SOME PROPERTIES

OF A LEBESGUE-STIELTJES INTEGRAL

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THESIS

Presented to the Graduate Council of the
North Texas State College in Partial
Fulfillment of the Requirements

For the Degree of

MASTER OF SCIENCE

By

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Denton, Texas

August, 1951

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CHAPTER I

INTRODUCTION

The Stieltjes integration was first introduced by Thomas Jan Stieltjes in a memoir of 1894 on the subject of continued fractions.[1] He defined the integration of a continuous function with respect to a monotone non-decreasing function, the integration being performed over an entire interval.[2]

It is the purpose of this paper to define a Lebesgue integral over a measurable set, the integration being performed with respect to a monotone non-decreasing function as in the Stieltjes integral, and to develop a few of the fundamental properties of such an integral. Much of the development and generalization of such an integral is credited to J. Radon, for whom the integral is often called the Radon or Lebesgue-Radon integral.[3] The definition of the integral given in this paper appears to be equivalent to

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[1]Jesse Douglas, Survey of the Theory of Integration, p. 19.

[2]T. H. Hildebrandt, "On Integrals Related to and Extensions of the Lebesgue Integrals", American Mathematical Society Bulletin, pp. 177-178.

[3]Stanislaw Saks, Theory of Integration, p. 67, Translated by L. C. Young.

that given by Hobson[1], so that we shall, as he does, refer to the integral as a Lebesgue-Stieltjes integral.

All functions used in this paper will be assumed to be single-valued and defined on some set E. By a set we shall in all cases mean a linear set of points. For convenience, we list now the majority of definitions to be employed.

<u>Definition 1.1</u>. If S_1 and S_2 are any two sets, then $S_1 + S_2$ will mean the set of all points which are in either S_1 or S_2 (or both).

<u>Definition 1.2</u>. If S_1 and S_2 are any two sets, then $S_1 \cdot S_2$ will mean the set of all points which are in both S_1 and S_2.

<u>Definition 1.3</u>. If S_1 and S_2 are any two sets, then $S_1 - S_2$ will mean the set of all points which are in S_1 but not in S_2.

<u>Definition 1.4</u>. An open interval (a,b) is the set of all points x such that $a < x < b$. A closed interval $[a,b]$ is the set of all points x such that $a \leq x \leq b$.

<u>Definition 1.5</u>. A point set S will be considered bounded if there exists an open interval (a,b) such that every element of S is a point in (a,b).

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[1] E. W. Hobson, <u>The Theory of Functions of a Real Variable</u>, p. 663.

<u>Definition</u> <u>1.6</u>. A function $f(x)$ will be considered bounded on the set E if there exists a constant $M > 0$ such that $|f(x)| \leq M$ for every x in E.

<u>Definition</u> <u>1.7</u>. A covering of a set S is a set K of open intervals such that every point of S is in at least one of the intervals of K.

<u>Definition</u> <u>1.8</u>. A set E is of <u>Lebesgue</u> <u>measure</u> <u>zero</u> if for every $\epsilon > 0$ there exists a covering of E with length sum $< \epsilon$.

<u>Definition</u> <u>1.9</u>. The greatest lower bound (g.l.b.) of a set E is the number L such that for every x in E

$$x \geq L$$

and such that for every $\epsilon > 0$ there exists an x' in E such that

$$x' < L + \epsilon.$$

<u>Definition</u> <u>1.10</u>. The least upper bound (l.u.b.) of a set E is the number U such that for every x in E

$$x \leq U$$

and such that for every $\epsilon > 0$ there exists an x' in E such that

$$x' > U - \epsilon.$$

<u>Definition</u> <u>1.11</u>. The g.l.b. of a function $f(x)$ on a set E is the number L such that for every x in E

$$f(x) \geq L$$

and such that for every $\epsilon > 0$ there exists an x' in E such that

$$f(x') < L + \epsilon.$$

Definition 1.12. The l.u.b. of a function $f(x)$ on a set E is the number U such that for every x in E

$$f(x) \leq U$$

and such that for every $\epsilon > 0$ there exists an x' in E such that

$$f(x') > U - \epsilon.$$

Definition 1.13. Let S be any bounded set contained in (a,b). Let $\{I_n\}$ be any covering of S and denote by $l(I_p)$ the length of the interval I_p. Then the g.l.b. of $\sum_n l(I_n)$ for all finite or denumerable coverings of S is defined to be the exterior measure of S and is denoted by $m_e(S)$. Denoting $\bar{S} = (a,b) - S$, we let the interior measure of S be represented by the symbol $m_i(S)$ and define

$$m_i(S) = b - a - m_e(\bar{S}).$$

If $m_i(S) = m_e(S)$, then we shall say that S is measurable and denote its measure by $m(S) = m_i(S) = m_e(S)$.[1]

Definition 1.14. Let S be an unbounded point set. Then we shall define S to be measurable if, for every positive integer n, the set $S_n = (-n,n)\cdot S$ is measurable. If S

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[1]H. P. Thielman, Lectures on the Theory of Functions, p. 73.

is measurable, we define

$$m(S) = \lim_{n \to \infty} m(S_n).$$

Definition 1.15. A function $f(x)$ is said to be measurable if, for every real number k, the set $E = \underset{x}{E}\left[f(x) > k\right]$ is measurable.

Definition 1.16. A function $g(x)$ is monotone non-decreasing over E if, for every two points $x_1 > x_2$ in E, $g(x_1) \geq g(x_2)$.

Definition 1.17. A function is of bounded variation if it can be represented as the difference of two bounded monotone non-decreasing functions.

Definition 1.18. A function $g(x)$ defined on a set E is said to be a Lipschitzian function or to satisfy a Lipschitz condition if there exists a constant M such that $\left|g(x_1) - g(x_2)\right| \leq M\left|x_1 - x_2\right|$ for all x_1 and x_2 in E.[1] It is noted that if $g(x)$ is monotone non-decreasing, the absolute value signs may be removed, provided $x_1 > x_2$.

Definition 1.19. If $g(x)$ is a monotone non-decreasing function on the set E and if for every measurable subset E^* of E the ordinate set $Y^* = \underset{y}{E}\left[y = g(x),\ x \text{ in } E^*\right]$ is measurable, then we shall say that $g(x)$ is Y-measurable.

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[1]Edward James McShane, Integration, p. 50.

Definition 1.20. A function $g(x)$ is absolutely continuous on E if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\sum |g(\beta_i) - g(\alpha_i)| < \epsilon$ for all finite or denumerable sets of non-overlapping (except on the end-points) intervals $\{I_n\} = [\alpha_n, \beta_n]$ whose length sum $\sum l(I_i) < \delta$ and such that for every i, α_i and β_i are points of E.[1]

We shall assume the truth of the following theorems.

Theorem 1.1. If S_1 and S_2 are any two measurable sets whose overlapping is a set of measure zero, then $S = S_1 + S_2$ is measurable and $m(S) = m(S_1) + m(S_2)$.

Theorem 1.2. If S_1 and S_2 are any two measurable sets, then $S = S_1S_2$ is measurable.

Theorem 1.3. Every null, finite, or denumerable set is of measure zero.

Theorem 1.4. If S_1 and S_2 are any two measurable sets, then $S = S_1 - S_2$ is measurable. If S_1 contains S_2, i.e., if S_2 is a subset of S_1, then $m(S) = m(S_1) - m(S_2)$.

Theorem 1.5. If $f_1(x)$ and $f_2(x)$ are any two measurable functions, then

(i) $F(x) = f_1(x) + f_2(x)$ is measurable,

(ii) $F(x) = f_1(x) - f_2(x)$ is measurable,

(iii) $F(x) = f_1(x) \cdot f_2(x)$ is measurable.

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[1] Compare ibid., pp. 47-48.

<u>Theorem 1.6</u>. Let g(x) be a bounded monotone non-decreasing function on the measurable set E. A necessary and sufficient condition that g(x) be Y-measurable is that for every subset E^* of E with measure zero, the ordinate set $Y^* = \underset{y}{E}\left[y = g(x),\ x \text{ in } E^*\right]$ be of measure zero.[1]

<u>Theorem 1.7</u>. If f(x) is any measurable function, the set $E = \underset{x}{E}\left[y' < f(x) \leq y''\right]$ is measurable for every $y' < y''$.

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[1]Hobson, <u>op</u>. <u>cit</u>., p. 341.

CHAPTER II

SOME PROPERTIES OF A LEBESGUE-STIELTJES INTEGRAL

We are now in a position to make our initial definition of the integral.

<u>Definition 2.1</u>. Let f(x) be measurable and bounded on the measurable set E. Let g(x) be a bounded, monotone non-decreasing, Y-measurable function over E. Let U denote an upper bound of f(x) over E and let L denote a lower bound, not the g.l.b., of f(x) on E. Let

$$\sigma : L = y_0 < y_1 < y_2 < \cdot \cdot \cdot < y_{n-1} < y_n = U$$

be any subdivision of $[L, U]$. Denote by E_i the set of points x at which $y_{i-1} < f(x) \leq y_i$ and let $Y_i = \underset{y}{E}[y = g(x), \text{ x in } E_i]$. Then

$$\overline{\sum}_{\sigma} :\sim: \sum_{i=1}^{n} y_i m(Y_i),$$

$$\underline{\sum}_{\sigma} :\sim: \sum_{i=1}^{n} y_{i-1} m(Y_i),$$

$$\overline{\int}_E :\sim: \text{g.l.b. of } \overline{\sum}_{\sigma} \text{ for all possible } \sigma,$$

and

$$\underline{\int}_E :\sim: \text{l.u.b. of } \underline{\sum}_{\sigma} \text{ for all possible } \sigma.$$

If $\overline{\int}_E = \underline{\int}_E$, then their common value will be defined to be the Lebesgue-Stieltjes integral of f(x) with respect to g(x) over E and is denoted by

$$\int_E f(x)\,dg(x).$$

<u>Theorem</u> <u>2.1</u>. The L-S integral is independent of the choice of an upper bound U and a lower bound, not the g.l.b., L.

Proof: Let f(x) be any single-valued function defined, measurable and bounded on the measurable set E. Let g(x) be any bounded, monotone non-decreasing, Y-measurable function over E. Choose any upper bound U of f(x) on E and let L be any lower bound which is not the g.l.b. of f(x) on E. Denote by U^* and L^* respectively the l.u.b. and the g.l.b. of f(x) on E. Then $U \geq U^*$, $L < L^*$.

Using the notation of the definition let σ be any subdivision of $[L,U]$. Choose the positive integer μ so that $y_{\mu-1} \leq L^* < y_\mu$ and a positive integer ν so that $y_{\nu-1} < U^* \leq y_\nu$. Let $y_0^* = L^*$, $y_1^* = y_\mu$, $y_2^* = y_{\mu+1}$, . . . , $y_{k-1}^* = y_{\nu-1}$, $y_k^* = U^*$. Then

$$E_i \;:\sim:\; \underset{x}{E}\left[y_{i-1} < f(x) \leq y_i\right],$$

$$E_i^* \;:\sim:\; \underset{x}{E}\left[y_{i-1}^* < f(x) \leq y_i^*\right],$$

$$Y_i \;:\sim:\; \underset{y}{E}\left[y = g(x),\ x \text{ in } E_i\right],$$

and

$$Y_i^* \;:\sim:\; \underset{y}{E}\left[y = g(x),\ x \text{ in } E_i^*\right].$$

Thus

$$\overline{\sum}_{\sigma} = y_1 m(Y_1) + \ldots + y_{\mu-1} m(Y_{\mu-1}) + y_{\mu} m(Y_{\mu}) + y_{\mu+1} m(Y_{\mu+1}) + \ldots + y_{\nu} m(Y_{\nu}) + y_{\nu+1} m(Y_{\nu+1}) + \ldots + y_n m(Y_n),$$

and

$$\underline{\sum}_{\sigma} = y_0 m(Y_1) + \ldots + y_{\mu-2} m(Y_{\mu-1}) + y_{\mu-1} m(Y_{\mu}) + y_{\mu} m(Y_{\mu+1}) + \ldots + y_{\nu-1} m(Y_{\nu}) + y_{\nu} m(Y_{\nu+1}) + \ldots + y_{n-1} m(Y_n).$$

Denote by Y_0^* the set of points $\underset{y}{E}\left[y = g(x),\ x \text{ in } E^{**}\right]$ where $E^{**} = \underset{x}{E}\left[f(x) = L^*\right]$. Let

$$\overline{\sum}_{\sigma^*} = y_1^* m(Y_0^*) + \sum_{i=1}^{k} y_i^* m(Y_i^*),$$

$$\underline{\sum}_{\sigma^*} = y_0^* m(Y_0^*) + \sum_{i=1}^{k} Y_{i-1}^* m(Y_i^*).$$

Since y_0^* and y_k^* bound the function $f(x)$ over E, we have

$$\begin{aligned}\overline{\sum}_{\sigma} &= y_1^* m(Y_0^* + Y_1^*) + y_2^* m(Y_2^*) + \ldots + y_{\nu} m(Y_k^*) \\ &= y_1^* m(Y_0^*) + y_1^* m(Y_1^*) + y_2^* m(Y_2^*) + \ldots \\ &\qquad + y_k^* m(Y_k^*) + (y_{\nu} - y_k^*) m(Y_k^*).\end{aligned}$$

But $y_{\nu} - y_k^* \geq 0$. Hence $\overline{\sum}_{\sigma} \geq \overline{\sum}_{\sigma^*}$. Since σ was an arbitrary subdivision of $[L,U]$, it follows that for every subdivision σ of $[L,U]$ there exists a subdivision σ^* of $[L^*,U^*]$ so that

$$\text{(1)} \qquad \overline{\sum}_{\sigma} \geq \overline{\sum}_{\sigma^*}.$$

Next, let σ^* be any subdivision of $[L^*,U^*]$ and let $y_0 = L$, $y_1 = y_0^* = L^*$, $y_2 = y_1^*$, . . . , $y_{k+1} = y_k^*$ and if $U > U^*$, let $y_{k+2} = U$. [If $U = U^*$, $y_{k+1} = U$]. Then

$$\overline{\sum}_{\sigma} = y_1 m(Y_1) + y_2 m(Y_2) + \ldots + y_{k+j} m(Y_{k+j}),$$

where $j = 1$ or 2, and

$$\begin{aligned}\overline{\sum}_{\sigma} &= y_0^* m(Y_0^*) + y_1^* m(Y_1^*) + \dots + y_k^* m(Y_k^*) \\ &= y_1^* m(Y_0^*) - (y_1^* - y_0^*) m(Y_0^*) + \sum_{i=1}^{k} y_i^* m(Y_i^*) \\ &= \overline{\sum}_{\sigma^*} - (y_1^* - y_0^*) m(Y_0^*) \leq \overline{\sum}_{\sigma^*},\end{aligned}$$

i.e., for any subdivision σ^* of $[L^*, U^*]$, there exists a subdivision σ of $[L, U]$ such that

$$\text{(2)} \qquad \overline{\sum}_{\sigma^*} \geq \overline{\sum}_{\sigma}.$$

Now

$$\overline{\int}_E = \text{g.l.b. for all possible } \sigma \text{ of } \overline{\sum}_{\sigma}.$$

Then from (1) it follows that

$$\overline{\int}_E \geq \text{g.l.b. for all possible } \sigma^* \text{ of } \overline{\sum}_{\sigma^*},$$

and from (2) we have

$$\overline{\int}_E \leq \text{g.l.b. for all possible } \sigma^* \text{ of } \overline{\sum}_{\sigma^*}.$$

It follows that

$$\overline{\int}_E = \text{g.l.b. for all possible } \sigma^* \text{ of } \overline{\sum}_{\sigma^*},$$

which is independent of the choice of L and U. In a similar fashion, it can be shown that

$$\underline{\int}_E = \text{l.u.b. for all possible } \sigma^* \text{ of } \underline{\sum}_{\sigma^*},$$

so that $\underline{\int}_E$ is also independent of the choice of L and U.

Thus the theorem follows.

Let f(x) be any bounded, measurable function on the measurable set E and let g(x) be any bounded, monotone non-decreasing, Y-measurable function defined on E. Referring to the notation of Definition 2.1, we state the following four theorems.

Theorem 2.2. If σ_1 and σ_2 are two subdivisions of $[L,U]$ such that every point of σ_1 is in σ_2, then $\overline{\sum}_{\sigma_2} \leq \overline{\sum}_{\sigma_1}$ and $\underline{\sum}_{\sigma_2} \geq \underline{\sum}_{\sigma_1}$.

First let σ be a subdivision of $[L,U]$ and denote by σ^* the subdivision of $[L,U]$ consisting of all the points of σ plus one extra point, say y^*. Then we shall denote

$$\sigma : L = y_0 < y_1 < \cdot \cdot \cdot < y_p < y_{p+1} < \cdot \cdot \cdot < y_{n-1} < y_n = U,$$

$$\sigma^* : L = y_0 < y_1 < \cdot \cdot \cdot < y_p < y^* < y_{p+1} < \cdot \cdot \cdot < y_{n-1} < y_n = U.$$

Let $E_i = \underset{x}{E}\left[y_{i-1} < f(x) \leq y_i\right]$, and let $E^* = \underset{x}{E}\left[y_p < f(x) \leq y^*\right]$, $E^{**} = \underset{x}{E}\left[y^* < f(x) \leq y_{p+1}\right]$. Let Y_i be the set of functional values for g(x) over E_i and let Y^* and Y^{**} be the set of functional values for g(x) over E^* and E^{**}, respectively. Then

$$\begin{aligned}\overline{\sum}_{\sigma^*} &= y_1 m(Y_1) + \cdot \cdot \cdot + y_p m(Y_p) + y^* m(Y^*) + y_{p+1} m(Y^{**}) \\ &\quad + y_{p+2} m(Y_{p+2}) + \cdot \cdot \cdot + y_n m(Y_n) \\ &= \overline{\sum}_{\sigma} + y^* m(Y^*) + y_{p+1} m(Y^{**}) - y_{p+1} m(Y_{p+1}) \\ &\leq \overline{\sum}_{\sigma} + y_{p+1} m(Y^*) + y_{p+1} m(Y^{**}) - y_{p+1} m(Y_{p+1})\end{aligned}$$

$$= \overline{\sum}_{\sigma} + y_{p+1}m(Y^* + Y^{**}) - y_{p+1}m(Y_{p+1})$$

$$= \overline{\sum}_{\sigma}.$$

Hence $\overline{\sum}_{\sigma^*} \leq \overline{\sum}_{\sigma}$. In a similar fashion, it can be shown that $\underline{\sum}_{\sigma^*} \geq \underline{\sum}_{\sigma}$.

Next, let σ_1 and σ_2 be any two subdivisions of $[L,U]$ such that σ_2 contains every point of σ_1. Let q denote the number of points in σ_1 and let q + i denote the number of points in σ_2. If i = 0, the theorem is obvious and if i = 1, it has been proved in the preceding paragraph. Let K denote the set of positive integers, i, for which the theorem is true. Then K contains 1. Suppose K contains k. Let σ_2 consist of all the points of σ_1 plus k + 1 additional points. Choose one of the k + 1 additional points, say x_0, and denote by σ_3 the subdivision consisting of all the points of σ_2 except x_0. Since K contains k, $\overline{\sum}_{\sigma_3} \leq \overline{\sum}_{\sigma_1}$ and $\underline{\sum}_{\sigma_3} \geq \underline{\sum}_{\sigma_1}$. But K also contains 1. Hence $\overline{\sum}_{\sigma_2} \leq \overline{\sum}_{\sigma_3}$ and $\underline{\sum}_{\sigma_2} \geq \underline{\sum}_{\sigma_3}$. Thus $\overline{\sum}_{\sigma_2} \leq \overline{\sum}_{\sigma_1}$ and $\underline{\sum}_{\sigma_2} \geq \underline{\sum}_{\sigma_1}$. Therefore K contains k + 1. By induction, K contains every positive integer and the theorem follows.

<u>Theorem 2.3</u>. If σ is any subdivision of $[L,U]$, then $\underline{\sum}_{\sigma} \leq \overline{\sum}_{\sigma}$.

For, let σ be any subdivision of $[L,U]$. Then

$$\overline{\sum}_{\sigma} - \underline{\sum}_{\sigma} = (y_1 - y_0)m(Y_1) + (y_2 - y_1)m(Y_2) + \cdots + (y_{n-1} - y_{n-2})m(Y_{n-1}) + (y_n - y_{n-1})m(Y_n) \geq 0.$$

<u>Theorem</u> <u>2.4</u>. If σ_1 and σ_2 are any two subdivisions of $[L,U]$, $\underline{\sum}_{\sigma_1} \leq \overline{\sum}_{\sigma_2}$.

Let σ_1 and σ_2 be any two subdivisions of $[L,U]$. Denote by σ_3 the subdivision consisting of all the points used in σ_1 and σ_2. Then by Theorem 2.2,

$$\underline{\sum}_{\sigma_1} \leq \underline{\sum}_{\sigma_3},$$

$$\overline{\sum}_{\sigma_3} \leq \overline{\sum}_{\sigma_2}.$$

By the preceding theorem, $\underline{\sum}_{\sigma_3} \leq \overline{\sum}_{\sigma_3}$. Hence

$$\underline{\sum}_{\sigma_1} \leq \overline{\sum}_{\sigma_2}.$$

<u>Theorem</u> <u>2.5</u>. $\overline{\int}_E \geq \underline{\int}_E$.

For suppose otherwise, i.e., suppose $\overline{\int}_E < \underline{\int}_E$. From the definition of the upper and lower integrals, there exists a σ_1 and a σ_2 such that

$$\underline{\sum}_{\sigma_1} > \underline{\int}_E - \epsilon,$$

$$\overline{\sum}_{\sigma_2} < \overline{\int}_E + \epsilon.$$

Choose $\epsilon = \frac{1}{4}(\underline{\int}_E - \overline{\int}_E)$. Then

$$\underline{\sum}_{\sigma_1} - \overline{\sum}_{\sigma_2} > \underline{\int}_E - \overline{\int}_E - 2\epsilon$$

$$= \left[\underline{\int}_E - \overline{\int}_E\right] - \tfrac{1}{2}\left[\underline{\int}_E - \overline{\int}_E\right]$$

$$= \tfrac{1}{2}\left[\underline{\int}_E - \overline{\int}_E\right]$$

$$> 0.$$

But from Theorem 2.4, $\underline{\sum}_{\sigma_1} - \overline{\sum}_{\sigma_2} \leq 0$. Therefore the supposition was false and the theorem follows.

Henceforth we shall say $[L,U]$ is <u>admissible</u> if L is a lower bound, not the g.l.b., and if U is an upper bound of f(x) on E.

<u>Theorem 2.6</u>. Every function bounded, single-valued and measurable over the measurable set E is L-S integrable over E with respect to a function bounded, monotone non-decreasing and Y-measurable over E.

Let $\epsilon > 0$ be arbitrarily chosen and using the notation of the definition 2.1, let

$$\sigma : L = y_0 < y_1 < y_2 < \cdots < y_{n-1} < y_n = U$$

be a subdivision of any admissible $[L,U]$ such that for every $1 \leq i \leq n$,

$$y_i - y_{i-1} < \frac{\epsilon}{m(Y) + 1},$$

where $Y = \underset{y}{E}[y = g(x),\ x \text{ in } E]$. Then

$$0 \leq \overline{\sum}_\sigma - \underline{\sum}_\sigma = \sum_{i=1}^{n} y_i m(Y_i) - \sum_{i=1}^{n} y_{i-1} m(Y_i)$$
$$= \sum_{i=1}^{n} (y_i - y_{i-1}) m(Y_i)$$

$$< \sum_{i=1}^{n} \frac{\epsilon}{m(Y) + 1} m(Y_i)$$

$$= \frac{\epsilon}{m(Y) + 1} m(Y)$$

$$< \epsilon.$$

But

$$\underline{\sum}_{\sigma} \le \underline{\int}_E \le \overline{\int}_E \le \overline{\sum}_{\sigma}.$$

Thus

$$0 \le \overline{\int}_E - \underline{\int}_E < \epsilon.$$

Since $\epsilon > 0$ was arbitrarily chosen, this inequality must hold for every $\epsilon > 0$. It follows that

$$\overline{\int}_E = \underline{\int}_E.$$

__Theorem 2.7.__ Let $f(x)$ be a bounded, single-valued, measurable function over the measurable set E. Let $g(x)$ be a bounded, monotone non-decreasing, Y-measurable function over E. Then a necessary and sufficient condition that $f(x)$ be L-S integrable with respect to $g(x)$ over E is that for every $\epsilon > 0$ and for every admissible $[L,U]$ there exists a subdivision σ of $[L,U]$ such that

$$\overline{\sum}_{\sigma} - \underline{\sum}_{\sigma} < \epsilon.$$

__Sufficiency:__ See Theorem 2.6.

__Necessity:__ Suppose that for every $\epsilon > 0$ and for every admissible $[L,U]$ there exists a subdivision σ of $[L,U]$

such that

$$\overline{\overline{\sum}}_{\sigma} - \underline{\underline{\sum}}_{\sigma} < \epsilon.$$

Choose an arbitrary $\epsilon > 0$ and any admissible $[L,U]$. Then there exists a subdivision σ of $[L,U]$ such that

$$\overline{\overline{\sum}}_{\sigma} - \underline{\underline{\sum}}_{\sigma} < \epsilon.$$

But

$$\overline{\int} \leq \overline{\overline{\sum}}_{\sigma}, \underline{\int} \geq \underline{\underline{\sum}}_{\sigma}, \text{ and } \overline{\int} \geq \underline{\int}.$$

Thus it follows that

$$\overline{\int} - \underline{\int} < \epsilon.$$

Since this result can be obtained for every preassigned $\epsilon > 0$, it must follow that

$$\overline{\int} = \underline{\int}.$$

Therefore $\int_E f(x)dg(x)$ exists. It has previously been shown that the L-S integral is independent of the choice of admissible $[L,U]$. The truth of the theorem follows.

<u>Theorem 2.8</u>. If $f(x)$ is bounded, single-valued, and measurable over the measurable set E, if $g(x)$ is a bounded, monotone non-decreasing function defined over E, and if $g(x)$ is Y-measurable over E, then

$$L \cdot m(Y) \leq \int_E f(x)dg(x) \leq U \cdot m(Y).$$

Let σ be any subdivision of $[L, U]$. Then

$$\overline{\sum}_\sigma = \sum_{i=1}^{n} y_i m(Y_i) \leq \sum_{i=1}^{n} Um(Y_i) = U\sum_{i=1}^{n} m(Y_i) = Um(Y),$$

and

$$\underline{\sum}_\sigma = \sum_{i=1}^{n} y_{i-1} m(Y_i) \geq \sum_{i=1}^{n} Lm(Y_i) = L\sum_{i=1}^{n} m(Y_i) = Lm(Y).$$

From Theorem 2.3,

$$\underline{\sum}_\sigma \leq \int_E f(x)dg(x) \leq \overline{\sum}_\sigma .$$

Therefore,

$$L\cdot m(Y) \leq \int_E f(x)dg(x) \leq Um(Y).$$

<u>Theorem 2.9</u>. If $f(x)$ is bounded, single-valued, and measurable over the measurable set E, if $g(x)$ is bounded, monotone non-decreasing, and Y-measurable over E, then $f(x)$ is L-S integrable with respect to $g(x)$ over any measurable subset of E.

Let M be any measurable subset of E. Since $f(x)$ is bounded over E, it is bounded over M. Because $f(x)$ is measurable over E, the set of points of E where $f(x) > k$, for any real k, is measurable. Since the set of points of M where $f(x) > k$ is the product of M and the set of points of E where $f(x) > k$, it follows that $\underset{x}{E}[f(x) > k, x \text{ in } M]$, for any real k, is measurable. Hence $f(x)$ is measurable over M. Clearly, $g(x)$ retains its boundedness and is monotone non-decreasing over M. Hence by Theorem 2.6,

$$\int_M f(x)dg(x) \text{ exists.}$$

Theorem 2.10. If $f(x)$ is bounded, measurable, and single-valued over the measurable set E, if $g(x)$ is bounded, monotone non-decreasing, and Y-measurable over E, and if E_1 and E_2 are non-overlapping measurable subsets of E such that $E = E_1 + E_2$, then

$$\int_{E_1} f(x)dg(x) \text{ and } \int_{E_2} f(x)dg(x)$$

both exist and

$$\int_E f(x)dg(x) = \int_{E_1} f(x)dg(x) + \int_{E_2} f(x)dg(x).$$

By the previous theorem $\int_{E_1} f(x)dg(x)$ and $\int_{E_2} f(x)dg(x)$ both exist.

Let $\epsilon > 0$ be arbitrarily chosen and choose an upper bound U and a lower bound L, not the greatest lower bound, of $f(x)$ on E. By Theorem 2.7, there exists a subdivision σ of $[L,U]$ such that $\overline{\sum_E}\sigma - \underline{\sum_E}\sigma < \epsilon$, where $\overline{\sum_E}\sigma$ and $\underline{\sum_E}\sigma$ are the upper and lower sums over E. Thus

$$\overline{\sum_E}\sigma = y_1 m(Y_1) + y_2 m(Y_2) + \cdot \cdot \cdot + y_n m(Y_n)$$

where $E_j = \underset{x}{E}\left[y_{j-1} < f(x) \le y_j\right] \cdot E$ and $Y_j = \underset{y}{E}\left[y = g(x), \text{ x in } E_j\right]$. Using similar notation,

$$\overline{\sum_{E_1}}\sigma = y_1 m(Y_{11}) + y_2 m(Y_{12}) + \cdot \cdot \cdot + y_n m(Y_{1n})$$

and

$$\overline{\sum_{E_2}}\sigma = y_1 m(Y_{21}) + y_2 m(Y_{22}) + \ . \ . \ . + y_n m(Y_{2n})$$

where

$$E_{ij} = \mathop{E}_{x}\left[y_{j-1} < f(x) \leq y_j\right] \cdot E_i$$

and

$$Y_{ij} = \mathop{E}_{y}\left[y = g(x),\ x \text{ in } E_{ij}\right] ,$$

$i = 1, 2$. Clearly, for $j = 1,2, \ . \ . \ . \ , n$,

$$E_j = E_{1j} + E_{2j}$$

and E_{1j}, E_{2j} are non-overlapping. Then it is clear that for $j = 1,2, \ . \ . \ . \ , n$,

$$Y_j = Y_{1j} + Y_{2j}.$$

By Theorem 1.6, Y_{1j} overlaps Y_{2j} on at most a set of measure zero. Therefore, by Theorem 1.1, for $j = 1,2, \ . \ . \ . \ , n$,

$$m(Y_j) = m(Y_{1j}) + m(Y_{2j}).$$

It follows that

$$\overline{\sum_{E}}\sigma = \overline{\sum_{E_1}}\sigma + \overline{\sum_{E_2}}\sigma.$$

In a similar fashion, it can be shown that

$$\underline{\sum_{E}}\sigma = \underline{\sum_{E_1}}\sigma + \underline{\sum_{E_2}}\sigma.$$

Also,

$$\underline{\sum_{E_1}}\sigma \leq \int_{E_1} f(x)\,dg(x) \leq \overline{\sum_{E_1}}\sigma,$$

and

$$\underline{\sum_{E_2}}\sigma \leq \int_{E_2} f(x)\,dg(x) \leq \overline{\sum_{E_2}}\sigma.$$

Adding inequalities gives

$$\underline{\sum_{E_1}}\sigma + \underline{\sum_{E_2}}\sigma \leq \int_{E_1} f(x)dg(x) + \int_{E_2} f(x)dg(x) \leq \overline{\sum_{E_1}}\sigma + \overline{\sum_{E_2}}\sigma,$$

or

$$\underline{\sum_{E}}\sigma \leq \int_{E_1} f(x)dg(x) + \int_{E_2} f(x)dg(x) \leq \overline{\sum_{E}}\sigma.$$

But $\underline{\sum_{E}}\sigma \leq \int_E f(x)dg(x) \leq \overline{\sum_{E}}\sigma$. It follows that

$$\left| \int_E f(x)dg(x) - \left[\int_{E_1} f(x)dg(x) + \int_{E_2} f(x)dg(x) \right] \right| < \epsilon.$$

Since $\epsilon > 0$ was arbitrarily chosen, it follows that

$$\int_E f(x)dg(x) = \int_{E_1} f(x)dg(x) + \int_{E_2} f(x)dg(x).$$

<u>Definition</u> <u>2.2</u>. Let $f(x)$ be bounded and measurable over the measurable set E. Let $G(x)$ be a function of bounded variation over E. Then $G(x) = g_1(x) - g_2(x)$ where both $g_1(x)$ and $g_2(x)$ are monotone non-decreasing functions. If $g_1(x)$ and $g_2(x)$ are both Y-measurable over E then we define

$$\int_E f(x)dG(x) = \int_E f(x)dg_1(x) + \int_E f(x)dg_2(x).$$

<u>Theorem</u> <u>2.11</u>. Let $g(x)$ be a bounded, monotone non-decreasing function over the measurable set E. A sufficient condition that, for every subset of measure zero, E^*, of E, the ordinate set $Y^* = \underset{y}{E}\left[y = g(x),\ x \text{ in } E^*\right]$ be a set of measure

zero, is that $g(x)$ be a Lipschitzian function over E.

Let $g(x)$ be a bounded, monotone non-decreasing Lipschitzian function over the measurable set E, and suppose E^* is a subset of E with measure zero. If E^* is null, finite, or denumerable, then $Y^* = \underset{y}{E}\left[y = g(x),\ x \text{ in } E^*\right]$ is at most a denumerable set, and hence is of measure zero. Suppose E^* is non-denumerable. Let $\epsilon > 0$ be arbitrarily chosen. Then there exists a sequence of intervals $\{I_n\}$ covering E^* and such that $\sum l(I_n) < \frac{\epsilon}{2M}$ where M is the constant such that

$$g(x_2) - g(x_1) < M(x_2 - x_1)$$

for every $x_1 < x_2$ in E. Consider the interval $I_p = (\alpha, \beta)$ of this sequence and denote E^*I_p by E_p. Suppose first that E_p is a non-denumerable set. Since $g(x)$ is bounded on E, $g(x)$ is bounded on E_p. Denote k_1 the g.l.b. of $g(x)$ on E_p and by k_2 the l.u.b. of $g(x)$ on E_p. If $k_1 = k_2$, then the set $Y_p = \underset{y}{E}\left[y = g(x),\ x \text{ in } E_p\right]$ is a single point and certainly may be covered by an interval J_p such that $l(J_p) \leq 2M(\beta - \alpha) = 2Ml(I_p)$. Suppose then that $k_2 > k_1$. Then for every x in E_p,

$$k_1 \leq g(x) \leq k_2$$

and there exists an x_1^* and an x_2^* in E such that

$$g(x_1^*) < k_1 + \delta\ ;\ g(x_2^*) > k_2 - \delta$$

where $\delta = \min\left[\frac{M\cdot(\beta - \alpha)}{4}, \frac{k_2 - k_1}{4}\right]$. It is clear that $g(x_1^*) < g(x_2^*)$. Thus $x_1^* < x_2^*$. We have

$$g(x_2^*) - g(x_1^*) \leq M|x_2 - x_1|.$$

But $|x_2 - x_1| < \beta - \alpha$. Hence

$$g(x_2^*) - g(x_1^*) < M(\beta - \alpha).$$

Then

$$(k_2 - \delta) - (k_1 + \delta) < M(\beta - \alpha),$$
$$(k_2 - k_1) - 2\delta < M(\beta - \alpha),$$

and

$$k_2 - k_1 < M(\beta - \alpha) + 2\delta,$$
$$\leq \frac{3}{2} M(\beta - \alpha).$$

Let $J_p = (k_1 - \delta, k_2 + \delta)$. Then it follows that $l(J_p) \leq 2Ml(I_p)$. But J_p covers the ordinate set Y_p. Since I_p was an arbitrarily chosen interval in the sequence $\{I_n\}$, it follows that for every interval I_s in this sequence, the ordinate set $Y_s = \underset{y}{E}[y = g(x), x \text{ in } E_s]$, where E_s is the set $I_s \cdot E$, may be covered with an interval J_s such that $l(J_s) \leq 2Ml(I_s)$. Thus there exists a covering $\{J_n\}$ of the ordinate set $Y^* = \underset{y}{E}[y = g(x), x \text{ in } E^*]$ with length sum λ such that

$$\lambda \leq 2M \sum l(I_n).$$

But $\sum l(I_n) < \frac{\epsilon}{2M}$. Hence

$$\lambda < 2M \cdot \frac{\epsilon}{2M} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, the theorem follows.

<u>Theorem 2.12</u>. If k is any real number, if f(x) is bounded and measurable over the measurable set E, and if g(x) is a bounded monotone non-decreasing, Y-measurable

function over E, then $\int_E kf(x)dg(x)$ exists and

$$\int_E kf(x)dg(x) = k\int_E f(x)dg(x).$$

Let k be any real number, let f(x) be bounded and measurable over the measurable set E, and let g(x) be bounded, monotone non-decreasing and Y-measurable over E. If F(x) = kf(x), then by previous theorems F(x) is bounded and measurable over E. By Theorem 2.6, $\int_E F(x)dg(x)$ exists.

If k = 0, then obviously $\int_E kf(x)dg(x) = k\int_E f(x)dg(x)$. Suppose $k > 0$. Choose some admissible $[L,U]$ for the function f(x) over E. Then $kL < F(x) \leq kU$ for every x in E and $[kL,kU]$ is an admissible bounding interval of F(x) on E.

Choose $n > \frac{2k(U - L)m(Y)}{\epsilon}$ where $Y = \underset{y}{E}[y = g(x),\ x \text{ in } E]$. Consider the two subdivisions

$$\sigma \text{ of } [L,U] : L < L + \tfrac{1}{n}(U - L) < L + \tfrac{2}{n}(U - L) < \ldots$$

$$< L + \tfrac{n}{n}(U - L) = U,$$

$$\sigma^* \text{ of } [kL,kU] : kL < kL + \tfrac{1}{n}(kU - kL) < kL + \tfrac{2}{n}(kU - kL)$$

$$< \ldots < kL + \tfrac{n}{n}(kU - kL) = kU.$$

Let $Y_i = \underset{y}{E}[y = g(x),\ x \text{ in } E_i]$ where

$$E_i = \underset{x}{E}\left[L + \frac{i - 1}{n}(U - L) < f(x) \leq L + \frac{i}{n}(U - L)\right]$$

and let $Y_i^* = \underset{y}{E}\left[y = g(x),\ x \text{ in } E_i^*\right]$ where

$$E_i^* = \underset{x}{E}\left[kL + \frac{i-1}{n}(kU - kL) < F(x) \leq kL + \frac{i}{n}(kU - kL)\right].$$

Using the notation previously employed, we have

$$\overline{\overline{\sum}}_\sigma - \underline{\underline{\sum}}_\sigma = \sum_{i=1}^{n} \frac{1}{n}(U - L)m(Y_i) = \frac{1}{n}(U - L)m(Y) < \frac{\epsilon}{2k},$$

and

$$\overline{\overline{\sum}}_{\sigma^*} - \underline{\underline{\sum}}_{\sigma^*} = \sum_{i=1}^{n} \frac{1}{n}(kU - kL)m(Y_i) = \frac{k}{n}(U - L)m(Y) < \frac{\epsilon}{2}.$$

Then it follows that

$$\left| \int_E f(x)dg(x) - \overline{\overline{\sum}}_\sigma \right| < \frac{\epsilon}{2k},$$

and

$$\left| \int_E F(x)dg(x) - \overline{\overline{\sum}}_{\sigma^*} \right| < \frac{\epsilon}{2}.$$

Now for every $1 \leq i \leq n$, $E_i = E_i^*$. Hence for every $1 \leq i \leq n$, $Y_i = Y_i^*$, and we have

$$\overline{\overline{\sum}}_{\sigma^*} = \sum_{i=1}^{n}\left[kL + \frac{i}{n}(kU - kL)\right] m(Y_i^*)$$

$$= \sum_{i=1}^{n}\left[kL + \frac{ik}{n}(U - L)\right]m(Y_i)$$

$$= k\sum_{i=1}^{n}\left[L + \frac{i}{n}(U - L)\right] m(Y_i)$$

$$= k \overline{\overline{\sum}}_\sigma.$$

Then

$$\left| \int_E F(x)dg(x) - k\int_E f(x)dg(x) \right| \leq \left| \int_E F(x)dg(x) - \overline{\sum}_{\sigma^*} \right|$$

$$+ \left| \overline{\sum}_{\sigma^*} - k\overline{\sum}_{\sigma} \right|$$

$$+ \left| k\overline{\sum}_{\sigma} - k\int_E f(x)dg(x) \right|$$

$$< \frac{\epsilon}{2} + 0 + k\left| \overline{\sum}_{\sigma} - \int_E f(x)dg(x) \right|$$

$$< \frac{\epsilon}{2} + k \cdot \frac{\epsilon}{2k}$$

$$= \epsilon.$$

Since $\epsilon > 0$ was arbitrary, it follows that

$$\int_E F(x)dg(x) = k\int_E f(x)dg(x).$$

Suppose $k < 0$. Then we may choose a lower bound L and an upper bound U of f(x) over E and further specify that L is not the g.l.b. and U is not the l.u.b., of f(x) on E. Then $[L,U]$ is an admissible bounding interval of f(x) on E and $[kU,kL]$ is an admissible bounding interval of $F(x) = kf(x)$ on E. With the obvious modifications, the proof of the preceding paragraph will suffice for the case where $k < 0$.

<u>Theorem 2.13</u>. If g(x) is a bounded, monotone non-decreasing, absolutely continuous function over the measurable set E, then g(x) is Y-measurable over E.

Let $g(x)$ be any bounded, monotone non-decreasing, absolutely continuous function on the measurable set E. Suppose E^* is any subset of E with measure zero. Choose $\epsilon > 0$. Then there exists a $\delta > 0$ such that $\sum |g(\beta_i) - g(\alpha_i)| < \frac{\epsilon}{4}$ for all finite or denumerable sets of non-overlapping (except on the end-points) intervals $\{I_n\} = [\alpha_n, \beta_n]$ whose length sum $\sum l(I_n) < \delta$ and where for each i, α_i and β_i are points of E^*. Since E^* is of measure zero, there exists a set of open intervals $\{I_n^*\}$ covering E^* and such that $\sum l(I_n^*) < \delta$. Now the point set sum $F = \sum I_n^*$ is an open set. Then $F = \sum J_n$ where $\{J_n\}$ is a sequence of non-overlapping open intervals. Clearly, $\{J_n\}$ covers E^* and $\sum l(J_n) < \delta$. Let us make each of the $\{J_n\}$ closed, i.e., if before $J_n = (\alpha_n, \beta_n)$, we now have $J_n = [\alpha_n, \beta_n]$. We now have a sequence of closed intervals $\{J_n\}$, which overlap at most on the end-points, and such that $\{J_n\}$ covers E^* and $\sum l(J_n) < \delta$. Let us first remove from $\{J_n\}$ all intervals which contain no point of E^* and consider the remaining intervals denoted by $\{J_n^*\}$. Let us remove from the sequence $\{J_n^*\}$ all intervals J_p^* such that the g.l.b. of $g(x)$ on J_p^* = the l.u.b. of $g(x)$ on J_p^*. Call this sequence of intervals $\{P_n\}$ and let the remaining intervals of $\{J_n^*\}$ be denoted by $\{R_n\}$. Then the set of points $Y^* = \underset{y}{E}[y = g(x), x \text{ in } E^* \cdot \sum P_n]$ is at most denumerable. Consequently Y^* is of measure zero may be covered with a sequence of intervals $\{Q_n^*\}$ such that $\sum l(Q_n^*) < \frac{\epsilon}{4}$. Next consider the sequence $\{R_n\}$. Let L_t be the g.l.b. of $g(x)$ on R_t and U_t be the l.u.b. of $g(x)$ on R_t. Then $L_t < U_t$ and there exists an x_{1t} and an x_{2t} in $E^* \cdot R_t$ such that $x_{1t} < x_{2t}$ and

$$g(x_{1t}) < L_t + \lambda_t,$$

$$g(x_{2t}) > U_t - \lambda_t,$$

where $\lambda_t = \min\left[\frac{\epsilon}{2^t + 2}, \frac{1}{4}(U_t - L_t)\right]$. Let $Z_t = \left[x_{1t}, x_{2t}\right]$. In exactly the same manner, construct an interval Z_n for every R_n. Then $\sum l(Z_n) < \delta$. Let $Q_s^{**} = (g(x_{2s}) + \lambda_s, g(x_{1s}) - \lambda_s)$. From the property of absolute continuity, we have

$$\sum \left|g(x_{2s}) - g(x_{1s})\right| < \frac{\epsilon}{4},$$

and from the definition of λ_t we see that

$$\sum 2\lambda_n \leq \frac{\epsilon}{2}.$$

Then $\sum l(Q_n^{**}) < \frac{3\epsilon}{4}$. But $\left\{Q_n^{**}\right\}$ covers the set

$$Y^{**} = \underset{y}{E}\left[y = g(x), \ x \text{ in } E^* \cdot \sum R_n\right].$$

Furthermore, the set $Y = \underset{y}{E}\left[y = g(x), \ x \text{ in } E^*\right]$ is the sum of Y^* and Y^{**}. Then the intervals $\left\{Q_n^*\right\}$ and $\left\{Q_n^{**}\right\}$ constitute a covering of Y of length sum $< \epsilon$. Since this result may be obtained for every preassigned $\epsilon > 0$, it follows that Y is of measure zero. Since E^* was an arbitrary subset of E with measure zero, it follows from Theorem 1.6 that $g(x)$ is Y-measurable over E.

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