ON UNIFORM CONVERGENCE

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CHAPTER I

INTRODUCTION

§1. Preliminary Remarks

1.1. Infinite series were first employed in the seventeenth century but little thought was given them concerning convergence or divergence. It was generally held, for example by Lagrange, that if the nth term of a series approached zero as n increased, the series was convergent even though Bernoulli had given an example which disproved this. The first mathematician to give a necessary and sufficient condition for convergence was Bolzano, but since his work remained almost unknown for a long period of time, our modern theories of convergence have come from the work of Cauchy and Abel.[1]

1.2. By an infinite sequence of real numbers, in notation $\{a_n\}$, we mean an ordered set of numbers which may be mated biunuquely with the set of positive integers and which are ordered like the natural order of the set of positive integers, i.e.,

$$a_1, a_2, a_3, \ldots, a_n, \ldots,$$

such that each value of n, $n = 1, 2, 3, \ldots$, determines a

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[1] E. W. Hobson, The Theory of Functions of a Real Variable, Vol. II, second edition revised, pp. 5-6.

unique term. When we say that a sequence <u>converges</u> to some number a, we mean that for n large enough, a_n is arbitrarily close to a. A precise, mathematical definition of convergence of a sequence of numbers is listed below.

1.3. <u>If for every positive number</u> ε, <u>no matter how small, there exists an integer</u> N <u>such that for</u> $n > N$

$$|a_n - a| < \varepsilon,$$

<u>the sequence</u> $\{a_n\}$ <u>is said to converge to</u> a, <u>in notation</u> $\{a_n\} \to a$.[2] That $\{a_n\}$ converges to a may also be written

$$\lim_{n \to \infty} a_n = a.$$

1.4. A (formal) series of numbers is a (formal) expression of term by term addition of a sequence. In other words, a series may be obtained from a sequence by replacing the commas with plus signs. We will use the symbol Σa_n to represent an infinite series, i.e.,

$$\Sigma a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots .$$

When we say that a <u>series converges</u> to some number A, we mean that if we take n large enough, the sum of the first n terms is arbitrarily close to A. A precise, mathematical definition of convergence of a series of numbers is listed below.

1.5. <u>If for every positive number</u> ε <u>there exists an integer</u> N <u>such that for</u> $n > N$

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[2] Cf. <u>ibid</u>., p. 1.

$$\left|[a_1 + a_2 + a_3 + \dots + a_n] - A\right| < \epsilon,$$

<u>the series</u> Σa_n <u>is said to converge to</u> A, <u>in notation</u> $\Sigma a_n = A$ <u>or</u> $\Sigma a_n \rightarrow A$.[3]

1.6. It is interesting to note that for every series there is a corresponding, closely related sequence. For example, we will define $\{s_n\}$ in terms of Σa_n as follows. Let

$$s_1 = a_1,\ s_2 = a_1 + a_2,\ \dots,\ s_n = a_1 + \dots + a_n,\ \dots .$$

Thus, we have a sequence $\{s_n\}$ which for every n is the sum of the first n terms of Σa_n, and s_n is called the nth partial sum of Σa_n. If we wish to construct a series whose nth partial sum is equal to the nth term of a given sequence $\{s_n\}$, we may define it as follows:

$$\Sigma a_n = s_1 + (s_2 - s_1) + \dots + (s_n - s_{n-1}) + \dots .$$

Thus, for every series Σa_n, there is associated the sequence $\{s_n\}$ of nth partial sums of Σa_n, and for every sequence $\{s_n\}$, there is an associated series whose nth partial sums are the terms of the sequence $\{s_n\}$. Furthermore, if $\{s_n\}$ is the sequence thus uniquely associated with Σa_n, $\{s_n\}$ is convergent if and only if Σa_n is convergent, and in case the two are convergent,

$$\Sigma a_n = \lim_{n \to \infty} s_n.$$

1.7. In reality sequences and series of numbers are

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[3]Cf. <u>ibid</u>.

special cases of sequences and series of functions where the functions are constants. For a sequence or a series of functions, we must speak of a domain of definition, i.e., the values of x for which $f_n(x)$ is defined for $n = 1, 2, \ldots$, and a domain of convergence, i.e., the values of x for which $\{f_n(x)\}$ or $\Sigma f_n(x)$ converge. In this paper, we will be concerned primarily with series of functions and a particular type of convergence which is described below. The purpose of this paper is to familiarize the reader with the concept of uniform convergence. In the main it is a compilation of material found in various references and revised to conform to standard notation.

§2. Definitions

1.8. In this paper the symbol :=: signifies "is defined to be", "means that", or "means". A domain of definition (of a function, series, etc.) :=: a set of real numbers (or points). We will, without loss of generalization, state the theorems and definitions of this paper in terms of intervals instead of a general domain of definition. A closed interval, in notation [a, b], consists of the set of points x such that $a \leq x \leq b$. An open interval, in notation (a, b), consists of the set of points x such that $a < x < b$. The function f(x) is bounded on [a, b] :=: there exists a finite number K such that $|f(x)| < K$ on [a, b]; similarly, f(x) is bounded from above on [a, b] :=: there exists a finite K such that

$f(x) < K$ on $[a, b]$. U is an upper bound of $f(x)$ on $[a, b]$ means that $f_n(x) \leq U$ on $[a, b]$. A sequence (series) has a <u>limit</u> (<u>sum</u>) $f(x)$:=: the sequence (series) "converges" to $f(x)$.

1.9. $F_n(x)$ is the <u>nth</u> <u>partial</u> <u>sum</u> of $\Sigma f_n(x)$:=: $F_n(x) = \Sigma f_n(x)$, $n = 1, 2, \ldots n$, i.e.,

$$F_n(x) = f_1(x) + f_2(x) + \ldots + f_n(x).$$

Unless otherwise stated, $F_n(x)$ will be the nth partial sum of $\Sigma f_n(x)$, and $G_n(x)$ will be the nth partial sum of $\Sigma g_n(x)$.

1.10. The <u>nth</u> <u>remainder</u> of $\Sigma f_n(x)$, denoted by $R_n(x)$, is

$$R_n(x) = f_{n+1}(x) + f_{n+2}(x) + \ldots .$$

Hence,

$$\Sigma f_n(x) = F_n(x) + R_n(x).$$

Unless otherwise stated, $R_n(x)$ will denote the nth remainder of $\Sigma f_n(x)$.

1.11. $\Sigma f_n(x)$ <u>converges</u> to $f(x)$, in notation $\Sigma f_n(x) \to f(x)$, on $[a, b]$:=: for every $\varepsilon > 0$ and for every x in $[a, b]$ there exists an N such that, for $n > N$,

$$|F_n(x) - f(x)| < \varepsilon.$$

$\Sigma f_n(x)$ <u>converges</u> <u>absolutely</u> :=: $\Sigma |f_n(x)|$ converges.

1.12. $\Sigma f_n(x)$ <u>converges</u> <u>uniformly</u> to $f(x)$, in notation $\Sigma f_n(x) \Rightarrow f(x)$, on $[a, b]$:=: for every $\varepsilon > 0$ there exists an N such that, for $n > N$ and for every x in $[a, b]$,

$$|F_n(x) - f(x)| < \varepsilon.$$

1.13. Although we are restricting our discussion to series, the concept of uniform convergence of a sequence is of equal importance. Specifically, the definition is as follows: $\{F_n(x)\}$ converges to $F(x)$ uniformly (in notation $\{F_n(x)\} \Rightarrow F(x)$):=: for every $\varepsilon > 0$ there exists an N such that, for $n > N$ and for every x in $[a, b]$,

$$|F_n(x) - F(x)| < \varepsilon.$$

We may extend our discussion in 1.6 by the statement: The series $\Sigma f_n(x) \Rightarrow f(x)$ on $[a, b]$ if and only if the sequence of nth partial sums of $\Sigma f_n(x)$ converges uniformly to $f(x)$ on $[a, b]$.

§3. Uniform Convergence

1.14. The main difference between convergence and uniform convergence is that convergence is a point property while uniform convergence is an interval property. In other words, if a series is convergent on $[a, b]$, it is merely convergent for each point of the interval. If it is uniformly convergent on $[a, b]$, it converges "in a uniform manner" over the entire interval.

1.15. Uniform convergence is a "stronger" type of convergence than ordinary convergence. Obviously, if a series is uniformly convergent on an interval, it is certainly convergent at every point of the interval. On the other hand, convergence on an interval $[a, b]$ does not necessarily imply uniform convergence on $[a, b]$. For, let

$$\Sigma f_n(x) = 1/x - 1/2x - 1/6x - \ldots - 1/n(n - 1)x - \ldots$$

on $(0, 1)$. Since $F_n(x) = 1/nx$ we have immediately that

$\Sigma f_n(x) \to 0$ on $(0, 1)$, but $\Sigma f_n(x)$ does not converge uniformly to zero on $(0, 1)$. For let $\varepsilon = 1/2$ and choose any N. For $n > N$ let $x = 1/n$; then,

$$|F_n(x) - f(x)| = |1/nx - 0| = 1 > \varepsilon.$$

Thus all of the conditions of the denial of uniform convergence are met.

1.16. As is suggested in 1.15, it may be of interest to give explicitly the following positive definition that $\Sigma f_n(x)$ does not converge uniformly to the function $f(x)$ on $[a, b]$ (in notation as indicated):

$\Sigma f_n(x) \not\rightrightarrows f(x)$ on $[a, b]$:=: there exists an $\varepsilon > 0$ such that, for every N there exists an $n > N$ and a point x_o in $[a, b]$ such that,

$$|F_n(x_o) - f(x_o)| \geq \varepsilon.$$

§ 4. Assumptions

1.17. It will be assumed in this paper that the reader knows the fundamentals of elementary analysis and a few elementary set properties. For example, it will be assumed that the reader is familiar with the concepts of continuity, differentiability, Lebesgue measure zero, and Riemann integration in the third chapter. Also some knowledge of series of numbers and ordinary convergence of series of functions is assumed. Listed below are some theorems without proofs that will be referred to in this paper.

1.18. _A necessary and sufficient condition that_

$\Sigma f_n(x) \rightarrow f(x)$, on $[a, b]$, i.e., $\lim_{n\to\infty} F_n(x) = f(x)$, is that for every $\varepsilon > 0$ and for x in $[a, b]$ there exists an N such that, for $n > N$ and $m > N$,

$$|F_n(x) - F_m(x)| < \varepsilon.$$

1.19. If $\Sigma a_n \rightarrow a$ and if for every n,

$$|b_n| \leq a_n,$$

then Σb_n converges.

1.20. Σa_n converges if and only if for every $\varepsilon > 0$ there exists an N such that for $n > N$,

$$|R_n| < \varepsilon,$$

where R_n is the nth remainder of Σa_n.

§5. Summary of Chapters

1.21. As we have seen, the purpose of Chapter I is two-fold. The first being to acquaint the reader with the concept of uniform convergence, and the second, a statement of assumptions. Now, in studying the property of uniform convergence, it might not be convenient or practical to always use the definition given in Chapter I. Further, it might not be possible to tell if a series is uniformly convergent without other definitions and tests. This is the reason for Chapter II. In it the reader will find other definitions of uniform convergence and statements of some of the more important tests listed in a systematic manner. Chapter III is an investigation of the behavior of uniform convergence. In it several fundamental theorems (concerning algebraic calculations, continuity, differentiability, and integrability) are stated along with their contra-positives and some related theorems. In Chapter IV we have a brief discussion of various generalizations of uniform convergence.

CHAPTER II

TESTS FOR UNIFORM CONVERGENCE

§6. Definition Tests

2.1. There are definitions for uniform convergence other than 1.12, and these definitions can be used for tests just as well as the theorems we normally think of as tests. Stated immediately below (2.2 and 2.3) are two such definitions, and it will be noted that in each, the conditions of the hypothesis are both necessary and sufficient for uniform convergence. This is not the case with any of the other tests below.

2.2. A necessary and sufficient condition that $\Sigma f_n(x) \Rightarrow f(x)$ on $[a, b]$ is that for every $\varepsilon > 0$ there exists an integer N such that, for every x in $[a, b]$ and for $n > N$ and $m > N$,

$$|F_n(x) - F_m(x)| < \varepsilon.$$

Proof of necessity. Choose $\varepsilon > 0$ arbitrarily. There exists an N such that, for $n > N$ and for every x in $[a, b]$,

$$|F_n(x) - f(x)| < \varepsilon/2.$$

Also, for $m > N$ and for every x in $[a, b]$,

$$|F_m(x) - f(x)| < \varepsilon/2.$$

Pick $m > N$ and $n > N$. Then, for every x in $[a, b]$,

$$|F_n(x) - F_m(x)| = |F_n(x) - f(x) + f(x) - F_m(x)|$$

$$\leq |F_n(x) - f(x)| + |F_m(x) - f(x)|$$
$$< \varepsilon/2 + \varepsilon/2$$
$$= \varepsilon.$$

Therefore, the condition is necessary.

<u>Proof of sufficiency</u>. Choose $\varepsilon > 0$ arbitrarily. There exists an N such that for $n > N$, $m > N$, and for every x in [a, b],

$$|F_n(x) - F_m(x)| < \varepsilon.$$

From 1.17 we see that $F_m(x)$ has a limit, say f(x). Choose $n > N$ and consider it fixed. Then,

$$\lim_{m \to \infty} |F_n(x) - F_m(x)| = |F_n(x) - f(x)| \leq \varepsilon.$$

Therefore, the condition is sufficient.

2.3. <u>A necessary and sufficient condition that $\Sigma f_n(x) \Rightarrow f(x)$ on [a, b] is that for every $\varepsilon > 0$ there exists an N such that, for $n > N$ and for every x in [a, b],</u>

$$|R_n(x)| < \varepsilon.[1]$$

<u>Proof of necessity</u>. Choose $\varepsilon > 0$ arbitrarily. There exists an N such that, for $n > N$ and for every x in [a, b],

$$|F_n(x) - f(x)| < \varepsilon.$$

Choose $n > N$. Then, for every x in [a, b],

$$|R_n(x)| = |R_n(x) + F_n(x) - F_n(x)|$$
$$= |\Sigma f_n(x) - F_n(x)|$$

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[1]Cf. R. Courant, <u>Differential and Integral Calculus</u>, Vol. I, translated by E. J. McShane, revised, p. 391.

$$= \left| f(x) - F_n(x) \right|$$
$$< \varepsilon.$$

Therefore, the condition is necessary.

<u>Proof of sufficiency</u>. Choose $\varepsilon > 0$ arbitrarily. There exists an integer N such that, for $n > N$ and for every x in $[a, b]$,

$$|R_n(x)| < \varepsilon.$$

Choose $n > N$. Then for every x in $[a, b]$,

$$\Sigma f_n(x) - F_n(x) = R_n(x).$$

Hence,

$$|\Sigma f_n(x) - F_n(x)| = |R_n(x)|.$$

Therefore,

$$|\Sigma f_n(x) - F_n(x)| < \varepsilon.$$

The theorem follows.

§7. The Weierstrass M-test

2.4. The Weierstrass M-test, commonly called the M-test, has the advantage of being applicable to many of the series in everyday use. Its disadvantage lies in the fact that it is sometimes exceedingly difficult to find a convergent series of positive terms to use in the comparison. The M-test is stated below.

2.5. <u>If for every</u> x <u>in</u> $[a, b]$,

$$|f_n(x)| \leq M_n,$$

<u>where</u> M_n <u>is a positive constant independent of</u> x, <u>and if</u>

ΣM_n *is convergent, then* $\Sigma |f_n(x)|$ *is uniformly convergent on* $[a, b]$.[2]

Proof. Denote by $R_n'(x)$ and K_n the nth remainders of $\Sigma |f_n(x)|$ and ΣM_n respectively, and choose $\varepsilon > 0$ arbitrarily. By 1.19 there exists an N such that for $n > N$

$$K_n < \varepsilon.$$

Since, for every n and for every x in [a, b]

$$|f_n(x)| \leq M_n,$$

we have that

$$R_n'(x) \leq K_n.$$

Now pick $n > N$. Then, for every x in [a, b],

$$\begin{aligned} |R_n'(x)| &= R_n'(x) \\ &\leq K_n \\ &< \varepsilon. \end{aligned}$$

2.6. The series,

$$\Sigma f_n(x) = 1 - 1/2 + 1/3 - \dots + (-1)^{n+1}/n + \dots$$

is known to converge uniformly in any interval [a, b], but the M-test fails to show it in this case. For,

$$\Sigma |f_n(x)| = 1 + 1/2 + 1/3 + 1/4 + \dots 1/n + \dots$$

which is known to be divergent. Hence, any series of constant terms, ΣM_n say, such that

$$|f_n(x)| \leq M_n$$

would be divergent.

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[2] T. J. Bromwich, *An Introduction to the Theory of Infinite Series*, second edition revised, p. 124.

2.7. The following test is a corollary of the M-test.

Let $\overline{u}_n$ denote the upper boundary of $|f_n(x)|$ on the interval [a, b]. If $\Sigma\overline{u}_n$ is convergent, then $\Sigma f_n(x)$ is uniformly convergent on [a, b] and is absolutely convergent for each point x in [a, b]. Further, $\Sigma|f_n(x)|$ is uniformly convergent on [a, b].[3]

Proof. $\Sigma\overline{u}_n$ is only a special series of constants such that for every n and for every x in [a, b],

$$|f_n(x)| \leq \overline{u}_n.$$

We have immediately from 2.5 that $\Sigma|f_n(x)|$ is uniformly convergent on [a, b], and hence $\Sigma f_n(x)$ is uniformly convergent and absolutely convergent on [a, b]. (Cf. 2.17).

2.8. A test which is more general and, in some cases, more useful than the M-test is as follows:

If the series $\Sigma u_n(x)$ is uniformly convergent on [a, b], and if for every n and for every x in [a, b]

$$|f_n(x)| \leq u_n(x),$$

then $\Sigma f_n(x)$ is uniformly convergent on [a, b].[4]

Remark. The proof of this test is analogous to the proof of 2.5 and will not be given here.

§8. Abel's Test

2.9. In order to prove Abel's test for uniform

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[3] Hobson, op. cit., p. 115.

[4] Cf. E. C. Titchmarsh, The Theory of Functions, p. 3.

convergence, we must use a lemma developed by him.[5] It is as follows:

<u>Lemma</u>. <u>If</u> $\{v_n\}$ <u>is a non-increasing sequence of positive terms, the sum of the first</u> p <u>terms of</u> $\Sigma a_n v_n$, <u>in notation</u> S_p, <u>lies between</u> Uv_1 <u>and</u> Lv_1, <u>where</u> U <u>and</u> L <u>are respectively the upper and lower limits of the sums</u>

$$a_1, a_1 + a_2, \dots, a_1 + a_2 + \dots + a_p.$$

<u>Proof</u>. Denote by A_n the nth partial sum of Σa_n. Then,

$$a_1 = A_1, a_2 = A_2 - A_1, \dots, a_p = A_p - A_{p-1}.$$

Using this notation we may write,

$$\begin{aligned} S_p &= A_1 v_1 + (A_2 - A_1)v_2 + \dots + (A_p - A_{p-1})v_p \\ &= A_1(v_1 - v_2) + A_2(v_2 - v_3) + \dots \\ &\quad + A_{p-1}(v_{p-1} - v_p) + A_p v_p. \end{aligned}$$

By hypothesis the factors $(v_1 - v_2)$, $(v_2 - v_3)$, ..., $(v_{p-1} - v_p)$, v_p are not negative. Since U and L are respectively the upper and lower limits of

$$A_1, A_2, A_3, \dots, A_p,$$

the sum S_p lies between

$$U(v_1 - v_2) + U(v_2 - v_3) + \dots + U(v_{p-1} - v_p) + Uv_p = Uv_1$$

and

$$L(v_1 - v_2) + L(v_2 - v_3) + \dots + L(v_{p-1} - v_p) + Lv_p = Lv_1.$$

Therefore,

$$Lv_1 \leq S_p \leq Uv_1.$$

The lemma follows.

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[5] Bromwich, <u>op</u>. <u>cit</u>., p. 57.

2.10. If we denote by K the maximum of $|L|$ and $|U|$ we may write further that

$$|S_p| \leq Kv_1.$$

2.11. Abel's test for uniform convergence is as follows: <u>If</u> $\Sigma g_n(x)$ <u>is uniformly convergent</u> (<u>not necessarily absolutely convergent</u>[6]) <u>on</u> [a, b], <u>if for any particular value of</u> x <u>in</u> [a, b] $f_n(x)$ <u>is positive, non-increasing with</u> n, <u>and if</u> $f_1(x)$ <u>is bounded from above for all</u> x <u>in</u> [a, b], <u>then</u> $\Sigma g_n(x)f_n(x)$ <u>is uniformly convergent on</u> [a, b].[7]

<u>Proof</u>. Let K be a positive, upper bound of $f_1(x)$. Denote by $F_n(x)$ and $S_n(x)$ respectively, the nth partial sums of $\Sigma g_n(x)$ and $\Sigma f_n(x)g_n(x)$ on [a, b], and choose $\varepsilon > 0$ arbitrarily. By 2.2 we have that there exists an N such that for $n > N$ and $m > N$

$$|F_n(x) - F_m(x)| < \varepsilon/K.$$

Choose $m > N$ and a positive integer p. Hence, for any x in [a, b],

$$\max\left[|g_{m+1}(x)|, |g_{m+1}(x) + g_{m+2}(x)|, \ldots, |g_{m+1}(x) + \ldots + g_{m+p}(x)|\right] < \varepsilon/K.$$

From 2.10 we have that

$$|S_{m+p}(x) - S_m(x)| < \varepsilon f_m(x)/K.$$

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[6] If $\Sigma g_n(x)$ were absolutely convergent on [a, b] and if g_n were independent of x, the M-test would apply. Furthermore, if $\Sigma|g_n(x)|$ were uniformly convergent, then the generalized M-test 2.8 would imply $\Sigma g_n(x)f_n(x)$ is uniformly convergent.

[7] Cf. Bromwich, <u>op</u>. <u>cit</u>., p. 125.

But, $f_m(x) \leq f_1(x) < K$. Hence,

$$|S_{m+p}(x) - S_m(x)| < \varepsilon K/K = \varepsilon.$$

The theorem follows.

2.12. Dirichlet gives the following theorem as a test for the convergence of a series of numbers.

If $\{b_n\}$ is a monotone, non-increasing sequence of positive numbers which tend to zero as n increases without limit and Σa_n is a convergent series, then $\Sigma a_n b_n$ is convergent.[8] From this we can determine an analogous test for uniform convergence which is somewhat similar to Abel's test.

2.13. *$\Sigma a_n f_n(x)$ is uniformly convergent on [a, b], if a_n is a positive function of n which tends monotonically to zero as n increases without limit and if there exists a constant K such that*

$$|F_n(x)| \leq K$$

for all values of n and for every x in [a, b].[9]

Proof. Choose $\varepsilon > 0$ arbitrarily. There exists an N such that for $n > N$

$$a_n < \varepsilon/(2K+1).$$

Choose $m > n > N$. Since for every n and for every x in [a, b], $|F_n(x)| < K$,

$$|f_n(x)|,\ |f_n(x) + f_{n+1}(x)|,\ |f_n(x) + \ldots + f_m(x)|,$$

are each less than 2K. By 2.10 we have that

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[8] Cf. G. H. Hardy, *A Course of Pure Mathematics*, sixth edition, p. 342.

[9] Titchmarch, *op. cit.*, p. 4.

$$|a_n f_n(x) + a_{n+1} f_{n+1}(x) + \dots + a_m f_m(x)| < K\varepsilon/(2K+1) < \varepsilon.$$

By 2.2 $\Sigma a_n f_n(x)$ is uniformly convergent on [a, b].

2.14. Dirichlet's test for uniform convergence is a generalization of 3.13. It is as follows:

<u>$\Sigma g_n(x) f_n(x)$ is uniformly convergent on [a, b] if the series $\Sigma f_n(x)$ is such that for all x in [a, b] and for every n,</u>

$$|F_n(x)| < K;$$

<u>if for every x in [a, b], $g_n(x)$ is positive and never increases with n; and as n increases without limit, $g_n(x)$ tends uniformly to zero for every x in [a, b].</u>[10]

2.15. The obvious disadvantage of the tests in this section is that they can not be used to test all functions which are uniformly convergent. Note, for instance, the following example. For every n and every x in [a, b] let

$$a_n = n$$

and

$$\Sigma f_n(x) = 1/2 + 1/2(2^2) + \dots + 1/n(2^n) + \dots .$$

Then,

$$\Sigma a_n f_n(x) = \Sigma(n) 1/n(2^n) = \Sigma 1/2^n = 1.$$

Hence, $\Sigma a_n f_n(x)$ is uniformly convergent on any interval [a, b], but these conditions do not satisfy all of the conditions for any of the tests in this section.

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[10] Bromwich, <u>op. cit.</u>, p. 125.

§9. Other Tests

2.16. _If $\Sigma f_n(x)$ is uniformly convergent on [a, b] and if $f_n(x) \geq 0$ for every n and for every x in [a, b], any series $\Sigma g_n(x)$ formed by rearranging the order of the terms of $\Sigma f_n(x)$ is also uniformly convergent on [a, b]._[11]

Proof. Denote by $F_n(x)$ and $G_n(x)$ respectively the nth partial sums of $\Sigma f_n(x)$ and $\Sigma g_n(x)$, and by $R_n(x)$, $T_n(x)$ their nth remainders. Choose $\varepsilon > 0$ arbitrarily. From 2.3 we have that there exists an N such that for $n > N$ and for every x in [a, b]

$$|R_n(x)| < \varepsilon.$$

Let $\Sigma g_n(x)$ be any rearrangement of $\Sigma f_n(x)$ and choose $n > N$. Denote by p an integer for which $G_p(x)$ contains the terms of $F_n(x)$; then $R_n(x)$ contains all of the terms of $T_p(x)$, and hence

$$|T_p(x)| \leq |R_n(x)| < \varepsilon.$$

The test follows from 2.3.

2.17. _If $\Sigma |f_n(x)|$ converges uniformly on [a, b], then $\Sigma f_n(x)$ also converges uniformly on [a, b]. Further, any rearrangement of $\Sigma f_n(x)$ is uniformly convergent._[12]

Proof. Choose $\varepsilon > 0$ arbitrarily. From 2.2 we have that there exists an N such that for $n > m > N$ and for every x in [a, b],

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[11] Cf. Hobson, _op. cit._, p. 115.

[12] Cf., _ibid._, p. 116.

$$|f_m(x)| + |f_{m+1}(x)| + \dots + |f_n(x)| < \varepsilon.$$

Choose $n > m > N$. Then for every x in $[a, b]$

$$\begin{aligned} &|f_m(x) + f_{m+1}(x) + \dots + f_n(x)| \\ &\leq |f_m(x)| + |f_{m+1}(x)| + \dots + |f_n(x)| \\ &< \varepsilon. \end{aligned}$$

Hence, by 2.2 $\Sigma f_n(x)$ is uniformly convergent on $[a, b]$. Let $\Sigma g_n(x)$ be any rearrangement of $\Sigma f_n(x)$. Since by 2.16 any rearrangement of $\Sigma|f_n(x)|$ is uniformly convergent on $[a, b]$, then $\Sigma|g_n(x)|$ converges uniformly on $[a, b]$, and, hence by the first part of 2.17, $\Sigma g_n(x)$ converges uniformly on $[a, b]$. The theorem follows.

2.18. Due to the work of Birkhoff a converse of 2.17 has been established. It is as follows.

_If_ $\Sigma f_n(x)$ _is uniformly convergent on_ $[a, b]$ _however its terms may be rearranged, then_ $\Sigma|f_n(x)|$ _is uniformly convergent on_ $[a, b]$.[13]

Proof. Suppose otherwise that under the conditions of the hypothesis $\Sigma|f_n(x)|$ is not uniformly convergent. Then from the denial of 2.2, there exists an $\varepsilon > 0$ such that, for every N, there exists an x in $[a, b]$ and an $n > m > N$ such that

$$|f_m(x)| + |f_{m+1}(x)| + \dots + |f_n(x)| \geq 2\varepsilon.$$

Let P = the sum of the positive terms in

$$f_m(x) + f_{m+1}(x) + \dots + f_n(x),$$

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[13] G. D. Birkhoff, "A Theorem Concerning Uniform Convergence", _Annals of Mathematics_, Vol. VI, (1904-1905), p. 90.

and let Q = the sum of the negative terms. Then,

$$P + |Q| \geq 2\varepsilon.$$

If we let $U = \max [P, |Q|]$, then

$$U \geq \varepsilon.$$

Choose $N_o = 0$. As above, there exists an x_o in $[a, b]$ and an $n_o > m_o > N_o$ such that,

$$U_o \geq \varepsilon.$$

Choose $N_1 = n_o$. There exists an x_1 in $[a, b]$ and an $n_1 > m_1 > N_1$ such that

$$U_1 \geq \varepsilon.$$

Choose $N_2 = n_1$. There exists an x_2 in $[a, b]$ and an $n_2 > m_2 > N_2$ such that

$$U_2 \geq \varepsilon.$$

In general choose $N_n = n_{n-1}$. There exists an x_n in $[a, b]$ and an $n_n > m_n > N_n$ such that

$$U_n \geq \varepsilon.$$

Denote by $U_o(x)$ the sum of the terms which are in the section $n = 1$ to $n = n_o$ of $\Sigma f_n(x)$ and also in U_o, i.e., $U_o(x)$ is the finite summation consisting of the terms whose subscripts are the same as the subscripts of the terms of U_o; then denote by $V_o(x)$ the sum of the terms which are in the above section but not in U_o. In general, denote $U_n(x)$ the sum of the terms which are in both the section $n = n_{n-1} + 1$ to n_n of $\Sigma f_n(x)$ and U_n; then, denote by $V_n(x)$ the sum of the terms which are in the section but not in U_n. Rearrange $\Sigma f_n(x)$ in such a way as to have all of the terms of each of the finite sums $U_o(x)$, $U_1(x)$, ... , $U_n(x)$, ... and $V_o(x)$, $V_1(x)$, ... , $V_n(x)$, ... grouped together as follows:

$$\Sigma f_n(x) = U_1(x) + V_1(x) + U_2(x) + V_2(x) + \cdots$$
$$+ U_n(x) + V_n(x) + \cdots .$$

Now choose any integer N. There exists two integers $p > q > N$ and an x_r in [a, b] such that,

$$|f_q(x_r) + f_{q+1}(x_r) + \cdots + f_p(x_r)| = U_r.$$

But U_r was so constructed that

$$U_r \geq \varepsilon.$$

This contradicts the hypothesis that every rearrangement of $\Sigma f_n(x)$ is uniformly convergent. Hence the theorem follows.

2.19. The ratio test for ordinary convergence is as follows:

The series $\Sigma f_n(x)$ is convergent on [a, b] if for every x in [a, b] there exists an r, $o < r < 1$, such that for every n,

$$\left|\frac{f_{n+1}(x)}{f_n(x)}\right| \leq r.$$

A very similar test for uniform convergence is listed below.

2.20. The series $\Sigma f_n(x)$ is uniformly convergent on [a, b] if $f_1(x)$ is bounded on [a, b] and if there exists an r, $o < r < 1$, such that for every x in [a, b] and for every n

$$\left|\frac{f_{n+1}(x)}{f_n(x)}\right| \leq r.$$ [14]

[14] Cf. Titchmarsh, *op. cit.*, p. 4.

<u>Proof</u>. Since for every x in [a, b] and for every n,

$$\left|\frac{f_{n+1}(x)}{f_n(x)}\right| \leq r,$$

$$r|f_n(x)| \geq |f_{n+1}(x)|.$$

Similarly

$$r^2|f_{n-1}(x)| \geq r|f_n(x)|,$$

and finally we have $r^n|f_1(x)| \geq r|f_n(x)|$

or

$$|f_n(x)| \leq r^{n-1}|f_1(x)|.$$

Let $M > 0$ be an upper bound of $f_1(x)$ on [a, b]. Then,

$$|f_n(x)| \leq r^{n-1}M.$$

But, since $r < 1$, $\Sigma r^{n-1}M$ is convergent. Hence, by 2.5 $\Sigma|f_n(x)|$ is uniformly convergent on [a, b].

2.21. <u>Remark</u>. The ratio tests fails when $r = 1$. For take the divergent series $\Sigma 1/n$. Clearly,

$$\left|\frac{f_{n+1}(x)}{f_n(x)}\right| = \frac{n}{n+1} \leq 1$$

for all n. Now take the convergent series $\Sigma(-1)^{n+1}/n$. Then

$$\left|\frac{f_{n+1}(x)}{f_n(x)}\right| = \frac{n}{n+1} \leq 1.$$

From this we see that if $r = 1$ the series may either converge uniformly or not.

CHAPTER III

SOME FUNDAMENTAL THEOREMS

§10. Calculations with Uniformly Convergent Series

3.1. *If* $\Sigma f_n(x) \Rightarrow f(x)$ *on* $[a, b]$ *and if* $\Sigma g_n(x) \Rightarrow g(x)$ *on* $[a, b]$, *then*

$$\Sigma[f_n(x) + g_n(x)] \Rightarrow f(x) + g(x)$$

on $[a, b]$.

Proof. Let $F_n(x)$, $G_n(x)$, $S_n(x)$ denote respectively the nth partial sum of $\Sigma f_n(x)$, $\Sigma g_n(x)$, $\Sigma[f_n(x) + g_n(x)]$. Let $\varepsilon > 0$ be chosen arbitrarily. Then there exists an integer N_1 such that for $n > N_1$ and for every x in $[a, b]$

$$|F_n(x) - f(x)| < \varepsilon/2.$$

There exists an integer N_2 such that for $n > N_2$ and for every x in $[a, b]$

$$|G_n(x) - g(x)| < \varepsilon/2.$$

Let $N = \max[N_1, N_2]$ and choose $n > N$. Then for every x in $[a, b]$

$$\begin{aligned} S_n(x) - [f(x) + g(x)] &= |F_n(x) + G_n(x) - f(x) - g(x)| \\ &\leq |F_n(x) - f(x)| + |G_n(x) - g(x)| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

The theorem follows.

3.2. <u>If</u> $\Sigma f_n(x) \Rightarrow f(x)$ <u>on</u> $[a, b]$ <u>and if</u> $\Sigma g_n(x) \Rightarrow g(x)$ <u>on</u> $[a, b]$, <u>then</u>

$$\Sigma[f_n(x) - g_n(x)] \Rightarrow f(x) - g(x)$$

<u>on</u> $[a, b]$.

For a proof, see 3.5.

3.3. <u>If</u> $\Sigma f_n(x) \Rightarrow f(x)$ <u>on</u> $[a, b]$ <u>and if</u> $u(x)$ <u>is defined and bounded on</u> $[a, b]$, <u>then</u> $\Sigma u(x)f_n(x) \Rightarrow u(x)f(x)$ <u>on</u> $[a, b]$.[1]

<u>Proof</u>. Since $u(x)$ is bounded on $[a, b]$, there exists a positive number K such that for every x in $[a, b]$

$$|u(x)| \leq K.$$

Let $F_n(x)$ be the nth partial sum of $\Sigma f_n(x)$ and let $S_n(x)$ be the nth partial sum of $\Sigma u(x)f_n(x)$. Now

$$\begin{aligned} S_n(x) &= u(x)f_1(x) + u(x)f_2(x) + \dots + u(x)f_n(x) \\ &= u(x)[f_1(x) + f_2(x) + \dots + f_n(x)] \\ &= u(x)F_n(x). \end{aligned}$$

Let $\varepsilon > 0$ be chosen arbitrarily. There exists an N such that for $n > N$ and for every x in $[a, b]$

$$|F_n(x) - f(x)| < \varepsilon/K.$$

Choose $n > N$. Then for every x in $[a, b]$

$$\begin{aligned} |S_n(x) - u(x)f(x)| &= |u(x)F_n(x) - u(x)f(x)| \\ &= |u(x)|\,|F_n(x) - f(x)| \\ &< K|F_n(x) - f(x)| \\ &< K\varepsilon/K = \varepsilon. \end{aligned}$$

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[1]K. Knopp, <u>Theory and Application of Infinite Series</u>, second German edition translated by R. C. Young, p. 337.

The theorem follows.

3.4. Corollary of 3.1 and 3.3. If the p series

$$\Sigma f_{n1}(x),\ \Sigma f_{n2}(x),\ \dots\ ,\ \Sigma f_{np}(x)$$

are such that

$$\Sigma f_{n1}(x) \Rightarrow f_1(x),\ \Sigma f_{n2}(x) \Rightarrow f_2(x),\ \dots\ ,\ \Sigma f_{np}(x) \Rightarrow f_p(x)$$

each on [a, b] and if the functions,

$$u_1(x),\ u_2(x),\ \dots\ ,\ u_p(x)$$

are each defined and bounded on [a, b], then the series $\Sigma g_n(x)$ where for every n,

$$g_n(x) = u_1(x)f_{n1}(x) + u_2(x)f_{n2}(x) + \dots + u_p(x)f_{np}(x),$$

is uniformly convergent on [a, b] and

$$\Sigma g_n(x) \Rightarrow u_1(x)f_1(x) + u_2(x)f_2(x) + \dots + u_p(x)f_p(x).$$

Proof. From 3.3 we have that

$$\Sigma u_1(x)f_{n1}(x),\ \Sigma u_2(x)f_{n2}(x),\ \dots\ ,\ \Sigma u_p(x)f_{np}(x)$$

each converge uniformly to

$$u_1(x)f_1(x),\ u_2(x)f_2(x),\ \dots\ ,\ u_p(x)f_p(x)$$

respectively on [a, b]. Then by repeated use of 3.1 we have that

$$\Sigma g_n(x) \Rightarrow u_1(x)f_1(x) + u_2(x)f_2(x) + \dots + u_p(x)f_p(x)$$

on [a, b].

3.5. Remark. An interesting special case of 3.4 is obtained when each of the functions $u_1(x),\ u_2(x),\ \dots\ ,\ u_p(x)$ are constants. This immediately gives us a proof of 3.2. For let $u_1(x) = 1$ and $u_2(x) = -1$, then we have

$$\Sigma[f_n(x) + (-1)g_n(x)] \Rightarrow f(x) + (-1)g(x)$$
$$= f(x) - g(x).$$

This is all that 3.2 states.

3.6. <u>If</u> $\Sigma f_n(x)$ <u>converges uniformly on</u> [a, b], <u>then there exists an</u> N <u>such that, for</u> $n > N$, $f_n(x)$ <u>is bounded on</u> [a, b] <u>and</u> $\{f_n(x)\}$ <u>converges uniformly to zero.</u>[2]

<u>Proof.</u> Let $\varepsilon > 0$ be chosen arbitrarily. By 2.3 there exists an N such that for $n > N$,

$$|R_n(x)| < \varepsilon/2.$$

Choose $n - 1 > N$. Then for every x in [a, b],

$$\begin{aligned} |f_n(x)| &= |R_{n-1}(x) - R_n(x)| \\ &\leq |R_{n-1}(x)| + |R_n(x)| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

Hence, $\{f_n(x)\}$ converges uniformly to zero, and, for $n > N$, $f_n(x)$ is bounded on [a, b].

3.7. <u>If</u> $\Sigma g_n(x)$ <u>is uniformly convergent on</u> [a, b] <u>and if</u> $\Sigma|f_n(x)|$ <u>is uniformly convergent on</u> [a, b] <u>then</u>

$$\Sigma|f_n(x)g_n(x)|, \; n = 1, 2, \ldots ,$$

<u>is uniformly convergent on</u> [a, b].

<u>Proof.</u> Let $R_n'(x)$ be the remainder of $\Sigma|f_n(x)|$ and let $T_n(x)$ be the remainder of $\Sigma|f_n(x)g_n(x)|$. Choose $\varepsilon > 0$ arbitrarily. By 2.3 there exists an N_1 such that for $n > N_1$ and for every x in [a, b]

$$R_n'(x) < \varepsilon.$$

From 3.6 there exists an N_2 such that for $n > N_2$ and for every x in [a, b]

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[2]Cf. <u>ibid.</u>, p. 338.

$$|g_n(x)| < 1.$$

Let $N = \max[N_1, N_2]$ and choose $n > N$. Then for every x in $[a, b]$

$$\begin{aligned} T_n(x) &= |f_{n+1}(x)g_{n+1}(x)| + |f_{n+2}(x)g_{n+2}(x)| + \dots \\ &< |f_{n+1}(x)| + |f_{n+2}(x)| + \dots \\ &< \varepsilon. \end{aligned}$$

The theorem follows by 2.3.

3.8. <u>Assume</u> $\Sigma f_n(x) \Rightarrow f(x)$, $\Sigma g_n(x) \Rightarrow g(x)$ <u>on</u> $[a, b]$; <u>and</u> $f(x)$ <u>and</u> $g(x)$ <u>are bounded on</u> $[a, b]$. <u>Then if</u>

$$S_n(x) = \Sigma f_n(x)g_m(x),\ n, m, = 1, 2, \dots, n,$$

<u>it follows that</u>

$$\{S_n(x)\} \Rightarrow f(x)g(x)$$

<u>on</u> $[a, b]$.

<u>Remark</u>. The finite summation,

$$S_n(x) = \Sigma f_n(x)g_m(x),\ n = 1, 2, \dots, n,$$

is often called the "nth square" of the infinite matrix consisting of the terms of the formal product

$$[\Sigma f_n(x)][\Sigma g_n(x)].$$

The theorem then proves that the sequence of nth squares of this infinite matrix converges uniformly to $f(x)g(x)$.

<u>Proof of</u> 3.8. Denote respectively by $F_n(x)$ and $G_n(x)$ the nth partial sums of $\Sigma f_n(x)$ and $\Sigma g_n(x)$. Let $K > 0$ and $L/2 > 0$ be respectively bounds of $|g(x)|$ and $|f(x)|$ on $[a, b]$. Choose $\varepsilon > 0$ arbitrarily. There exists an N_1 such that, for $n > N_1$ and for every x in $[a, b]$,

$$|F_n(x)| < L.$$

There exists an N_2 such that, for $n > N_2$ and for every x in [a, b],

$$|F_n(x) - f(x)| < \varepsilon/2K.$$

There exists an N_3 such that for $n > N_3$ and for every x in [a, b]

$$|G_n(x) - g(x)| < \varepsilon/2K.$$

Let $N = \max[N_1, N_2, N_3]$ and choose $n > N$. Then, for every x in [a, b]

$$\begin{aligned} |S_n(x) - f(x)g(x)| &= |F_n(x)G_n(x) - f(x)g(x)| \\ &= |F_n(x)G_n(x) - F_n(x)g(x) \\ &\quad + F_n(x)g(x) - f(x)g(x)| \\ &\leq |F_n(x)|\,|G_n(x) - g(x)| \\ &\quad + |g(x)|\,|F_n(x) - f(x)| \\ &< L\varepsilon/2L + K\varepsilon/2K \\ &= \varepsilon. \end{aligned}$$

The theorem follows.

§11. On Continuity

3.9. <u>If</u> $\Sigma f_n(x) \Rightarrow f(x)$ <u>on</u> [a, b], <u>and if for every</u> n, $f_n(x)$ <u>is continuous at</u> x_o in [a, b], <u>then</u> f(x) <u>is continuous at</u> x_o.[3]

<u>Proof</u>. Since the sum of any finite number of continuous functions is continuous, we have that $F_n(x)$ is continuous at $x = x_o$. Choose $\varepsilon > 0$ arbitrarily. There exists an N such that, for $n > N$ and for every x in [a, b],

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[3] Cf. <u>ibid</u>., p. 338.

$$|F_n(x) - f(x)| < \varepsilon/3.$$

Hence,

$$|F_n(x_o) - f(x_o)| < \varepsilon/3.$$

From continuity, we have that there exists a $\delta > 0$ such that for $|x - x_o| < \delta$

$$|f_n(x) - f_n(x_o)| < \varepsilon/3.$$

Choose $n > N$ and an x such that $|x - x_o| < \delta$. Then

$$\begin{aligned} |f(x) - f(x_o)| &= |f(x) - F_n(x) + F_n(x) - f(x_o)| \\ &\leq |f(x) - F_n(x)| + |F_n(x) - f(x_o)| \\ &= |f(x) - F_n(x)| + |F_n(x) - F_n(x_o) \\ &\quad + F_n(x_o) - f(x_o)| \\ &\leq |f(x) - F_n(x)| + |F_n(x) - F_n(x_o)| \\ &\quad + |F_n(x_o) - f(x_o)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon. \end{aligned}$$

The theorem follows.

3.10. An interesting corollary of 3.9 comes from letting $f_n(x)$, for every n, be continuous for every x_o in $[a, b]$, (i.e., for every n, $f_n(x)$ is continuous on $[a, b]$). The result is as follows:

If $\Sigma f_n(x) \Rightarrow f(x)$ on $[a, b]$, and if for every n, $f_n(x)$ is continuous on $[a, b]$, then $f(x)$ is continuous on $[a, b]$.[4]

Remark. The condition of ordinary convergence in place of uniform convergence in 3.10 is not enough to assure

[4] Ibid., p. 339.

continuity of the limit function. For, look at the following example. Let $\Sigma f_n(x)$ be a series whose nth partial sum is defined as follows on the interval [0, 1]:

$$F_n(x) = \begin{cases} 1 - nx \text{ for } 0 \leq x \leq 1/n \\ 0 \text{ for } 1/n < x \leq 1 \end{cases}$$

Then, the limit function $f(x) = 1$ for $x = 0$ and zero for $x \neq 0$.

3.11. The theorem of 3.9 has two useful contra-positives. They are stated immediately below.

(a) _If_ $\Sigma f_n(x) \Rightarrow f(x)$ _on_ [a, b] _and if_ $f(x)$ _is discontinuous for some point_ x_o _in_ [a, b], _then for every_ N _there exists an_ $n > N$ _such that_ $f_n(x)$ _is discontinuous at_ x_o.

(b) _If_ $\Sigma f_n(x) \rightarrow f(x)$ _on_ [a, b]; _if for every_ n, $f_n(x)$ _is continuous at_ x_o _in_ [a, b]; _and if_ $f(x)$ _is discontinuous at_ x_o, _then_ $\Sigma f_n(x) \not\Rightarrow f(x)$ _on_ [a, b].

3.12. By using the limit concept of continuity, we may state the theorem of 3.10 in the following alternative form:

_If_ $\Sigma f_n(x) \Rightarrow f(x)$ _on_ [a, b] _and if for every_ n, $f_n(x)$ _is continuous on_ [a, b], _then for every_ x_o _in_ [a, b]

$$\lim_{x \to x_o} [\Sigma f_n(x)] = \Sigma[\lim_{x \to x_o} f_n(x)].[5]$$

3.13. We state two converses of 3.10 neither of which is true. They are stated below and examples are given which disprove them.

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[5] _Ibid._, p. 339.

(a) *If* $\Sigma f_n(x) \Rightarrow f(x)$ *on* $[a, b]$ *and if* $f(x)$ *is continuous on* $[a, b]$, *then there exists an* N *such that for* $n > N$, $f_n(x)$ *is continuous on* $[a, b]$.

An example which disproves this theorem is as follows: Let $\Sigma f_n(x)$ be a series such that for x in $[0, 1]$

$$f_n(x) = \begin{cases} 0 \text{ for } 0 \leq x < 1/2 \\ 1/n \text{ for } 1/2 \leq x \leq 1 \end{cases}.$$

Obviously $\Sigma f_n(x) \Rightarrow 0$ on $[0,1]$, but $f_n(x)$ is discontinuous at $x = 1/2$ for every n. Hence, the theorem is false.

(b) *If* $\Sigma f_n(x) \rightarrow f(x)$ *on* $[a, b]$; *if for every* n, $f_n(x)$ *is continuous on* $[a, b]$; *and if* $f(x)$ *is continuous on* $[a, b]$, *then* $\Sigma f_n(x) \Rightarrow f(x)$ *on* $[a, b]$.

An example which disproves this theorem is as follows: Let $\Sigma f_n(x)$ be a series such that $F_n(x) = nx/(1 + n^2x^2)$ on $[0, 1]$. Obviously, $\Sigma f_n(x)$ converges to zero on $[0, 1]$ and, therefore, we have $f_n(x)$ continuous for every n and x in $[0, 1]$ and also its limit function $f(x) = 0$ is continuous on $[0, 1]$. But choose $\varepsilon = 1/4$ and then pick N arbitrarily. There exists an $n > N$ and an x in $[0, 1]$ such that $nx = 1$, i.e., $n = 1/x$. Then

$$\begin{aligned} |F_n(x) - f(x)| &= |nx/(1 + n^2x^2) - 0| \\ &= 1/(1 + 1) = 1/2 > \varepsilon. \end{aligned}$$

Hence $\Sigma f_n(x) \not\Rightarrow f(x)$ on $[0, 1]$.

§12. On Integration

3.14. In order to prove the main theorem on integration of uniformly convergent series (see 3.15), we will first assume the two theorems stated immediately below.

(a) *If* $y = f(x)$ *is bounded on* $[a, b]$, *then a necessary and sufficient condition for* $f(x)$ *to be R-integrable is that the set of points of discontinuity of* $f(x)$ *on* $[a, b]$ *be of Lebesgue measure zero.*

(b) *If both* $f(x)$ *and* $g(x)$ *are R-integrable on* $[a, b]$, *and if, for every* x *in* $[a, b]$, $f(x) \leq g(x)$, *then*

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

3.15. *If* $\Sigma f_n(x) \Rightarrow f(x)$ *on* $[a, b]$ *and if for every* n, $f_n(x)$ *is R-integrable on* $[a, b]$, *then* $f(x)$ *is also R-integrable on* $[a, b]$, *and*

$$\int_a^b f(x)\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx + \dots ,$$

i.e.,

$$\int_a^b \Sigma f_n(x)\,dx = \Sigma\Big[\int_a^b f_n(x)\,dx\Big].$$

Proof. Let Z_n denote the set of points x of $[a, b]$ at which $f_n(x)$ is discontinuous. Then let

$$Z = Z_1 + Z_2 + \dots .$$

If x_o is not a point of Z, then each of the functions $f_n(x)$ are continuous at x_o, and by 3.9, $f(x)$ is continuous at x_o. Then, since the functions $f_n(x)$ are all continuous on $[a, b] - Z$, $f(x)$ is continuous on $[a, b] - Z$. But Z, being the sum of a denumerable number of sets of measure zero, is also of measure zero. Therefore, $f(x)$ is discontinuous on at most a set of measure zero. Hence by (a) of 3.14, $f(x)$ is R-integrable on $[a, b]$.

Let $\varepsilon > 0$ be chosen arbitrarily. There exists an N such that, for $n > N$ and for every x in [a, b],

$$-\varepsilon/(b-a) < F_n(x) - f(x) < \varepsilon/(b-a).$$

By (b) of 3.14 we have that

$$-\int_a^b \varepsilon/(b-a)dx < \int_a^b F_n(x)dx - \int_a^b f(x)dx < \int_a^b \varepsilon/(b-a)dx.$$

Hence,

$$-\varepsilon(b-a)/(b-a) < \int_a^b F_n(x)dx - \int_a^b f(x)dx < \varepsilon(b-a)/(b-a),$$

and

$$-\varepsilon < \int_a^b F_n(x)dx - \int_a^b f(x)dx < \varepsilon.$$

Therefore,

$$\Sigma\int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

3.16. In the above theorem, the condition of uniform convergence is necessary. For, consider the following example: Let $\Sigma f_n(x)$ be a series such that on [0, 1]

$$F_n(x) = \begin{cases} 4n^2x \text{ for } 0 \leq x < 1/2n \\ -4n^2x + 4n \text{ for } 1/2n \leq x \leq 1/n. \\ 0 \text{ for } 1/n < x \leq 1 \end{cases}$$

Since $f_n(x)$ is continuous on [0, 1] it is R-integrable and we see at once that $\Sigma f_n(x) \to 0$ on [a, b]. But, for every n we see that $\int_0^1 F_n(x)dx = 1$. Hence, we have that

$$\Sigma\int_0^1 f_n(x)dx = 1,$$

and

$$\int_0^1 \Sigma f_n(x)dx = 0.$$

Hence,

$$\Sigma\int_0^1 f_n(x)dx \neq \int_0^1 \Sigma f_n(x)dx.$$

§13. On Differentiability

3.17. We shall first prove an auxiliary theorem on an interchange of limits needed for the main theorem (see 3.18) in this section.

<u>If</u> $\{f_n(x)\} \Rightarrow f(x)$ <u>on an interval</u> (a, b) <u>containing</u> x_o <u>and if for every</u> n, $\lim_{x \to x_o} f_n(x) = a_n$ <u>then</u> $\{a_n\}$ <u>converges and</u>

$$\lim_{n \to \infty} [\lim_{x \to x_o} f_n(x)] = \lim_{x \to x_o} [\lim_{n \to \infty} f_n(x)].$$

<u>Proof</u>. First consider the sequence $\{a_n\}$. Choose $\varepsilon > 0$. Since $\{f_n(x)\} \Rightarrow f(x)$, there exists an $N > 0$ such that for $n > N$, $m > N$, and for every x in (a, b)

$$|f_m(x) - f_n(x)| < \varepsilon/3.$$

Since $\lim_{x \to x_o} f_n(x) = a_n$, there exists a $\delta_1 > 0$ such that for $0 < |x - x_o| < \delta_1$

$$|a_n - f_n(x)| < \varepsilon/3.$$

Likewise, there exists a $\delta_2 > 0$ such that for $0 < |x - x_o| < \delta_2$

$$|a_m - f_m(x)| < \varepsilon/3.$$

Let $\delta = \min [\delta_1, \delta_2]$. Then if $m > N$, $n > N$, and $0 < |x - x_o| < \delta$

$$\begin{aligned} |a_m - a_n| &= |a_m - f_m(x) + f_m(x) - f_n(x) + f_n(x) - a_n| \\ &\leq |a_m - f_m(x)| + |f_m(x) - f_n(x)| + |f_n(x) - a_n| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Hence, by 1.18, $\{a_n\}$ converges. Call its limit a. We have thus proved that $\lim_{n \to \infty} [\lim_{x \to x_o} f_n(x)] = a$.

Again choose $\varepsilon > 0$. There exists an N_1 such that for

$n > N_1$

$$|a - a_n| < \varepsilon/3.$$

There exists an N_2 such that for $n > N_2$ and for every x in (a, b)

$$|f_n(x) - f(x)| < \varepsilon/3.$$

There exists a $\Delta > 0$ such that for $0 < |x - x_0| < \Delta$

$$|a_n - f_n(x)| < \varepsilon/3.$$

Choose $M = \max [N_1, N_2]$. Then for $n > M$ and for $0 < |x - x_0| < \Delta$

$$\begin{aligned} |a - f(x)| &= |a - a_n + a_n - f_n(x) + f_n(x) - f(x)| \\ &\leq |a - a_n| + |a_n - f_n(x)| + |f_n(x) - f(x)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Hence $\lim\limits_{x \to x_0} f(x) = a$, and the theorem follows.

3.18. <u>Assume</u> $f_n(x)$, $n = 1, 2, \ldots$, <u>is differentiable on</u> (a, b); <u>and</u> $\Sigma f_n(x) \to f(x)$ <u>and</u> $\Sigma f'(x) \Rightarrow g(x)$ <u>on</u> (a, b); <u>then</u> f(x) <u>is differentiable on</u> (a, b) <u>and</u>

$$f'(x) = g(x),$$

i.e., $f'(x) = \Sigma f_n'(x)$.

<u>Proof</u>. Let $F_n(x)$, $F_n'(x)$ denote respectively the nth partial sum of $\Sigma f_n(x)$, $\Sigma f_n'(x)$. In sequence notation, we have given that $\{F_n(x)\} \to f(x)$ and $F_n'(x) \Rightarrow g(x)$ on (a, b), and we are to prove that f(x) is differentiable and

$$f'(x) = \lim_{n \to \infty} F_n'(x)$$

for x in (a, b).

Consider the function

$$D(n, h) = \frac{F_n(x_o + h) - F_n(x_o)}{h}$$

where x_o is a fixed point in (a, b). We shall examine the double limits $\lim_{h\to 0}[\lim_{n\to\infty} D(n, h)]$ and $\lim_{n\to\infty}[\lim_{h\to 0} D(n, h)]$; we may apply 3.17 if we can show that $D(n, h)$ converges uniformly as $n \to \infty$. Now

$$D(p, h) - D(q, h)$$

$$= \frac{F_p(x_o + h) - F_p(x_o)}{h} - \frac{F_q(x_o + h) - F_q(x_o + h)}{h}$$

$$= \frac{[F_p(x_o + h) - F_q(x_o + h)] - [F_p(x_o) - F_q(x_o)]}{h}$$

$$= F'_p(x_o + \theta h) - F'_q(x_o + \theta h),$$

where $|\theta| < 1$. (The last equality is obtained from the theorem of the mean for differentiation). Choose $\varepsilon > 0$. Then, since $F'_n(x)$ is uniformly convergent, there exists an N such that for $p > N$, $q > N$, and $(x_o + \theta h)$ in (a, b),

$$|F'_p(x_o + \theta h) - F'_q(x_o + \theta h)| < \varepsilon.$$

Hence, by 2.2, $D(n, h)$ converges uniformly as $n \to \infty$, and, of course,

$$D(n, h) \Rightarrow \frac{f(x_o + h) - f(x)}{h}.$$

Also, by assumption, for every x_o in (a, b)

$$\lim_{h\to 0} D(n, h) = \lim_{h\to 0} \frac{F_n(x_o + h) - F_n(x_o)}{h}$$

$$= F'_n(x_o),$$

and

$$\lim_{n \to \infty} F_n'(x_o) = g(x_o).$$

Hence,

$$\lim_{n \to \infty} [\lim_{h \to 0} D(n, h)] = g(x_o).$$

Now by 3.17, $\lim_{h \to 0} [\lim_{n \to \infty} D(n, h)]$ exists and equals $g(x_o)$.

But since x_o was an arbitrary point in (a, b), for every x in (a, b), we thus have

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x + h) - f(x)}{h}\right] = g(x).$$

The theorem follows.

3.19. In the above theorem, the condition that the derivatives converge uniformly in the interval is necessary even though the series may converge uniformly in the interval. For, consider the series $\Sigma f_n(x)$ whose nth partial sums equal $(1/n)\sin(nx)$ for every n. It is quite evident that the series converges uniformly on $[0, \pi]$, and further, $f_n'(x)$ exists for every n and for every x in $[0, \pi]$. Upon differentiating term by term, we obtain the series $\Sigma f_n'(x)$ whose nth partial sums equal $\cos(nx)$. But it is easily shown that $\Sigma f_n'(x)$ is not even convergent on an everywhere dense subset of $[0, \pi]$.

§14. On Power Series

3.20. Among all classes of series of functions $\Sigma f_n(x)$, the class of series called <u>power series</u> is usually considered the most important. By definition, a power series is of the form

$$\Sigma a_n x^n = a_o + a_1 x + \dots + a_n x^n + \dots .$$

In connection with our present paper, it is of special importance that uniform convergence exist automatically for power series. The following theorem states more precisely the property to which we have just alluded:

(a) <u>If R is the radius of convergence of $\Sigma a_n x^n$, then the series $\Sigma a_n x^n$ converges absolutely and uniformly in every interval which lies strictly inside [- R, R], i.e., for any interval [a, b] such that $-R < a < b < R$.</u>

Another theorem we shall need below is the following:

(b) <u>The series $\Sigma n a_n x^{n-1} = a_1 + 2a_2 x + \dots + n a_n x^{n-1} + \dots$ has the same interval of convergence as $\Sigma a_n x^n$. Hence, the series obtained from $\Sigma a_n x^n$ by termwise differentiation is uniformly convergent in every interval which lies strictly inside [- R, R].</u>

3.21. <u>If the interval of convergence of $\Sigma a_n x^n$ is [- R, R] and if [a, b] lies strictly inside this interval, then</u>

$$\Sigma \int_a^b a_n x^n dx = \int_a^b [\Sigma a_n x^n] dx.$$

<u>Proof</u>. The theorem follows directly from 3.15 and (a) of 3.20.

3.22. <u>If $\Sigma a_n x^n$ is a given power series and if x_o is any point in (- R, R), the interval of convergence of the series, then $f(x) = \Sigma a_n x^n$ is differentiable at x_o, and</u>

$$f'(x_o) = \Sigma n a_n x^{n-1}.$$

<u>Proof</u>. The theorem follows at once from 3.18 and (b) of 3.20.

CHAPTER IV

GENERALIZATIONS OF UNIFORM CONVERGENCE

§15. A Family of Functions With a Continuous Parameter y

4.1. The concept of uniform convergence of series and sequences of functions may be generalized in several directions. One such generalization may be made in terms of the function $f(x, y)$ of two variables.

4.2. For convenience, assume $f(x, y)$ is defined in the closed rectangle in two-space, say in R: $a \leq x \leq b$, $c \leq y \leq d$. Let y_0 be a point in $[c, d]$, and let $g(x)$ be a given function defined on $[a, b]$.

$f(x, y)$ converges uniformly to $g(x)$ on $[a, b]$ as $y \rightarrow y_0$ (in notation, $f(x, y) \Rightarrow g(x)$ as $y \rightarrow y_0$) :=: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, for $0 < |y - y_0| < \delta$ and for every x in $[a, b]$, $|f(x, y) - g(x)| < \varepsilon$.

It is to be noted that $f(x, y)$ need not be defined for $y = y_0$. In particular, y_0 may be equal to c or d, in which case the approach will be from only one side of the line $y = y_0$.

4.3. Apropos the concept of 4.2 we prove the following generalization of 3.17:

If $f(x, y)$, defined in R: $0 < x < b$, $0 < y < d$, is such *that* $\lim_{y \to 0} f(x, y)$ *exists and equals* $g(x)$ *on* $(0, b)$, $\lim_{x \to 0} f(x, y)$ *exists and equals* $h(y)$ *on* $(0, d)$, *the convergence of* $f(x, y)$ *to* $g(x)$ *is uniform as* $y \to 0$ *and* $\lim_{x \to 0} g(x)$ *exists and equals* c, *then the* $\lim_{y \to 0} h(y)$ *exists and equals* c, i.e.,

$$\lim_{x \to 0} g(x) = \lim_{y \to 0} h(y)$$

or, *in other words*,

$$\lim_{x \to 0} [\lim_{y \to 0} f(x, y)] = \lim_{y \to 0} [\lim_{x \to 0} f(x, y)].$$

Proof. Choose $\varepsilon > 0$. Since $f(x, y) \rightrightarrows g(x)$ as $y \to 0$, there exists a $\delta > 0$ such that for every $0 < y < \delta$ and for every x in $(0, b)$

$$|f(x, y) - g(x)| < \varepsilon/3.$$

Since $\lim_{x \to 0} g(x) = c$, there exists a $\Delta > 0$ such that for $0 < x < \Delta$

$$|g(x) - c| < \varepsilon/3.$$

Now choose y_1 such that $0 < y_1 < \delta$. Since, for every y of this type, $\lim_{x \to 0} f(x, y) = h(y)$, there exists an x_1 with $0 < x_1 < \Delta$ for which

$$|f(x_1, y_1) - h(y_1)| < \varepsilon/3.$$

Hence,

$$|h(y_1) - c| = |h(y_1) - f(x_1, y_1) + f(x_1, y_1) - g(x_1) + g(x_1) - c|$$

$$\le |h(y_1) - f(x_1, y_1)| + |f(x_1, y_1) - g(x_1)| + |g(x_1) - c| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Thus, $\lim_{y \to 0} h(y)$ exists and equals c. The theorem follows.

Remark. In connection with the last proof, let (x, y) be an arbitrary point in the rectangle R: $0 < x < \delta$, $0 < y < \Delta$. Then

$$|f(x, y) - c| = |f(x, y) - g(x) + g(x) - c|$$
$$\le |f(x, y) - g(x)| + |g(x) - c|$$
$$< \varepsilon/3 + \varepsilon/3 < \varepsilon.$$

Hence, under the hypotheses of the last theorem, we may conclude that as (x, y) goes independently to zero

$$\lim f(x, y)$$

exists and equals c.

§16. Uniform Convergence on an Arbitrary Set

4.4. Up to this point, we have restricted our domain of definition of the functions $f_n(x)$ of $\Sigma f_n(x)$ to an interval. It is readily seen that our definition of 1.12 may be extended to the case when the domain of definition of the functions $f_n(x)$ is an arbitrary (linear) set. This generalized definition may be stated as follows:

$\Sigma f_n(x)$ converges uniformly on S (in notation $\Sigma f_n(x) \Rightarrow f(x)$) :=: for every $\varepsilon > 0$ there exists an N such that, for $n > N$ and for every x in S,

$$|F_n(x) - f(x)| < \varepsilon.$$

4.5. If the set S is a single point or even a finite set of points, it is readily seen that ordinary convergence $\Sigma f_n(x)$ on S implies uniform convergence on S.

4.6. $\Sigma f_n(x)$ is <u>uniformly convergent in a neighborhood of</u> x_o :=: there exists an open interval (a, b) containing x_o such that $\Sigma f_n(x)$ is uniformly convergent on (a, b).

4.7. $\Sigma f_n(x)$ <u>converges uniformly to</u> f(x) <u>in an infinitesimal neighborhood</u> of x_o :=: for every $\varepsilon > 0$ there exists an open interval (a, b) containing x_o and an N such that, for $n > N$ and for every x in (a, b),

$$|F_n(x) - f(x)| < \varepsilon.$$

<u>Remark</u>. It is to be noted that in this definition the neighborhood (a, b) of x_o as well as the integer N depends upon ε.

4.8. W. H. Young has proved the following theorem:

<u>If</u> $\Sigma f_n(x)$ <u>converges uniformly to</u> f(x) <u>in an infinitesimal neighborhood of every point</u> x <u>in</u> [a, b], <u>then</u> $\Sigma f_n(x)$ <u>converges uniformly to</u> f(x) <u>in</u> [a, b].[1]

<u>Proof</u>. Choose $\varepsilon > 0$. For every point x in [a, b] there exists an open interval, say I_x, and an integer, say N_x, such that for $n > N_x$ and for every x which is in I_x (and in [a, b])

$$|F_n(x) - f(x)| < \varepsilon.$$

By the Borel covering theorem, there exists a finite number

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[1] Bromwich, <u>op. cit.</u>, p. 139.

of these intervals I_x which also cover [a, b]. Suppose there are M of these intervals. Corresponding to the M subscripts x of these intervals, there are M positive integers N_x, say N(1), ... , N(M). Let N be the maximum of these M positive integers. Then for n > N and for every x in [a, b]

$$|F_n(x) - f(x)| < \varepsilon.$$

The theorem follows.

§ 17. Quasi-uniform Convergence

4.9. There has been considered in the mathematical literature a number of types of convergence which are less stringent than uniform convergence. The words "simple," "approximate," and "quasi" are often used in naming such types of convergence.

4.10. $\Sigma f_n(x)$ <u>converges</u> <u>semi-quasi-uniformly</u> to f(x) in [a, b] :=: for every $\varepsilon > 0$ and for every N there exists an n > N such that, for every x in [a, b],

$$|F_n(x) - f(x)| < \varepsilon.$$

The following theorem may be readily proved:

<u>If</u> $\Sigma f_n(x)$ <u>converges</u> <u>semi-quasi-uniformly</u> <u>to</u> f(x) <u>on</u> [a, b], <u>then</u> <u>there</u> <u>exists</u> <u>a</u> <u>sub-sequence</u> <u>of</u> $\{F_n(x)\}$ <u>which</u> <u>converges</u> <u>uniformly</u> <u>to</u> f(x) <u>on</u> [a, b], i.e., $\Sigma f_n(x)$ <u>converges</u> <u>uniformly</u> <u>to</u> f(x) <u>if</u> <u>brackets</u> <u>are</u> <u>inserted</u> <u>at</u> <u>appropriate</u> <u>places</u>.

4.11. Even though semi-quasi-uniform convergence may be considered as an approximate of uniform convergence, and in particular is not implied from ordinary convergence (on an interval), it does not necessarily imply ordinary convergence. An example to show this is

$$\Sigma f_n(x) = 1 + 1 + 2 + 1/2 + \dots + n + 1/n + \dots .$$

For this reason, the following notion would seem to be more important in the applications:

$\Sigma f_n(x)$ converges quasi-uniformly to $f(x)$ on $[a, b]$:=: $\Sigma f_n(x)$ is both convergent and semi-quasi-uniformly convergent on $[a, b]$.

BIBLIOGRAPHY

Birkhoff, George D., "A Theorem Concerning Uniform Convergence," Annals of Mathematics, VII (1904-1905), 90-92.

Bromwich, T. J. I'a., An Introduction to the Theory of Infinite Series, second edition revised, St. Martin's Street, London, Macmillan and Co., Limited, 1931.

Courant, R., Differential and Integral Calculus, Vol. I, revised edition, New York, Interscience Publishers, Inc., 1937.

Hardy, G. H., A Course of Pure Mathematics, sixth edition, Cambridge, Cambridge University Press, 1933.

Hobson, E. W., The Theory of Functions of a Real Variable and the Theory of Fourier's Series, Vol. II, second edition, revised, Cambridge, Cambridge University Press, 1926.

Knopp, Konrad, Theory and Application of Infinite Series, translated by R. C. Young, London and Glasgow, Blackie and Son Limited, 1928.

Titchmarsh, E. C., The Theory of Functions, Oxford at the Clarendon Press, 1932.