = A(1/x_1) + A(x_2).$$

But

$$A\left(\frac{x_1x_2 - 1}{x_1 + x_2}\right) = -p - A\left(\frac{x_1 + x_2}{x_1x_2 - 1}\right)$$

and

$$-A\left(\frac{x_1 + x_2}{x_1x_2 - 1}\right) = A\left(\frac{x_1 + x_2}{1 - x_1x_2}\right).$$

Also $A(1/x_2) = -p - A(x_2)$. Hence

$$-p + A\left(\frac{x_1 + x_2}{1 - x_1x_2}\right) - p - A(x_2) = A(x_1)$$

and

$$A\left(\frac{x_1 + x_2}{1 - x_1x_2}\right) = A(x_1) + A(x_2) + 2p.$$

CHAPTER III

CERTAIN RELATED FUNCTIONS

<u>Theorem 3.1</u>: $A(x)$ has a single-valued inverse function $T(x)$ defined, continuous, monotone increasing, and differentiable on $(-p,p)$. Furthermore, for every x in $(-p,p)$, $T'(x) = 1 + T^2(x)$.

Let a be any point in $(-p,p)$. Since $\lim_{y\to+\infty} A(y) = p$, there exists a y_1 such that $A(y_1) > a$; since the $\lim_{y\to-\infty} A(y) = -p$, there exists a y_2 such that $A(y_2) < a$. Since $A(y)$ is continuous, there exists a point b between y_2 and y_1 such that $A(b) = a$. Furthermore because $A(y)$ is monotone increasing, b is unique. We define $T(a)$ to be b. Since a was an arbitrary point in $(-p,p)$, $T(x)$ may be defined in this manner on $(-p,p)$. Thus $T(x) = y$, where $A(y) = x$, so that $T\left[A(y)\right] = y$ and $A\left[T(x)\right] = x$.

Let $x_1 < x_2$ be any two points in $(-p,p)$. Consider $[r,s]$ such that $-p < r < x_1 < x_2 < s < p$. Let $c = T(r)$ and $d = T(s)$. Since $A(y)$ on $[c,d]$ is continuous, monotone increasing, differentiable, and $A'(y) \neq 0$; $T(x)$ is continuous, monotone increasing, and differentiable on $[r,s]$. Since the points x_1 and x_2 were arbitrarily chosen in $(-p,p)$, $T(x)$ is continuous, monotone increasing, and differentiable on

$(-p,p)$. We know that $A'(b)$ exists and is not zero. Thus $T'(a) = \frac{1}{A'(b)} = 1 + T^2(a)$. Since a was arbitrarily chosen, this completes the proof.

<u>Theorem 3.2</u>: The $\lim_{x \to p^-} T(x) = +\infty$; $\lim_{x \to -p^+} T(x) = -\infty$.

Let $M > 0$ be arbitrarily chosen. Let $d = p - A(M)$. Then $0 < p - d < p$. Choose any x such that $p - d < x < p$. Since $T(p - d) = M$, $T(x) > M$. Thus $\lim_{x \to p^-} T(x) = +\infty$. Similarly we may show that $\lim_{x \to -p^+} T(x) = -\infty$.

<u>Definition 3.1</u>: $S(x) = \frac{T(x)}{\sqrt{1 + T^2(x)}}$ for every x in $(-p,p)$; $S(x) = 1$ at $x = p$; $S(x) = -1$ at $x = -p$.

<u>Definition 3.2</u>: $C(x) = \frac{1}{\sqrt{1 + T^2(x)}}$ for every x in $(-p,p)$; $C(x) = 0$ at $x = p$; $C(x) = 0$ at $x = -p$.

<u>Theorem 3.3</u>: $S(x)$ is continuous and monotone increasing on $[-p,p]$.

Since $T(x)$ is continuous on $(-p,p)$ and $\sqrt{1 + T^2(x)} \neq 0$, then it follows from Theorem 1.9 that $S(x)$ is continuous on $(-p,p)$. Let $x_1 < x_2$ be any two points in $(-p,p)$. Then $T(x_1) < T(x_2)$. There exists a point a such that $x_1 < a < x_2$ and $S'(a) = \frac{S(x_2) - S(x_1)}{x_2 - x_1}$. Since $S'(a) = \frac{1}{\sqrt{1 + T^2(a)}} > 0$ and $x_2 - x_1 > 0$, then $S(x_2) - S(x_1) > 0$. Hence $S(x_1) < S(x_2)$ and $S(x)$ must be monotone increasing on $(-p,p)$. Obviously, for every x in $(-p,p)$,

$$S(-p) = -1 < \frac{T(x)}{\sqrt{1 + T^2(x)}} < 1 = S(p).$$

Hence S(x) is monotone increasing in $[-p,p]$. Now

$$\lim_{x \to p^-} S(x) = \lim_{x \to p^-} \frac{1}{\sqrt{\frac{1}{T^2(x)} + 1}} = 1 = S(p);$$

$$\lim_{x \to -p^+} S(x) = \lim_{x \to -p^+} \frac{-1}{\sqrt{\frac{1}{T^2(x)} + 1}} = -1 = S(-p).$$

Hence S(x) is continuous on the left at $x = p$ and continuous on the right at $x = -p$. The theorem follows.

Theorem 3.4: C(x) is continuous on $[-p,p]$; C(x) is monotone increasing on $[-p,0]$ and monotone decreasing on $[0,p]$.

Since T(x) is continuous on $(-p,p)$ and $\sqrt{1 + T^2(x)} \neq 0$, then it follows from Theorem 1.9 that C(x) is continuous on $(-p,p)$. Let $x_1 < x_2$ be any two points in $(-p,0]$. Then $T(x_1) < T(x_2)$ and $T(x_1) < 0$, $T(x_2) \leq 0$. There exists a point a such that $x_1 < a < x_2$ and $C'(a) = \frac{C(x_2) - C(x_1)}{x_2 - x_1}$. Since $C'(a) = -\frac{T(a)}{\sqrt{1 + T^2(a)}}$ and $T(a) < 0$, then $C'(a) > 0$ in $(-p,0]$. Since $x_2 - x_1 > 0$, then $C(x_2) - C(x_1) > 0$. Hence $C(x_1) < C(x_2)$ and C(x) must be monotone increasing on $(-p,0]$. Obviously, for every x in $(-p,0]$,

$$C(-p) = 0 < \frac{1}{\sqrt{1 + T^2(x)}}.$$

Hence C(x) is monotone increasing on $[-p,0]$.

Let $x_3 < x_4$ be any two points in $[0,p)$. Then $T(x_3) < T(x_4)$ and $T(x_3) \geqq 0$, $T(x_4) > 0$. There exists a point b such that $x_3 < b < x_4$ and $C'(b) = \frac{C(x_4) - C(x_3)}{x_4 - x_3}$. Since $C'(b) = -\frac{T(b)}{\sqrt{1 + T^2(b)}}$ and $T(b) > 0$, then $C'(b) < 0$ in $[0,p)$. Since $x_4 - x_3 > 0$, then $C(x_4) - C(x_3) < 0$. Hence $C(x_3) > C(x_4)$ and $C(x)$ must be monotone decreasing on $[0,p)$. Obviously for every x in $[0,p)$,

$$\frac{1}{\sqrt{1 + T^2(x)}} > 0 = C(p).$$

Hence $C(x)$ is monotone decreasing on $[0,p]$. Now

$$\lim_{x \to p^-} C(x) = \lim_{x \to p^-} \frac{1}{\sqrt{1 + T^2(x)}} = 0 = C(p);$$

$$\lim_{x \to -p^+} C(x) = \lim_{x \to -p^+} \frac{1}{\sqrt{1 + T^2(x)}} = 0 = C(-p).$$

Hence $C(x)$ is continuous on the left at $x = p$ and continuous on the right at $x = -p$. The theorem follows.

<u>Theorem 3.5</u>: $S(x)$ is differentiable on $[-p,p]$ and $S'(x) = C(x)$ for every x in $[-p,p]$.

Since $T(x)$ is differentiable on $(-p,p)$ and $\sqrt{1 + T^2(x)} \neq 0$, then it follows from Theorem 1.15 that $S(x)$ is differentiable on $(-p,p)$ and $S'(x) = \frac{1}{\sqrt{1 + T^2(x)}} = C(x)$. Since $C(x)$ is continuous at $x = p$, then for every $e > 0$, there exists a $d > 0$ such that, for every $|x - p| < d$, $|C(x) - C(p)| < e$. Choose x such that $p - d < x < p$. Then since x is continuous at x and at p and differentiable on (x,p), there exists an a in (x,p) and $S'(a) = \frac{S(x) - S(p)}{x - p}$. Since $S'(a) = C(a)$, then

$$\left| C(p) - \frac{S(x) - S(p)}{x - p} \right| = |C(p) - S'(a)|$$
$$= |C(p) - C(a)| < e.$$

Hence $S(x)$ is differentiable at $x = p$ and $S'(p) = C(p)$. Similarly we may show that the theorem holds for $x = -p$.

<u>Theorem 3.6</u>: $C(x)$ is differentiable on $[-p,p]$ and $C'(x) = -S(x)$ for every x in $[-p,p]$.

Since $T(x)$ is differentiable on $(-p,p)$ and $\sqrt{1 + T^2(x)} \neq 0$, then it follows from Theorem 1.15 that $C(x)$ is differentiable on $(-p,p)$ and $C'(x) = -\frac{T(x)}{\sqrt{1 + T^2(x)}} = -S(x)$. Since $S(x)$ is continuous at $x = p$, then for every $e > 0$, there exists a $d > 0$ such that, for every $|x - p| < d$, $|S(x) - S(p)| < e$. Choose x such that $p - d < x < p$. Then since $C(x)$ is continuous at x and at p and differentiable on (x,p), there exists an a in (x,p) and $C'(a) = \frac{C(x) - C(p)}{x - p}$. Since $C'(a) = -S(a)$, then

$$\left| -S(p) - \frac{C(x) - C(p)}{x - p} \right| = |-S(p) - C'(a)|$$
$$= |-S(p) + S(a)| < e.$$

Hence $C(x)$ is differentiable at $x = p$ and $C'(p) = -S(p)$. Similarly we may show that the theorem holds for $x = -p$.

<u>Theorem 3.7</u>: For every x in $(-p,p)$, $T(-x) = -T(x)$.

Let $y = T(x)$, then $A(y) = x$. $A(-y) = -A(y) = -x$, then $T(-x) = -y$. Therefore $T(-x) = -T(x)$.

<u>Corollary 3.7.1</u>: $T(0) = 0$.

<u>Theorem 3.8</u>: For every x in $[-p,p]$, $S(-x) = -S(x)$.

Let x be any point in $(-p,p)$. Then

$$S(-x) = \frac{T(-x)}{\sqrt{1 + T^2(-x)}} = \frac{-T(x)}{\sqrt{1 + T^2(x)}} = -S(x).$$

If $x = p$, then $-S(x) = -S(p) = -1 = S(-p)$. If $x = -p$, then $-S(x) = -S(-p) = 1 = S(p)$.

<u>Theorem 3.9</u>: For every x in $[-p,p]$, $C(-x) = C(x)$.

Let x be any point in $(-p,p)$. Then

$$C(-x) = \frac{1}{\sqrt{1 + T^2(-x)}} = \frac{1}{\sqrt{1 + T^2(x)}} = C(x).$$

If $x = p$, then $C(x) = C(p) = 0 = C(-p)$. If $x = -p$, then $C(x) = C(-p) = 0 = C(p)$.

<u>Theorem 3.10</u>: For every $x > 0$ in $(-p.p)$, $T(p-x) = \frac{1}{T(x)}$; and for every $x < 0$ in $(-p.p)$, $T(-p - x) = \frac{1}{T(x)}$.

Let $x = A(y)$. Then $T(x) = y$ and $\frac{1}{T(x)} = \frac{1}{y}$. If

$x > 0$, then $A(y) > 0$ and $y > 0$. Hence $A(1/y) = p - A(y)$, $1/y = T[p - A(y)]$, and $\frac{1}{T(x)} = T(p - x)$.

If $x < 0$, then $A(y) < 0$ and $y < 0$. Hence $A(1/y) = -p - A(y)$, $1/y = T[-p - A(y)]$, and $\frac{1}{T(x)} = T(-p - x)$.

<u>Theorem 3.11</u>: For every $x \geq 0$ in $[-p,p]$, $S(p - x) = C(x)$; for every $x \leq 0$ in $[-p,p]$, $S(-p - x) = -C(x)$.

Let $0 < x < p$. Then

$$S(p - x) = \frac{T(p - x)}{\sqrt{1 + T^2(p - x)}}$$
$$= \frac{\frac{1}{T(x)}}{\sqrt{1 + \left[\frac{1}{T(x)}\right]^2}} .$$

But $T(x) > 0$; therefore

$$S(p - x) = \frac{1}{\sqrt{1 + T^2(x)}} = C(x).$$

If $x = 0$, then $S(p - x) = S(p) = 1 = C(0)$. If $x = p$, then $S(p - x) = S(0) = 0 = C(p)$. Hence the theorem holds for any $x \geq 0$ in $[-p,p]$.

Now let $-p < x < 0$. Then

$$S(-p - x) = \frac{T(-p - x)}{\sqrt{1 + T^2(-p - x)}}$$
$$= \frac{\frac{1}{T(x)}}{\sqrt{1 + \left[\frac{1}{T(x)}\right]^2}} .$$

But $T(x) < 0$; therefore

$$S(-p - x) = -\frac{1}{\sqrt{1 + T^2(x)}} = -C(x).$$

If $x = 0$, then $S(-p - x) = S(-p) = -1 = -C(0)$. If $x = -p$, then $S(-p - x) = S(0) = 0 = -C(-p)$. Hence the theorem holds for any $x \leq 0$ in $[-p,p]$.

Theorem 3.12: For every $x \geq 0$ in $[-p,p]$, $C(p - x) = S(x)$; for every $x \leq 0$ in $[-p,p]$, $C(-p - x) = -S(x)$.

Let $0 < x < p$. Then

$$C(p - x) = \frac{1}{\sqrt{1 + T^2(p - x)}}$$

$$= \frac{1}{\sqrt{1 + \left[\frac{1}{T(x)}\right]^2}} .$$

But $T(x) > 0$, therefore

$$C(p - x) = \frac{T(x)}{\sqrt{1 + T^2(x)}} = S(x).$$

If $x = 0$, then $C(p - x) = C(p) = 0 = S(0)$. If $x = p$, then $C(p - x) = C(0) = 1 = S(p)$. Hence the theorem holds for any $x \geq 0$ in $[-p,p]$.

Now let $-p < x < 0$. Then

$$C(-p - x) = \frac{1}{\sqrt{1 + T^2(-p - x)}}$$

$$= \frac{1}{\sqrt{1 + \left[\frac{1}{T(x)}\right]^2}}.$$

But $T(x) < 0$; therefore

$$C(-p - x) = - \frac{T(x)}{\sqrt{1 + T^2(x)}} = -S(x).$$

If $x = 0$, then $C(-p - x) = C(-p) = 0 = -S(0)$. If $x = -p$, then $C(-p - x) = C(0) = 1 = -S(-p)$. Hence the theorem holds for any $x \leq 0$ in $[-p,p]$.

<u>Theorem 3.13</u>: For every x in $[-p,p]$, $S^2(x) + C^2(x) = 1$.

Let x be any point in $(-p,p)$. Then

$$S^2(x) + C^2(x) = \left[\frac{T(x)}{\sqrt{1 + T^2(x)}}\right]^2 + \left[\frac{1}{\sqrt{1 + T^2(x)}}\right]^2 = 1.$$

If $x = p$ or $-p$, it is obvious that $S^2(x) + C^2(x) = 1$.

<u>Theorem 3.14</u>: For every x_1 and x_2 in $(-p,p)$, $\frac{T(x_1) + T(x_2)}{1 - T(x_1)T(x_2)} = T(x_1 + x_2)$, $T(x_1 + x_2 - 2p)$, or $T(x_1 + x_2 + 2p)$ according as $-p < x_1 + x_2 < p$, $x_1 + x_2 > p$, or $x_1 + x_2 < -p$.

Let $y_1 = T(x_1)$, $y_2 = T(x_2)$. Then $A(y_1) = x_1$, $A(y_2) = x_2$ and $\frac{T(x_1) + T(x_2)}{1 - T(x_1)T(x_2)} = \frac{y_1 + y_2}{1 - y_1 y_2}$.

Case I. Let $-p < x_1 + x_2 < p$. Then $-p < A(y_1) + A(y_2) < p$ and $-p - A(y_1) < A(y_2) < p - A(y_1)$. We may have $y_1 > 0$,

$y_1 < 0$, or $y_1 = 0$. If $y_1 > 0$, $A(1/y_1) = p - A(y_1)$. Then $A(1/y_1) > A(y_2)$ and $1/y_1 > y_2$. If $y_1 < 0$, $A(1/y_1) = -p - A(y_1)$. Then $A(1/y_1) < A(y_2)$ and $1/y_1 < y_2$. Hence

$$A\left(\frac{y_1 + y_2}{1 - y_1 y_2}\right) = A(y_1) + A(y_2);$$

$$T\left[A\left(\frac{y_1 + y_2}{1 - y_1 y_2}\right)\right] = T\left[A(y_1) + A(y_2)\right];$$

$$\frac{T(x_1) + T(x_2)}{1 - T(x_1)T(x_2)} = T(x_1 + x_2).$$

If $y_1 = 0$, then $A(y_1) = 0$ and the same equation obviously holds.

Case II. Let $x_1 + x_2 > p$. Then $A(y_1) + A(y_2) > p$ and $A(y_2) > p - A(y_1)$. Also $y_1 > 0$ and $A(1/y_1) = p - A(y_1)$. Thus $A(y_2) > A(1/y_1)$ and $y_2 > 1/y_1$. Hence

$$A\left(\frac{y_1 + y_2}{1 - y_1 y_2}\right) = A(y_1) + A(y_2) - 2p;$$

$$T\left[A\left(\frac{y_1 + y_2}{1 - y_1 y_2}\right)\right] = T\left[A(y_1) + A(y_2) - 2p\right];$$

$$\frac{T(x_1) + T(x_2)}{1 - T(x_1)T(x_2)} = T(x_1 + x_2 - 2p).$$

Case III. Let $x_1 + x_2 < -p$. Then $A(y_1) + A(y_2) < -p$ and $A(y_2) < -p - A(y_1)$. Also $y_1 < 0$ and $A(1/y_1) = -p - A(y_1)$. Thus $A(y_2) < A(1/y_1)$ and $y_2 < 1/y_1$. Hence

$$A\left(\frac{y_1 + y_2}{1 - y_1 y_2}\right) = A(y_1) + A(y_2) + 2p;$$

$$T\left[A\left(\frac{y_1 + y_2}{1 - y_1 y_2}\right)\right] = T\left[A(y_1) + A(y_2) + 2p\right];$$

$$\frac{T(x_1) + T(x_2)}{1 - T(x_1)T(x_2)} = T(x_1 + x_2 + 2p).$$

Theorem 3.15: For every x_1 and x_2 in $[-p,p]$, $S(x_1)C(x_2) + S(x_2)C(x_1) = S(x_1 + x_2)$, $-S(x_1 + x_2 - 2p)$, or $-S(x_1 + x_2 + 2p)$ according as $-p \leq x_1 + x_2 \leq p$, $x_1 + x_2 > p$, or $x_1 + x_2 < -p$.

Case I. Let $-p \leq x_1 + x_2 \leq p$. First we will consider $-p < x_1 + x_2 < p$. Then

$$S(x_1 + x_2) = \frac{T(x_1 + x_2)}{\sqrt{1 + T^2(x_1 + x_2)}}$$

$$= \frac{\dfrac{T(x_1) + T(x_2)}{1 - T(x_1)T(x_2)}}{\sqrt{1 + \left[\dfrac{T(x_1) + T(x_2)}{1 - T(x_1)T(x_2)}\right]^2}}.$$

Referring to Theorem 3.14, we see that for $-p < x_1 + x_2 < p$, $A(1/y_1) > A(y_2)$ if $y_1 > 0$, $A(1/y_1) < A(y_2)$ if $y_1 < 0$, and $A(y_1) = 0$ if $y_1 = 0$. If $y_1 > 0$, $1/y_1 > y_2$. Then $1 - y_1 y_2 > 0$ and $1 - T(x_1)T(x_2) > 0$. If $y_1 < 0$, $1/y_1 < y_2$. Then $1 > y_1 y_2$, $1 - y_1 y_2 > 0$, and $1 - T(x_1)T(x_2) > 0$. If $y_1 = 0$, then $1 - y_1 y_2 > 0$ and $1 - T(x_1)T(x_2) > 0$. Hence

$$S(x_1 + x_2) = \frac{T(x_1) + T(x_2)}{\sqrt{[1 - T(x_1)T(x_2)]^2 + [T(x_1) + T(x_2)]^2}}$$

$$= \frac{T(x_1)}{\sqrt{1 + T^2(x_1)}} \cdot \frac{1}{\sqrt{1 + T^2(x_2)}}$$

$$+ \frac{T(x_2)}{\sqrt{1 + T^2(x_2)}} \cdot \frac{1}{\sqrt{1 + T^2(x_1)}}$$

$$= S(x_1)C(x_2) + S(x_2)C(x_1).$$

We will now consider $x_1 + x_2 = p$. Then $x_2 \leq p$ and $x_2 = p - x_1$. Thus $p - x_1 \leq p$ and $x_1 \geq 0$. For $x_1 \geq 0$, $S(x_2) = S(p - x_1) = C(x_1)$ and $C(x_2) = C(p - x_1) = S(x_1)$. Hence $S(x_1)C(x_2) + S(x_2)C(x_1) = S^2(x_1) + C^2(x_1) = 1 = S(p)$ $= S(x_1 + x_2)$.

Now consider $x_1 + x_2 = -p$. Then $-p \leq x_2$ and $x_2 = -p - x_1$. Thus $-p \leq -p - x_1$ and $x_1 \leq 0$. For $x_1 \leq 0$, $S(x_2) = S(-p - x_1) = -C(x_1)$ and $C(x_2) = C(-p - x_1) = -S(x_1)$. Hence $S(x_1)C(x_2) + S(x_2)C(x_1) = -S^2(x_1) - C^2(x_1) = -1$ $= S(-p) = S(x_1 + x_2)$.

Case II. Let $x_1 + x_2 > p$. If x_1 and x_2 are in $(-p,p)$, then

$$-S(x_1 + x_2 - 2p) = -\frac{T(x_1 + x_2 - 2p)}{\sqrt{1 + [T(x_1 + x_2 - 2p)]^2}}$$

$$= -\frac{\dfrac{T(x_1) + T(x_2)}{1 - T(x_1)T(x_2)}}{\sqrt{1 + \left[\dfrac{T(x_1) + T(x_2)}{1 - T(x_1)T(x_2)}\right]^2}}.$$

Referring to Theorem 3.14, we see that if $x_1 + x_2 > p$,

$y_1 > 0$ and $A(1/y_1) < A(y_2)$. Then $1/y_1 < y_2$, $1 - y_1 y_2 < 0$, and $1 - T(x_1)T(x_2) < 0$. Hence

$$-S(x_1 + x_2 - 2p) = \frac{T(x_1) + T(x_2)}{\sqrt{[1 - T(x_1)T(x_2)]^2 + [T(x_1) + T(x_2)]^2}}$$

$$= S(x_1)C(x_2) + S(x_2)C(x_1).$$

We will now consider $x_1 = p$ and x_2 in $(0,p]$. Then $S(x_1)C(x_2) + S(x_2)C(x_1) = C(x_2)$ and $-S(x_1 + x_2 - 2p) = -S(-p - x_2) = S(p - x_2) = C(x_2)$.

Case III. Let $x_1 + x_2 < -p$. If x_1 and x_2 are in $(-p,p)$, then

$$-S(x_1 + x_2 + 2p) = -\frac{T(x_1 + x_2 + 2p)}{\sqrt{1 + [T(x_1 + x_2 + 2p)]^2}}$$

$$= -\frac{\dfrac{T(x_1) + T(x_2)}{1 - T(x_1)T(x_2)}}{\sqrt{1 + \left[\dfrac{T(x_1) + T(x_2)}{1 - T(x_1)T(x_2)}\right]^2}}.$$

Referring to Theorem 3.14, we see that if $x_1 + x_2 < -p$, $y_1 < 0$ and $A(1/y_1) > A(y_2)$. Then $1/y_1 > y_2$, $y_1 y_2 > 1$, $1 - y_1 y_2 < 0$, and $1 - T(x_1)T(x_2) < 0$. Hence

$$-S(x_1 + x_2 + 2p) = \frac{T(x_1) + T(x_2)}{\sqrt{[1 - T(x_1)T(x_2)]^2 + [T(x_1) + T(x_2)]^2}}$$

$$= S(x_1)C(x_2) + S(x_2)C(x_1).$$

We will now consider $x_1 = -p$ and x_2 in $[-p,0)$. Then

$S(x_1)C(x_2) + S(x_2)C(x_1) = -C(x_2)$ and $-S(x_1 + x_2 + 2p)$ $= -S(p + x_2) = S(-p - x_2) = -C(x_2)$.

<u>Theorem 3.16</u>: For every x_1 and x_2 in $[-p,p]$, $C(x_1)C(x_2) - S(x_1)S(x_2) = C(x_1 + x_2)$, $-C(x_1 + x_2 - 2p)$, or $-C(x_1 + x_2 + 2p)$ according as $-p \leq x_1 + x_2 \leq p$, $x_1 + x_2 > p$, or $x_1 + x_2 < -p$.

Case I. Let $-p \leq x_1 + x_2 \leq p$. First we will consider $-p < x_1 + x_2 < p$. Then

$$C(x_1 + x_2) = \frac{1}{\sqrt{1 + T^2(x_1 + x_2)}}$$

$$= \frac{1}{\sqrt{1 + \left[\frac{T(x_1) + T(x_2)}{1 - T(x_1)T(x_2)}\right]^2}}.$$

Referring to Theorem 3.15, we see that if $-p < x_1 + x_2 < p$, then $1 - T(x_1)T(x_2) > 0$. Hence

$$C(x_1 + x_2) = \frac{1 - T(x_1)T(x_2)}{\sqrt{[1 - T(x_1)T(x_2)]^2 + [T(x_1) + T(x_2)]^2}}$$

$$= \frac{1}{\sqrt{1 + T^2(x_1)}} \cdot \frac{1}{\sqrt{1 + T^2(x_2)}}$$

$$- \frac{T(x_1)}{\sqrt{1 + T^2(x_1)}} \cdot \frac{T(x_2)}{\sqrt{1 \quad T^2(x_2)}}$$

$$= C(x_1)C(x_2) - S(x_1)\,S(x_2).$$

We will now consider $x_1 + x_2 = p$. Then $x_2 \leq p$ and $x_2 = p - x_1$. Thus $p - x_1 \leq p$ and $x_1 \geq 0$. For $x_1 \geq 0$, $S(x_2) = S(p - x_1) = C(x_1)$ and $C(x_2) = C(p - x_1) = S(x_1)$.

Hence $C(x_1)C(x_2) - S(x_1)\,S(x_2) = C(x_1)S(x_1) - S(x_1)C(x_1)$

$= 0 = C(p) = C(x_1 + x_2)$.

Now consider $x_1 + x_2 = -p$. Then $-p \leq x_2$ and $x_2 = -p - x_1$. Thus $-p \leq -p - x_1$ and $x_1 \leq 0$. For $x_1 \leq 0$, $S(x_2) = S(-p - x_1)$

$= -C(x_1)$ and $C(x_2) = C(-p - x_1) = -S(x_1)$. Hence

$C(x_1)C(x_2) - S(x_1)S(x_2) = -C(x_1)S(x_1) + S(x_1)C(x_1) = 0$

$= C(-p) = C(x_1 + x_2)$.

Case II. Let $x_1 + x_2 > p$. If x_1 and x_2 are in $(-p,p)$, then

$$-C(x_1 + x_2 - 2p) = -\frac{1}{\sqrt{1 + [T(x_1 + x_2 - 2p)]^2}}$$

$$= -\frac{1}{\sqrt{1 + \left[\dfrac{T(x_1) + T(x_2)}{1 - T(x_1)T(x_2)}\right]^2}}.$$

Referring to Theorem 3.15, we see that if $x_1 + x_2 > p$, $1 - T(x_1)T(x_2) < 0$. Hence

$$-C(x_1 + x_2 - 2p) = \frac{1 - T(x_1)T(x_2)}{\sqrt{[1 - T(x_1)T(x_2)]^2 + [T(x_1) + T(x_2)]^2}}$$

$$= C(x_1)C(x_2) - S(x_1)S(x_2).$$

We will now consider $x_1 = p$ and x_2 in $(0,p]$. Then $C(x_1)C(x_2) - S(x_1)S(x_2) = -S(x_2)$ and $-C(x_1 + x_2 - 2p)$

$= -C(-p + x_2) = -C(p - x_2) = -S(x_2)$.

Case III. Let $x_1 + x_2 < -p$. If x_1 and x_2 are in $(-p,p)$, then

$$-C(x_1 + x_2 + 2p) = -\frac{1}{\sqrt{1 + [T(x_1 + x_2 + 2p)]^2}}$$

$$= -\frac{1}{\sqrt{1 + \left[\frac{T(x_1) + T(x_2)}{1 - T(x_1)T(x_2)}\right]^2}} .$$

Referring to Theorem 3.15, we see that if $x_1 + x_2 < -p$, then $1 - T(x_1)T(x_2) < 0$. Hence

$$-C(x_1 + x_2 + 2p) = \frac{1 - T(x_1)T(x_2)}{\sqrt{[1 - T(x_1)T(x_2)]^2 + [T(x_1) + T(x_2)]^2}}$$

$$= C(x_1)C(x_2) - S(x_1)S(x_2).$$

We will now consider $x_1 = -p$ and x_2 in $[-p,0)$. Then $C(x_1)C(x_2) - S(x_1)\,S(x_2) = S(x_2)$ and $-C(x_1 + x_2 + 2p)$ $= -C(p + x_2) = -C(-p - x_2) = S(x_2)$.

CHAPTER IV

THE TRIGONOMETRIC FUNCTIONS OF A REAL VARIABLE

Definition 4.1: Let x be any real number in $(-p,p)$. Then, for every integer n, $\tan(x + 2np)$ is defined to be $T(x)$. If x is an odd multiple of p, tan x is not defined.

Theorem 4.1: The function tan x is differentiable and continuous for any real value of x other than an odd multiple of p, and tan x is discontinuous if x is an odd multiple of p. Furthermore, $D_x \tan x\Big]_{x=a} = 1 + \tan^2 a$.

Let a be any real number not an odd multiple of p. Then there exists a point k in $(-p,p)$ and an integer n such that $a = k + 2np$. Also $\tan(k + 2np) = T(k)$ and $T'(k) = 1 + T^2(k)$. Let $e > 0$ be chosen. Then there exists a $d > 0$ such that for $0 < |x - k| < d$,

$$\left| 1 + T^2(k) - \frac{T(x) - T(k)}{x - k} \right| < e.$$

Choose any x such that $0 < |x - a| < d$. Consider the number $x_0 = x - 2np$. Then $|x_0 - k| = |x - 2np - (a - 2np)| = |x - a|$. Thus $0 < |x_0 - k| < d$. We have

$$\left|1 + \tan^2 a - \frac{\tan x - \tan a}{x - a}\right|$$

$$= \left|1 + T^2(k) - \frac{T(x_0) - T(k)}{(x_0 + 2np) - (k + 2np)}\right|$$

$$= \left|1 + T^2(k) - \frac{T(x_0) - T(k)}{x_0 - k}\right| < e.$$

Hence tan x is differentiable for any real value of x other than an odd multiple of p and $D_x \tan x\Big]_{x=a} = 1 + \tan^2 a$.

For any real value of x not an odd multiple of p, tan x is differentiable and therefore continuous. Since tan x is not defined for odd multiples of p, it is discontinuous at those points.

<u>Definition 4.2</u>: Let x be any real number in $[-p,p]$. Then sin (x + 2np) is defined to be S(x) or -S(x) according as the integer n is even or odd.

<u>Definition 4.3</u>: Let x be any real number in $[-p,p]$. Then cos (x + 2np) is defined to be C(x) or -C(x) according as the integer n is even or odd.

<u>Theorem 4.2</u>: For any real value of x, sin x is continuous and differentiable. Furthermore, $D_x \sin x\Big]_{x=a} = \cos a$.

Let a be any real number. Then there exists a point k in $[-p,p]$ and an integer n such that $a = k + 2np$. If n is an even integer, then sin (k + 2np) = S(k), S'(k) = C(k), and cos (k + 2np) = C(k). There exists a $d > 0$ such that , for $0 < |x - k| < d$,

$$\left|C(k) - \frac{S(x) - S(k)}{x - k}\right| < e.$$

Choose any x such that $0 < |x - a| < d$. Consider the number $x_o = x - 2np$. Then $|x_o - k| = |x - 2np - (a - 2np)|$ $= |x - a|$. Hence $0 < |x - k| < d$. We have

$$\left|\cos a - \frac{\sin x - \sin a}{x - a}\right|$$
$$= \left|C(k) - \frac{S(x_o) - S(k)}{(x_o + 2np) - (k + 2np)}\right|$$
$$= \left|C(k) - \frac{S(x_o) - S(k)}{x_o - k}\right| < e.$$

Thus sin x is differentiable when n is an even integer and $D_x \sin x\Big]_{x=a} = \cos a$. Similarly we can show that sin x is differentiable and $D_x \sin x\Big]_{x=a} = \cos a$ when n is an odd integer.

The function sin x is continuous for all real values of x since it is differentiable at those points.

<u>Theorem 4.3</u>: For any real value of x, cos x is continuous and differentiable. Furthermore, $D_x \cos x\Big]_{x=a} = -\sin a$.

Let a be any real number. Then there exists a point k in $[-p,p]$ and an integer n such that $a = k + 2np$. If n is an even integer then $\cos (k + 2np) = C(k)$, $C'(k) = -S(k)$, and $\sin (k + 2np) = S(k)$. There exists a $d > 0$ such that for $0 < |x - k| < d$,

$$\left| -S(k) - \frac{C(x) - C(k)}{x - k} \right| < e.$$

Choose any x such that $0 < |x - a| < d$. Consider the number $x_0 = x - 2np$. Then $|x_0 - k| = |x - 2np - (a - 2np)|$ $= |x - a|$. Hence $0 < |x_0 - k| < d$. We have

$$\left| - \sin a - \frac{\cos x - \cos a}{x - a} \right|$$
$$= \left| -S(k) - \frac{C(x_0) - S(k)}{(x_0 + 2np) - (k + 2np)} \right|$$
$$= \left| -S(k) - \frac{C(x_0) - C(k)}{x_0 - k} \right| < e.$$

Thus cos x is differentiable when n is an even integer and $D_x \cos x\big]_{x=a} = -\sin a$. Similarly we may show that cos x is differentiable and $D_x \cos x\big]_{x=a} = -\sin a$ when n is an odd integer.

The function cos x is continuous for all real values of x since it is differentiable at those points.

<u>Theorem 4.4</u>: For all real values of x,

$$\sin^2 x + \cos^2 x = 1.$$

Let x be any real number. Then there exists a point k in $[-p,p]$ and an integer n such that $x = k + 2np$. Thus $\sin(k + 2np) = S(k)$ or $-S(k)$ and $\cos(k + 2np) = C(k)$ or $-C(k)$ according as the integer n is even or odd. Hence $\sin^2 x + \cos^2 x = S^2(k) + C^2(k) = 1$.

Corollary 4.4.1: For all real values of x, sin x $= \pm\sqrt{1 + \cos^2 x}$ and $\cos x = \pm\sqrt{1 + \sin^2 x}$.

Theorem 4.5: For any real value of x not an odd multiple of p, $\frac{\sin x}{\cos x} = \tan x$.

Let x be any real number not an odd multiple of p. Then there exists a point k in (-p,p) and an integer n such that x = k + 2np. If n is an even integer, $\sin(k + 2np) = S(k)$, $\cos(k + 2np) = C(k)$, and $\tan(k + 2np) = T(k)$. Thus

$$\frac{\sin x}{\cos x} = \frac{S(k)}{C(k)} = \frac{\frac{T(k)}{\sqrt{1 + T^2(k)}}}{\frac{1}{\sqrt{1 + T^2(k)}}} = T(k) = \tan x.$$

Similarly we may show that the theorem holds if n is an odd integer.

Corollary 4.5.1: For any real value of x not an odd multiple of p, $\sin x = \cos x \tan x$.

Definition 4.4: For any real value of x not an even multiple of p, $\cot x = \frac{\cos x}{\sin x}$; for x an even multiple of p, cot x is not defined.

Theorem 4.6: The function cot x is differentiable and continuous for any real value of x not an even multiple of p, and cot x is discontinuous if x is an even multiple of p. Furthermore, $D_x \cot x\Big]_{x=a} = -\frac{1}{\sin^2 a}$.

Since, for any real value of x not an even multiple of p, $\sin x \neq 0$ and both sin x and cos x are differentiable, then it follows from Theorem 1.15 that cot x is differentiable. Also $D_x \cot x\Big]_{x=a} = \frac{-\sin^2 a - \cos^2 a}{\sin^2 a} = \frac{-(\sin^2 a + \cos^2 a)}{\sin^2 a}$ $= -\frac{1}{\sin^2 a}$.

The function cot x is continuous for all real values of x not an even multiple of p since it is differentiable at those points. Since cot x is not defined for even multiples of p, it is discontinuous at those points.

<u>Definition 4.5</u>: For any real value of x not an odd multiple of p, $\sec x = \frac{1}{\cos x}$; for x an odd multiple of p, sec x is not defined.

<u>Theorem 4.7</u>: The function sec x is differentiable and continuous for any real value of x not an odd multiple of p, and sec x is discontinuous for x an odd multiple of p. Furthermore, $D_x \sec x\Big]_{x=a} = \sec a \tan a$.

Since, for every real value of x not an odd multiple of p, $\cos x \neq 0$ and cos x is differentiable, then it follows from Theorem 1.15 that sec x is differentiable. Also $D_x \sec x\Big]_{x=a} = \frac{\sin a}{\cos^2 a} = \frac{1}{\cos a} \cdot \frac{\sin a}{\cos a} = \sec a \tan a$.

The function sec x is continuous for all real values of x not an odd multiple of p since it is differentiable at those points. Since sec x is not defined for odd multiples of p, it is discontinuous at those points.

Definition 4.6: For any real value of x not an even multiple of p, $\csc x = \frac{1}{\sin x}$; for x an even multiple of p, csc x is not defined.

Theorem 4.8: The function csc x is differentiable and continuous for any real value of x not an even multiple of p, and csc x is discontinuous if x is an even multiple of p. Furthermore, $D_x \csc x\Big]_{x=a} = -\csc a \cot a$.

Since, for any real value of x not an even multiple of p, $\sin x \neq 0$ and sin x is differentiable, then it follows from Theorem 1.15 that csc x is differentiable. Also

$$D_x \csc x\Big]_{x=a} = -\frac{\cos a}{\sin^2 a} = -\frac{1}{\sin a} \cdot \frac{\cos a}{\sin a} = -\csc a \cot a.$$

The function csc x is continuous for all real values of x not an even multiple of p since it is differentiable at those points. Since csc x is not defined for even multiples of p, it is discontinuous at those points.

Theorem 4.9: For any real value of x not a multiple of p, $\cot x = \frac{1}{\tan x}$.

Let x be any real number not a multiple of p. Then

$$\frac{1}{\tan x} = \frac{1}{\frac{\sin x}{\cos x}} = \frac{\cos x}{\sin x} = \cot x.$$

Theorem 4.10: For any real number not an odd multiple of p, $\sec^2 x = \tan^2 x + 1$.

Let x be any real number not an odd multiple of p. Then $\tan^2 x + 1 = \frac{\sin^2 x}{\cos^2 x} + 1 = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$ $= \sec^2 x$.

Theorem 4.11: For all real values of x not an even multiple of p, $\csc^2 x = 1 + \cot^2 x$.

Let x be any real number not an even multiple of p. Then $1 + \cot^2 x = 1 + \frac{\cos^2 x}{\sin^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$ $= \csc^2 x$.

Theorem 4.12: For all real values of x not an odd multiple of p, $\tan(-x) = -\tan x$.

Let x be any real number not an odd multiple of p. Then there exists a point k in (-p,p) and an integer n such that $x = k + 2np$. Also $\tan(k + 2np) = T(k)$. From Theorem 3.7, we have $T(-k) = -T(k)$. Then $\tan(-x) = -\tan x$.

Theorem 4.13: For all real values of x,

$$\sin(-x) = -\sin x.$$

Let x be any real number. Then there exists a point k in [-p,p] and an integer n such that $x = k + 2np$. If n is an even integer, $\sin(k + 2np) = S(k)$. From Theorem 3.8, we have $S(-k) = -S(k)$. Therefore $\sin(-x) = S(-k) = -S(k)$ $= -\sin x$.

Similarly the theorem holds if n is an odd integer.

Theorem 4.14: For all real values of x,

$$\cos(-x) = \cos x.$$

Let x be any real number. Then there exists a point k in $[-p,p]$ and an integer n such that $x = k + 2np$. If n is an even integer, $\cos(k + 2np) = C(k)$. From Theorem 3.9, we have $C(-k) = C(k)$. Therefore $\cos(-x) = C(-k) = C(k) = \cos x$.

Similarly we may show that the theorem holds if n is an odd integer.

Theorem 4.15: For any real value of x not an even multiple of p, $\cot(-x) = -\cot x$.

Let x be any real number not an even multiple of p. Then $\cot(-x) = \frac{\cos(-x)}{\sin(-x)} = \frac{\cos x}{-\sin x} = -\cot x$.

Theorem 4.16: For any real value of x not an even multiple of p, $\csc(-x) = -\csc x$.

Let x be any real number not an even multiple of p. Then $\csc(-x) = \frac{1}{\sin(-x)} = \frac{1}{-\sin x} = -\csc x$.

Theorem 4.17: For any real value of x not an odd multiple of p, $\sec(-x) = \sec x$.

Let x be any real number not an odd multiple of p. Then $\sec(-x) = \frac{1}{\cos(-x)} = \frac{1}{\cos x} = \sec x$.

<u>Theorem 4.18</u>: For all real values of x_1 and x_2, except where x_1, x_2, or $x_1 + x_2$ is an odd multiple of p,

$$\tan (x_1 + x_2) = \frac{\tan x_1 + \tan x_2}{1 - \tan x_1 \tan x_2}.$$

Let x_1 and x_2 be any two real numbers not an odd multiple of p. We may determine two points k_1 and k_2 in $(-p,p)$ and two integers n_1 and n_2 such that $x_1 = k_1 + 2n_1p$, and $x_2 = k_2 + 2n_2p$. We may also determine a k_3 in $(-p,p)$ and $n_3 = 0$, 1, or -1 such that $k_1 + k_2 = k_3 + 2n_3p$ according $-p < k_1 + k_2 < p$, $k_1 + k_2 > p$, or $k_1 + k_2 < -p$. Then for $x_1 + x_2$ not an odd multiple of p, we have $x_1 + x_2 = (k_1 + k_2) + (n_1 + n_2)2p = k_3 + (n_1 + n_2 + n_3)2p$. Also $\tan (x_1 + x_2) = T(k_3)$, $\tan x_1 = T(k_1)$, and $\tan x_2 = T(k_2)$. Hence

$$\frac{\tan x_1 + \tan x_2}{1 - \tan x_1 \tan x_2} = \frac{T(k_1) + T(k_2)}{1 - T(k_1)T(k_2)}.$$

From Theorem 3.14,

$$\frac{T(k_1) + T(k_2)}{1 - T(k_1)T(k_2)} = T(k_1 + k_2),\ T(k_1 + k_2 - 2p),\ \text{or } T(k_1 + k_2 + 2p),$$

according as $-p < k_1 + k_2 < p$, $k_1 + k_2 > p$, or $k_1 + k_2 < -p$. But $T(k_1 + k_2) = T(k_3)$ for $-p < k_1 + k_2 < p$, $T(k_1 + k_2 - 2p) = T(k_3)$ for $k_1 + k_2 > p$, and $T(k_1 + k_2 + 2p) = T(k_3)$ for $k_1 + k_2 < -p$. Hence

$$\frac{\tan x_1 + \tan x_2}{1 - \tan x_1 \tan x_2} = \tan (x_1 + x_2).$$

<u>Theorem 4.18</u>: For all real values of x_1 and x_2, $\sin (x_1 + x_2) = \sin x_1 \cos x_2 + \sin x_2 \cos x_1$.

Let x_1 and x_2 be any two real numbers. We may determine two points k_1 and k_2 in $[-p,p]$ and two integers n_1 and n_2 such that $x_1 = k_1 + 2n_1p$, and $x_2 = k_2 + 2n_2p$. We may also determine a k_3 in $[-p,p]$ and $n_3 = 0$, 1, or -1 such that $k_1 + k_2 = k_3 + 2n_3p$ according as $-p \leq k_1 + k_2 \leq p$, $k_1 + k_2 > p$, or $k_1 + k_2 < -p$. Then $x_1 + x_2 = (k_1 + k_2) + (n_1 + n_2)2p = k_3 + (n_1 + n_2 + n_3)2p$. Also $\sin (x_1 + x_2) = S(k_3)$ or $-S(k_3)$ according as $(n_1 + n_2 + n_3)$ is an even or odd integer; $\sin x_1 = S(k_1)$ or $-S(k_1)$ and $\cos x_1 = C(k_1)$ or $-C(k_1)$ according as n_1 is an even or odd integer; $\sin x_2 = S(k_2)$ or $-S(k_2)$ and $\cos x_2 = C(k_2)$ or $-C(k_2)$ according as n_2 is an even or odd integer.

Case I. Let n_1 and n_2 be even integers, or let n_1 and n_2 be odd integers. Then

$$\sin x_1 \cos x_2 + \sin x_2 \cos x_1 = S(k_1)C(k_2) + S(k_2)C(k_1).$$

Also $\sin (x_1 + x_2) = S(k_3)$ if $n_3 = 0$ and $-S(k_3)$ if $n_3 = 1$ or -1. From Theorem 3.15,

$$S(k_1)C(k_2) + S(k_2)C(k_1) = S(k_1 + k_2),\ -S(k_1 + k_2 - 2p),$$
$$\text{or } -S(k_1 + k_2 + 2p),$$

according as $-p \leq k_1 + k_2 \leq p$, $k_1 + k_2 > p$, or $k_1 + k_2 < -p$. But $S(k_1 + k_2) = S(k_3)$ for $-p \leq k_1 + k_2 \leq p$; $-S(k_1 + k_2 - 2p) = -S(k_3)$ for $k_1 + k_2 > p$; $-S(k_1 + k_2 + 2p) = -S(k_3)$ for $k_1 + k_2 < -p$. Hence if n_1 and n_2 are even integers, or if n_1 and n_2 are odd integers,

$$\sin x_1 \cos x_2 + \sin x_2 \cos x_1 = \sin (x_1 + x_2).$$

Case II. Let n_1 be an even integer and n_2 an odd integer, or let n_1 be an odd integer and n_2 an even integer. Then

$$\sin x_1 \cos x_2 + \sin x_2 \cos x_1 = - \left[S(k_1)C(k_2) + S(k_2)C(k_1)\right].$$

Also $\sin (x_1 + x_2) = -S(k_3)$ if $n_3 = 0$ and $S(k_3)$ if $n_3 = 1$ or -1. From Theorem 3.15,

$$- \left[S(k_1)C(k_2) + S(k_2)C(k_1)\right] = -S(k_1 + k_2),\ S(k_1 + k_2 - 2p),\ \text{or } S(k_1 + k_2 + 2p),$$

according as $-p \leq k_1 + k_2 \leq p$, $k_1 + k_2 > p$, or $k_1 + k_2 < -p$. But $-S(k_1 + k_2) = -S(k_3)$ for $-p \leq k_1 + k_2 \leq p$, $S(k_1 + k_2 - 2p) = S(k_3)$ for $k_1 + k_2 > p$, and $S(k_1 + k_2 + 2p) = S(k_3)$ for $k_1 + k_2 < -p$. Hence if n_1 is an even integer and n_2 an odd integer, or if n_1 is an odd integer and n_2 an even integer,

$$\sin x_1 \cos x_2 + \sin x_2 \cos x_1 = \sin (x_1 + x_2).$$

<u>Theorem 4.19</u>: For all real values of x_1 and x_2, $\cos (x_1 + x_2) = \cos x_1 \cos x_2 - \sin x_1 \sin x_2$.

Let x_1 and x_2 be any two real numbers. We may determine two points k_1 and k_2 in $[-p,p]$ and two integers n_1 and n_2 such that $x_1 = k_1 + 2n_1p$ and $x_2 = k_2 + 2n_2p$. We may also determine a k_3 in $[-p,p]$ and $n_3 = 0$, 1, or -1 such that $k_1 + k_2 = k_3 + 2n_3p$ according as $-p \leq k_1 + k_2 \leq p$, $k_1 + k_2 > p$, or $k_1 + k_2 < -p$. Then $x_1 + x_2 = (k_1 + k_2) + (n_1 + n_2)2p = k_3 + (n_1 + n_2 + n_3)2p$. Also $\cos(x_1 + x_2) = C(k_3)$ or $-C(k_3)$ according as $(n_1 + n_2 + n_3)$ is an even or odd integer; $\sin x_1 = S(k_1)$ or $-S(k_1)$ and $\cos x_1 = C(k_1)$ or $-C(k_1)$ according as n_1 is an even or odd integer; $\sin x_2 = S(k_2)$ or $-S(k_2)$ and $\cos x_2 = C(k_2)$ or $-C(k_2)$ according as n_2 is an even or odd integer.

Case I. Let n_1 and n_2 be even integers, or n_1 and n_2 be odd integers. Then

$$\cos x_1 \cos x_2 - \sin x_1 \sin x_2 = C(k_1)C(k_2) - S(k_1)S(k_2).$$

Also $\cos(x_1 + x_2) = C(k_3)$ if $n_3 = 0$ and $-C(k_3)$ if $n_3 = 1$ or -1. From Theorem 3.16,

$$C(k_1)C(k_2) - S(k_1)S(k_2) = C(k_1 + k_2),\ -C(k_1 + k_2 - 2p),\ \text{or } -C(k_1 + k_2 + 2p),$$

according as $-p \leq k_1 + k_2 \leq p$, $k_1 + k_2 > p$, or $k_1 + k_2 < -p$. But $C(k_1 + k_2) = C(k_3)$ for $-p \leq k_1 + k_2 \leq p$, $-C(k_1 + k_2 - 2p) = -C(k_3)$ for $k_1 + k_2 > p$, and $-C(k_1 + k_2 + 2p) = -C(k_3)$ for $k_1 + k_2 < -p$. Hence if n_1 and n_2 are even integers, or n_1

and n_2 are odd integers,

$$\cos x_1 \cos x_2 - \sin x_1 \sin x_2 = \cos (x_1 + x_2).$$

Case II. Let n_1 be an even integer and n_2 be an odd integer, or let n_1 be an odd integer and n_2 an even integer. Then

$$\cos x_1 \cos x_2 - \sin x_1 \sin x_2 = - \left[C(k_1)C(k_2) - S(k_1)S(k_2)\right].$$

Also $\cos (x_1 + x_2) = -C(k_3)$ if $n_3 = 0$ and $C(k_3)$ if $n_3 = 1$ or -1. From Theorem 3.16,

$$-\left[C(k_1)C(k_2) - S(k_1)S(k_2)\right] = -C(k_1 + k_2),\ C(k_1 + k_2 - 2p),\ \text{or } C(k_1 + k_2 + 2p),$$

according as $-p \leq k_1 + k_2 \leq p$, $k_1 + k_2 > p$, or $k_1 + k_2 < -p$. But $-C(k_1 + k_2) = -C(k_3)$ for $-p \leq k_1 + k_2 \leq p$; $C(k_1 + k_2 - 2p) = C(k_3)$ for $k_1 + k_2 > p$; $C(k_1 + k_2 + 2p) = C(k_3)$ for $k_1 + k_2 < -p$. Hence if n_1 is an even integer and n_2 an odd integer, or if n_1 is an odd integer and n_2 an even integer,

$$\cos x_1 \cos x_2 - \sin x_1 \sin x_2 = \cos (x_1 + x_2).$$

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