THE DEVELOPMENT OF THE NATURAL NUMBERS

BY MEANS OF THE PEANO POSTULATES

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. ADDITION AND ORDERING</td>
<td>3</td>
</tr>
<tr>
<td>III. SUBTRACTION, MULTIPLICATION, AND DIVISION</td>
<td>19</td>
</tr>
<tr>
<td>IV. DEFINITION OF NUMBER</td>
<td>32</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>34</td>
</tr>
</tbody>
</table>
CHAPTER I

INTRODUCTION

Preceding the development of the natural numbers, we shall consider the following definitions, assumptions, and theorems as necessary.

Definition 1.1. By a set we mean any finite or infinite aggregate or collection $S$ of definite, separate objects. The objects will be called the elements of the set.\footnote{H. P. Thielman, Lectures on the Theory of Functions of Real Variables, p. 14.}

We shall consider a set $S$ to be defined when, given any object $e$, it is possible to determine whether or not $e$ belongs to $S$.

Definition 1.2. A function on a set $A$ to a set $B$ is defined to mean a correspondence by which to each element of $A$, there corresponds a uniquely determined element of $B$.

We shall consider a mathematical system $[S, f(x)]$, consisting of a set $S$ and a function $f(x)$ defined on $S$ to $S$. This mathematical system will be assumed to have the following properties:
Postulate 1. There exists a unique element $k$ in $S$ such that, for every element $x$ in $S$, $f(x)$ is not $k$.

Postulate 2. For every pair of elements $e_1$ and $e_2$ of $S$, $f(e_1)$ and $f(e_2)$ are the same element only if $e_1$ and $e_2$ are the same element.

Postulate 3. If $T$ is any set of elements of $S$ having the properties

I. $T$ contains $k$, and
II. for every element $x$ in $T$, $T$ also contains $f(x)$,
then $T$ contains all elements of $S$.

Definition 1.3. If $a$ and $b$ are elements of $S$, $a = b$ will mean that $a$ and $b$ are the same element.

The following three theorems will be given as obvious.

Theorem 1.1. For every element $e$ of $S$, $e = e$.

Theorem 1.2. For every pair of elements $a$ and $b$ of $S$, if $a = b$, then $b = a$.

Theorem 1.3. If $a$, $b$, and $c$ are elements of $S$ and if $a = b$ and $b = c$, then $a = c$. 
CHAPTER II

ADDITION AND ORDERING

Definition 2.1. For every element a of S,
\[ a + k \models f(a), \]
and
\[ a + f(b) \models f(a + b). \]

Theorem 2.1. If a and b are elements of S and if \( a \neq b \), then
\[ f(a) \neq f(b). \]

Proof: Suppose the contrary, i.e., that \( f(a) = f(b) \). Then, by Postulate 2, \( a = b \). But this is a contradiction; hence \( f(a) \neq f(b) \).

Theorem 2.2. For every element a of S
\[ f(a) \neq a. \]

Proof: Let T denote the set of all elements e of S for which \( f(e) \neq e \). By Postulate 1,
\[ f(k) \neq k, \]
hence k belongs to T. Let t be any element of T. Then \( f(t) \neq t \). By Theorem 2.1,
\[ f[f(t)] \neq f(t), \]
and it follows that T contains \( f(t) \). By Postulate 3, T
contains all elements of $S$. Hence for every element $a$ of $S$, $f(a) \neq a$.

**Theorem 2.3.** If $x \neq k$, then there exists an element $u$ of $S$ such that $x = f(u)$.

**Proof:** Let $T$ be the set consisting of the element $k$ and each $x$ for which there exists such a $u$. For any $x$ in $S$, $f(x) \neq k$ by Postulate 1. Let $t$ be any element of $T$ other than $k$. With $u$ denoting the element $t$, we have

$$f(t) = f(u),$$

and $f(t)$ belongs to $T$. Hence, by Postulate 3, $T$ contains all elements of $S$. That is, for $x \neq k$, there exists an element $u$ of $S$ such that $x = f(u)$.

**Theorem 2.4.** Addition is defined and unique for every pair of elements $a$ and $b$ of $S$.

**Proof:** Let $a$ be any fixed but arbitrary element of $S$. Let $T$ denote the set of all elements $e$ of $S$ for which $a + e$ is defined and unique. Now $T$ contains $k$, for $a + k$ is defined uniquely to be $f(a)$. Let $t$ be any element of $T$. Then $a + t$ is defined and unique. Since, by Definition 2.1, $a + f(t)$ is defined and unique, it follows that $T$ also contains $f(t)$. By Postulate 3, $T$ contains all elements of $S$. That is, $a + b$ is defined and unique for all elements $b$ of $S$. But $a$ was an arbitrary element of $S$; hence $a + b$ is defined and unique for every pair of elements $a$ and $b$ of $S$. 
Theorem 2.5. Addition is associative; i.e., for any three elements $a$, $b$, $c$ of $S$,

$$(a + b) + c = a + (b + c).$$

Proof: Let $a$ and $b$ be any two fixed but arbitrary elements of $S$. Let $T$ denote the set of all elements $e$ of $S$ for which $(a + b) + e = a + (b + e)$.

Now

$$(a + b) + k = f(a + b),$$

and

$$f(a + b) = a + f(b)$$

by Definition 2.1. Hence, by Theorem 1.3,

$$(a + b) + k = a + f(b).$$

On the other hand,

$$a + (b + k) = a + f(b),$$

and

$$a + f(b) = a + (b + k).$$

Hence, by Theorem 1.3,

$$(a + b) + k = a + (b + k),$$

and $k$ belongs to $T$. Now let $t$ be any element of $T$. Then

$$(a + b) + t = a + (b + t).$$

Now

$$(a + b) + f(t) = f[(a + b) + t]$$

by Definition 2.1, and

$$f[(a + b) + t] = f[a + (b + t)]$$

from the construction of the set $T$. Hence, by Theorem 1.3,

$$(a + b) + f(t) = f[a + (b + t)].$$
Also
\[ a + \lfloor b + f(t) \rfloor = a + f(b + t) \]
by Definition 2.1, and
\[ a + f(b + t) = f\lceil a + (b + t) \rceil. \]
Then
\[ a + \lfloor b + f(t) \rfloor = f\lceil a + (b + t) \rceil \]
by Theorem 1.3. By Theorem 1.2
\[ f\lceil a + (b + t) \rceil = a + \lfloor b + f(t) \rfloor. \]
By Theorem 1.3, it follows that
\[ (a + b) + f(t) = a + \lfloor b + f(t) \rfloor, \]
and \( T \) contains \( f(t) \). By Postulate 3, \( T \) contains all elements of \( S \), and
\[ (a + b) + c = a + (b + c) \]
is true for every choice of \( c \). But since \( a \) and \( b \) were arbitrary, it follows that
\[ (a + b) + c = a + (b + c) \]
is true for every choice of \( a, b, \) and \( c \).

**Lemma 2.1.** For every element \( x \) of \( S \),
\[ x + k = k + x. \]

**Proof:** Let \( T \) denote the set of all elements \( e \) of \( S \) for which
\[ e + k = k + e. \]
Now \( T \) contains \( k \), for
\[ k + k = k + k. \]
Let \( t \) be any element of \( T \). Then
\[
t + k = k + t.
\]
Now
\[
f(t) + k = f[f(t)]
= f(t + k)
= f(k + t)
= k + f(t).
\]
Hence \( T \) contains \( f(t) \). By Postulate 3, \( T \) contains all elements of \( S \); that is, for every \( x \) in \( S \)
\[
x + k = k + x.
\]

**Theorem 2.6.** Addition is commutative; i.e., for every pair of elements \( a \) and \( b \) of \( S \),
\[
a + b = b + a.
\]

**Proof:** Let \( a \) be any fixed but arbitrary element of \( S \).

Let \( T \) denote the set of all elements \( e \) of \( S \) for which
\[
a + e = e + a.
\]

By Lemma 2.1, \( T \) contains \( k \). Let \( t \) be any element of \( T \). Then
\[
a + t = t + a.
\]
Now
\[
a + f(t) = f(a + t)
\]
by Definition 2.1, and
\[
f(a + t) = f(t + a)
\]
from the construction of the set \( T \). Hence
\[
a + f(t) = f(a + t)
\]
by Theorem 1.3. Also, by Definition 2.1,
\[ f(t + a) = (t + a) + k, \]

and, by Theorem 2.5,
\[ (t + a) + k = t + (a + k). \]

From the construction of the set \( T \),
\[ t + (a + k) = t + (k + a). \]

By Theorem 2.5,
\[ t + (k + a) = (t + k) + a. \]

But
\[ (t + k) + a = f(t) + a. \]

By repeated application of Theorems 1.2 and 1.3, it follows that
\[ a + f(t) = f(t) + a, \]
and \( f(t) \) belongs to \( T \). Hence, by Postulate 3, \( T \) contains all elements of \( S \). Hence
\[ a + b = b + a \]
holds for every choice of \( b \). But since \( a \) was arbitrary, it follows that
\[ a + b = b + a \]
holds for every choice of \( a \) and \( b \).

**Theorem 2.7.** For every pair of elements \( a \) and \( b \) of \( S \),
\[ b \neq a + b. \]

**Proof:** Let \( a \) be any fixed but arbitrary element of \( S \). Let \( T \) consist of all elements \( e \) of \( S \) for which
\[ e \neq a + e. \]

Since, by Postulate 1, \( k \neq a + k = f(a), k \) belongs to \( T \).
Let \( t \) be any element of \( T \). Then
\[
\begin{align*}
t \neq a + t.
\end{align*}
\]

It follows from Theorem 2.1 that
\[
\begin{align*}
f(t) \neq f(a + t) \\
= a + f(t).
\end{align*}
\]
Hence \( T \) contains \( f(t) \). By Postulate 3, \( T \) contains all elements of \( S \). Hence
\[
b \neq a + b
\]
for every choice of \( b \). But since \( a \) was arbitrary,
\[
b \neq a + b
\]
for every choice of \( a \) and \( b \).

**Theorem 2.8.** For every three elements \( a, b, \) and \( c \) of \( S \), if \( a \neq b \), then
\[
a + c \neq b + c.
\]

**Proof:** Let \( a \) be any fixed element of \( S \), and let \( b \) be a fixed element of \( S \) but such that \( a \neq b \). Let \( c \) be an arbitrary element of \( S \). Denote by \( T \) the set of all elements \( e \) of \( S \) for which
\[
a + e \neq b + e.
\]
Since \( a \neq b \), it follows from Theorem 2.1 that \( f(a) \neq f(b) \).
But, by Definition 2.1, \( f(a) = a + k \) and \( f(b) = b + k \). Hence
\[
a + k \neq b + k,
\]
and \( k \) belongs to \( T \). Let \( t \) be any element of \( T \). Then
\[
a + t \neq b + t.
\]
By Theorem 2.1,
\[ f(a + t) \neq f(b + t). \]

But, from Definition 2.1, \( f(a + t) = a + f(t) \), and \( f(b + t) = b + f(t) \). Hence
\[ a + f(t) \neq b + f(t), \]
and \( T \) contains \( f(t) \). By Postulate 3, \( T \) contains all elements of \( S \). Thus, if \( a \neq b \),
\[ a + c \neq b + c \]
holds for every choice of \( c \). But since \( a \) and \( b \) were arbitrary, it follows that if \( a \neq b \), then
\[ a + c \neq b + c \]
holds for every choice of \( a, b, \) and \( c \).

**Theorem 2.9.** If \( a, b, \) and \( c \) are elements of \( S \) and if \( a = b \), then
\[ a + c = b + c. \]

**Proof:** Let \( T \) denote the set of all elements \( e \) of \( S \) for which
\[ a + e = b + e. \]

Since
\[ a = b, \]
then, by Postulate 2,
\[ f(a) = f(b). \]
But
\[ f(a) = a + k \]
and
\[ f(b) = b + k. \]
Hence
\[a + k = b + k\]
and \(k\) belongs to \(T\). Let \(t\) be any element of \(T\). Then
\[a + t = b + t.\]
By Postulate 2,
\[f(a + t) = f(b + t),\]
or
\[a + f(t) = b + f(t),\]
and \(T\) contains \(f(t)\). By Postulate 3, \(T\) contains all elements of \(S\). That is, if \(a = b\), then
\[a + c = b + c.\]

**Theorem 2.10.** If \(a, b,\) and \(c\) are elements of \(S\) and if \(a + b = b + c\), then \(a = c\).

**Proof:** Suppose the contrary; i.e., that
\[a \neq c.\]
Then, by Theorem 2.8,
\[a + b \neq c + b\]
\[= b + c.\]
But this is a contradiction; hence
\[a = c.\]

**Definition 2.2.** If \(a\) and \(b\) are elements of \(S\), \(a > b\)
will mean that there exists an element \(c\) such that \(a = b + c\).
By \(a < b\) we shall mean that \(b > a\).

**Theorem 2.11.** If \(a\) and \(b\) are any two elements of \(S\),
then one and only one of the relationships

\[ a > b, \ a = b, \text{ or } a < b, \]

must hold.

**Proof:** Since by Theorem 2.7, \( b \neq a + b \) for every
pair of elements \( a \) and \( b \) of \( S \), \( b > a \) and \( b = a \) cannot hold. Similarly, \( a > b \) and \( a = b \) cannot hold. Also, \( a > b \) and \( a < b \) would imply

\[ a = b + c \]

and

\[ b = a + d \]

for some pair of elements \( c \) and \( d \). This yields

\[
\begin{align*}
  a &= b + c \\
  &= (a + d) + c \\
  &= a + (d + c) \\
  &= (d + c) + a. 
\end{align*}
\]

This contradicts Theorem 2.7; hence \( a > b \) and \( a < b \) cannot
hold. Therefore, at most one of the relationships \( a > b \),
\( a = b \), or \( a < b \) can hold.

Next, let \( a \) be any fixed but arbitrary element of \( S \).
Let \( T \) be the set of elements \( e \) of \( S \) for which one of the
relationships \( a > e \), \( a = e \), or \( a < e \) must hold. Suppose \( e = k \). Then, either

\[ a = k \]

or, by Theorem 2.3, there exists an element \( u \) of \( S \) such
that \( a = f(u) \). This yields

\[ a = f(u) = u + k = k + u. \]
Hence
\[ a > k, \]
and \( T \) contains \( k \). Next, let \( t \) be any element of \( T \). Then \( a > t, a = t, \) or \( a < t \). Suppose \( a > t \). Then
\[ a = t + u \]
for some element \( u \) of \( S \). Suppose \( u = k \). Then
\[ a = t + k \]
\[ = f(t), \]
and \( T \) contains \( f(t) \). Suppose \( u \neq k \). Then
\[ u = w + k \]
for some element \( w \). Thus
\[ a = t + (w + k) \]
\[ = (w + k) + t \]
\[ = w + (k + t) \]
\[ = w + (t + k) \]
\[ = w + f(t) \]
\[ = f(t) + w. \]
Hence
\[ a > f(t), \]
and \( T \) contains \( f(t) \). Next suppose \( a = t \). Then
\[ f(t) = t + k \]
\[ = a + k, \]
and
\[ f(t) > a, \]
or
\[ a < f(t), \]
and $T$ contains $f(t)$. Finally suppose $a < t$. Then

$$t > a$$

so that

$$t = a + v$$

for some element $v$. Hence, by Postulate 2,

$$f(t) = f(a + v) = a + f(v),$$

and

$$f(t) > a.$$  

In any case, $f(t)$ belongs to $T$. Thus $a > b$, $a = b$, or $a < b$ for any element $b$, but since $a$ was arbitrary, one of the relationships $a > b$, $a = b$, or $a < b$ must hold for every choice of $a$ and $b$.

**Theorem 2.12.** If $a > b$ and $b \geq c$, then $a > c$.

Proof: Suppose first that $b = c$. Since

$$a > b$$

$$a = b + d,$$

for some element $d$. Then

$$a = c + d.$$  

Hence

$$a > c.$$  

Suppose next that $b > c$. Then

$$a = b + d$$

and

$$b = c + f.$$
for some two elements \( d \) and \( f \) of \( S \). Then
\[
\begin{align*}
a &= b + d \\
&= (c + f) + d \\
&= c + (f + d).
\end{align*}
\]

Hence
\[
a > c.
\]

**Theorem 2.13.** If \( a \geq b \) and \( b > c \), then \( a > c \).

**Proof:** Suppose first that \( a > b \). Then, by Theorem 2.12,
\[
a > c.
\]
Suppose next that \( a = b \). Then, since \( b > c \),
\[
b = c + f
\]
for some element \( f \) of \( S \). Then
\[
a = c + f,
\]
and it follows that
\[
a > c.
\]

**Theorem 2.14.** For any two elements \( a \) and \( b \) of \( S \),
\[
a + b > a.
\]

**Proof:**
\[
a + b = a + b
\]
by Theorem 1.1. Hence, by Definition 2.2,
\[
a + b > a.
\]

**Theorem 2.15.** If \( a, b, \) and \( c \) are elements of \( S \) and if \( a > b \), then
\[
a + c > b + c.
\]
Proof: Since \( a > b \),

\[ a = b + d \]

for some element \( d \) of \( S \). Then

\[ a + c = (b + d) + c \]
\[ = (d + b) + c \]
\[ = d + (b + c) \]
\[ = (b + c) + d. \]

Hence, by Definition 2.2,

\[ a + c > b + c. \]

Theorem 2.16. If \( a, b, \) and \( c \) are elements of \( S \) and if \( a + c > b + c \), then \( a > b \).

Proof: We shall prove the contrapositive: If \( a \nless b \), then \( a + c \nless b + c \), or if \( a \less b \), then \( a + c \less b + c \). Suppose first that \( a = b \). Then by Theorem 2.9, we have \( a + c = b + c \). Suppose next that \( a < b \). Then \( b > a \). By Theorem 2.15, \( b + c > a + c \). Thus

\[ a + c < b + c. \]

Therefore, if \( a + c > b + c \), then \( a > b \).

Theorem 2.17. If \( a, b, c, \) and \( d \) are elements of \( S \) and if \( a > b \) and \( c > d \), then \( a + c > b + d \).

Proof: By Theorem 2.15,

\[ a + c > b + c, \]

and
\[ b + c = c + b \]
\[ > d + b \]
\[ = b + d. \]

Hence, by Theorem 2.12,
\[ a + c > b + d. \]

**Theorem 2.13.** For every element \( x \) in \( S \),
\[ x \geq k. \]

**Proof:** Either
\[ x = k, \]

or
\[ x \neq k. \]

If
\[ x \neq k, \]
by Theorem 2.3, there exists an element \( u \) of \( S \) such that
\[ x = f(u). \]

But
\[ f(u) = u + k \]
\[ = k + u \]
\[ > k. \]

Hence
\[ x \geq k. \]

**Theorem 2.19.** If \( a \) and \( b \) are elements of \( S \) and if
\[ a > b, \]
then
\[ a \geq b + k. \]
Proof: Since
\[ a > b \]
\[ a = b + c, \]
for some element \( c \) of \( S \). But, by Theorem 2.18,
\[ c \geq k. \]

Hence
\[ a \geq b + k. \]

**Theorem 2.20.** If \( a \) and \( b \) are elements of \( S \) and if
\[ a < b + k, \]
then
\[ a \leq b. \]

Proof: Suppose the contrary; that is,
\[ a > b. \]
Then, by Theorem 2.19,
\[ a \geq b + k. \]
But this is a contradiction; hence
\[ a \leq b. \]

**Theorem 2.21.** In every non-empty subset of \( S \), there is a smallest element.

Proof: Let \( N \) be any non-empty subset of \( S \), and let \( M \) be the set of all elements \( x \) such that \( x \leq \) every element of \( N \). By Theorem 2.18, the set \( M \) contains the element \( k \). Not every element \( x \) belongs to \( M \); in fact, for each \( y \) of \( N \) the element \( f(y) = y + k \) does not belong to \( M \), since
\[ y + k > y. \]
Hence there exists an element $m$ in $M$ such that $f(m)$ does not belong to $M$; for otherwise, every element of $S$ would belong to $M$, by Postulate 3. Now $m$ belongs to $N$, for otherwise we would have for each $n$ of $N$
\[ m < n, \]
and, by Theorem 2.19,
\[ m + k \leq n. \]
This places $f(m) = m + k$ in the set $M$, contrary to fact. Hence $m$ is the smallest element in $N$. 
CHAPTER III

SUBTRACTION, MULTIPLICATION, AND DIVISION

**Definition 3.1.** If \( a \) and \( b \) are elements of \( S \), \( a - b \)
will be defined to be any element \( c \) such that
\[
b + c = a.
\]

**Theorem 3.1.** Subtraction of \( b \) from \( a \) is defined if
and only if \( a > b \).

**Proof:** Obvious.

**Theorem 3.2.** When defined, \( a - b \) is unique.

**Proof:** Suppose \( a - b \) is defined and \( a = b + c \). Let
\( d \) be any element of \( S \) not equal to \( c \). By Theorem 2.11,
either \( d < c \) or \( d > c \). Suppose that
\[
d < c.
\]
Then
\[
c > d,
\]
and
\[
c = d + f,
\]
where \( f \) is some element of \( S \). Then
\[
b + c = b + (d + f)
\]
\[
= (b + d) + f.
\]
By Definition 2.2,
\[
b + c > b + d.
\]
By Theorem 2.13,
\[ a > b + d. \]

By Theorem 2.11,
\[ a \neq b + d, \]
and \( d \) is not a possible answer to \( a - b \). Suppose next that \( d > c \).

Then, by Theorem 2.15,
\[ d + b > c + b, \]
or
\[ b + d > b + c. \]

Then, by Theorem 2.12,
\[ b + d > a, \]
and
\[ a < b + d. \]

By Theorem 2.11,
\[ a \neq b + d, \]
and again \( d \) is not an answer to \( a - b \).

**Definition 3.2.** If \( a \) and \( b \) are elements of \( S \),
\[ a \cdot k : \sim: a \]
and
\[ a \cdot f(b) : \sim: ab + a. \]

**Theorem 3.3.** Multiplication is defined and unique for every pair of elements \( a \) and \( b \) of \( S \).

Proof: Let \( a \) be any fixed but arbitrary element of \( S \).
Let $T$ be the set of all elements $e$ of $S$ for which $a \cdot e$ is defined and unique. Since $a \cdot k$ is uniquely defined, $T$ contains $k$. Let $t$ be any element of $T$. Then $a \cdot t$ is uniquely defined. Since addition is defined and unique,

$$at + a$$

would be a unique element of $S$. Hence, by Definition 3.2, $a \cdot f(t)$ belongs to $T$. By Postulate 3, $T$ contains all elements of $S$. That is, $ab$ is uniquely defined for all elements $b$ of $S$. But since $a$ was arbitrary, $a \cdot b$ is defined uniquely for every pair of elements $a$ and $b$ of $S$.

**Theorem 3.4.** Multiplication is distributive; i.e., if $a$, $b$, and $c$ are elements of $S$, then

$$a(b + c) = ab + ac.$$  

**Proof:** Let $a$ and $b$ be fixed but arbitrary elements of $S$. Let $T$ be the set of all elements $e$ of $S$ for which

$$a(b + e) = ab + ae.$$  

Now

$$a(b + k) = a \cdot f(b)$$

$$= ab + a$$

$$= ab + ak,$$

and $T$ contains $k$. Let $t$ be any element of $T$. Then

$$a(b + t) = ab + at.$$  

Also,

$$a[b + f(t)] = a \cdot f(b + t) = a(b + t) + a$$

$$= (ab + at) + a = ab + (at + a)$$

$$= ab + a \cdot f(t).$$
Hence \( f(t) \) belongs to \( T \). By Postulate 3, \( T \) contains all elements of \( S \); that is,
\[
a(b + c) = ab + ac
\]
holds for every choice of \( c \). But since \( a \) and \( b \) were arbitrary,
\[
a(b + c) = ab + ac
\]
holds for every choice of \( a, b, \) and \( c \).

**Theorem 3.5.** Multiplication is associative; i.e., if \( a, b, \) and \( c \) are elements of \( S \), then
\[
(ab)c = a(bc).
\]

**Proof:** Let \( a \) and \( b \) be fixed but arbitrary elements of \( S \). Let \( T \) be the set of all elements \( e \) of \( S \) for which
\[
(ab)e = a(be).
\]

Now
\[
(ab)k = ab
= a(b \cdot k),
\]
and \( k \) belongs to \( T \). Let \( t \) be any element of \( T \). Then
\[
(ab)t = a(bt).
\]

Also,
\[
(ab) \cdot f(t) = (ab)t + ab
= a(bt) + ab
= a(bt + b)
= a[b \cdot f(t)].
\]

Hence \( T \) contains \( f(t) \). By Postulate 3, \( T \) contains all elements of \( S \). That is,
\[(ab)c = a(bc)\]
holds for every choice of \(c\). But since \(a\) and \(b\) were arbitrary,
\[(ab)c = a(bc)\]
holds for every choice of \(a, b,\) and \(c\).

**Theorem 3.6.** Alternate form for the Distributive Law, i.e., if \(a, b,\) and \(c\) are elements of \(S\), then
\[(a + b)c = ac + bc\.

**Proof:** Let \(a\) and \(b\) be fixed but arbitrary elements of \(S\). Denote by \(T\) the set of all elements \(e\) of \(S\) for which
\[(a + b)e = ae + be\.

Now
\[(a + b)k = a + b = ak + bk,\]
and \(k\) belongs to \(T\). Let \(t\) be any element of \(T\). Then
\[(a + b)t = at + bt.\]
Also,
\[
(a + b)f(t) = (a + b)(t + k) = (at + bt) + (a + b) \\
= [(at + bt) + a] + b = [a + (at + bt)] + b \\
= [(a + at) + bt] + b = [(at + a) + bt] + b \\
= [af(t) + bt] + b = af(t) + (bt + b) \\
= af(t) + bf(t).
\]
Hence \(f(t)\) belongs to \(T\). By Postulate 3, \(T\) contains all elements of \(S\). That is
\[(a + b)c = ac + bc\]
holds for every choice of \(c\). But since \(a\) and \(b\) were arbitrary,
\[(a + b)c = ac + bc\]

holds for every choice of \(a, b,\) and \(c.\)

**Lemma 3.1.** For every element \(x\) in \(S,\)
\[kx = xk.\]

**Proof:** Let \(T\) consist of all elements \(e\) of \(S\) for which
\[ek = ke.\]

Now \(T\) contains \(k\) for
\[k \cdot k = k \cdot k.\]

Let \(t\) be any element of \(T.\) Then
\[tk = kt.\]

Now, by Definition 3.2,
\[f(t) \cdot k = f(t),\]

and
\[k \cdot f(t) = kt + k\]
\[= f(kt)\]
\[= f(tk)\]
\[= f(t).\]

Hence
\[f(t) \cdot k = k \cdot f(t),\]

and \(f(t)\) belongs to \(T.\) By Postulate 3, \(T\) contains all
elements of \(T.\) Therefore, for every element \(x\) of \(S\)
\[kx = xk.\]

**Theorem 3.7.** Multiplication is commutative; i.e., for
every pair of elements \(a\) and \(b\) of \(S,\)
\[ab = ba.\]
Proof: Let \( b \) be any fixed but arbitrary element of \( S \). Denote by \( T \) the set of all elements \( e \) of \( S \) for which 
\[
eb = be.
\]
By Lemma 3.1, \( T \) contains \( k \). Let \( t \) be any element of \( T \). Then 
\[
tb = bt.
\]
Now
\[
b \cdot f(t) = bt + b
= tb + b.
\]
And
\[
f(t) \cdot b = (t + k)b
= tb + kb
= tb + bk
= tb + b.
\]
Hence
\[
b \cdot f(t) = f(t) \cdot b,
\]
and \( f(t) \) belongs to \( T \). By Postulate 3, \( T \) contains all elements of \( S \). That is,
\[
ab = ba
\]
holds for every choice of \( a \). But since \( b \) was arbitrary,
\[
ab = ba
\]
holds for every choice of \( a \) and \( b \).

**Theorem 3.2.** If \( a \), \( b \), and \( c \) are elements of \( S \) and if \( a > b \), then
\[
ac > bc.
\]

Proof: Since \( a > b \),
\[
a = b + d,
\]
for some element \( d \) of \( S \). Then
\[
ac = (b + d)c = bc + dc > bc.
\]

**Theorem 3.9.** If \( a, b, c, \) and \( d \) are elements of \( S \) such that \( a > b \) and \( c > d \), then
\[
ac > bd.
\]

**Proof:** Since
\[
a > b,
\]
we have, by Theorem 3.8,
\[
ac > bc.
\]
And since
\[
c > d,
\]
we have
\[
ob > db,
\]
or
\[
bc > bd.
\]
Hence, by Theorem 2.12,
\[
ac > bd.
\]

**Theorem 3.10.** If \( a, b, \) and \( c \) are elements of \( S \) and if \( a = b \), then
\[
ac = bc.
\]

**Proof:** By Definition 1.3, \( a \) and \( b \) are the same element. By Theorem 3.3 multiplication is unique. Hence \( ac \) is the
same element as $bc$, or

$$ac = bc.$$  

**Theorem 3.11.** If $a$, $b$, $c$ are elements of $S$ and if $ac = bc$, then $a = b$.

Proof: We shall prove the contrapositive: If $a \neq b$, then $ac \neq bc$. Let $T$ consist of all elements $e$ of $S$ for which $ae \neq be$.

Since $ak = a$ and $bk = b$, we have

$$ak \neq bk,$$

and $k$ belongs to $T$. Let $t$ be any element of $T$. Then $at \neq bt$.

Since $a \neq b$, then either $a > b$ or $a < b$. Suppose that $a > b$.

Then, by Theorem 3.8,

$$af(t) > bf(t).$$

Hence

$$af(t) \neq bf(t),$$

and $f(t)$ belongs to $T$. By Postulate 3, $T$ contains all elements of $S$. Suppose next that $a < b$. Then $b > a$. By the first part of this proof, we have

$$bf(t) \neq af(t),$$

or

$$af(t) \neq bf(t),$$

so that again $T$ contains $f(t)$. Then $T$ contains all elements of $S$. But this is a contradiction; hence if $ac = bc$, then $a = b$. 

**Definition 3.3.** If $a$ and $b$ are elements of $S$, $a \div b$ will mean any element $c$ of $S$ with the property that $c \cdot b = a$.

**Theorem 3.12.** Division is unique when it is possible.

Proof: Suppose that $a = cb$, where $a$, $b$, and $c$ are elements of $S$. Choose any element $d$ of $S$ such that $d \neq c$.

Either $d > c$, or $d < c$.

Suppose first that $d > c$.

Then $d = c + f$, where $f$ is some element of $S$. Then

$$db = (c + f)b$$
$$= b(c + f)$$
$$= bc + bf$$
$$> bc.$$

But $bc = a$;

hence $db > a$. 
Therefore \( db \neq a \).

Suppose next that \( d < c \).

Then \( c > d \),

and \( c = d + g \),

for some element \( g \) of \( S \). Then

\[
\begin{align*}
    cb &= (d + g)b \\
    &= b(d + g) \\
    &= bd + bg \\
    &> bd.
\end{align*}
\]

Hence \( cb > bd \),

and \( bd < cb \).

Therefore \( bd \neq a \),

and \( a \div b \) can have at most one answer.

That division is sometimes possible, sometimes not, may be shown by example:

**Example 1:**

\[ f(k) \div k = f(k), \]

since
\[ f(k) \cdot k = k \cdot f(k) \]
\[ = k \cdot k + k \]
\[ = k + k \]
\[ = f(k). \]

**Example 2:** The operation, \( k \div f(k) \), is impossible. We shall first show that \( k \) is not a possible answer. Now
\[ k \cdot f(k) = k \cdot k + k \]
\[ = k + k \]
\[ > k. \]

Next let \( c \) be any element of \( S \) other than \( k \). Then
\[ c = f(d), \]
for some element \( d \) of \( S \). Then
\[ c \cdot f(k) = f(d) \cdot f(k) \]
\[ = f(d) \cdot k + f(d) \]
\[ = f(d) + f(d) \cdot k \]
\[ > f(d) \]
\[ > k. \]

**Definition 3.4.** If \( a \div b \) is possible, we shall say that \( b \) is a factor of \( a \) or that \( a \) is a multiple of \( b \). We also say that \( b \) divides \( a \).
CHAPTER IV
DEFINITION OF NUMBER

Definition 4.1. Let a set $S_1$ be given with an operation $0_1$ for its elements. Let a set $S_2$ be given with an operation $0_2$ for its elements. Let a correspondence $T_1$ denoted by $\leftrightarrow$ be given between the elements of $S_1$ and the elements of $S_2$ such that if $a_1$ and $a_2$ are elements of $S_1$ and $b_1$ and $b_2$ are elements of $S_2$ and $a_1 \leftrightarrow b_1$ and $a_2 \leftrightarrow b_2$ then $a_1 0_1 a_2 \leftrightarrow b_1 0_2 b_2$. Such a correspondence is called an isomorphism.¹

We have assumed the existence of at least one mathematical system $[S, f(x)]$ satisfying the Peano postulates, and we have recognized that there might be many such systems. In fact, the system we have been considering was merely assumed to be one such system. Hence the preceding theorems and definitions are valid in any such system. Furthermore, any two such systems are isomorphic.

We now define the natural numbers or positive integers or positive whole numbers, as they are variously called, as follows:

1: the class consisting of all the $k$'s, i.e., consisting of all the unique elements $k$ such that for any $x$ in

¹H. P. Thielman, Lectures on the Theory of Functions of Real Variables, p. 4.
the particular system \( f(x) \) is not \( k \). The unique element \( k \) in any particular system will be a representative or representation of "1".

Supposing \( n \) to be defined, we define \( n + 1 \) to be the class of \( f(r) \)'s where \( r \) represents \( n \) in a particular system. By Postulate 3, every element of the set \( S \) will be used in this process, and since any particular number system is isomorphic with the set \( S \), every number in a given system will also be used.

Thus we may think of the following as representations of \( 1, 1 + 1, (1 + 1) + 1, \) etc.:

\[
1, 2, 3, 4, 5, \ldots ,
\]
\[
I, II, III, IV, V, \ldots ,
\]
\[
eins, zwei, drei, vier, funf, \ldots ,
\]
\[
uno, dos, tres, cuatro, cinco, \ldots ,
\]
\[
un, deux, trois, quatre, cinq, \ldots .
\]
BIBLIOGRAPHY


