CONTRIBUTIONS TO DESCRIPTIVE SET THEORY

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Assume AD+V=L(R). In the first chapter, let $W^1_1$ denote the club measure on $\omega_1$. We analyze the embedding $j_{W^1_1}\restr \text{HOD}$ from the point of view of inner model theory. We use our analysis to answer a question of Jackson-Ketchersid about codes for ordinals less than $\omega_\omega$. In the second chapter, we provide an indiscernibles analysis for models of the form $L[T_n,x]$. We use our analysis to provide new proofs of the strong partition property on $\delta^{1_2}(2n+1)$.
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Foremost, I would like to thank my parents. Their support has laid a strong foundation for my life.

Secondly, I would like to thank Steve Jackson. He has spent an incredible amount of time teaching me this math.
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1.1. Introduction

Assume $V = L(\mathbb{R}) + AD$. Given a measure $\mu$ on $\kappa \in ON$ and an inner model $M$, one can define the "external ultrapower" of $M$ by $\mu$, $Ext(M, \mu)$, to be the model with universe $\{[f]_\mu : f : \kappa \to M\}$ and the $\epsilon$ relation defined in the usual ultrapower way. Note that the universe of $Ext_\mu(M)$ consists of all functions in $L(\mathbb{R})$ from $\kappa$ into $M$. If $M \models AC$ then the proof of Wos’ Theorem works to show that $j_\mu : M \to Ext_\mu(M)$ (defined by $j_\mu(a) = [C_a]_\mu$ where $C_a$ is the constant function with output $a$) is an elementary embedding.

External ultrapowers of $HOD$ are interesting to study for a variety of reasons. Firstly, they seem like the most natural way to study the large cardinal structure of $HOD$. The reader may recall that Woodin used external ultrapowers in his proof that $\delta_1^2$ is strong to $\Theta$, see [8]. Secondly, it is interesting to consider external ultrapowers from the point of view of inner model theory, as we do in this paper. One can ask questions such as "is the external ultrapower embedding the map of an iteration tree?" or "what initial segments of the external ultrapower extender are on the $HOD$ sequence?"

Clearly, these two reasons for studying external ultrapowers are related. It is our point of view that a systematic study of external ultrapowers from the inner model theory point of view will be necessary to uncover the large cardinal structure of HOD and how that structure interacts with the structural theory of $L(\mathbb{R})$.

In section 2 we will provide an analysis below $\omega_2$ of the external ultrapower of HOD via the club measure on $\omega_1$. In section 3 we apply our techniques from section 2 to answer a question of Jackson-Ketchersid about codes for ordinals below $\omega_\omega$. Section 4 we extend our analysis of $j_{W_1^1}$ to the extent we are able. In section 5 we prove some miscellaneous facts about $j_{W_1^1}$.

We will assume that the reader is familiar with inner model theory for finitely many woodin cardinals and, in particular, the $HOD^{L(\mathbb{R})}$ analysis. We refer the reader to [10] for
background and notation.

We are thankful to Hugh Woodin who originally directed our attention to this subject at the 2014 inner model theory conference in Berkeley. We are also thankful to Steve Jackson for his advice along the way.

1.2. The External Ultrapower of HOD by $W_1$ below $\omega_2$

In the notation of [4], we let $W_1$ denote the club measure on $\omega_1$. In this section we provide an analysis of the embedding $j_{W_1} : HOD \to Ext(HOD, W_1)$ below $\omega_2$. In particular, we show that the $(\omega_1, \omega_2)$ extender derived from $j_{W_1}$ is the map of an iteration tree. Our analysis will rely on the following result of Steel (we direct the reader to [11] for a sketch of the proof):

**Theorem 1.1.** $HOD$ and $Ext(HOD, W_1)$ have a successful comparison.

Let $\mu = W_1 \cap HOD$. We define the iterated ultrapower in the standard fashion:

1. $Ult^0(HOD, \mu) = HOD$
2. $Ult^{\alpha+1}(HOD, \mu) = Ult(HOD, i_\alpha(\mu))$
3. For $\alpha$ a limit ordinal, $Ult^{\alpha}(HOD, \mu) = \text{DirLim}_{\beta<\alpha} Ult^\beta(HOD, \mu)$

where $i_\alpha : HOD \to Ult^\alpha(HOD, \mu)$ is the canonical embedding. We also define for $\alpha < \beta$

$i_{\alpha, \beta} : Ult^\alpha(HOD, \mu) \to Ult^\beta(HOD, \mu)$ to be the canonical embedding.

Before giving a finer analysis of $j_{W_1}$, we provide a coarser analysis of $Ext(HOD, W_1)$:

**Theorem 1.2.** $Ext(HOD, W_1) \upharpoonright \omega_2 = Ult^{\omega_2}(HOD, \mu) \upharpoonright \omega_2$.

**Proof.** By Theorem 1.1, let

$$HOD \stackrel{T}{\longrightarrow} M$$

$$Ext(HOD, W_1) \stackrel{S}{\longrightarrow} N$$

be a successful comparison of HOD and $Ext(HOD, W_1)$ via the trees $T$ and $S$. Note that, by the minimality of HOD, $M=N$. Hence if $b$ is the branch through $T$ leading to $M$ and
c is the branch through $S$ leading to $N$, then neither $b$ nor $c$ drop. Our proof will be an induction on the first $\omega_2$ steps of the comparison. At stage $\alpha < \omega_2$ we show the following:

(i): At stage $\alpha$ of the comparison, $Ext(HOD, W_1^1)$ does not move (i.e. $E^S_{\alpha} = \emptyset$).

(ii): $E^T_{\alpha} = i_\alpha(\mu)$ and $M^T_{\alpha+1} = Ult^{\alpha+1}(HOD, \mu)$

It is enough to verify the induction hypotheses at $\alpha$ a successor ordinal. Assume we have verified the induction hypotheses for $\beta < \alpha < \omega_2$, we will now verify the induction hypotheses at $\alpha$.

First, observe that if $Ext(HOD, W_1^1)$ moves at all in the comparison, then the first extender applied has length $> \omega_2$. This is because $Ext(HOD, W_1^1) \models \lnot \omega_2$ is the least measurable cardinal”. This means that if $E$ is an extender on the $Ext(HOD, W_1^1)$ sequence with $\text{length}(E) < \omega_2$ and we apply $E$ to $Ext(HOD, W_1^1)$, then at that stage in the comparison we drop in model. Further, we will be dropping to a point below the first total extender on the $Ext(HOD, W_1^1)$ sequence. This means that the rest of the extenders used on $S$ are not total over $Ext(HOD, W_1^1)$, so that any branch through $S$ must drop.

Similarly, if $E^T_{\alpha} \neq \emptyset$, then $\text{length}(E^T_{\alpha}) \geq \text{length}(i_\alpha(\mu))$ because, by induction, $i_\alpha(\mu)$ is the least total extender on the $M^T_{\alpha}$ sequence. This shows that, if $\gamma$ is the least place of disagreement between $M^T_{\alpha}$ and $Ext(HOD, W_1^1)$, then $\gamma \geq \text{length}(i_\alpha(\mu))$

Next, we show that $Ext(HOD, W_1^1)$ does not have anything indexed at $\text{length}(i_\alpha(\mu))$. Suppose toward contradiction that $Ext(HOD, W_1^1)$ had an extender $E$ indexed at $\text{length}(i_\alpha(\mu))$, then $E = i_\alpha(\mu)$, or else $Ext(HOD, W_1^1)$ moves in the comparison with an extender of length $< \omega_2$. So both $Ext(HOD, W_1^1)$ and $M^T_{\alpha}$ have $i_\alpha(\mu)$ on their sequence. This implies that the last model, $M$, has $i_\alpha(\mu)$ on its sequence. In addition, $M$ does not think $i_\alpha(\mu)$ is total because $M \models \lnot \omega_2$”. However, this is impossible because $M$ is an iterate of $M^T_{\alpha}$, and it is impossible for $M^T_{\alpha}$ to add subsets of $i_\alpha(\omega_1)$ by an iteration tree.

So far we have shown that at stage $\alpha$, $Ext(HOD, W_1^1)$ does not move (which verifies
(ii) and that \( \text{length}(E^T_\alpha) = \text{length}(i_\alpha(\mu)) \). Thus \( E^T_\alpha = i_\alpha(\mu) \) and, by induction hypothesis (ii), \( i_\alpha(\mu) \) is not total over \( M^T_\beta \) for any \( \beta < \alpha \). Thus \( M^T_{\alpha+1} = \text{Ult}^{\alpha+1}(HOD, \mu) \) which verifies (ii).

We have shown that in the first \( \omega_2 \) many steps of the comparison, \( \text{Ext}(HOD, W_1^1) \) does not move while HOD iterates \( \mu \). Both \( \text{Ult}^{\omega_2}(HOD, \mu) \) and \( \text{Ext}(HOD, W_1^1) \) have \( \omega_2 \) as their least measurable cardinal, so that, for the rest of the comparison, all extenders used have critical point \( \geq \omega_2 \). This shows that \( \text{Ext}(HOD, W_1^1) \upharpoonright \omega_2 = \text{Ult}^{\omega_2}(HOD, \mu) \upharpoonright \omega_2 \). □

We have shown that \( \text{Ext}(HOD, W_1^1) \upharpoonright \omega_2 = \text{Ult}^{\omega_2}(HOD, \mu) \upharpoonright \omega_2 \), but we would also like to show that the associated embeddings are the same. Let \( E \) be the \( (\omega_1, \omega_2) \) extender derived from \( j_{W_1^1} \) and let \( \{\xi_\alpha\}_{\alpha<\gamma} \) enumerate its generators. Let \( \mu_\alpha \) be the measure on \( \xi_\alpha \) derived from the embedding \( \text{Ult}(HOD, E \upharpoonright \xi_\alpha) \rightarrow \text{Ult}(HOD, E \upharpoonright (\xi_\alpha + 1)) \). We complete the analysis of \( j_{W_1^1} \upharpoonright \omega_2 \) by proving the following:

**Theorem 1.3.** Using the notation above, \( \gamma = \omega_2 \), \( \xi_\alpha = i_\alpha(\omega_1) \), and \( \mu_\alpha = i_\alpha(\mu) \).

**Proof.** We will make frequent use of the following Lemma of Jackson-Ketchersid:

**Lemma 1.4.** Let \( \Gamma \) be a proper class of ordinals and let \( M \) be a non-dropping iterate of \( HOD \). Let \( \kappa \) be a cardinal in \( M \) so that there are no total extenders in \( M \) overlapping \( \kappa \). Then for every \( A \subset \kappa, A \in M \), there is a \( B \) that is definable in \( M \) from ordinals in \( \Gamma \cup \kappa \) such that \( A = B \cap \kappa \).

**Proof.** Let \( N = \text{Hull}^M(\kappa \cup \Gamma) \). Then \( N \) is iterable, so \( M \) and \( N \) have a successful comparison. Say

\[
M \xrightarrow{\mathcal{T}} P
\]

\[
N \xrightarrow{\mathcal{S}} Q
\]

Again, by the minimality of HOD, \( P = Q \). If \( i_\mathcal{T} : M \rightarrow P, i_\mathcal{S} : N \rightarrow Q \) are the iteration maps, then \( \text{crit}(i_\mathcal{T}), \text{crit}(i_\mathcal{S}) \geq \kappa \) (or else all branches through \( \mathcal{T}, \mathcal{S} \) would drop), so that \( (2^\kappa)^M = (2^\kappa)^P = (2^\kappa)^Q = (2^\kappa)^N \). □
We begin the proof of the theorem. We know that $\xi_0 = \omega_1$ and $\mu_0 = \mu$.

Let $\alpha < \omega_2$ and suppose that for $\beta < \alpha$ we have shown that $\xi_\beta = i_\beta(\omega_1)$ and $\mu_\beta = i_\beta(\mu)$. We first show that $\xi_\alpha = i_\alpha(\omega_1)$:

If $\alpha$ is a limit ordinal then $i_\alpha(\omega_1) = \operatorname{sup}_{\beta < \alpha} (i_\beta(\omega_1))$. Further, it is always the case that $\xi_\alpha \geq \operatorname{sup}_{\beta < \alpha} (\xi_\beta)$, so that $\xi_\alpha \geq i_\alpha(\omega_1)$. By induction hypothesis, $\text{Ult}^\alpha(HOD, \mu) = \text{Ult}(HOD, E \upharpoonright \xi_\alpha)$ so that $\text{Ult}(HOD, E \upharpoonright \xi_\alpha) \models "i_\alpha(\omega_1) is the least measurable cardinal"$. Whereas $\text{Ult}(HOD, E) \models "\omega_2 is the least measurable cardinal"$. This implies that if $k : \text{Ult}(HOD, E \upharpoonright \xi_\alpha) \to \text{Ext}(HOD, W_1^1)$, then $k = i_\alpha(\omega_1)$, i.e. $\xi_\alpha = i_\alpha(\omega_1)$.

If $\alpha + 1$ is a successor ordinal, we show $\xi_{\alpha + 1} = i_{\alpha + 1}(\omega_1)$ by generalizing the Jackson-Ketchersid argument of [11].

$\xi_{\alpha + 1}$ is the critical point of the embedding $k : \text{Ult}(HOD, E \upharpoonright \xi_{\alpha + 1}) \to \text{Ext}(HOD, W_1^1)$. As before, it is immediate that $\xi_{\alpha + 1} \leq i_{\alpha + 1}(\omega_1)$ (as $\text{Ult}(HOD, E \upharpoonright \xi_\alpha)$ thinks $i_{\alpha + 1}(\omega_1)$ is the least measurable cardinal). By induction hypothesis, $\text{Ult}^{\alpha + 1}(HOD, \mu) = \text{Ult}(HOD, E \upharpoonright \xi_{\alpha + 1})$. Our analysis of the comparison of HOD and $\text{Ext}(HOD, W_1^1)$ shows that, in fact, $\text{Ult}^{\alpha + 1}(HOD, \mu)$ and $\text{Ext}(HOD, W_1^1)$ have a successful comparison. Let

$$
\begin{align*}
\text{Ult}(HOD, E \upharpoonright \xi_{\alpha + 1}) & \xrightarrow{T} M \\
\downarrow k & \\
\text{Ext}(HOD, W_1^1) & \xrightarrow{\xi} N
\end{align*}
$$

be the comparison. As before, the minimality of HOD implies that $M = N$. Let $i_T : \text{Ult}(HOD, E \upharpoonright \xi_{\alpha + 1}) \to M$, $i_S : \text{Ext}(HOD, W_1^1) \to N$ be the associated embeddings, and let $\Gamma$ be a proper class of ordinals fixed by $i_T, i_S$, and $k$.

Next we will show that $\xi_{\alpha + 1} = i_{\alpha + 1}(\omega_1)$. We first show that $\forall \xi < i_{\alpha + 1}(\omega_1)$, $\xi$ is definable in $\text{Ult}(HOD, E \upharpoonright \xi_{\alpha + 1})$ from ordinals in $i_\alpha(\omega_1) \cup \{i_\alpha(\omega_1)\} \cup \Gamma$. To see this, observe that $\xi = i_{\alpha, \alpha + 1}(f)(i_\alpha(\omega_1))$ for some $f : i_\alpha(\omega_1) \to i_\alpha(\omega_1), f \in \text{Ult}^\alpha(HOD, \mu)$. By Lemma 1.4, there is some $A$ definable in $\text{Ult}^\alpha(HOD, \mu)$ from $i_\alpha(\omega_1) \cup \Gamma$ such that $f = A \cap i_\alpha(\omega_1)$. Then $i_{\alpha, \alpha + 1}(A)$ is definable in $\text{Ult}^{\alpha + 1}(HOD, \mu) = \text{Ult}(HOD, E \upharpoonright \xi_{\alpha + 1})$ by the same definition, so that $\xi = i(A)(i_\alpha(\omega_1))$ is definable in $\text{Ult}(HOD, E \upharpoonright \xi_{\alpha + 1})$ from ordinals in $i_\alpha(\omega_1) \cup \{i_\alpha(\omega_1)\} \cup \Gamma$. 

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Now, assume toward contradiction that $\xi_{\alpha+1} < i_{\alpha+1}(\omega_1)$, then $\xi_{\alpha+1} = \tau^{\text{Ult}(HOD,E|\xi_{\alpha+1})}(\bar{\beta})$
where $\bar{\beta}$ is some sequence of ordinals fixed by $k, i_\tau$, and $i_\mathcal{S}$. Recall that $\text{crit}(i_\tau) = i_{\alpha+1}(\omega_1)$ so that $i_\tau(\xi_{\alpha+1}) = \xi_{\alpha+1}$. Then $\xi_{\alpha+1} = i_\tau(\xi_{\alpha+1}) = \tau^M(\bar{\beta}) = \tau^N(\bar{\beta}) = i_\mathcal{S}(k(\xi_{\alpha+1})) > \xi_{\alpha+1}$ a contradiction.

We proceed to show that $\mu_{\alpha+1} = i_{\alpha+1}(\mu)$. Suppose not, then there is some $A \subset i_{\alpha+1}(\omega_1)$, $A \in \text{Ult}(HOD,E|\xi_{\alpha+1})$ so that $A \in \mu_\alpha$, $A \notin i_\alpha(\mu)$. Fix such an $A$. Lemma 1.4 implies that there is a $B$ which is definable in $\text{Ult}(HOD,E|\xi_\alpha)$ from ordinals in $\xi_\alpha \cup \Gamma$ with $A = B \cap_{\alpha+1}(\omega_1)$, say $B = \tau^{\text{Ult}(HOD,E|\xi_{\alpha+1})}(\bar{\beta})$ (note that $\bar{\beta}$ is fixed by all of the relevant embeddings). But then $i_{\alpha+1}(\omega_1) \in i_\mathcal{S}(k(B)) = \tau^N(\bar{\beta}) = \tau^M(\bar{\beta}) = i_\tau(B)$ and $i_{\alpha+1}(\omega_1) \notin i_\tau(B)$, a contradiction. This proves that $i_\alpha(\mu) = \mu_\alpha$.

$\square$

1.3. Coding Ordinals Below $\omega_\omega$ by functions in HOD

To complete this chapter, we show how the techniques of the previous sections can be used to answer a question of Jackson-Ketchersid. While the statement of Theorem 1.6 is purely descriptive set theoretic, its proof requires significant inner model theory (and no other proof is known). This adds to the growing list of descriptive set theoretic statements that can be proved using techniques from inner model theory (see [1], [7], and [9] for further examples).

**Definition 1.5.** Let $f \in HOD$, $f : \omega^n_1 \to \omega_1$ for some $n \in \omega$. We say "there is a gap at $f$" iff

$$\sup(\{[g]_{W^n_1} : g \in HOD \land [g] < [f]\}) < [f]_{W^n_1}$$

Jackson-Ketchersid in [11] showed that for $f \in HOD$, $f : \omega_1 \to \omega_1$, there is no gap at $f$. I.e. viewing functions $f : \omega_1 \to \omega_1$ as representing ordinals $< \omega_2$ via $W^n_1$, the ordinals represented by HOD functions constitute an initial segment of $\omega_2$. In what follows, we provide an analysis of the $f : \omega^n_1 \to \omega_1$ such that there is a gap at $f$. 

6
Theorem 1.6. Let $f \in HOD$, $f : \omega_1^n \to \omega_1$. Then $f$ begins a gap iff $cof^{Ult(HOD,\mu^n)}([f]_\mu^n) \in \{i_1(\omega_1), \ldots, i_{n-1}(\omega_1)\}$

Proof. We prove the $n = 2$ case.

Let $F$ be the extender that comes from iterating $i_\mu(\mu)$ $\omega_2$-many times. Our proof will rely on the model $Ult(Ult(HOD,\mu \times \mu), F)$, so we first prove some facts about that model.

First, note that $F$ is a complete extender over both $Ult(HOD,\mu)$ and $Ult(HOD,\mu \times \mu)$. So, ostensibly, $F$ gives rise to two embeddings, $i_F^{\mu}$ and $i_F^{\mu \times \mu}$. However, due to the fact that $Ult(HOD,\mu)$ and $Ult(HOD,\mu \times \mu)$ have the same $i_\mu(\omega_1)$-sequences, for $A \in Ult(HOD,\mu) \cap Ult(HOD,\mu \times \mu)$, $i_F^{\mu}(A) = i_F^{\mu \times \mu}(A)$. So, in what follows, we will refer to both $i_F^{\mu}$ and $i_F^{\mu \times \mu}$ as $i_F$.

Lemma 1.7. $Ult(Ult(HOD,\mu \times \mu), F) = Ult^{\omega_2+1}(HOD,\mu)$

Proof. First, note that $Ult(Ult(HOD,\mu), F) = Ult^{\omega_2}(HOD,\mu)$. Next, let $M$ be an initial segment of $Ult(HOD,\mu \times \mu)$. Then $i_F(M)$ is an initial segment of $Ult(Ult(HOD,\mu \times \mu), F)$. Thus, it is sufficient to show that $i_F(M)$ is also an initial segment of $Ult^{\omega_2+1}(HOD,\mu)$. To this end, note that

$Ult(HOD,\mu) \models " M is an initial segment of my ultrapower via $i_1(\mu)$"$

so that

$Ult^{\omega_2}(HOD,\mu) \models " i_F(M) is an initial segment of my ultrapower via $i_{\omega_2}(\mu)$"

which implies that $i_F(M)$ is an initial segment of $Ult^{\omega_2+1}(HOD,\mu)$. □

There is a canonical embedding $k : Ult(HOD,\mu \times \mu) \to Ext(HOD, W_1^2)$, namely $k([f]_{\mu \times \mu}) = [f]_{W_1^2}$. The aforementioned Jackson-Ketchersid result implies that $crit(k) = i_1(\omega_1)$, and it is not difficult to see that $k(i_1(\omega_1)) = \omega_2$.

Lemma 1.8. Let $F'$ be the $(i_1(\omega_1), \omega_2)$ extender derived from $k$. Then $F' = F$.

Proof. The proof is very similar to the proof of Theorem 1.3, we provide a sketch. Let $\{\xi_\alpha\}_{\alpha<\gamma}$ enumerate in increasing order the generators of $F'$. We show by induction on $\alpha$ that

(i): $\xi_\alpha = i_{1+\alpha}(\omega_1)$
(ii): if $\mu_\alpha$ is the measure derived from $\text{Ult}(HOD, F' \upharpoonright \xi_\alpha) \rightarrow \text{Ult}(HOD, F')$, then

$$\mu_\alpha = \iota_{1+\alpha}(\mu).$$

Assume we have shown that the induction hypotheses hold for $\beta < \alpha$, we now verify them at $\alpha$.

First note that Theorem 1.1 implies that $HOD$ and $\text{Ext}(HOD, W^2_1)$ have a successful comparison, and the proof of Theorem 1.2 shows that in the first $\omega_3$-many steps of this comparison, $\text{Ext}(HOD, W^2_1)$ doesn’t move while $HOD$ iterates $\mu$. Thus $\text{Ult}^{\omega_3+1}(HOD, \mu)$ and $\text{Ext}(HOD, W^2_1)$ have a successful comparison. Thus, we have the following picture

\[
\begin{array}{cccc}
\text{Ult}(\text{Ult}(HOD, \mu \times \mu), F) & \xrightarrow{\tau} & M \\
\text{Ult}(HOD, \mu \times \mu) & \xrightarrow{i_F(\xi_\alpha)} & \text{Ult}(\text{Ult}(HOD, \mu \times \mu), F' \upharpoonright \xi_\alpha) \\
\text{Ext}(HOD, W^2_1) & \xrightarrow{k_2} & N
\end{array}
\]

. with $k_1$ and $k_2$ the canonical factor maps. Again, the minimality of $HOD$ implies that $M = N$. By definition $\xi_\alpha = \text{crit}(k_2)$. Thus, in order to verify the induction hypothesis at $\alpha$, we must show that $\text{crit}(k_2) = \text{crit}(k_1)$ (recall that $\text{crit}(k_1) = \iota_{1+\alpha}(\omega_1)$) and that the measures derived from these embeddings are the same. As before, fix a proper class of ordinals $\Gamma$ that is fixed by $k_1, k_2, \iota_\tau$, and $i_S$.

The fact that $k(\iota_1(\omega_1)) = \omega_2$ implies that $\text{crit}(k_2) \leq \iota_{1+\alpha}(\omega_1)$. If $\alpha$ were a limit ordinal, then it is clear that $\text{crit}(k_2) = \iota_{1+\alpha}(\omega_1)$ because it is always the cause that $\xi_\alpha \geq \sup_{\beta < \alpha}(\xi_\beta)$. Thus, we may assume that $\alpha = \beta + 1$ is a successor ordinal. Assume toward contradiciton that $\text{crit}(k_2) < \iota_{1+\alpha}(\omega_1)$. We will derive a contradiction by showing that all $\xi < \iota_{1+\alpha}(\omega_1)$ are definable in $\text{Ult}(\text{Ult}(HOD, \mu \times \mu), F' \upharpoonright \xi_\alpha)$ from ordinals in $\Gamma \cup \iota_{1+\alpha}(\omega_1)$. For suppose this were the case, say $\xi = \tau^{\text{Ult}(\text{Ult}(HOD, \mu \times \mu), F' \upharpoonright \xi_\alpha)}(\bar{\beta})$ and recall that $\iota_{1+\alpha}(\omega_1) = \text{crit}(k_1)$. Then $\xi = \iota_\tau(k_1(\xi)) = \tau^M(\bar{\beta}) = i_S(k_2(\xi)) > \xi$ a contradiction.

We complete our proof of induction hypothesis (i) by showing that all $\xi < \iota_{1+\alpha}(\omega_1)$ are definable in $\text{Ult}(\text{Ult}(HOD, \mu \times \mu), F' \upharpoonright \xi_\alpha)$ from ordinals in $\Gamma \cup \iota_{1+\alpha}(\omega_1)$. For ease of notation, let $P = \text{Ult}(\text{Ult}(HOD, \mu \times \mu), F' \upharpoonright \xi_\alpha)$. Because $\alpha = \beta + 1$ is a successor ordinal,
\( P = \text{Ult}(P_0, i_{1+\beta}(\mu)) \) where \( P_0 = \text{Ult}(\text{Ult}(\text{HOD}, \mu \times \mu), F \upharpoonright \xi_\beta) \). Say \( \xi = [g]_{i_{1+\beta}(\mu)} \). Then Lemma 1.4 implies that there is a \( B \in P_0 \) that is definable in \( P_0 \) from ordinals in \( \Gamma \cup i_{1+\beta}(\omega_1) \) such that \( g = B \cap i_{1+\beta}(\omega_1) \). Then \( \xi = i_{\mu+\beta}(B)(i_{1+\beta}(\omega_1)) \), so that \( \xi \) is definable in \( P \) from ordinals in \( \Gamma \cup i_{1+\alpha}(\omega_1) \) as required.

The proof of induction hypothesis (ii) at \( \alpha \) is a standard application of Lemma 1.4. For assume that the measures on \( i_{1+\alpha}(\omega_1) \) derived from \( k_1 \) and \( k_2 \) were not the same and let \( i_{1+\alpha}(\omega_1) \) witness this. Say \( i_{1+\alpha}(\omega_1) \in k_1(A) \) but \( i_{1+\alpha}(\omega_1) \notin k_2(A) \). By Lemma 1.4 let \( B \in P \) be definable in \( P \) from ordinals in \( \Gamma \cup i_{1+\alpha}(\omega_1) \) so that \( A = B \cap i_{1+\alpha}(\omega_1) \). Say \( B = \tau^P(\bar{\beta}) \). Then \( i_{1+\alpha}(\omega_1) \in i_\pi(k_1(B)) = \tau^M(\bar{\beta}) = i_S(k_2(B)) \) but \( i_{1+\alpha}(\omega_1) \notin i_S(k_2(B)) \) a contradiction.

Recalling that \( \text{Ult}(\text{Ult}(\text{HOD}, \mu \times \mu), F) = \text{Ult}^{\omega_2+1}(\text{HOD}, \mu) \), we have the following picture:

\[
\begin{array}{ccc}
\text{HOD} & \xrightarrow{i_{\mu \times \mu}} & \text{Ult}(\text{HOD}, \mu \times \mu) \\
\downarrow j_W & & \downarrow \downarrow i_F \\
\text{Ext}(\text{HOD}, W_1^2) & \xleftarrow{k'} & \text{Ult}^{\omega_2+1}(\text{HOD}, \mu)
\end{array}
\]

We claim that \( \text{crit}(k') = i_F(i_2(\omega_1)) \). Assume we have shown this, and let \( f \in \text{HOD}, f : \omega_1^2 \to \omega_1 \). Then \( i_F([f]_{\mu \times \mu}) < \text{crit}(k') \), so that \( i_F([f]_{\mu \times \mu}) = [f]_W \). This implies that \( f \) begins a gap iff \( i_F \) is discontinuous at \( [f]_{\mu \times \mu} \), i.e. \( f \) begins a gap iff \( \text{cof}^{\text{Ult}(\text{HOD}, \mu \times \mu)}([f]_{\mu \times \mu}) = i_1(\omega_1) \).

We complete the proof by showing \( \text{crit}(k') = i_F(i_2(\omega_1)) \). Assume toward contradiction that \( \text{crit}(k') = \xi < i_F(i_2(\omega_1)) \). Let \( \mathcal{T} \) and \( \mathcal{S} \) be the trees from the previous lemma comparing \( \text{Ult}^{\omega_2+1}(\text{HOD}, \mu) \) with \( \text{Ext}(\text{HOD}, W_1^2) \). As in Lemma 1.9, let \( \Gamma \) be a proper class of ordinals fixed by \( k', i_\pi \), and \( i_\mathcal{S} \). Let \( \pi : M \cong \text{Hull}^{\text{Ult}^{\omega_2+1}(\text{HOD}, \mu)}(\Gamma \cup \xi) \) be the uncollapse map. If the critical point of \( \pi \) is not \( \xi \), then we get a contradiction because \( \xi \) is definable in \( \text{Ult}^{\omega_2+1}(\text{HOD}, \mu) \) from ordinals that are fixed by \( k', i_\pi \), and \( i_\mathcal{S} \), \( \xi \) is not moved by \( i_\pi \), but \( \xi \) is moved by \( i_\mathcal{S} \circ k \). So we may assume that \( \text{crit}(\pi) = \xi \). Corollary 1.10 then shows
that the measure on $\xi$ coming from $\pi$ is in $\text{Ext}(HOD, W_2^1)$. But $\text{Ult}^{\omega_2+1}(HOD, \mu)$ and $\text{Ext}(HOD, W_2^1)$ agree up through $i_F(i_2(\omega_1))$, contradicting the fact that $\xi$ is not measurable in $\text{Ult}^{\omega_2+1}(HOD, \mu)$. This completes the proof of the $n = 2$ case.

The general case is similar. For example, for the $n = 3$ case, one would analyze the embedding $k : \text{Ult}(HOD, \mu^3) \to \text{Ext}(HOD, W_3^1)$ and show that the $(i_1(\omega_1), \omega_3)$ extender derived from $k$ comes from iterating $i_1(\mu) \omega_2$-many times, and then iterating the image of $i_2(\mu) \omega_3$-many times. Thus, if $f : \omega_3 \to \omega_1$, $f \in HOD$ and $[f]_{\mu^3}$ is a point of discontinuity of $k$, then $\text{cof}^{\text{Ult}(HOD, \mu^3)}([f]_{\mu^3}) \in \{i_1(\omega_1), i_2(\omega_1)\}$. □

1.4. $j_{W_1^1}$ above $\omega_2$

Let $\kappa^1 \in \text{ON}$ be least such that $\kappa^1$ has Mitchell order 1 in $HOD$. Let $\delta = \sup(j_{W_1^1}'' \kappa^1)$. In this section, we analyze the $(\omega_1, \delta)$ extender derived from $j_{W_1^1}$. Throughout this section, we will assume that $M^\#_\omega$ exists. Our use of $M^\#_\omega$ will be restricted to Corollary 1.10 below.

While the reader may think that the following Lemma and Corollary are a bit abstract, he/she will see that they are exactly what we need at a critical point in the argument. The following lemma and it’s corollary are due to Steve Jackson.

**Lemma 1.9.** Let

$$
\begin{align*}
HOD & \xrightarrow{\tau} P \\
\text{Ext}(HOD, W_1^1) & \xrightarrow{S} P
\end{align*}
$$

be the comparison. Then there is proper class of ordinals $\Gamma$ such that $\Gamma$ is fixed by $j_{W_1^1}$ and all of the extenders used in $T$ such that $\Gamma \cap \delta \in \text{Ext}(HOD, W_1^1)$ for all $\delta \in \text{ON}$.

**Proof.** Standard generic coding arguments show that $\text{Ext}(HOD, W_1^1) = HOD^{L(\mathbb{R}_G)}$ where $G$ is generic for the Levy collapse of $\omega_1$ to $\omega$ (see [6]). Thus, it is enough to show that there is a definable proper class in $L(\mathbb{R}_G)$ such that $\Gamma$ is fixed by $j_{W_1^1}$ and all of the extenders used in $T$. Let $L(\mathbb{R}_G)[H]$ be the further collapse to make the reals countable. Standard forcing arguments show that it is enough to get such a definable in $\Gamma$ in $L(\mathbb{R}_G)[H]$. In $L(\mathbb{R})[H]$ we can define the proper class of ordinals fixed by all extenders appearing in countable
mice. Recalling that the $j_{W_1}$ extender is in $HOD$, and that $HOD$ is a countable mouse in $L(\mathbb{R}, \check{\kappa})[H]$, it is clear that the preceding definition works.

**Corollary 1.10.** Let $E$ be an extender derived from $j_{W_1}$ and let $k : \text{Ult}(HOD, E) \to Ext(HOD, W_1)$ be the factor map. Let $\xi = \text{crit}(k)$ and let $\Gamma$ be a proper class of ordinals fixed by $k$ such that $\Gamma \cap \delta \in Ext(HOD, W_1)$ for all $\delta \in ON$. Assume further that there are no total extenders in $\text{Ult}(HOD, E)$ overlapping $\xi$. Let $\pi : M \cong \text{Hull}^{\text{Ult}(HOD, E)}(\xi \cup \Gamma)$ be the uncollapse map and assume that $\xi = \text{crit}(\pi)$. Then the measure on $\xi$ derived from $\pi$ is in $Ext(HOD, W_1)$.

**Proof.** Let $\nu$ denote the measure on $\xi$ coming from $\pi$. Then for $A \subset \xi$ and $A \in \text{Ult}(HOD, E)$ we have that $A \in \nu$ if and only if for all $B$ with $A = B \cap \xi$ such that $B$ is definable in $\text{Ult}(HOD, E)$ from ordinals in $\xi \cup \Gamma$ it is the case that $\xi \in B$. Because ordinals less that $\xi$ and $\Gamma$ are both fixed by $k$, it is also true that $A \in \nu$ if and only if for all $B$ with $A = B \cap \xi$ such that $B$ definable in $Ext(HOD, W_1)$ from ordinals in $\xi \cup \Gamma$, it is the case that $k(\xi) \in B$.

Now, using the fact that $M^{\xi}_0$ exists, let $\gamma \gg \xi$ be such that $\gamma$ is an indiscernible of $Ext(HOD, W_1)$ and all subsets of $\xi$ are definable in $Ext(HOD, W_1)$ from ordinals in $\xi \cup \Gamma$ (here we use Lemma 1.4 and the fact that no total extenders overlap $\xi$). Thus $A \in \nu$ if and only if for all $B$ with $A = B \cap \xi$ such that $B$ is definable in $Ext(HOD, W_1)$ from ordinals in $\xi \cup (\Gamma \cap \gamma)$, it is the case that $k(\xi) \in B$. This shows that $\nu$ is definable in $Ext(HOD, W_1)$ so that $\nu \in Ext(HOD, W_1)$ as required.

Before we begin our analysis of $j_{W_1}$, we first need to prove some facts about the structure of $Ext(HOD, W_1)$ and $j_{W_1}$.

**Theorem 1.11.** Let $\kappa$ be a measurable cardinal in $Ext(HOD, W_1)$, then $\text{cof}^{L(\mathbb{R})}(\kappa) \geq \omega_2$.

**Proof.** Let $\kappa$ be measurable in $Ext(HOD, W_1)$. By Steel’s Dichotomy, $k = [f]$ where $\forall_{W_1} \alpha \text{ cof}^{L(\mathbb{R})}(f(\alpha)) \geq \omega_1$. Let $h$ be s.t. $h(\alpha) = \text{cof}^{L(\mathbb{R})}(f(\alpha))$. We prove the theorem through the following two claims.

**Claim 1.12.** $\text{cof}^{L(\mathbb{R})}([h]) \geq \omega_2$
Proof. If $\forall_{W_1}^* \alpha \ h(\alpha) = \omega_1$, then the claim is immediate, as $[h] = \omega_2$. So we may assume that $\forall_{W_1}^* \alpha \ h(\alpha) > \omega_1$ is some $L(\mathbb{R})$-regular cardinal. Assume toward contradiction that $\exists (\gamma_\beta)_{\beta < \omega_1}$ such that $sup(\gamma_\beta) = [h]$. Let $F$ be the $\Sigma^1_2$ function s.t. if $x \in WO$ and $F(x) = y$, then $y$ is the code for some function $f_y$ with $[f_y] = \gamma_{|x|}$ and $\forall \alpha \ f_y(\alpha) < h(\alpha)$. Let $g$ be the generic coding function and, for $\beta, \alpha < \omega_1$ and $s_0 \in \beta^{<\omega}$, define

$$G(\beta, s_0, \alpha) = \begin{cases} \gamma_0 & \text{if } \exists \gamma \ \forall_{N_{s_0}}^* s \ (F(g(\beta^s))(\alpha) = \gamma \land \gamma = \gamma_0) \\ 0 & \text{otherwise} \end{cases}$$

Then for all $\beta < \omega_1$,

(i): $\forall \alpha \ sup_{s_0} G(\beta, s_0, \alpha) < h(\alpha)$

(ii): $[\alpha \mapsto sup_{s_0} G(\beta, s_0, \alpha)] \geq \gamma_\beta$

(i) is clear from the fact that $cof^{L(\mathbb{R})}(h(\alpha)) > \omega_1$.

To see (ii), fix $\beta < \omega_1$, let $[f_\beta] = \gamma_\beta$, and assume toward contradiction that $A \in W_1^I$ is such that $\forall \alpha \in A \ sup_{s_0} G(\beta, s_0, \alpha) < f_\beta(\alpha)$. But $\forall s \ \exists \alpha \in A \ F(g(\beta^s))(\alpha) = f_\beta(\alpha)$. Thus, by the additivity of category, $\exists \alpha_0 \in A \ \exists s_0 \in \beta^{<\omega} \ \forall_{N_{s_0}}^* s \ F(g(\beta^s))(\alpha_0) = f_\beta(\alpha_0)$. i.e. $G(\beta, s_0, \alpha_0) = f_\beta(\alpha_0)$, a contradiction.

By the fact that $\forall_{W_1}^* \alpha \ cof^{L(\mathbb{R})} h(\alpha) > \omega_1$ is a $L(\mathbb{R})$ regular cardinal, (i) implies that $\forall_{W_1}^* \alpha \ sup_{s_0, s_0} G(\beta, s_0, \alpha) < h(\alpha)$. Further, (ii) implies that $\forall \beta_0 \ [\alpha \mapsto sup_{s_0, s_0} G(\beta, s_0, \alpha)] > \gamma_{\beta_0}$. But this contradicts that $sup(\gamma_\beta) = [h]$, thus $cof^{L(\mathbb{R})}([h]) \geq \omega_2$ as required.

Claim 1.13. $cof^{L(\mathbb{R})}([f]) = [h]$

Proof. It suffices to show that $f$ has uniform cofinality $h$. i.e., there is a function $H$ such that $\forall_{W_1}^* \alpha \ sup_{\beta < h(\alpha)} H(\alpha, \beta) = f(\alpha)$. We argue for the existence of $H$ via a generic coding argument similar to the previous claim. Let $F$ be the $\Sigma_2^1$ function s.t. $\forall x \in WO$, if $F(x) = y$ then $y$ is the code of some function $f_y : h(|x|) \to f(|x|)$ witnessing that $f(|x|)$ has cofinality $h(|x|)$. Let $g$ be the generic coding function and for $\alpha < \omega_1$, $\beta < h(\alpha)$, and $s_0 \in (\alpha^{<\omega}$ define

$$G(\alpha, s_0, \beta) = \begin{cases} \gamma_0 & \text{if } \exists \gamma \ \forall_{N_{s_0}}^* s \ (F(g(\alpha^s))(\beta) = \gamma \land \gamma = \gamma_0) \\ 0 & \text{otherwise} \end{cases}$$

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We claim that \( \forall \alpha < \omega_1 \sup_{s_0, \beta} G(\alpha, s_0, \beta) = f(\alpha) \). Fix \( \delta < f(\alpha) \), we will show \( \sup_{s_0, \beta} G(\alpha, s_0, \beta) \geq \delta \). By definition, \( \forall^* s \exists \beta < h(\alpha) \exists \gamma > \delta F(g(\alpha^\gamma s))(\beta) = \gamma \). Thus, by the additivity of category, \( \exists \beta_0 < h(\alpha) \exists \gamma_0 > \delta \exists s_0 \forall^*_s F(g(\alpha^\gamma s))(\beta_0) = \gamma_0 \). I.e. \( G(\alpha, s_0, \beta_0) = \gamma_0 \). Thus \( \sup_{s_0, \beta} G(\alpha, s_0, \beta) \geq \delta \) as required.

\[ \square \]

**Theorem 1.14.** Let \( \xi < \Theta \) and let \( j_\xi : Ult(HOD, E \upharpoonright \xi) \to Ext(HOD, W_1^1) \) be the canonical embedding. For \( [\eta, f]_{E \upharpoonright \xi} \in Ult(HOD, E \upharpoonright \xi) \) (\( \eta \in \xi^\omega \) and \( f \in HOD \)) define \( \pi_\xi([\eta, f]) \) by

\[ \forall^*_{W_1^1} \alpha \pi_\xi([\eta, f]_{E \upharpoonright \xi})(\alpha) = f(\eta(\alpha)) \]

Then \( \pi_\xi = j_\xi \).

**Proof.** Define \( \pi : Ult(HOD, E) \to Ext(HOD, W_1^1) \) by

\[ \forall^*_{W_1^1} \alpha \pi([\eta, f]_E)(\alpha) = f(\eta(\alpha)) \]

We show that \( \pi = id \). The theorem then follows as \( \pi_\xi([\eta, f]_{E \upharpoonright \xi}) = \pi([\eta, f]_E) = [\eta, f]_E \).

We first show \( \pi \) is well defined. Let \( [\eta_1, f_1] = [\eta_2, f_2] \), then \( \forall^*_{E_{\eta_1 \cup \eta_2}} \alpha f'_1(\bar{\alpha}) = f'_2(\bar{\alpha}) \) where \( f'_1, f'_2 \) are \( f_1, f_2 \) with the appropriate dummy variables added. Let \( \eta = \eta_1 \cup \eta_2 \). By definition, \( A \in E_\eta \leftrightarrow \eta \in j_{W_1^1}(A) \). I.e. \( A \in E_\eta \leftrightarrow \forall^*_{W_1^1} \alpha \eta(\alpha) \in A \). Thus \( \forall^*_{W_1^1} \alpha f'_1(\eta(\alpha)) = f'_2(\eta(\alpha)) \) and \( \pi_\xi([\eta_1, f_1]) = \pi_\xi([\eta_2, f_2]) \) as required.

To see that \( \pi = id \), observe that \( \pi(\gamma) = \pi([\gamma, id]) \) and \( \forall^*_{W_1^1} \alpha \pi([\gamma, id]) = \gamma(\alpha) \), thus \( \pi(\gamma) = \gamma \).

\[ \square \]

**Definition 1.15.** Let \( \mu \) be a measure with critical point \( \kappa \). Let \( f : \kappa \to ON \). We define \( \sup^*_\mu(f) \), the almost everywhere sup of \( f \), to be \( \sup(\{ \gamma : \forall^*_{\mu} \alpha f(\alpha) \leq \gamma \}) \).

Recall, we defined \( \delta = \sup(j_{W_1^1}(\kappa^1)) \) for \( \kappa^1 \) the least cardinal of Mitchell order 1 in HOD. Let \( E \) be the \( (\omega_1, \delta) \) extender coming from \( j_{W_1^1} \) and let \( \{ \xi_\alpha \}_{\alpha < \gamma} \) enumerate its generators. Let \( \mu_\alpha \) be the measure on \( \xi_\alpha \) derived from \( j_\alpha : Ult(HOD, E \upharpoonright \xi_\alpha) \to Ext(HOD, W_1^1) \).

The following theorem analyzes \( E \):
Theorem 1.16. Let

\[ HOD \xrightarrow{T} M \]

\[ Ext(HOD,W_1^1) \xrightarrow{S} N \]

be the comparison. Let \( \kappa_\alpha = \text{crit}(E^T_\alpha) \). Then, for \( \alpha < \delta \)

1: (i) \( \kappa^{\alpha+1} \) is the least measurable greater than \( \kappa_\alpha \) in \( M_{\alpha+1}^T \) and \( E^T_{\alpha+1} \) is the order 0 measure on \( \kappa_{\alpha+1} \) in \( M_{\alpha+1}^T \).

(ii) \( \xi_{\alpha+1} = \kappa_{\alpha+1} \) and \( \mu_{\alpha+1} = E^T_{\alpha+1} \)

If \( \alpha \) is a limit, let \( \kappa = \sup_{\beta < \alpha}(\kappa_\beta) \).

2: If there is a single measure that is iterated cofinally many times in \( T \upharpoonright \alpha \) (so \( \kappa \) is measurable in \( M_\alpha^T \)) and \( \kappa \) is not measurable in \( Ext(HOD,W_1^1) \), then

(i) \( \kappa_\alpha = \kappa \) and \( E^T_\alpha \) is the order 0 measure on \( \kappa \) in \( M_\alpha^T \).

(ii) \( \xi_\alpha = \kappa_\alpha \) and \( \mu_\alpha = E^T_\alpha \).

3: If there is a single measure that is iterated cofinally many times in \( T \upharpoonright \alpha \) and \( \kappa \) is measurable in \( Ext(HOD,W_1^1) \), then

(i) \( \kappa_\alpha \) is the least measurable greater than \( \kappa \) in \( M_\alpha^T \) and \( E^T_\alpha \) is the order 0 measure on \( \kappa_\alpha \) in \( M_\alpha^T \).

(ii) \( \xi_\alpha = \kappa_\alpha \) and \( \mu_\alpha = E^T_\alpha \).

4: If there is not a single measure that is iterated cofinally many times in \( T \upharpoonright \alpha \), then

(i) \( \kappa_\alpha \) is the least measurable \( \geq \kappa \) in \( M_\alpha^T \) and \( E^T_\alpha \) is the order 0 measure on \( \kappa_\alpha \) in \( M_\alpha^T \).

(ii) \( \xi_\alpha = \kappa_\alpha \) and \( \mu_\alpha = E^T_\alpha \).

5: \( E^S_\alpha = \emptyset \).

Proof. 1: By induction. We assume that the theorem holds for \( \beta < \alpha + 1 \), thus \( M_{\alpha+1}^T = \text{Ult}(HOD,E \upharpoonright \xi_\alpha+1) \).

Let \( \kappa'_{\alpha+1} \) be the least measurable in \( M_{\alpha+1}^T \) greater than \( \kappa_\alpha \). We first show that \( \xi_{\alpha+1} \geq \kappa'_{\alpha+1} \).
By induction hypothesis, $M^{T}_{\alpha+1} = \text{Ult}(HOD, E \upharpoonright \xi_{\alpha+1})$ and $\text{Ult}(HOD, E \upharpoonright \xi_{\alpha+1}), Ext(HOD, W^1_1)$ have a successful comparison, say

$$\text{Ult}(HOD, E \upharpoonright \xi_{\alpha+1}) \xrightarrow{T'} M$$

$$k \downarrow$$

$$Ext(HOD, W^1_1) \xrightarrow{S'} M$$

with $k$ the canonical factor map.

Let $\Gamma$ be a proper class of ordinals fixed by all of the above embeddings.

We first observe that $\kappa'_{\alpha+1} \leq \text{crit}(i^\tau_T), \text{crit}(i^\tau_S)$. It is clear that $\kappa'_{\alpha+1} \leq \text{crit}(i^\tau_T)$ because $\kappa'_{\alpha+1}$ is the least measurable greater than $\kappa_{\alpha}$. Further, it is also the case the $Ext(HOD, W^1_1)$ has no measurables in-between $\kappa_{\alpha}$ and $\kappa'_{\alpha+1}$. This is true because $Ult(HOD, E \upharpoonright \xi_{\alpha+1}) \models "\text{there are no measurables in-between } \kappa_{\alpha} \text{ and } \kappa'_{\alpha+1}.$" So that, by elementarity, $Ext(HOD, W^1_1) \models "$\text{there are no measurables in-between } k(\kappa_{\alpha}) \text{ and } k(\kappa'_{\alpha+1})"$ and $k(\kappa_{\alpha}) = \kappa_{\alpha}$ and $k(\kappa'_{\alpha+1}) \geq \kappa'_{\alpha+1}$. This shows that $\kappa'_{\alpha+1} \leq \text{crit}(i^\tau_S)$.

To show that $\xi_{\alpha+1} \geq \kappa'_{\alpha+1}$, we show that all $\xi < \kappa'_{\alpha+1}$ are definable in $\text{Ult}(HOD, E \upharpoonright \xi_{\alpha+1})$ from ordinals in $\kappa_{\alpha} \cup \{\kappa_{\alpha}\} \cup \Gamma$. Fix such a $\xi$ and let $i : M^T_{\alpha} \rightarrow M^T_{\alpha+1}$ be the ultrapower map from applying $\mu_{\alpha}$. Then $\xi = i(f)(\kappa_{\alpha})$ for some $f : \kappa_{\alpha} \rightarrow \kappa_{\alpha} f \in M^T_{\alpha}$. By Lemma 1.4, there is an $A$ definable in $M^T_{\alpha}$ from $\kappa_{\alpha} \cup \Gamma$ such that $f = A \cap \kappa_{\alpha}$, so that $\xi = i(A)(\kappa_{\alpha})$ is definable in $M^T_{\alpha+1}$ from $\kappa_{\alpha} \cup \{\kappa_{\alpha}\} \cup \Gamma$.

Now suppose toward contradiction that $\xi_{\alpha+1} < \kappa'_{\alpha+1}$, then $\xi_{\alpha+1} = T^{\text{Ult}(HOD,E \upharpoonright \xi_{\alpha+1})}(\bar{\beta})$ for some $\bar{\beta}$ fixed by all of the relevant embeddings. Then $\xi_{\alpha+1} = i^\tau_T(\xi_{\alpha+1}) = T^M(\bar{\beta}) = i^S(k(\xi_{\alpha+1})) > \xi_{\alpha+1}$, a contradiction.

It remains to be shown that $\xi_{\alpha+1} = \kappa'_{\alpha+1}$, that $\kappa'_{\alpha+1} = \kappa_{\alpha+1}$, and that $E^T_{\alpha+1}$ is the order 0 measure on $\kappa_{\alpha+1}$ in $M^T_{\alpha+1}$. To prove all of these, it suffices to show that $\kappa'_{\alpha+1}$ is not measurable in $Ext(HOD, W^1_1)$. For example, if $\kappa'_{\alpha+1}$ is not measurable in $Ext(HOD, W^1_1)$, then $\xi_{\alpha+1} = \kappa'_{\alpha+1}$, for if $\xi_{\alpha+1} > \kappa'_{\alpha+1}$, then by elementarity $\kappa'_{\alpha+1}$ would be measurable in $Ext(HOD, W^1_1)$, a contradiction. Similarly, if $\kappa'_{\alpha+1}$ is not measurable in $Ext(HOD, W^1_1)$, then it is straightforward to see that $\kappa'_{\alpha+1} = \kappa_{\alpha+1}$ and the (by now) standard Jackson Ketchersid argument shows that $E^T_{\alpha+1}$ is the order 0 measure on $\kappa_{\alpha+1}$ in $M^T_{\alpha+1}$.
We conclude our proof of (1) by showing that $\kappa'_{\alpha+1}$ is not measurable in $\text{Ext}(HOD, W_1)$. Our strategy is to show that $\text{cof}^L(\kappa'_{\alpha+1}) \leq \omega_1$, so that by Theorem 1.11, $\kappa'_{\alpha+1}$ cannot be measurable in $\text{Ext}(HOD, W_1)$.

It is clear that $\kappa'_{\alpha+1} \leq i_T, so that $\text{cof}^M (\kappa'_{\alpha+1}) = \omega_1$, so that by Theorem 1.11, $\kappa'_{\alpha+1}$ cannot be measurable in $\text{Ext}(HOD, W_1)$.

Theorem 1.14 shows that $[\bar{\xi}, f]_E = [g]_{W_1}$ for some $g$ such that $\forall \beta g(\beta) = f(\bar{\xi}(\beta))$, so it is also the case that $\forall \beta \text{cof}^L[g]_{W_1} \in \{\omega, \omega_1\}$. Putting all this together, we get that $\text{cof}^L(\kappa'_{\alpha+1}) = \text{cof}^L([g]_{W_1}) \leq \omega_1$, so that $\kappa = \sup_{\beta < \alpha} (\kappa_\beta)$ is measurable in $M_\alpha^T$ and not measurable in $\text{Ext}(HOD, W_1)$. It is clear to see that $E_\alpha^T$ is then the order 0 measure on $\kappa$ in $M_\alpha^T$ (so $\kappa_\alpha = \kappa$) and the standard Jackson-Ketchersid argument shows that $\kappa_\alpha = \xi_\alpha$ and $\mu_\alpha = E_\alpha^T$.

We will divide statement (4) of the theorem into two cases. Statement (4a) will be statement (4) of the theorem with the additional hypothesis that $\kappa$ is not measurable in $M_\alpha^T$. Note that, in this situation, (4a) claims that $\kappa_\alpha$ is the least measurable in $M_\alpha^T$ that is strictly bigger than $\kappa$.

Statement (4b) will be statement (4) of the theorem with the additional hypothesis that $\kappa$ is measurable in $M_\alpha^T$. Note that, in this situation, (4b) claims that $\kappa_\alpha = \kappa$.

We prove (3) and (4a) together through the following series of claims.

Let $\alpha < \delta$ be a limit ordinal such that either they hypotheses of (3) or (4a) are satisfied at $\alpha$. Let $\kappa = \sup_{\beta < \alpha} (\kappa_\beta)$

Claim 1: $\xi_\alpha > \kappa$

It is clear that $\xi_\alpha \geq \kappa$ because, by induction hypothesis, $\xi_\beta = \kappa_\beta$ for $\beta < \alpha$. Assume toward contradiction that $\xi_\alpha = \kappa$, we will derive a contradiction in the cases (3) and (4a).
(3): The assumptions of (3) imply that $\kappa$ is measurable in both $M^T_\alpha$ and in $Ext(HOD,W_1^1)$. We first show that these two measures are the same. For suppose this were not the case. Our induction hypothesis implies that $M^T_\alpha = Ult(HOD,E \upharpoonright \xi_\alpha)$ and that $M^T_\alpha, Ext(HOD,W_1^1)$ have a successful comparison, say

$$M^T_\alpha \xrightarrow{\tau'} M$$

$$k \downarrow$$

$$Ext(HOD,W_1^1) \xrightarrow{S'} M$$

with $k$ the canonical factor map.

If the measures on $\kappa$ in $M^T_\alpha$ and $Ext(HOD,W_1^1)$ are not the same, then $\text{crit}(i_{\tau'}) = \text{crit}(i_{S'}) = \kappa$ but the measures derived from these embeddings differ. Further, our induction hypothesis implies that $\xi_\alpha \geq \kappa$. Let $A \subset \kappa$ be measured large by $i_{\tau'}$ and small by $i_{S'}$. By Lemma 1.4, there is a $B$ with $B \cap \kappa = A$ such that $B$ is definable in $Ult(HOD,E \upharpoonright \xi_\alpha)$ by ordinals in $\kappa \cup \Gamma$ where $\Gamma$ is a proper class fixed by all the relevant embeddings. Say $B = \tau^{Ult(HOD,E(\xi_\alpha))}(\bar{\beta})$. Then $\kappa \in i_{\tau'}(B) = \tau^M(\bar{\beta})$ but $\kappa \notin i_{S'}(k(B)) = \tau^M(\bar{\beta})$, a contradiction.

We have shown that the measures on $\kappa$ in $M^T_\alpha$ and $Ext(HOD,W_1^1)$ are the same. This means that $\text{crit}(i_{\tau'}), \text{crit}(i_{S'}) > \kappa$. We proceed to show that $\xi_\alpha > \kappa$. While it is clear that $\xi_\alpha \geq \kappa$ (because $\kappa = \sup_{\beta < \alpha}(\xi_\beta)$), it is not as clear that $\xi_\alpha \neq \kappa$. Assume toward contradiction that this were the case, that $\xi_\alpha = \kappa$. Then $\kappa$ is the critical point of the factor map $k$ (see above diagram).

First, we observe that the measure on $\kappa$ derived from $k$ is different than the order 0 measure on $\kappa$ in $M^T_\alpha$. We will use $\mu$ to denote the order 0 measure on $\kappa$ in $M^T_\alpha$. If the measure derived from $k$ were equal to $\mu$, then we would have the following diagram

$$Ult(M^T_\alpha, \mu)$$

$$\xrightarrow{i_\mu}$$

$$\xrightarrow{k'}$$

$$M^T_\alpha \xrightarrow{k} Ext(HOD,W_1^1)$$

with canonical factor map $k'$. Because $\mu$ is an order 0 measure, $\kappa$ is not measurable in $Ult(M^T_\alpha, \mu)$. But $\text{crit}(k') = \xi_{\alpha+1} > \kappa$, so that $\kappa$ is not measurable in $Ext(HOD,W_1^1)$, a
Let $\Gamma$ be a proper class of ordinals as in Lemma 1.9 and let $\pi : M \cong \text{Hull}^{M_\alpha^T}(\Gamma \cup \kappa)$ be the uncollapse map. If $\kappa$ is not the critical point of $\pi$, then we get a contradiction as $\kappa$ is definable by ordinals fixed by $k, i_T$, and $i_S$. Thus we may assume that $\kappa$ is the critical point of $\pi$. Corollary 1.10 then shows that the measure on $\kappa$ derived from the uncollapse map is in $\text{Ext}(\text{HOD}, W_{11})$. This means that the measure on $\kappa$ from the uncollapse map is the same as the order 0 measure on $\kappa$ in $\text{Ext}(\text{HOD}, W_{11})$. We also know that $\text{Ext}(\text{HOD}, W_{11})$ and $M_\alpha^T$ have the same measure on $\kappa$, so that the measure coming from the uncollapse map is the same as the measure on $\kappa$ coming from the uncollapse map.

Let $A \subset \kappa$, $A \in M_\alpha^T$ be so that $A$ is measured small by $\pi$ but large by $k$. Then $\pi(A) = B$ is definable in $M_\alpha^T$ from ordinals in $\kappa \cup \Gamma$ and $\kappa \notin B$. Say $B = \tau^{M_\alpha^T}(\beta)$ with $\beta$ fixed by all of the relevant embeddings. Then $\kappa \notin i_T(B) = \tau(M(\beta) = i_S(k(B))$, but $\kappa \in i_S(k(B))$. This completes the proof of claim 1 for (3).

(4a): Under the hypotheses of (4a), $\kappa$ is not measurable in $M_\alpha^T$. Assume first that $\kappa$ is not measurable in $\text{Ext}(\text{HOD}, W_{11})$. Let $\Gamma$ be a proper class of ordinals as guaranteed by 1.9 and let $\pi : M \cong \text{Hull}^{M_\alpha^T}(\Gamma \cup \kappa)$ be the uncollapse map. As before, if $\kappa$ is not the critical point of $\pi$, then $\kappa$ is in fact definable in $M_\alpha^T$ from ordinals that are fixed by $k, i_T$, and $i_S$ and we can get a contradiction. Thus, we may assume that $\kappa$ is the critical point of $\pi$. Corollary 1.10 then shows that measure derived from $\pi$ is in $\text{Ext}(\text{HOD}, W_{11})$. But $\kappa$ is not measurable in $\text{Ext}(\text{HOD}, W_{11})$, a contradiction.

If $\kappa$ is measurable in $\text{Ext}(\text{HOD}, W_{11})$, then again look at $\pi : M \cong \text{Hull}^{M_\alpha^T}(\Gamma \cup \kappa)$ for $\Gamma$ some proper class of ordinals fixed by all the relevant embeddings. As before, if $\kappa$ is not the critical point of $\pi$, then $\kappa$ is definable in $M_\alpha^T$ from ordinals that are fixed by all the embeddings and we can get a contradiction because $\kappa$ is not moved by $i_T$, but $\kappa$ is moved by $i_T \circ k$. If $\kappa$ is the critical point of $\pi$ then the fact that $\kappa$ is not measurable in $M_\alpha^T$ implies that the measure on $\kappa$ derived form $\pi$ is an order 0 measure. Similarly, because $\kappa$ is measurable in $\text{Ext}(\text{HOD}, W_{11})$, the measure on $\kappa$ derived from $k$ must have mitchell order $> 0$. Thus the measures on $\kappa$ coming from $k$ and $\pi$ are not the same and one can proceed as in (3).
above. This completes the proof of claim 1.

Let $\kappa'_\alpha$ be the least measurable in $M^T_\alpha$ that is bigger than $\kappa$.

**Claim 2:** $\xi_\alpha \geq \kappa'_\alpha$

In cases (3) and (4a) we know that $\text{crit}(i_T) \geq \kappa'_\alpha$ (in case (4a) this is immediate, in case (3) it follows from the fact that $M^T_\alpha$ and $\text{Ext}(HOD,W_1^1)$ have the same measures on $\kappa$). Assume toward contradiction that $\xi_\alpha < \kappa'_\alpha$. Then Corollary 1.10 implies that the measure on $\xi_\alpha$ coming from $k$ is in $\text{Ext}(HOD,W_1^1)$. But $\text{Ext}(HOD,W_1^1)$ and $M^T_\alpha$ agree up through $\kappa'_\alpha$, so that the measure on $\xi_\alpha$ coming from $k$ is in $M^T_\alpha$, contradicting the fact that $\xi_\alpha$ is not measurable in $M^T_\alpha$. This completes the proof of claim 2.

**Claim 3:** $\kappa'_\alpha$ is not measurable in $\text{Ext}(HOD,W_1^1)$.

As before, we show that $\text{cof}^{L(R)}(\kappa'_\alpha) \leq \omega_1$. Theorem 1.11 will then imply that $\kappa'_\alpha$ cannot be measurable in $\text{Ext}(HOD,W_1^1)$.

First note that if $\kappa'_\alpha$ has a preimage in $HOD$, say $\tilde{\kappa}'_\alpha$, then $\tilde{\kappa}'_\alpha < \omega_2$ because of the length of the extender we are considering. Further, the map $i_\alpha : HOD \to M^T_\alpha$ is continuous at $\tilde{\kappa}'_\alpha$ (we have been iterating measures below $\kappa$). Thus, we see that $\text{cof}^{L(R)}(\kappa'_\alpha) \leq \omega_1$ when $\kappa'_\alpha$ has a preimage in $HOD$.

We may assume that $\kappa'_\alpha$ does not have a preimage in $HOD$. Let $\beta + 1 < \alpha$ be least such that $M^T_{\beta+1}$ has a preimage of $\kappa'_\alpha$, say $\bar{\kappa}'_\alpha$. As before, the map $i_{\beta+1,\alpha} : M^T_{\beta+1} \to M^T_\alpha$ is continuous at $\bar{\kappa}'_\alpha$, so it is sufficient to show that $\text{cof}^{L(R)}(\bar{\kappa}'_\alpha) \leq \omega_1$. $M^T_{\beta+1} = \text{Ult}(M^T_\beta,\mu_\beta)$, so we may pick some $f \in M^T_\beta$ with $\text{dom}(f) = \kappa_\beta$ so that $[f]_{\mu_\beta} = \bar{\kappa}'_\alpha$. Note that $f$ is not constant, or else $\kappa'_\alpha$ would have a preimage in an earlier model.

**Case 1:** $\forall^*_{\mu_\beta}\gamma f(\gamma) < \kappa_\beta$

The proof of this case is very similar to the proof of (1) above.

The case hypothesis implies that $\text{cof}^{M^T_\beta}([f]_{\mu_\beta}) = (\kappa^+_\beta)^{M^T_\beta}$. It is also the case that $(\kappa^+_\beta)^{M^T_\beta}$ is the least measurable in $M^T_\beta$. Thus, in order to complete the proof of Claim 3 under this case hypothesis, it is sufficient to show that $\text{cof}^{L(R)}((\kappa^+_\beta)^{M^T_{\beta+1}}) \leq \omega_1$. Recall that $M^T_{\beta+1} = \text{Ult}(HOD,E \upharpoonright \xi_{\beta+1})$ and that $\xi_{\beta+1} > (\kappa^+_\beta)^{M^T_{\beta+1}}$. This means that if $([\xi,h]_{E \upharpoonright \xi_{\beta+1}})^{M^T_{\beta+1}} = [\bar{\xi},h]_{E \upharpoonright \xi_{\beta+1}}$, then $[\xi,h]_{E \upharpoonright \xi_{\beta+1}} = [\bar{\xi},h]_E$. Theorem 1.14 tells us that $[\bar{\xi},h]_E = [g]_{W_1^1}$ where $\forall^*_{W_1^1}\gamma g(\gamma)$ is a
HOD successor cardinal. Thus $\forall^*_\omega \gamma \; \text{cof}^L(g(\gamma)) = \omega$, so the uniform cofinality analysis says that $\text{cof}^L([g]_{\omega_1}) \in \{\omega, \omega_1\}$. Recalling that $[g]_{\omega_1} = (\kappa^+_\beta)^{M_{\beta+1}}$, we see that we have completed the proof of Claim 3 under the case hypothesis.

**Case 2:** $\forall^*_\mu^\beta \gamma \; f(\gamma) > \kappa^\beta$.

Let $\delta = \sup^*_\mu^\beta (f)$. The logic of the proof will be very similar to the logic in the case 1 hypothesis, but now $\delta$ will play the role of $\kappa^\beta$. That is, we will first show that $\text{cof}^{M^T_\beta}([f]_{\mu^\beta}) = (\delta^+)^{M^T_\beta}$. Secondly, we will show that $(\delta^+)^{M^T_\beta} = (\delta^+)^{M^T_{\beta+1}}$. Lastly, we will complete the proof by demonstrating that $\text{cof}^L((\delta^+)^{M^T_{\beta+1}}) \leq \omega_1$.

In this paragraph, we show that $\text{cof}^{M^T_\beta}([f]_{\mu^\beta}) = (\delta^+)^{M^T_\beta}$. First observe that $\delta < [f] < (\delta^+)^{M^T_\beta}$. Further, $\text{cof}^{M^T_\beta}(\delta) = \kappa^\beta$, so it suffices to show that $\text{cof}^{M^T_\beta}([f]) \geq \delta$. Assume toward contradiction that $\text{cof}^{M^T_\beta}([f]) = \rho < \delta$. Without loss of generality, we may assume that $\forall \gamma \; f(\gamma) > \rho$. It is also the case that $\forall^*_\mu^\beta \gamma \; f(\gamma)$ is regular in $M^T_\beta$. Now fix a sequence of functions $\{f_\eta\}_{\eta < \rho}$ in $M^T_\beta$ such that $\forall \eta < \rho \; \forall \gamma \; f_\eta(\gamma) < f(\gamma)$ and $\sup_{\eta < \rho}(f_\eta) = [f]$. If we then define $f'(\gamma) = \sup_{\eta < \rho}(f_\eta(\gamma))$, then by the regularity of $f(\gamma)$, $f'(\gamma) < f(\gamma)$ for almost all $\gamma$, so that $[f'] < [f]$. But $\forall \eta < \rho \; [f'] > [f_\eta]$, a contradiction to our assumption that $\sup_{\eta < \rho}(f_\eta) = [f]$.

In this paragraph, we show that $(\delta^+)^{M^T_\beta} = (\delta^+)^{M^T_{\beta+1}}$. First, we will construct an injection from $(2^\delta)^{M^T_\beta}$ into $(2^\delta)^{M^T_{\beta+1}}$. Let $h \in M^T_\beta$ be such that $[h]_{\mu^\beta} = \delta$. For every $A \subset \delta$ with $A \in M^T_\beta$, define the function $g_A$ with $\text{dom}(g_A) = \kappa^\beta$ by $g_A(\gamma) = A \cap h(\gamma)$. Then $[g_A]_{\mu^\beta} \subset \delta$. It is also clear that for $A, B \subset \delta$, $A \neq B$, then $[g_A] \neq [g_B]$. This shows that $(2^\delta)^{M^T_\beta}$ injects into $(2^\delta)^{M^T_{\beta+1}}$. Our injection is in $M^T_\beta$, so the fact that $(\delta^+)^{M^T_\beta}$ is a cardinal in $M^T_\beta$ implies that $(\delta^+)^{M^T_\beta} \leq (\delta^+)^{M^T_{\beta+1}}$. But it is impossible that $(\delta^+)^{M^T_\beta} < (\delta^+)^{M^T_{\beta+1}}$ because $M^T_{\beta+1}$ is an inner model of $M^T_\beta$, so if the inequality were strict then $M^T_\beta$ would collapse its own $\delta^+$, a contradiction. Thus $(\delta^+)^{M^T_\beta} = (\delta^+)^{M^T_{\beta+1}}$.

To complete the proof of the claim, we show that $\text{cof}^L((\delta^+)^{M^T_{\beta+1}}) \leq \omega_1$. The function $i_{\beta+1,\alpha} : M^T_{\beta+1} \to M^T_\alpha$ is continuous at $(\delta^+)^{M^T_{\beta+1}}$, so it is sufficient to show that $\text{cof}^L(i_{\beta+1,\alpha}((\delta^+)^{M^T_{\beta+1}})) \leq \omega_1$. $i_{\beta+1,\alpha}((\delta^+)^{M^T_{\beta+1}}) = [\xi, h]_{E|\xi}$, for some $\xi < \xi_\alpha$. Further, $\xi_\alpha \geq \kappa'_\alpha > i_{\beta+1,\alpha}((\delta^+)^{M^T_{\beta+1}})$ so that $[\xi, h]_{E|\xi_\alpha} = [\xi, h]_E$. Theorem 1.14 shows that $[\xi, h]_E = \omega_1$. Therefore, we have completed the proof of Claim 3 under the case hypothesis.
$[g]_{W_1}$ where $\forall W_1 \gamma g(\gamma)$ is a HOD successor cardinal, so that Theorem 1.14, Steel’s analysis, and Jackson’s uniform cofinality analysis show that $\text{cof}^{L(R)}([\xi, g]_E) \leq \omega_1$. This shows that $\text{cof}^{L(R)}((\delta^+)_{M_{\delta+1}}) \leq \omega_1$ and completes the proof of claim 3.

In view of the preceding claims, we complete the proofs of (3) and (4a). In both cases, we know that $M_{\alpha}^T$ and $\text{Ext}(HOD, W_1^1)$ do not disagree with respect to any measures on $\kappa$ and $\kappa'_{\alpha}$ is the least measurable in $M_{\alpha}^T$ that is bigger than $\kappa$. We showed that $\xi_{\alpha} \geq \kappa'_{\alpha}$ so that, by elementarity, $\text{Ext}(HOD, W_1^1)$ does not have any measurables inbetween $\kappa$ and $\kappa'_{\alpha}$. Further, we showed that $\kappa'_{\alpha}$ is not measurable in $\text{Ext}(HOD, W_1^1)$ so that the measure on $\kappa'_{\alpha}$ is indeed the least place of disagreement and $\kappa_{\alpha} = \kappa'_{\alpha}$. It is also clear that $\xi_{\alpha} = \kappa'_{\alpha}$, for if $\xi_{\alpha} > \kappa'_{\alpha}$ then elementarity of the factor map would imply that $\kappa'_{\alpha}$ were measurable in $\text{Ext}(HOD, W_1^1)$, a contradiction. Finally, the proof of (3) and (4a) is completed by showing that the order 0 measure on $\kappa_{\alpha}$ in $M_{\alpha}^T$ is the same as the measure derived from $j_{\alpha} : \text{Ult}(HOD, E \upharpoonright \xi_{\alpha}) \rightarrow \text{Ext}(HOD, W_1^1)$ using Lemma 1.4.

We complete our proof of the theorem by proving (4b). Under the hypothesis of (4b), $\kappa = \sup_{\beta < \alpha}(\kappa_{\beta})$ is measurable in $M_{\alpha}^T$, and the measure on $\kappa$ has not been iterated cofinally many times in $T \upharpoonright \alpha$. As in the proofs of the previous cases, it is sufficient to show that $\kappa$ is not measurable in $\text{Ext}(HOD, W_1^1)$.

Under the hypothesis of (4b), our proof that $\kappa$ is not measurable in $\text{Ext}(HOD, W_1^1)$ breaks up into two cases.

**Case 1:** The measure on $\kappa$ has not been iterated at all in $T \upharpoonright \alpha$.

In this case, we again look at the first place that $\kappa$ has a preimage. Say $\kappa$ has a preimage in $M_{\beta+1}^T$ which we will call $\bar{\kappa}$. The fact that $\kappa$ has not been iterated at all in $T \upharpoonright \alpha$ implies that the map $i_{\beta+1, \alpha} : M_{\beta+1}^T \rightarrow M_{\alpha}^T$ is continuous at $\bar{\kappa}$. Say $\bar{\kappa} = [f]_{\mu_{\beta}}$ and let $\delta = \sup_{\mu_{\beta}}(f)$. As in the proof of claim 3 above, we may complete the proof of (4b) under the Case 1 hypothesis by showing

1. $\text{cof}^{M_{\delta}^T}(\bar{\kappa}) = (\delta^+)_{M_{\delta}^T}$
2. $(\delta^+)_{M_{\delta}^T} = (\delta^+)_{M_{\delta+1}^T}$
3. $\text{cof}^{L(R)}((\delta^+)_{M_{\delta+1}^T}) \leq \omega_1$. 

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The reader may consult the proof of claim 3 above to verify that we have proved these things.

**Case 2:** The measure on $\kappa$ has been iterated in $\mathcal{T} \upharpoonright \alpha$.

In this case, while the measure on $\kappa$ has been iterated in $\mathcal{T} \upharpoonright \alpha$, the case (4) hypothesis imply that it has not been iterated cofinally many times in $\mathcal{T} \upharpoonright \alpha$. That is, there is some $\rho < \alpha$ such that all iterations of the measure on $\kappa$ in $\mathcal{T} \upharpoonright \alpha$ actually occur in $\mathcal{T} \upharpoonright \rho$.

We claim that there is a last place in $\mathcal{T} \upharpoonright \alpha$ where the measure on $\kappa$ has been iterated. That is, there is an $M_{\beta}^T$ that contains a preimage of $\kappa$, say $\bar{\kappa}$, such that $M_{\beta+1}^T$ is the ultrapower of $M_{\beta}^T$ via its measure on $\bar{\kappa}$, and for all $\gamma$ with $\beta < \gamma < \alpha$, the measure on $\kappa$ is not applied at stage $\gamma$. For suppose this were not the case. That would mean that there is an $\eta < \alpha$ such that the measure on $\kappa$ is applied cofinally many times in $\mathcal{T} \upharpoonright \eta$ but the measure on $\kappa$ is not applied at any stage $\gamma$ with $\eta \leq \gamma < \alpha$. Fix such an $\eta$. Then $\bar{\kappa} = \sup_{\beta < \eta} \kappa_{\beta}$ is a preimage of $\kappa$ in $M_{\eta}^T$. By our choice of $\eta$, we know that the measure applied at stage $\eta$ is not the measure on $\bar{\kappa}$, so it must be some measure with critical point greater than $\bar{\kappa}$. But then, our analysis of the comparison shows that, for all $\gamma$ with $\eta < \gamma < \alpha$, the measure applied at stage $\gamma$ will also have critical point greater than $\bar{\kappa}$. This contradicts the fact that $\kappa = i_{\eta, \alpha}(\bar{\kappa})$ and $\kappa = \sup_{\beta < \alpha} (\kappa_{\beta})$.

Thus, there is a last place in $\mathcal{T} \upharpoonright \alpha$ where the measure on $\kappa$ has been iterated, let us call this place $\beta$. Then $\kappa_{\beta}$ is a preimage of $\kappa$ in $M_{\beta}^T$, so that $i_{\beta, \beta+1}(\kappa_{\beta})$ is a preimage of $\kappa$ in $M_{\beta+1}^T$. Further, the fact that $\beta$ is the last place that the measure on $\kappa$ has been iterated implies that the map $i_{\beta+1, \alpha} : M_{\beta+1}^T \rightarrow M_{\alpha}^T$ is continuous at $i_{\beta, \beta+1}(\kappa_{\beta})$. Thus it is sufficient to show that $\text{cof}^{L(\mathbb{R})}(i_{\beta, \beta+1}(\kappa_{\beta})) \leq \omega_1$. As in the proof of claim 3, we see that $\text{cof}^{M_{\beta}^T}(i_{\beta, \beta+1}(\kappa_{\beta})) = (\kappa_{\beta}^+)^{M_{\beta}^T}$. It is also the case that $(\kappa_{\beta}^+)^{M_{\beta}^T} = (\kappa_{\beta}^+)^{M_{\beta+1}^T}$. So it is sufficient to show that $\text{cof}^{L(\mathbb{R})}((\kappa_{\beta}^+)^{M_{\beta+1}^T}) \leq \omega_1$. Applying the same arguments from before, we see that $(\kappa_{\beta}^+)^{M_{\beta+1}^T} = [\xi, h]_{E|_{\xi_{\beta+1}}}$. The fact that $\xi_{\beta+1} > (\kappa_{\beta}^+)^{M_{\beta+1}^T}$ means that $[\xi, h]_{E|_{\xi_{\beta+1}}} = [\xi, h]_E$. Theorem 1.14 and Steel’s analysis then show that $[\xi, h]_E = [g]_{W_1^1}$ where $W_1^1 \gamma \text{cof}^{L(\mathbb{R})}(g(\gamma)) \leq \omega_1$. The uniform cofinality analysis shows that for such a $g$, $\text{cof}^{L(\mathbb{R})}([g]_{W_1^1}) \leq \omega_1$. This completes the proof of (4b) under the case 2 hypothesis.

□
1.5. Miscellaneous Facts

In this section we first use Theorem 1.14 to prove Theorem 1.17. We then provide an analogous theorem to Theorem 1.14 with respect to $W_1^2$.

**Theorem 1.17.** Let $\kappa$ be a HOD cardinal such that $\text{cof}^{L(\mathbb{R})}(\kappa) \neq \omega_1$ and let $c_\kappa : \omega_1 \to \text{ON}$ denote the constant $\kappa$ function. Then $(\omega_1, c_\kappa)$ is never a generator of $E$. That is, there is no $\xi$ such that $\xi = [\omega_1, c_\kappa]_{E|\xi}$ and $\xi = \text{crit}(j_\xi)$.

**Proof.** Assume toward contradiction that there is such a $\xi$ so that $\xi = [\omega_1, c_\kappa]_{E|\xi}$ and $\xi = \text{crit}(j_\xi)$. Let $f \in L(\mathbb{R})$ be such that $[f]_{W_1^2} = \xi$. Then theorem 1.14 implies that

(i): $\forall^*_{W_1^2} \alpha f(\alpha) < c_\kappa(\alpha) = \kappa$

(ii): If $\eta < \xi$ and $g \in \text{HOD}$, $g : \text{spt}(\eta) \to \text{ON}$ are such that $[\eta, g]_{E|\xi} < \xi$, then $\forall^*_{W_1^2} g(\eta(\alpha)) < f(\alpha)$.

But this is impossible because $\text{cof}^{L(\mathbb{R})}(\kappa) \neq \omega_1$, together with (i), implies that $\exists \delta < \kappa$ such that $\forall^*_{W_1^2} f(\alpha) < \delta$. Thus we may define $g : \omega_1 \to \text{ON}$ by $g(\alpha) = \delta$ and $[\omega_1, g]$ violates (ii).

We used our analysis of $j_\xi : \text{Ult}(\text{HOD}, E \restriction \xi) \to \text{Ext}(\text{HOD}, W_1^2)$ (Theorem 1.14) to describe which constant functions cannot be generators. We now provide a similar analysis of the factor maps coming from $k : \text{Ext}(\text{HOD}, W_1^1) \to \text{Ext}(\text{HOD}, W_1^2)$ to show which functions cannot be generators. Toward our analysis of $k$, we prove a few lemmas:

**Lemma 1.18.** Let $g : \omega_1 \times \omega_1 \to \text{ON}$ and $\xi$ be such that $[g]_{W_1^2} = \xi$. Define $f : \omega_1 \to \text{ON}$ by $f(\alpha) = [\beta \mapsto g(\alpha, \beta)]_{W_1^1}$. Then $[f]_{W_1^1} = \xi$.

**Proof.** Suppose not and let $g$ be $W_1^2$ least so that the lemma fails. Let $f$ be defined as in the statement of the lemma, and let $h : \omega_1 \to \text{ON}$ be such that $[h]_{W_1^3} = [g]_{W_1^2}$. Note that $[f] > [h]$.

**Case 1:** $g$ has uniform cofinality $\omega$.

Let $l : \omega_1 \times \omega_1 \times \omega \to \text{ON}$ witness that $g$ has uniform cofinality $\omega$ (i.e. $g(\alpha, \beta) = \sup_n l(\alpha, \beta, n)$). For each $n \in \omega$, define $g_n : \omega_1 \times \omega_1 \to \text{ON}$ by $g_n(\alpha, \beta) = l(\alpha, \beta, n)$. Then for
all \( n \in \omega, \forall_{\omega_1}^* \alpha h(\alpha) > [\beta \mapsto g_n(\alpha, \beta)], \) which implies that \( \forall_{\omega_1}^* \alpha h(\alpha) \geq [\beta \mapsto \sup_n g_n(\alpha, \beta)]. \) But this is impossible, because \( \forall \alpha f(\alpha) = [\beta \mapsto \sup_n g_n(\alpha, \beta)], \) and (as noted above) \( [f] > [h]. \)

**Case 2:** \( g \) has uniform cofinality \( \alpha. \)

This is similar to case 1. Let \( l : \omega_1 \times \omega_1 \times \omega_1 \rightarrow ON \) witness that \( g \) has uniform cofinality \( \alpha \) (i.e. \( g(\alpha, \beta) = \sup_{\gamma < \alpha} l(\alpha, \beta, \gamma) \)). For each \( \gamma < \omega_1 \), define \( g_\gamma : \omega_1 \times \omega_1 \rightarrow ON \) by

\[
g_\gamma(\alpha, \beta) = \begin{cases} l(\alpha, \beta, \gamma) & \text{if } \gamma < \alpha \\ 0 & \text{otherwise} \end{cases}
\]

Then \( \forall_{\omega_1}^* \alpha \forall \gamma < \alpha h(\alpha) > [\beta \mapsto g_\gamma(\alpha, \beta)], \) which implies that \( \forall_{\omega_1}^* \alpha h(\alpha) \geq [\beta \mapsto \sup_{\gamma < \alpha} g_\gamma(\alpha, \beta)]. \) But, again, this is impossible because \( \forall \alpha f(\alpha) = [\beta \mapsto \sup_{\gamma < \alpha} g_\gamma(\alpha, \beta)] \) and \( [f] > [h]. \)

**Case 3:** \( g \) has uniform cofinality \( > \alpha. \)

By hypothesis, for all \( g' : \omega_1 \times \omega_1 \rightarrow ON \) such that \( [g'] < [g], \) if we define \( f' : \omega_1 \rightarrow \omega_1 \) by \( f'(\alpha) = [\beta \mapsto g'(\alpha, \beta)], \) then \( [f'] < [h]. \) We will derive a contradiction by constructing a \( g' \) such that \( [g'] < [g] \) and \( [f'] > [h]. \)

To this end, let \( r : \omega_1 \rightarrow ON \) be such that \( [h] < [r] < [f]. \) Let \( F \) be the \( \Sigma_2^1 \) function such that \( \forall x \in WO, F(x) = y \) implies that \( y \) is the code of a function \( f_y : \omega_1 \rightarrow \omega_1 \) such that \( [f_y] = r(|x|) \) and \( \forall \beta f_y(\beta) < g(|x|, \beta). \) Let \( G \) be the generic coding function and for \( \alpha, \beta < \omega_1 \) and \( s_0 \in \alpha^{<\omega}, \) define

\[
g'(\alpha, s_0, \beta) = \begin{cases} \gamma_0 & \text{if } \exists \gamma \forall_{N_{s_0}} s (F(G(\alpha^s))(\beta) = \gamma \land \gamma = \gamma_0) \\ 0 & \text{otherwise} \end{cases}
\]

and let \( g'(\alpha, \beta) = \sup_{s_0 \in \alpha^{<\omega}} g'(\alpha, s_0, \beta). \)

To see that \( [g'] < [g] \) assume not. Then \( \forall \alpha \forall \beta \forall s_0 \in \alpha^{<\omega} g'(\alpha, s_0, \beta) < g(\alpha, \beta) \) but
\[
\forall_{\omega_1}^* \alpha \forall_{\omega_1}^* \beta \sup_{s_0 \in \alpha^{<\omega}} g'(\alpha, s_0, \beta) = g(\alpha, \beta),
\]
which implies that \( g \) has uniform cofinality \( \alpha, \) a contradiction (there is a uniform bijection \( \alpha \rightarrow \alpha^{<\omega} \)). Letting \( f' : \omega_1 \rightarrow ON \) be defined by \( f'(\alpha) = [\beta \mapsto g'(\alpha, \beta)], \) then it is clear that \( [f'] > [h] \) because \( \forall \alpha f'(\alpha) \geq r(\alpha). \) This concludes case 3. \( \square \)
**Lemma 1.19.** Let \( k : \text{Ext}(\text{HOD}, W_1^1) \to \text{Ext}(\text{HOD}, W_1^1 \times W_1^1) \) be define by \( k([f]) = [g] \) where \( g(\alpha, \beta) = f(\alpha) \). Then the extender derived from \( k \) is \( i_E(E) \).

**Proof.**

\[
\begin{array}{ccc}
\text{HOD} & \xrightarrow{j_{W_1^1} = i_E} & \text{Ext}(\text{HOD}, W_1^1) \\
& \downarrow{j_{W_1^1}} & \\
\text{Ext}(\text{HOD}, W_1^1 \times W_1^1) & \xleftarrow{k = i_E(E)} & \\
\end{array}
\]

Assume \( \xi \in k(A) \). If \( \xi = [g_1]_{W_1^1} \) and \( A = [f_1]_{W_1^1} \), then we must show that \( \forall_{W_1^1}^* \alpha f_1(\alpha) \in E_{g_1(\alpha)} \).

That \( \xi \in k(A) \) means that if \( \xi = [g_2]_{W_1^2} \) and \( f_2 : \omega_1 \times \omega_1 \to \text{HOD} \) is defined by \( f_2(\alpha, \beta) = f_1(\alpha) \), then \( \forall_{W_1^1}^* \alpha \forall_{W_1^1}^* \beta g_2(\alpha, \beta) \in f_2(\alpha, \beta) \). Recalling that \( f_2(\alpha, \beta) = f_1(\alpha) \), this means that

\( \forall_{W_1^1}^* \alpha [\beta \mapsto g_2(\alpha, \beta)]_{W_1^1} \in [\beta \mapsto f_1(\alpha)]_{W_1^1} \)

If we define \( g_1 : \omega_1 \to ON \) by \( g_1(\alpha) = [\beta \mapsto g_2(\alpha, \beta)]_{W_1^1} \), then Lemma 1.18 implies that \( [g_1]_{W_1^1} = \xi \). Substituting this into (\( \star \)), we get that \( \forall_{W_1^1}^* \alpha g_1(\alpha) \in j_{W_1^1}(f_1(\alpha)) \), i.e. \( \forall_{W_1^1}^* \alpha f_1(\alpha) \in E_{g_1(\alpha)} \) as required. \( \square \)

**Theorem 1.20.** Let \( \xi < \Theta \) and let \( j_\xi : \text{Ult}(\text{Ext}(\text{HOD}, W_1^1), i_E(E) \upharpoonright \xi) \to \text{Ult}(\text{Ext}(\text{HOD}, W_1^1), i_E(E)) \) be the canonical embedding. Let \( [\eta, f]_{i_E(E) \upharpoonright \xi} \in \text{Ult}(\text{Ext}(\text{HOD}, W_1^1), i_E(E) \upharpoonright \xi) \) (\( \eta \in \xi^{<\omega} \) and \( f \in \text{Ext}(\text{HOD}, W_1^1) \)), and say \( (\eta, f) = [\rho, g]_E \). Then \( \forall_{E_\rho}^* \alpha g(\alpha) \) is a pair \( (\eta_\alpha, f_\alpha) \) such that \( [\rho, \alpha \mapsto \eta_\alpha]_E = \eta \) and \( [\rho, \alpha \mapsto f_\alpha]_E = f \). Define \( \pi_\xi : \text{Ult}(\text{Ext}(\text{HOD}, W_1^1), i_E(E) \upharpoonright \xi) \to \text{Ult}(\text{Ext}(\text{HOD}, W_1^1), i_E(E)) \) by

\[
\forall_{W_1^1}^* \alpha \forall_{W_1^1}^* \beta \pi_\xi([\eta, f]_{i_E(E) \upharpoonright \xi})(\alpha, \beta) = f_{\rho(\alpha)}(\eta_{\rho(\alpha)}(\beta))
\]

Then \( \pi_\xi = j_\xi \).

**Proof.** The logic of the proof is identical to that of Theorem 1.14. We first show \( \pi_\xi \) is well-defined. To that end, we will first show \( \pi_\xi \) is invariant under the choice of the \( [\rho, g] \). That is, let \( [\eta, f]_{i_E(E) \upharpoonright \xi} \in \text{Ult}(\text{Ext}(\text{HOD}, W_1^1), i_E(E) \upharpoonright \xi) \) (\( \eta \in \xi^{<\omega} \) and \( f \in \text{Ext}(\text{HOD}, W_1^1) \)), and say \( (\eta, f) = [\rho_1, g_1]_E \) and \( (\eta, f) = [\rho_2, g_2]_E \). By expanding \( g_1, g_2 \) by dummy variables if necessary, we may assume \( \rho_1 = \rho_2 = \rho \). So that for \( \forall_{E_\rho}^* \alpha \), both \( g_1(\alpha) \) and \( g_2(\alpha) \) are a
pair, say \((\eta_1^\alpha, f_1^\alpha)\) and \((\eta_2^\alpha, f_2^\alpha)\) (resp.) with \(\eta_1^\alpha = \eta_2^\alpha\) and \(f_1^\alpha = f_2^\alpha\). Thus, for \(\forall^*_E \alpha\), we may unambiguously refer to \(\eta^\alpha, f^\alpha\) and \(\pi_\xi\) is invariant under the choice of \((\rho, g)\).

Next, assume \([\eta_1, f_1]_{i_E(E)} = [\eta_2, f_2]_{i_E(E)}\). Again, we may assume that \(\eta_1 = \eta_2 = \eta\). Say \((\eta, f_1) = [\rho, g_1]_E\) and \((\eta, f_2) = [\rho, g_2]_E\). For \(\forall^*_E \alpha\), let \((\eta_1^\alpha, f_1^\alpha)\) denote the pair \(g_1(\alpha)\), and likewise for \(g_2\). Then

\[
Ext(HOD, W_1^1) \models \forall^*_E (\eta, f_1)(\beta) = f_2(\beta)
\]

means that

\[
HOD \models \forall^*_E \alpha \forall^*_E \alpha \beta f_1^\alpha(\beta) = f_2^\alpha(\beta)
\]

That is,

\[
\forall^*_E \alpha \forall^*_E \alpha \beta f_1^\alpha(\eta_1^\alpha(\beta)) = f_2^\alpha(\eta_2^\alpha(\beta))
\]

Which means that \(\pi_\xi\) is well-defined.

Next, define \(\pi : \text{Ult}(Ext(HOD, W_1^1), i_E(E)) \to Ext(HOD, W_1^2)\) the same way that \(\pi_\xi\) was defined. We complete the proof by showing that \(\pi\) is the identity. Consider \([\xi, id]_{i_E(E)}\), then \((\xi, id) = [\rho, g]_E\) where \(\forall^*_E \alpha, g(\alpha)\) is a pair \((\eta_\alpha, f_\alpha)\) such that \(f_\alpha = id\) and \(\forall^*_E \alpha \eta_\alpha(\xi) = \xi\).

So that

\[
\forall^*_E \alpha \forall^*_E \alpha \beta \pi([\xi, id])(\alpha, \beta) = id(\eta_\alpha(\beta))
\]

i.e. \(\forall^*_E \alpha \pi(\xi)(\alpha) = \eta_\alpha(\alpha)\) so that \(\pi(\xi) = \xi\) as required. \(\square\)
2.1. Introduction

Our goal is to present a theory of indiscernibles for the model $L[T_2]$ and to use that theory to give a new proof of the weak partition property on $\delta^4_3$. This represents joint work with Steve Jackson.

Martin originally proved the strong partition property on $\omega_1$ using the theory of indiscernibles for models of the form $L[x]$, for $x$ a real (see [5]). This theory, in conjunction with the fact that every subset of $\omega_1$ is in $L[x]$ for some real $x$, provided a good coding for subsets of $\omega_1$, and this coding was sufficient to prove the strong partition property. Becker and Kechris subsequently proved that every subset of $\delta_{2n+1}$ was in $L[T_{2n+1},x]$, for some real $x$, and the hope was that Martin’s techniques would generalize to these models, proving the strong partition property on $\delta_{2n+1}$. Unfortunately, a theory of indiscernibles for the models $L[T_n,x]$ was not forthcoming. Led by Kunen, other methods were developed for providing good codings of subsets of $\delta_{2n+1}$. These methods, which hinge on an analysis of measures on $\delta_{2n+1}$, crystallized into Jackson’s theory of descriptions. This theory suffices to prove the strong partition property on all $\delta_{2n+1}$, as well as for odd projective ordinals in projective-like hierarchies bounded by some wadge rank (below the first inaccessible?). It is hoped that the methods presented here might be generalized to prove the strong partition property for odd projective ordinals in scaled projective-like hierarchies below the supremum of the suslin cardinals.

The proof we will present will be in the spirit of Martin’s original proof. Jackson’s recent result that every weakly homogeneous tree has a stabilization (i.e. the tree $T$ can be restricted to large sets such that if $S$ is the tree constructed from the Martin-Solovay construction over $T$, then the leftmost branches of $S$ are a scale), allows one to replace the $L[T_n,x]$ of Becker-Kechris with $L[S_n,x]$ where $S_n$ is a homogeneous tree. The fact that $S_n$ is homogeneous allows one to develop a theory of indiscernibles for the models $L[S_n,x]$, which
in turn provide good codings. We now proceed with the proof.

2.2. Type-1 Description Review

We review some basic notation. The reals \( \mathbb{R} \) will be identified with the Baire space \( \omega^\omega \). By a tree on a set \( A \) we mean a subset of \( A^{<\omega} \) closed under initial segments. If \( T \) is a tree on \( \omega \times R \), \( p[T] = \{ x \in \omega : \exists r \in R^\omega \forall n(x|n,r|n) \in T \} \). If \( s_0, t_0, s_1, t_1 \in \omega^{<\omega} \), we write \((s_0, t_0) < (s_1, t_1)\) to mean \( s_0 \subset t_0 \) and \( s_1 \subset t_1 \). For \( T \) a tree we define the Kleene-Brouwer ordering on \( T <_{KB} \) (\(<_{KB} \) when \( T \) is clear from context) for \( s,t \in T \) by

\[
s <_{KB} t \leftrightarrow (t \subset s) \lor \exists n(s|n = t|n \land s(n) < t(n))
\]

note that if \( T \) is a tree on a wellfounded set, then \( T \) is wellfounded iff \(<_{KB} \) is a wellfounded relation. For \( T \) a tree on \( A \times B \), we define \( T_a \) for \( a \in A^{<\omega} \) to be \( \{ b \in B^{<n}|(a, b) \in T \} \).

Likewise, for \( x \in \omega^\omega \) define \( T_x = \bigcup_{n \in \omega} T_{x|n} \). Finally, we fix a bijection \( \pi : \omega \to \omega^{<\omega} \) such that if \( s \subset t \), then \( \pi^{-1}(s) < \pi^{-1}(t) \).

**Definition 2.1.** If \( T \) is a tree on \( \omega \times \omega \times \kappa \), we say \( T \) is weakly homogeneous if there are measures \( \{ \mu_{s,t} | s, t \in \omega^{<\omega} \land |s| = |t| \} \) such that

(i): \( \mu_{s,t} \) is a measure on \( \kappa^{|s|} \) such that \( \mu_{s,t}(T_{s,t}) = 1 \).

(ii): if \( s_0 \subset s_1 \) and \( t_0 \subset t_1 \) then \( \mu_{s_1,t_1} \) projects onto \( \mu_{s_0,t_0} \)

(iii): if \( x, y \in \omega^\omega \) are such that \( \exists \alpha \in \kappa^{<\omega} \forall n((x \upharpoonright n, y \upharpoonright n, \alpha \upharpoonright n) \in T \) and \( \{ A_n \}_{n \in \omega} \) are such that \( A_n \in \mu_{x|n,y|n} \), then \( \exists \alpha \in \kappa^{<\omega} \forall n(\alpha \upharpoonright n \in A_n) \)

**Definition 2.2.** If \( T \) is a weakly homogeneous tree on \( \omega \times \omega \times \kappa \) as witnessed by \( \{ \mu_{s,t} \} \), then we define the Martin-Solovay construction over \( T \) to be a tree \( S \) projecting to \( p[T]^c \) by

\( (s, \alpha) \in S \leftrightarrow \exists f : \prod_{i \leq \xi}(\{ \pi(i) \}) \times T_{\pi(i),\pi(i)} \to \kappa^+ \) such that \( f \) is order preserving with respect to the KB ordering on \( (\{ \pi(i) \}) \times T_{\pi(i),\pi(i)} \) (we order tuples lexicographically) and \( \forall i < n(\alpha(i) = [f^\pi(i)]_{\mu_{s,i},\pi(i)}) \). Where, for \( (\pi(i), \alpha) \in T, f^\pi(i)(\alpha) = f((\pi(i), \alpha)) \)

**Definition 2.3.** If \( T \) is a tree on \( \omega^n \times \kappa \), then we say \( T \) is good if \( \forall (s, \alpha) \in \omega^{<\omega}((s, \alpha) \in T \to \forall n < |\alpha|(\alpha(0) > \alpha(n))) \)
For example, we can arrange for the Schoenfield tree for any $\Sigma^1_2$ set to be good. Notice that if $T$ is a good, weakly-homogeneous tree, then we can take the $f$ as in Definition 2.2 to be into $\kappa$. Further, if $\kappa$ has the strong partition property then we can define measures \( \{\nu_s\}_{s \in \omega \prec \omega} \) on the Martin-Solovay tree $S$ as follows:

\[ A \in \nu_s \iff (\exists C \subset \kappa \text{ club such that } \forall \alpha \in \kappa^{|s|} (if (s, \alpha) \in S and the function } f \text{ that witness } (s, \alpha) \in S \text{ as in Definition 2.2 can be taken so that } f \text{ is of the correct type and } f : T_s \to C, \text{ then } \alpha \in A) \]

It is not difficult to see that in this situation, the $\{\nu_s\}$ witness that $S$ is homogenous.

We recall some definitions from [4]. $W^1_1$ is the club measure on $\omega_1$ and $W^m_1$ is its m-fold product. The following definitions come from Definition 4.25 and the paragraphs following it in [4]. Using the notation above, these definitions describe how the measures $\{\nu_s\}$ on $S$ come from the measures $\{\mu_{s,t}\}$ on $T$. Recall that in this context, we are only interested in permutations $f$ of $\{0, 1, ..., n-1\}$ such that $f(0) = n-1$.

**Definition 2.4.** A type-1 tree of uniform cofinalities (of depth $n$) is a function $R$ satisfying the following:

(i): $(p_1, i_1) \in \text{dom}(R)$ for $0 \leq i_1 \leq a$ for some integer $a$, and $p_1$ = the unique permutation of length 1, namely $p_1 = (1)$. for $i_1 = 0$, $R((p_1, i_1)) = (s)$, and for $i_1 \geq 0$, $R((p_1, i_1))$ is either $(\omega)$, or a permutation $p_2$ of length 2 (hence $p_2 = (2,1)$).

Also, $(p_1, i_1)$ is maximal in $\text{dom}(R)$ iff $R((p_1, i_1)) = (\omega)$ or $(s)$ (there are the only values permitted therefore if $m=n$). $R((p_1, i_1, ..., p_m, i_m)) = (s)$ iff $i_m = 0$.

(ii): In general, $\text{dom}(R)$ consists of tuples $(p_1, i_1, ..., p_m, i_m, m \leq n$, and such a tuple is maximal in $\text{dom}(R)$ iff $R((p_1, i_1, ..., p_m, i_m)) = (\omega)$ or $(s)$ (there are the only values permitted therefore if $m=n$). $R((p_1, i_1, ..., p_m, i_m)) = (s)$ iff $i_m = 0$. if $R((p_1, i_1, ..., p_m, i_m)) \neq (\omega)$ or $(s)$, then $R((p_1, i_1, ..., p_m, i_m))$ is a permutation $p_{m+1}$ immediately extending $p_m$. In this case we have $(p_1, i_1, ..., p_m, i_m, m+1) \in \text{dom}(R)$ for some integers $0 \leq i_{m+1} \leq a (a \geq 0$ and depends on $(p_1, i_1, ..., p_m, i_m, p_{m+1}))$.

**Definition 2.5.** For $R$ a type-1 tree of uniform cofinalities, we define $<^{R}$ to be the lexicographic ordering on sequences $(\alpha_1, i_1, ..., i_{m-1}, \alpha_m, i_m)$. 29
(i): \( \alpha_1, ..., \alpha_m < \omega_1 \)

(ii): \((\alpha_1, ..., \alpha_m)\) is of type \(p_m\) where \((p_1, ..., p_m)\) is the unique sequence such that \((p_1, i_1, ..., p_m, i_m) \in \text{dom}(R)\) (Here we say \((\alpha_1, ..., \alpha_m, i_m)\) is of type \((p_1, i_1, ..., p_m, i_m)\)).

**Definition 2.6.** Say a function \(f: \text{dom}(<^R) \to \omega_1\) is of type \(R\) if it is order-preserving and

(i): \(f((\alpha_1, i_1, ..., \alpha_m, i_m))\) has uniform cofinality \(\omega\) if either \((\alpha_1, i_1, ..., \alpha_m, i_m)\) has successor rank in \(<^R\) or if \(R((p_1, i_1, ..., p_m, i_m)) = (\omega)\) (for \((\alpha_1, i_1, ..., \alpha_m, i_m)\) of type \((p_1, i_1, ..., p_m, i_m)\)).

(ii): Otherwise, \(f((\alpha_1, i_1, ..., \alpha_m, i_m)) = \sup\{f(\bar{s}) : \bar{s} <^R (\alpha_1, i_1, ..., \alpha_m, i_m)\}\)

**Definition 2.7.** To each tree of uniform cofinalities \(R\) we associate a measure \(M^R\) by

\[ A \in M^R \iff \text{There is a club } C \subset \omega_1 \text{ such that } \forall f : \text{dom}(<^R) \to C \text{ of type } R \ [f]_R \in A \]

where \([f]_R = (\ldots, [f(p_1, i_1, ..., p_m, i_m)]_{W_{1}^{n}}, \ldots)\)

where \(f(p_1, i_1, ..., p_m, i_m)(\alpha_1, ..., \alpha_m) = f((\alpha_1, i_1, ..., \alpha_m, i_m))\) (here \((\alpha_1, ..., \alpha_m)\) appear in the correct order).

Now, assume we are applying the Martin Solovay construction (Definition 2.2) to a weakly homogeneous tree \(T\) such that \(p[T]\) is a \(\Sigma^1_2\) complete set. Then the \(\prod_{i \leq s}(\{\pi(i)\} \times T_{s}[\pi(i), \pi(i)]\) of Definition 2.2 is precisely \(\text{dom}(<^R)\) for some tree of uniform cofinalities \(R\). Thus the measures \(\{\nu_s\}\) on the resultant tree \(S\) are all measures of the form \(M^R\).

Finally, we will need one result from the descriptions analysis of [4]. Let \(R\) be a tree of uniform cofinalities and let \(f : \text{dom}(<^R) \to \omega_1\) be of type \(R\). Let \((\alpha_1, ..., \alpha_n) \in \omega^n\), we define \(f([\alpha_1, ..., \alpha_n])\) to be the set of all possible ways to apply \(f\) to a subset of \((\alpha_1, ..., \alpha_n)\). Formally, let \(f([\alpha_1, ..., \alpha_n])\) be the set

\[ \{\beta : \exists m \leq n \exists i_1, j_1, ..., i_m, j_m \leq n (\alpha_{j_1}, i_1, ..., \alpha_{j_m}, i_m) \in \text{dom}(<^R) \land f((\alpha_{j_1}, i_1, ..., \alpha_{j_m}, i_m)) = \beta\} \]

**Theorem 2.8.** Let \(R\) be a tree of uniform cofinalities and let \(F : \text{dom}(M^R) \to \omega_n\). Then there is a \(g : \omega_m^1 \to \omega_1\) \((m\) is the number of descriptions defined over \(R\) relative to \(W_{1}^{n-1}\)) s.t.

\[ \forall_{M^R}[f] \ F([f]) = [\bar{\alpha} \mapsto g(f[\bar{\alpha}])]_{W_{1}^{n-1}} \]
2.3. Indiscernibles for $L[T_2, x]$

For $T$ a weakly homogeneous tree, we say $T$ is stable iff if $S$ is the tree coming from the Martin Solovay construction over $T$, then the leftmost branches of $S$ are a scale. I.e. for $x \in p[S]$, if $\bar{\alpha}$ is the leftmost branch of $S_x$, then the functions $\phi_n(x) = \bar{\alpha}(n)$ are a scale on $p[S]$. By [3], for every weakly homogeneous tree $T$ on $\omega \times \omega \times \kappa$, there is a weakly homogeneous $T'$ on $\omega \times \omega \times \kappa$ such that $p[T] = p[T']$ and $T'$ is stable.

Fix a stable tree $T$ on $\omega \times \omega \times \omega_1$ such that $p[T]$ is a $\Sigma^1_2$ complete set. Let $T$ be weakly homogeneous as witnessed by the measures $\{\mu_{s,t}\}_{(s,t) \in \omega \times \omega}^\omega$ (recall each $\nu_{s,t}$ is of the form $W^m_1$ for some $m$). Let $S_2$ come from the Martin Solovay construction over $T$ and be homogeneous as witnessed by the measures $\{\mu_s\}_{s \in \omega}^\omega$ (recall each $\mu_s$ is of the form $M^R$ for some tree of uniform cofinalities $\mathcal{R}$). Fix a real $x \in \omega^\omega$, for the rest of this section we will develop a theory of indiscernibles for the model $L[S_2, x]$.

**Definition 2.9.** Let $\Gamma$ be a proper class of ordinals with $\min(\Gamma) > \omega_\omega$ and let $C \subset \omega_1$ be club. We say $(\Gamma, C)$ is an indiscernible pair in the language $\mathcal{L}$ iff

(i): For all $\alpha_1, \ldots, \alpha_n < \omega_\omega$, for all $(\gamma_1, \ldots, \gamma_m)$, $(\gamma'_1, \ldots, \gamma'_m)$ increasing tuples from $\Gamma$, and for all $(n+m)$-ary $\mathcal{L}$ formulas $\phi$,

$$L[S_2, x] \models \phi(\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_m) \iff L[S_2, x] \models \phi(\alpha_1, \ldots, \alpha_n, \gamma'_1, \ldots, \gamma'_m)$$

(ii): For all type-1 trees of uniform cofinalities $\mathcal{R}$, for all $f_1, f_2 : \text{dom}(<_\mathcal{R}) \to C$ of type $\mathcal{R}$, for all $\bar{\gamma} \in \Gamma^\omega$, and for all $\mathcal{L}$-formulas $\phi$,

$$L[S_2, x] \models \phi([f_1], \bar{\gamma}) \iff L[S_2, x] \models \phi([f_2], \bar{\gamma})$$

**Definition 2.10.** For $A \subset \omega_1$ we will write $A^\dagger$, the ”lift-up of $A$”’, to denote the set

$$\{\alpha : \exists f : \omega^m_1 \to A \land [f]_{W^m_1} = \alpha\}$$

In general, for $(\Gamma, C)$ an indiscernible pair, we will be interested in $\mathcal{H} = \text{hull}^L[S_2, x](\Gamma \cup C^\dagger)$. The transitive collapse of $\mathcal{H}$ will be some model $L[S, x]$ and the natural question is whether $S = S_2$. This motivates the following definition and lemma:
Definition 2.11. For $(\Gamma, C)$ an indiscernible pair, we will say $(\Gamma, C)$ is a full pair if there is a countable language $\mathcal{L}$ (containing $\{\epsilon, S_2\}$ ) and a set $A \subseteq \omega_1$ such that $\text{hull}_L^{[S_2, x]}(\Gamma \cup C^\uparrow) \cap \omega_\omega = A^\uparrow$.

Lemma 2.12. Let $(\Gamma, C)$ be a full pair, and let $M = L[S, x]$ be the transitive collapse of $\mathcal{H} = \text{hull}_L^{[S_2, x]}(\Gamma \cup C^\uparrow)$. Then $M = L[S_2, x]$.

Proof. $S = S_2^M$ is the collapse of $S_2 \cap \mathcal{H}$. I.e. $S$ is the collapse of the set

\[ \{(s, \alpha) : (s, \alpha) \in S_2 \land \alpha \in (A^\uparrow)^\omega\} \]

This means that, if $S_2$ comes from the Martin Solovay construction applied to $T$, then $S$ is the transitive collapse of the tree $S'$ where $S'$ comes from the Martin Solovay construction applied to $T$ except that the functions $f$ as in Definition 2.2 are required to be into $A$. Note that the transitive collapse of $A$ is $\omega_1$.

Let $\rho : A \to \omega_1$ and $\theta : \mathcal{H} \to L[S, x]$ be the collapse maps. For $f : \omega_1^\omega \to A$ let $(\rho(f))(\alpha) = \rho(f(\alpha))$. Note that

\[ (\star) \quad \forall f : \omega_1^\omega \to A, [\rho(f)]_1^\omega = \theta([f]_1^\omega) \]

Above, we observed that $S$ is the transitive collapse of the tree $S'$. By $(\star)$, the transitive collapse of $S'$ is the same as the Martin Solovay construction applied to $T$ with the functions $f$ as in Definition 2.2 required to be into the transitive collapse of $A$, which is the full Martin Solovay construction. Thus $S = S_2$.

\[ \square \]

Theorem 2.13. Let $\Gamma$ satisfy property (i) of an indiscernible pair. Then there is a $C \subseteq \omega_1$ such that $(\Gamma, C)$ is a full indiscernible pair.

Proof. Let $\Gamma$ be as above. The construction of $C$ will be an $\omega$ length induction. At each stage $n$ we will have a countable language $\mathcal{L}_n$ containing $\{\epsilon, S_2\}$, a club $C_n \subseteq \omega_1$, and a countable set of functions $\mathcal{F}_n \subseteq \omega_\omega_1$ such that the following properties hold:

(i): $(\Gamma, C_n)$ is an indiscernible pair in the language $\mathcal{L}_n$
(ii): $\mathcal{L}_{n+1}$ is $\mathcal{L}_n$ with countable many function symbols $\{g_m\}_{m \in \omega}$ added for each function $g \in \mathcal{F}_n$.

(iii): For every term $\tau$ in the language $\mathcal{L}_n$, for all $\gamma_1 < \ldots < \gamma_n \in \Gamma$, and for all trees of uniform cofinalities $\mathcal{R}$ such that $\forall^*_{\lambda \mathcal{R}}[f] \tau([f], \gamma_1, \ldots, \gamma_n) < \omega_{m+1}$ there exists a function $g \in \mathcal{F}_n$ such that for all $f : \text{dom}(\langle \mathcal{R} \rangle) \rightarrow C_n$

$$\tau([f], \gamma_1, \ldots, \gamma_n) = [\bar{\alpha} \mapsto g(f[\bar{\alpha}])]_{\omega_1}^n$$

(iv): $\mathcal{F}_n$ is closed under composition

Although (iv) is not necessary, it will make aspects of our proof notationally easier. Our construction will also guarantee that $C_{n+1} \subseteq C_n$.

We expand upon (ii): for $g \in \mathcal{F}_n$, $g : \omega_1^n \rightarrow \omega_1$, for each $l \in \omega$ we add a function $g_l$ to $\mathcal{L}_{n+1}$ to be interpreted as a function $g_l : \omega_1^m \rightarrow \omega_{m+1}$ by

$$g_l([f_1], \ldots, [f_m]) = [\bar{\alpha} \mapsto g(f_1(\bar{\alpha}), \ldots, f_m(\bar{\alpha}))]_{\omega_1}^n$$

This is independent of the choices of representatives $f_1, \ldots, f_m$.

Assume we have such a construction and set $C = \bigcap_{n \in \omega} C_n$, $\mathcal{L} = \bigcup_{n \in \omega} \mathcal{L}_n$, $\mathcal{F} = \bigcup_{n \in \omega} \mathcal{F}_n$. We show $C$ is a full set of indiscernibles. Specifically, let $A$ be the closure of $C$ under the functions in $\mathcal{F}$, we will show that $Hull_{\mathcal{L}[S^2, x]}(\Gamma \cup C^\uparrow) \cap \omega_\omega = A^\uparrow$.

Let $\mathcal{H} = Hull_{\mathcal{L}[S^2, x]}(\Gamma \cup C^\uparrow)$.

First we show that $\mathcal{H} \cap \omega_\omega \subseteq A^\uparrow$. To see this, let $\beta \in \mathcal{H} \cap \omega_\omega$. Then $\beta = \tau([f], \gamma_1, \ldots, \gamma_n)$ for some tree of uniform cofinalities $\mathcal{R}$ and $f : \text{dom}(\langle \mathcal{R} \rangle) \rightarrow C$, $\gamma_1 < \ldots < \gamma_n \in \Gamma$ and $\tau$ a term in $\mathcal{L}_m$. By (iii), there is a $g \in \mathcal{F}_m$ such that $\beta = [\bar{\alpha} \mapsto g(f[\bar{\alpha}])]_{\omega_1}^n$. Thus $\beta \in g[C]^\uparrow$ and clearly $g[C] \subseteq A$.

Next, we show $\mathcal{H} \cap \omega_\omega \supseteq A^\uparrow$. To this end, let $f : \omega_1^n \rightarrow A$ be such that $[f]$ is not representable by a $f' : \omega_1^n \rightarrow C$. A simple partition argument shows that, WLOG, there is a single $g \in \mathcal{F}$ such that $f : \omega_1^n \rightarrow g[C]$. Say $\text{dom}(g) = \omega_1^m$. For $i < m$, define $f_i : \omega_1^n \rightarrow C$ by $f_i(\bar{\alpha})$ is the $i$th coordinate of $g^{-1}(f(\bar{\alpha}))$. Then

$$[f] = [\bar{\alpha} \mapsto g(f_1(\bar{\alpha}), \ldots, f_m(\bar{\alpha}))]_{\omega_1}^n = g_n([f_1], \ldots, [f_m])$$
$g_n$ is a term in $\mathcal{L}_{n+1}$ and $[f_1], \ldots, [f_m] \in C^\dagger$, so $[f] \in \text{Hull}_L^{\mathcal{L}[S_2]}(C^\dagger)$ as required.

Finally, we describe how to construct $\{\mathcal{L}_n\}, \{\mathcal{C}_n\}, \{\mathcal{F}_n\}$ so that (i)-(iv) hold. Set $\mathcal{L}_1 = \{\epsilon, S_2\}$ $\mathcal{F}_0 = \emptyset$. By the strong partition property on $\mathcal{L}_1$, for each formula $\psi$ in $\mathcal{L}_n$, for each tree of uniform cofinalities $\mathcal{R}$, and for each $m \in \omega$, let $C_{\psi, \mathcal{R}, m} \subset \omega_1$ be club such that for all $f_1, f_2 : \text{dom}(<_\mathcal{R}) \to C_{\psi, \mathcal{R}, m}$ of type $\mathcal{R}$ and for all $\gamma_1 < \ldots < \gamma_m \in \Gamma$

$$L[S_2, x] \models \psi([f_1], \gamma_1, \ldots, \gamma_m) \iff L[S_2, x] \models \psi([f_2], \gamma_1, \ldots, \gamma_m)$$

Further, by theorem 2.35, for each tree of uniform cofinalities $\mathcal{R}$, and $\gamma_1 < \ldots < \gamma_m \in \Gamma$, and each term $\tau$ in $\mathcal{L}_n$ such that $\forall \mathcal{R} \alpha \tau(\alpha, \gamma_1, \ldots, \gamma_m) \prec \omega_\omega$, let $g^{\tau, \mathcal{R}, m} : \omega_1^\tau \to \omega_1$ and $C_{\tau, \mathcal{R}, m} \subset \omega_1$ club be so that for all $f : \text{dom}(<_\mathcal{R}) \to C_{\tau, \mathcal{R}, m}$ of type $\mathcal{R}$

$$\tau([f], \gamma_1, \ldots, \gamma_m) = [\alpha \mapsto g^{\tau, \mathcal{R}, m}(f[\alpha])]_{W_1^\dagger}$$

Then set $\mathcal{C}_n = \bigcap_{\psi, \mathcal{R}, m} C_{\psi, \mathcal{R}, m} \cap \bigcap_{\tau, \mathcal{R}, m} C_{\tau, \mathcal{R}, m}$ and let $\mathcal{F}_n$ the be the closure under compositions of $\mathcal{F}_{n-1} \cup \bigcup_{\tau, \mathcal{R}} \{g^{\tau, \mathcal{R}, m}\}$. To complete the induction, set $\mathcal{L}_{n+1} = \mathcal{L}_n \cup \bigcup_{\tau, \mathcal{R}} \{g^{\tau, \mathcal{R}, m}\}_{i \in \omega}$.

It is clear that this construction has the desired properties.

2.4. The Weak Partition Property on $\delta_3^1$

Our goal in this section is to show that the weak partition property holds at $\delta_3^1$. We first show how to adapt the main theorem of [2] to our current situation.

Let $\{\phi_n\}$ be the scale on $p[S_2]$ coming from the leftmost branch, we now compute the complexity of the $\phi_n$.

**Theorem 2.14.** $\{\phi_n\}$ is a $\Sigma_3^1$ scale on $p[S_2]$.

**Proof.** We must show that the relations $<_n^*$ and $\leq_n^*$ are $\Sigma_3^1$ where

$$x <^*_n y \iff (x \in p[S_2] \land y \notin p[S_2]) \lor (\phi_n(x) < \phi_n(y))$$

$$x \leq^*_n y \iff (x \in p[S_2] \land y \notin p[S_2]) \lor (\phi_n(x) \leq \phi_n(y))$$

We will compute the relation $\leq^*_n$, the proof for $<_n$ is nearly identical.
For \( x \in \omega^\omega \) and \((s, \alpha) \in (\omega \times \kappa)^{<\omega}\), we will say "the tree \( T_x \) below \((s, \alpha)\)" to refer to the set \( \{(s', \alpha') : (s \cup s', \alpha \cup \alpha') \in T_x\} \). Examining Definition 2.2, it is clear that

\[
x \leq_n y \iff (x \in p[S_2] \land y \notin p[S_2]) \lor \forall^*_{W^*_1} \pi(x) \bar{\alpha} (\text{The tree } T_x \text{ below } (\pi(n), \sigma_1(\bar{\alpha})) \text{ embeds into the tree } T_y \text{ below } (\pi(n), \sigma_2(\bar{\alpha})))
\]

where \( \sigma_1, \sigma_2 \) are the unique permutations of length \(|\pi(n)|\) such that \((\pi(n), \sigma_1(\bar{\alpha})) \in T_x \) and \((\pi(n), \sigma_2(\bar{\alpha})) \in T_y \).

We will first show that "the tree \( T_x \) below \((\pi(n), \sigma_1(\bar{\alpha})) \) embeds into the tree \( T_y \) below \((\pi(n), \sigma_2(\bar{\alpha})) \)" is \( \Delta^1_3 \), and then show that \( \Delta^1_3 \) is closed under the quantifier \( \forall^*_{W^*_1} \).

To this end, first notice that the coding lemma gives us a \( \Sigma^1_2 \) coding of subsets of \( \omega_1 \)(better codings exist, but we will not need them here). That is, there is a map \( \phi : \omega^\omega \to P(\omega_1) \) such that

(i): \( \forall A \subset \omega_1 \text{ there is a real } z \text{ s.t. } \phi(z) = A. \)

(ii): \( \forall \alpha \{ z : \alpha \in \phi(z) \} \in \Sigma^1_2 \)

We can view such a coding as coding subsets of \( (\omega \times \omega_1)^{<\omega} \).

To say "\( z \) codes an embedding from the tree \( T_x \) into the tree \( T_y \) is to say

\[
\forall (s, \alpha) \in (\omega \times \omega_1)^{<\omega} \exists! (t, \beta) \in (\omega \times \omega_1)^{<\omega} (\phi(z)((s, \alpha), (t, \beta)))
\]

\[
\land \forall (s_0, \alpha_0), (s_1, \alpha_1), (t_0, \beta_0), (t_1, \beta_1) \in (\omega \times \omega_1)^{<\omega}
\]

\[
((\phi((s_0, \alpha_0), (t_0, \beta_0)) \land \phi((s_1, \alpha_1), (t_1, \beta_1)) \land ((s_0, \alpha_0) <_{T_x}^t (s_1, \alpha_1))) \rightarrow (t_0, \beta_0) <_{T_y}^{t_1} (t_1, \beta_1)))
\]

hence is \( \Delta^1_3 \) (\( \Delta^1_3 \) is closed under unions and intersections of length \( \omega_1 \)). The above easily generalizes to show that "\( z \) codes an embedding from the tree \( T_x \) below \((\pi(n), \sigma_1(\bar{\alpha})) \) into the tree \( T_y \) below \((\pi(n), \sigma_2(\bar{\alpha})) \)" is also \( \Delta^1_3 \).

"The tree \( T_x \) below \((\pi(n), \sigma_1(\bar{\alpha})) \) embeds into the tree \( T_y \) below \((\pi(n), \sigma_2(\bar{\alpha})) \)" can be computed as \( \Sigma^1_3 \) by \( \exists z \) ( \( z \) codes an embedding from the tree \( T_x \) below \((\pi(n), \sigma_1(\bar{\alpha})) \) into the tree \( T_y \) below \((\pi(n), \sigma_2(\bar{\alpha})) \)). It can also be computed as \( \Pi^1_3 \) by \( \forall z \) ( \( z \) does not compute an embedding from the tree \( T_y \) below \((\pi(n), \sigma_2(\bar{\alpha})) \) into a proper initial segment of the tree \( T_x \) below \((\pi(n), \sigma_1(\bar{\alpha})) \)).
Thus, it remains to be shown that $\Delta_3^1$ is closed under the quantifier $\forall_{\omega_1}^*$. We will show it is closed under $\forall_{\omega_1}^*$, the general case is only a complication of notation. To this end, let $\{A_\alpha : \alpha < \omega_1\}$ be a sequence of $\Delta_3^1$ sets. It suffices to show that $\forall_{\omega_1}^* A_\alpha$ is $\Sigma_3^1$, as an identical computation would show that $\neg(\forall_{\omega_1}^* A_\alpha)(= \forall_{\omega_1}^* (A_\alpha)^c)$ is $\Sigma_3^1$. Let $T'$ denote the Kunen tree. Then

$$\forall_{\omega_1}^* A_\alpha(x \in A_\alpha) \leftrightarrow \exists z((T'_z \text{ is wellfounded}) \land \forall \alpha < \omega_1((\alpha \text{ is closed under } T'_z) \rightarrow x \in A_\alpha))$$

hence is $\exists^R(\Pi_2^1 \land \Delta_3^1)$ which is $\Sigma_3^1$. \hfill $\Box$

A trivial modification to $S_2$ ensures that, for

$$((a_0, \ldots, a_n), (\xi_0, \ldots, \xi_n)), ((a_0, \ldots, a_n), (\xi'_0, \ldots, \xi'_n)) \in S_2((\xi_n < \xi'_n) \rightarrow \forall i < n(\xi_i \leq \xi'_i))$$

For $x,y \in p[S_2]$ and $n,m \in \omega$, define $(n, x) <_{S_2} (m, y)$ iff $(n = m \land \phi_n(x) < \phi_n(y))$. Then $<_{S_2} \in \Sigma_3^1$ and is well-founded of rank $\omega_\omega$. The following is a generalization of the main result of [2].

**Theorem 2.15.** For every $A \subseteq \omega_\omega$ there is a real $x$ such that $A \in L[S_2, x]$.

**Proof.** The argument is identical to that of [2], so we will only show how to modify the definitions to fit our situation. The only differences are that (i) we will use $<_{S_2}$ to code ordinals instead of $\phi_0$ (ii) $S_2$ is not the tree of a scale. We replace the following definitions in [2]:

$$T^n = \{(m, (a_0, \ldots, a_n), (\xi_0, \ldots, \xi_n)) : ((a_0, \ldots, a_n), (\xi_0, \ldots, \xi_n)) \in S_2 \land (n < m \lor \xi_m \leq \eta(m))\}$$

where $\eta(m)$ is such that if $|(m, x)|_{<_{S_2}} = \eta$ then $x(m) = \eta(m)$. If there is no such $x$ then $\eta(m) = \omega_1$. note that, as in [??], $(m, y) \in p[T^n] \rightarrow ((m, y)|_{<_{S_2}} \leq \eta)$

For all $n \in \omega$ let $Q_n \subseteq \omega \times \mathbb{R} \times (\omega \times \omega^{n+1} \times \omega^{n+1})$ be the following set:

$$\{(m_0, x, m_1, (a_0, \ldots, a_n), (\xi_0, \ldots, \xi_n)) : x \in p[S_2] \land (m_1, (a_0, \ldots, a_n), (\xi_0, \ldots, \xi_n)) \in T^{(m_0, x)}_{<_{S_2}}\}$$

As in [2], let $Q_n^*$ be the code set of $Q_n$ where the $i$th coordinate is encoded using $\phi_i$. Formally, $Q_n^* \subseteq \omega \times \mathbb{R} \times (\omega \times \omega^{n+1} \times \mathbb{R}^{n+1})$ is the set of tuples

$$\{(m_0, x, m_1, (a_0, \ldots, a_n), (z_0, \ldots, z_n)) : \forall i \leq n(z_i \in p[S_2]) \land$$

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\[(m_0, x, m_1, (a_0, ..., a_n), (\phi_0(z_0), ..., \phi_n(z_n))) \in Q_n\}\]

The main complication in our generalization is that, in [2], their T (our \(S_2\)) is the tree of a scale. This allows them to compute \(Q^*_n\) as \(\Sigma^1_3\) in an easy way. However, we are able to use the homogeneity of \(S_2\) to give a similar, albeit more complicated, computation. Namely,

\[(m_0, x, m_1, (a_0, ..., a_n), (z_0, ..., z_n)) \in Q^*_n \iff x \in p[S_2] \land \forall i \leq n(z_i \in p[S_2]) \land \exists y \in p[S_2] \land \forall i \leq n(y(i) = a_i) \land (\phi_0(y), ..., \phi_n(y)) \text{ has the same R-type as } (\phi_0(z_0), ..., \phi_n(z_n)) \land (m_1, y) <_{S_2} (m_0, x)]\]

Where \((\alpha_0, ..., \alpha_n)\) has the same R-type as \((\beta_0, ..., \beta_n)\) iff there is a tree of uniform cofinalities \(R\) and functions \(f_1, f_2 : dom(<_R) \to \omega_1\) of type \(R\) such that \([f_1]_R = (\alpha_0, ..., \alpha_n)\) and \([f_2]_R = (\beta_0, ..., \beta_n)\). A straightforward computation shows that this is \(\Sigma^1_3\); thus \(Q^*_n\) is \(\Sigma^1_3\). Finally, given \(A \subset \omega_\omega\), let \(A^*\) be the codeset for \(A\) using \(<_{S_2}\) and let \(x\) be such that \(A^*, <_{S_2}, Q^*_n \in \Sigma^1_3(x)\). The argument of [2] shows that \(A \in L[S_2, x]\).

\[\square\]

We now prove the weak partition property holds at \(\delta^1_3\).

Fix a countable language \(L\) containing \(\{\epsilon, S_2, x\} \cup \{\gamma_i\}_{i \in \omega}\) such that, for each \(n\), \(L\) has countably many \(n\)-ary function symbols \((\{\gamma_i\}_{i \in \omega} \text{ is to be interpreted as an increasing sequence in } \Gamma)\). Note that for any full set of indiscernibles \(C\) as constructed above, we can assume that \(C\) is a full set of indiscernibles as witnessed by \(L\) under some appropriate interpretation of \(L\).

Note that for any sequence of ordinals \(\alpha \in \omega^\omega\) there is a smallest tree of uniform cofinalities \(R_{\alpha}\) such that there is a function \(f : dom(<_{R_{\alpha}}) \to \omega_1\) of type \(R_{\alpha}\) and \(\alpha\) is a subset of \([f]\).

For \(C\) a full set of indiscernibles for \(L[S_2, x]\), we can code the \(L\)-theory of \(Hull^L_{L[S_2, x]}(C^\uparrow \cup \Gamma)\) by a real \(y\). Specifically (viewing \(y\) as a subset of \(\omega\)), for a tree of uniform cofinalities \(R\) and for a formula \(\psi\) in the language \(L\), we put \(< R, \psi > \in y\) iff \(\psi\) is defined on \(dom(M^R)\) and for \(f : dom(<R) \to C, L[S_2, x] \models \psi([f])\). In this situation, we say \(y\) ”is a sharp for \(L[S_2, x]\)".

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For y a real, we say "y looks like a sharp" iff y codes a consistent theory via the coding above and y$\models$ ZFC + V=L[S]. Note that to say "y looks like a sharp" is $\Delta^1_1$.

For y a real that looks like a sharp, we build the model $\mathcal{M}_y$ in the following way: the universe of $\mathcal{M}_y$, $M_y$, has elements of the form $<\tau, \mathcal{R}, f>$ where $\tau$ is an $\mathcal{L}$-function and $f : dom(<\mathcal{R}) \to \omega_1$ is a function of type $\mathcal{R}$. Given $<\tau_1, \mathcal{R}_1, f_1>, ..., <\tau_n, \mathcal{R}_n, f_n>$ in the universe of $\mathcal{M}_y$ and $\psi$ an n-ary $\mathcal{L}$-formula

$$\mathcal{M}_y \models \psi(<\tau_1, \mathcal{R}_1, f_1>, ..., <\tau_n, \mathcal{R}_n, f_n>) \leftrightarrow <\mathcal{R}([f_1], ..., [f_n]), \psi'> \in y$$

Where, if $f:dom(<\mathcal{R}([f_1], ..., [f_n])) \to \omega_1$ is such that $([f_1], ..., [f_n]) \subset [f]$, then $\psi'([f]) \leftrightarrow \psi(\tau_1([f_1], ..., \tau_n([f_n])))$.

We make some observations about $\mathcal{M}_y$.

For $x \in \omega^\omega$ and $\mathcal{R}$ a tree of uniform cofinalities, to say "x codes a function $f_x : dom(<\mathcal{R}) \to \omega_1$ of type $\mathcal{R}$" is $\Delta^1_3$ as it can be defined using ordinal quantifiers over $\omega_1$ and our $\Sigma^1_2$ coding of subsets of $\omega_1$. In addition, for $<\alpha_1, ..., \alpha_n>$ a sequence of ordinals less than $\omega_\omega$, $\mathcal{R}$ a tree of uniform cofinalities, and $x \in \omega^\omega$ that codes $f_x : dom(<\mathcal{R}) \to \omega_1$, to say "$[f_x] = <\alpha_1, ..., \alpha_n>$" is also $\Delta^1_3$.

Observe also, that for triples $<\tau_1, \mathcal{R}_1, x_1>, ..., <\tau_n, \mathcal{R}_n, x_n>$ and $\psi$ an $\mathcal{L}$-formula, to say "$x_i$ codes a function $f_{x_i}$ : Dom(<$\mathcal{R}_i$) $\to \omega_1$ of type $\mathcal{R}_i$ and $\mathcal{M}_y \models \psi(<\tau_1, f_{x_1}, ..., <\tau_n, f_{x_n}>)$" is $\Delta^1_3$ by a similar computation (here we again use that $\Delta^1_3$ is closed under $\forall^*_W$).

**Definition 2.16.** For y a real that looks like a sharp, quadruples $<\tau_1, \mathcal{R}_1, x_1>, ..., <\tau_n, \mathcal{R}_n, x_n>$ and $<\tau_{n+1}, \mathcal{R}_{n+1}, \alpha_{n+1}>, ..., <\tau_m, \mathcal{R}_m, \alpha_m>$, with $\alpha_i$ a sequence or ordinals less than $\omega_\omega$ of type $\mathcal{R}_i$, and $\psi$ an $\mathcal{L}$ formula, we write

$$\mathcal{M}_y \models \psi(<\tau_1, \mathcal{R}_1, x_1>, ..., <\tau_n, \mathcal{R}_n, x_n>, <\tau_{n+1}, \mathcal{R}_{n+1}, \alpha_{n+1}>, ..., <\tau_m, \mathcal{R}_m, \alpha_m>)$$

if and only if

$$\exists x_1, ..., x_m ([f_{x_i}] = \bar{\alpha}_i) \land$$

$$\mathcal{M}_y \models \psi(<\tau_1, \mathcal{R}_1, x_1>, ..., <\tau_n, \mathcal{R}_n, x_n>, <\tau_{n+1}, \mathcal{R}_{n+1}, x_{n+1}>, ..., <\tau_m, \mathcal{R}_m, x_m>)$$

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For \( y, \psi, \langle \tau_{n+1}, \mathcal{R}_{n+1}, \bar{\alpha}_{n+1} \rangle, \ldots, \langle \tau_{m}, \mathcal{R}_m, \bar{\alpha}_m \rangle \) as above, the set
\[
A = \{ \langle \tau_1, \mathcal{R}_1, x_1 \rangle, \ldots, \langle \tau_n, \mathcal{R}_n, x_n \rangle : \}
\]
\[
\mathcal{M}_y \models \psi(\langle \tau_1, \mathcal{R}_1, x_1 \rangle, \ldots, \langle \tau_n, \mathcal{R}_n, x_n \rangle, \langle \tau_{n+1}, \mathcal{R}_{n+1}, \bar{\alpha}_{n+1} \rangle, \ldots, \langle \tau_m, \mathcal{R}_m, \bar{\alpha}_m \rangle)
\]
is \( \Delta^1_3 \). It is \( \Sigma^1_3 \) by definition, it can also be computed as \( \Pi^1_3 \) by
\[
\forall x_1, \ldots, x_m ([f_x] = \bar{\alpha}_i) \rightarrow
\]
\[
\mathcal{M}_y \models \psi(\langle \tau_1, \mathcal{R}_1, x_1 \rangle, \ldots, \langle \tau_n, \mathcal{R}_n, x_n \rangle, \langle \tau_{n+1}, \mathcal{R}_{n+1}, x_{n+1} \rangle, \ldots, \langle \tau_m, \mathcal{R}_m, x_m \rangle)
\]
Lastly we note that, while there may be ill founded \( \mathcal{M}_y \), every \( \mathcal{M}_y \) contains an initial
segment of the ordinals, which we denote by \( WFP(\mathcal{M}_y) \). We prove the following lemma:

**Lemma 2.17.** Fix \( \beta < \delta^1_3 \). Let \( A_\beta = \{ (\tau, \mathcal{R}, x, y) : y \text{ looks like a sharp and } \beta \in WFP(\mathcal{M}_y) \text{ and } | \tau, \mathcal{R}, f_x |_{\mathcal{M}_y} = \beta \} \). Then \( A_\beta \in \Delta^1_3 \).

**Proof.** Our proof is by induction on \( \beta \).

For \( \beta = 0 \), \( \langle \tau, \mathcal{R}, x, y \rangle \in A_\beta \) iff \( y \) looks like a sharp and \( \mathcal{M}_y \models \langle \tau, \mathcal{R}, x \rangle = 0 \).
(Here, we are using that 0 is definable). By observations preceding the lemma, this is in \( \Delta^1_3 \).

For \( \beta = \alpha + 1 \) a successor ordinals,
\[
\langle \tau, \mathcal{R}, x, y \rangle \in A_\beta \leftrightarrow y \text{ looks like a sharp and } \exists \langle \tau', \mathcal{R}', x' \rangle
\]
\[
\langle \tau', \mathcal{R}', x', y \rangle \in A_\alpha \text{ and } \mathcal{M}_y \models \langle \tau', \mathcal{R}', x' \rangle + 1 = \langle \tau, \mathcal{R}, x \rangle
\]
So \( A_\beta \in \Sigma^1_3 \). Similarly,
\[
\langle \tau, \mathcal{R}, x, y \rangle \in A_\beta \leftrightarrow y \text{ looks like a sharp and } \forall \langle \tau', \mathcal{R}', x' \rangle
\]
\[
\langle \tau', \mathcal{R}', x', y \rangle \in A_\alpha \implies \mathcal{M}_y \models \langle \tau', \mathcal{R}', x' \rangle + 1 = \langle \tau, \mathcal{R}, x \rangle
\]
So \( A_\beta \in \Pi^1_3 \), hence \( \Delta^1_3 \).

For \( \beta \) a limit ordinal
\[
\langle \tau, \mathcal{R}, x, y \rangle \in A_\beta \leftrightarrow y \text{ looks like a sharp and}
\]

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\[| < \tau, \mathcal{R}, x > |_{\mathcal{M}_y} \leq \beta \text{ and } \neg(| < \tau, \mathcal{R}, x > |_{\mathcal{M}_y}) < \beta\]

We show that both of these conjuncts are in $\Delta^1_3$.

\[| < \tau, \mathcal{R}, x > |_{\mathcal{M}_y} \leq \beta \leftrightarrow \forall \alpha_1, ..., \alpha_n < \omega \omega \ \forall \mathcal{R}' \ \forall \tau'\]

\[(\forall x' \ (x' \text{ codes a function of type } \mathcal{R}') \text{ such that } [f_{x'}] = < \alpha_1, ..., \alpha_n > \text{ and }\]

\[\mathcal{M}_y =< \tau', \mathcal{R}', f_{x'} > < < \tau, \mathcal{R}, x > ) \]

\[\rightarrow \exists \alpha < \beta ( < \tau', \mathcal{R}', f_{x'} >, y) \in A_\alpha\]

So is $\Pi^1_3$. Similarly,

\[| < \tau, \mathcal{R}, x > |_{\mathcal{M}_y} \leq \beta \leftrightarrow \forall \bar{\alpha} < \omega \ \forall \mathcal{R}' \ \forall \tau'\]

\[(\mathcal{M}_y =< \tau', \mathcal{R}', \bar{\alpha} > < < \tau, \mathcal{R}, x > ) \text{ or }\]

\[(\mathcal{M}_y =< \tau', \mathcal{R}', \bar{\alpha} > < < \tau, \mathcal{R}, x > \text{ and } \exists \alpha < \beta < \tau', \mathcal{R}', \bar{\alpha} >, y ) \in A_\alpha\]

So is $\Sigma^1_3$, hence $\Delta^1_3$.

We finish our proof by showing $\neg(| < \tau, \mathcal{R}, x > |_{\mathcal{M}_y} < \beta)$ is $\Delta^1_3$. But this is straightforward because $\{< < \tau, \mathcal{R}, x >, y > : | < \tau, \mathcal{R}, x > |_{\mathcal{M}_y} < \beta\} = \bigcup_{\beta' < \beta} A_\beta$, so is $\Delta^1_3$.

\[\square\]

We now provide our coding:

**Theorem 2.18.** There is a coding $\Phi : \omega^\omega \rightarrow P(\omega)$ such that

(i): $\forall A \subset \omega^\omega$ there is a real $x$ such that $\Phi(x) = A$.

(ii): $\forall \alpha < \omega \omega \ \{x : \alpha \in \Phi(x) \} \in \Delta^1_3$.

**Proof.** For $x \in \omega \omega$ define $\Phi(x)$ by

\[\alpha \in \Phi(< \psi, < \tau, x >, y >) \leftrightarrow (x \text{ codes a function } f_x, \psi \text{ is an } \mathcal{L}\text{-formula and } y \text{ is a real that looks like a sharp}) \text{ and } (\mathcal{M}_y \models \psi(\alpha, < \tau, \mathcal{R}, f_x >))\]

Of course, by "$\mathcal{M}_y \models \psi(\alpha, < \tau, \mathcal{R}, f_x >)$" we mean "$\exists < \tau', \mathcal{R}', x' > (| < \tau', \mathcal{R}', x' > |_{\mathcal{M}_y} = \alpha) \wedge \mathcal{M}_y \models \psi(< \tau', \mathcal{R}', x' >, < \tau, \mathcal{R}, x >))$" or, equivalently, "$\forall < \tau', \mathcal{R}', x' > (| < \tau', \mathcal{R}', x' > |_{\mathcal{M}_y} = \alpha) \wedge \mathcal{M}_y \models \psi(< \tau', \mathcal{R}', x' >, < \tau, \mathcal{R}, x >))$".
τ', R', x' > |_{\mathcal{M}_y} = \alpha \rightarrow \mathcal{M}_y \models \psi(< \tau, R, z >))." Hence (ii) holds. By Theorem 2.15 property (i) also holds.

□

By [4], Theorem 2.18 implies that the weak partition property holds at $\delta_3^1$.

Recall from the theory of indiscernibles for $L[x]$ that the set of reals $\{y : \exists x (y = x^x)\}$ is $\Pi_2^1$. We conclude this section by generalizing this result to sharps for $L[S_2, x]$.

**Theorem 2.19.** The set of reals $A = \{y : \exists x (y \text{ is a sharp for } L[S_2, x])\}$ is $\Pi_3^1$.

**Proof.** $y \in A$ iff $y$ looks like a sharp, $\mathcal{M}_y$ is well-founded, and $S_2^{\mathcal{M}_y} = S_2$. Using our coding from Theorem 2.18, to say "$S_2^{\mathcal{M}_y} = S_2$" is $\Delta_3^1$. Further $\mathcal{M}_y$ is well founded iff $\forall z( z \text{ does not code an } \omega \text{-sequence of elements of } \mathcal{M}_y \text{ that is } \mathcal{M}_y \text{-decreasing})$, hence is $\Pi_3^1$ as desired. □

2.5. Type-2 Descriptions

Let $d$ be a description defined relative to $K_1, \ldots, K_n$, then the $\delta$ such that

$$\forall^*_{K_1} \alpha_1, \ldots, \forall^*_{K_n} \alpha_n \delta(\alpha_1, \ldots, \alpha_n) = h(\alpha_1, \ldots, \alpha_n, d)$$

is well defined. Further, if $\mathcal{R}$ is a type-2 tree of uniform cofinalities and $f : dom(<_{\mathcal{R}}) \rightarrow \delta_3^1$, then we may use $f$ as a final "lift-up" of the description. I.e. letting $g$ be one of the component functions of $f$, we may define $\delta$ to be s.t.

$$\forall^*_{K_1} \alpha_1, \ldots, \forall^*_{K_n} \alpha_n \delta(\alpha_1, \ldots, \alpha_n) = g(i)(h(\alpha_1, \ldots, \alpha_n, d))$$

where $g(i)$ is the $i$th invariant of $g$. For $f$ and $d$ as above, we define $E(f, d)$ to be the ordinal $\delta$ as above. For $K_1, \ldots, K_n$ a sequence of measures on $\omega_\omega$, we define $E^{K_1, \ldots, K_n}(f) = \{ E(f, d) : d \text{ is a description defined relative to } K_1, \ldots, K_n \}$. The following is a straightforward fact:

**Theorem 2.20.** Let $\mathcal{R}$ be a type-2 tree of uniform cofinalities and $f : dom(<_{\mathcal{R}}) \rightarrow C$ for some club $C \subset \delta_3^1$. Let $(K_1, \ldots, K_n)$ be some sequence of measures on
Then for all $\delta \in E^{K_1,\ldots,K_n}(f)$,
\[
\forall_{K_1}^{\ast} \alpha_1, \ldots, \forall_{K_n}^{\ast} \alpha_n \\delta(\alpha_1, \ldots, \alpha_n) \in C
\]

We prove a new fact in the theory of descriptions:

**Theorem 2.21.** Let $A \subset \delta_1^{1}, \delta < \delta_1^{1}$, and $r_1, \ldots, r_n$ be type-1 trees of uniform cofinalities such that
\[
\forall_{M_{r_1}}^{\ast} \alpha_1 \ldots \forall_{M_{r_n}}^{\ast} \alpha_n \\delta(\alpha_1, \ldots, \alpha_n) \in A
\]
Then there exists an $f : \omega^{m} \rightarrow A$, and descriptions defined relative to $M_{r_1}, \ldots, M_{r_n}, d_1, \ldots, d_m$ such that
\[
\forall_{M_{r_1}}^{\ast} \alpha_1 \ldots \forall_{M_{r_n}}^{\ast} \alpha_n \\
\delta(\alpha_1, \ldots, \alpha_n) = f(d_1(\alpha_1, \ldots, \alpha_n), \ldots, d_m(\alpha_1, \ldots, \alpha_m))
\]

**Proof.** The proof is repeated application of the following lemma:

**Lemma 2.22.** Let $r_1, \ldots, r_n$ be type-1 trees of uniform cofinalities, $f : \omega^{m} \rightarrow \delta_1^{1}$, and $d_1, \ldots, d_m$ be descriptions defined relative to $M_{r_1}, \ldots, M_{r_n}$. Then for all type 1 trees $r_{n+1}$, there is an $m' > m$ a function $f' : \omega^{m'} \rightarrow \delta_1^{1}$ and descriptions $d_{m+1}, \ldots, d_{m'}$ defined relative to $M_{r_1}, \ldots, M_{r_n}, M_{r_{n+1}}$ such that
\[
\forall_{M_{r_1}}^{\ast} \alpha_1 \ldots \forall_{M_{r_n}}^{\ast} \alpha_n \forall_{M_{r_{n+1}}}^{\ast} \alpha_{n+1} \\
f(d_1(\alpha_1, \ldots, \alpha_n), \ldots, d_m(\alpha_1, \ldots, \alpha_n))(\alpha_{n+1}) = \\
f'(d_1(\alpha_1, \ldots, \alpha_n), \ldots, d_m(\alpha_1, \ldots, \alpha_n), d_{m+1}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}), \ldots, d_{m'}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}))
\]

**Proof.** Let $r_1, \ldots, r_n, f,$ and $d_1, \ldots, d_m$ be as in the statement of the lemma. Let $r_{n+1}$ be a type-1 tree of uniform cofinalities. We will construct an $f'$ and $d_{m+1}, \ldots, d_{m'}$ such that the conclusion of the lemma holds.

Define $f_1 : \omega^{m+1} \rightarrow \delta_1^{1}$ by
\[
f_1(\beta_1, \ldots, \beta_m, \beta_{m+1}) = \mu \gamma \text{ such that } j_{M_{r_{n+1}}}^{M_{r_{n+1}}}(\gamma) \geq f(\beta_1, \ldots, \beta_m)
\]
and let $d_{m+1}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) = \alpha_{n+1}$. Then

$$\forall^*_{M_{r_1}} \alpha_1 \ldots \forall^*_{M_{r_n}} \alpha_n \forall^*_{M_{r_{n+1}}} \alpha_{n+1}$$

$$f(d_1(\alpha_1, \ldots, \alpha_n), \ldots, d_m(\alpha_1, \ldots, \alpha_n))(\alpha_{n+1}) \leq f_1(d_1(\alpha_1, \ldots, \alpha_n), \ldots, d_m(\alpha_1, \ldots, \alpha_n), d_{m+1}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}))$$

In general, at stage $l \in \omega$, we will have a function $f_l : \omega^{m+l} \to \delta^1_3$ and descriptions $d_{m+1}, \ldots, d_{m+l}$ such that

$$\forall^*_{M_{r_1}} \alpha_1 \ldots \forall^*_{M_{r_n}} \alpha_n \forall^*_{M_{r_{n+1}}} \alpha_{n+1}$$

$$f(d_1(\alpha_1, \ldots, \alpha_n), \ldots, d_m(\alpha_1, \ldots, \alpha_n))(\alpha_{n+1}) \leq f_l(d_1(\alpha_1, \ldots, \alpha_n), \ldots, d_m(\alpha_1, \ldots, \alpha_n), d_{m+1}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}), \ldots, d_{m+l}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}))$$

In the above equation, if equality holds, then we are done (set $f' = f_l$). If the inequality is strict, then we define an $f_{l+1} : \omega^{l+1} \to \delta^1_3$ and a description $d_{l+1}$ such that

$$f_{l+1}(d_1(\alpha_1, \ldots, \alpha_n), \ldots, d_m(\alpha_1, \ldots, \alpha_n), d_{m+1}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}), \ldots, d_{m+l+1}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}))$$

and

$$f_l(d_1(\alpha_1, \ldots, \alpha_n), \ldots, d_m(\alpha_1, \ldots, \alpha_n), d_{m+1}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}), \ldots, d_{m+l}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})) <$$

$$f_{l+1}(d_1(\alpha_1, \ldots, \alpha_n), \ldots, d_m(\alpha_1, \ldots, \alpha_n), d_{m+1}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}), \ldots, d_{m+l+1}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}))$$

It is clear that this process must terminate at some point, as otherwise the $(f_i)_{i \in \omega}$ would define an infinite decreasing sequence of ordinals in the iterated ultrapower by $M_{r_1}, \ldots, M_{r_n}, M_{r_{n+1}}$.

It remains to be shown that, given an $f_l$ and $d_{m+1}, \ldots, d_{m+l}$ such that the inequality in $(*)$ is strict, we can construct an $f_{l+1}$ and $d_{m+l+1}$ such that (1) and (2) hold. Let $d$, a description defined relative do $M_{r_1}, \ldots, M_{r_n}$, be such that
has uniform cofinality \( d(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) \) and let \( g \) witness this. i.e.
\[
\forall^*_{M_{r_1}} \alpha_1 \cdots \forall^*_{M_{r_n}} \alpha_n \forall^*_{M_{r_{n+1}}} \alpha_{n+1}
\]
\[
f_l(d_1(\alpha_1, \ldots, \alpha_n), \ldots, d_m(\alpha_1, \ldots, \alpha_n), d_{m+1}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}), \ldots, d_{m+l}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})) = \\
\sup_{\beta < d(\alpha_1, \ldots, \alpha_{n+1})} g(d_1(\alpha_1, \ldots, \alpha_n), \ldots, d_m(\alpha_1, \ldots, \alpha_n), d_{m+1}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}), \ldots, d_{m+l}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}), \beta)
\]
Let \( d' \) be the least description defined relative to \( M_{r_1}, \ldots, M_{r_n}, M_{r_{n+1}} \) less than \( d \) such that
\[
\forall^*_{M_{r_1}} \alpha_1 \cdots \forall^*_{M_{r_n}} \alpha_n \forall^*_{M_{r_{n+1}}} \alpha_{n+1}
\]
\[
f(d_1(\alpha_1, \ldots, \alpha_n), \ldots, d_m(\alpha_1, \ldots, \alpha_n))(\alpha_{n+1}) < \\
g(d_1(\alpha_1, \ldots, \alpha_n), \ldots, d_m(\alpha_1, \ldots, \alpha_n), d_{m+1}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}), \ldots, d_{m+l}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}), d'(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}))
\]
Setting \( d_{m+l+1} = \mathcal{L}(d') \), the fundamental lemma on descriptions (see ??) says that there is an \( h \) such that
\[
\forall^*_{M_{r_1}} \alpha_1 \cdots \forall^*_{M_{r_n}} \alpha_n \forall^*_{M_{r_{n+1}}} \alpha_{n+1}
\]
\[
f(d_1(\alpha_1, \ldots, \alpha_n), \ldots, d_m(\alpha_1, \ldots, \alpha_n))(\alpha_{n+1}) \leq \\
g(d_1(\alpha_1, \ldots, \alpha_n), \ldots, d_m(\alpha_1, \ldots, \alpha_n), d_{m+1}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}), \ldots, d_{m+l}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}), h(d_{m+l+1}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}))
\]
Setting \( f_{l+1}(\beta_1, \ldots, \beta_{m+l+1}) = g(\beta_1, \ldots, \beta_{m+l}, h(\beta_{m+l+1})) \), it is clear that properties (1) and (2) hold.

**Definition 2.23.** Let \( A \subset \delta_3^1 \). We define the lift-up of \( A \), \( A^\uparrow \), to be the set
\[
\{ [f]_{M_r} : r \text{ a type-1 tree of uniform cofinalities and } f : \text{dom}(M_r) \to A \}
\]
Further, we define the nth lift-up of \( A \) \( A^\uparrow_n \) by
\[
1.: \ A^{\uparrow,0} = A \\
2.: \ A^{\uparrow,n+1} = (A^{\uparrow,n})^\uparrow
\]
Finally, we define the iterated lift-up of $A$, $A^{↑,ω}$ by

$$A^{↑,ω} = \bigcup_{n \in ω} A^{↑,n}$$

For every sequence of ordinals $\delta$ in $C^{↑,ω}$, we can find functions $f_i : \omega^{m_i} \rightarrow C$, measures $M_{r_1}, ..., M_{r_{m_i}}$ and descriptions defined relative to these measures, $d_1^i, ..., d_{m_i}^i$, that witness Theorem 2.21 with respect to $δ_i$. For some type-2 tree of uniform cofinalities $R$, we can put all of the $f_i$ together into a single function $f : dom(<_R) \rightarrow C$ of type $R$. This tree $R$, together with the measures and descriptions, describes how the sequence $\delta$ was "built up" from $C$. This motivates the following definition:

**Definition 2.24.** Let $R$ be a type-2 tree of uniform cofinalities. We define an $R$-system, $S$, to be a triple of functions, $(h, g_1, g_2)$, such that $h : m \rightarrow \{b : b$ is a branch in $R\}$ for some integer $m$ and for all $i < m$,

1. $g_1(i) = (r_{b_1}^i, ..., r_{b_n}^i)$ is a sequence of type-1 trees of uniform cofinalities.
2. $g_2(i) = (d_{b_1}^i, ..., d_{b_l}^i)$ is a sequence of descriptions defined relative to $M_{r_1}^{\delta_b^1}, ..., M_{r_n}^{\delta_b^n}$ of length $l$, where $l$ is the length of the branch $h(i)$.

For $f : dom(<_R) \rightarrow \delta_3$ a function of type $R$, and for $S = (h, g_1, g_2)$ an $R$-system as above, we define the sequence of ordinals $[f^S] = (\delta_0, ..., \delta_{m-1})$ by

$$\forall_{M_{r_1}^{\delta}} \alpha_1 ... \forall_{M_{r_n}^{\delta}} \alpha_n$$

$$\delta_i(\alpha_1, ..., \alpha_n) = f^{h(i)}(d_{b_1}^i(\alpha_1, ..., \alpha_n), ..., d_{b_l}^i(\alpha_1, ..., \alpha_n))$$

where $f^{h(i)}$ is the component function of $f$ corresponding to the branch $h(i)$.

For $f$ a function, we define $\delta_f = sup(range(f))$. The main element of the description theory we will need is

**Lemma 2.25.** Let $F : (\delta_3^n) \rightarrow \delta_3^1$. Then for every type-2 tree of uniform cofinalities, $R$, for every $R$-system $S$, for every type-1 tree of uniform cofinalities, $r$, there exists measures on
$\forall_\omega, (K_1, ..., K_n)$, and functions $(r_1, ..., r_n)$ and $(\tau_1, ..., \tau_{n+1})$ such that

$$\forall_{M^\omega}[f] R(f) \forall_{r_1 \beta} \tau_1([f], \beta) = \tau_1([f], \beta)$$

$$\forall_{M^\omega}[f] \forall_{r_1 \beta} \forall_{r_2 \alpha_1} \tau_2([f], \beta, \alpha_1) = \tau_2([f], \beta, \alpha_1)$$

$$\forall_{M^\omega}[f] \forall_{r_1 \beta} \forall_{r_2 \alpha_1} \forall_{r_2 \alpha_2} \tau_3([f], \beta, \alpha_1, \alpha_2) = \tau_3([f], \beta, \alpha_1, \alpha_2)$$

$$\forall_{M^\omega}[f] \forall_{r_1 \beta} \forall_{r_2 \alpha_1} \forall_{r_3 \alpha_2} \tau_4([f], \beta, \alpha_1, \alpha_2) = \tau_4([f], \beta, \alpha_1, \alpha_2)$$

$$\forall_{M^\omega}[f] \forall_{r_1 \beta} \forall_{r_2 \alpha_1} \forall_{r_3 \alpha_2} \forall_{r_4 \alpha_3} \tau_5([f], \beta, \alpha_1, \alpha_2, \alpha_3) = \tau_5([f], \beta, \alpha_1, \alpha_2, \alpha_3)$$

2.6. Indiscernibles for $L[T_3, x]$

**Definition 2.26.** Let $L$ be a language and $C \subset \delta^3_1$ be club. We will say that $C$ witnesses Lemma 2.25 with respect to $L$ if for every $L$ skolem function $F$ with codomain $\delta^3_1$, for every type-2 tree of uniform cofinalities, $R$, for every $R$-system $S$, for every type-1 tree of uniform cofinalities, $r$, there exists measures on $\omega^\omega$, $(K_1, ..., K_n)$, and functions $(r_1, ..., r_n)$ and $(\tau_1, ..., \tau_{n+1})$ such that the conclusion of Lemma 2.25 holds, where the $\forall_{M^\omega}[f]$ quantifier may be interpreted as $\forall f : Dom(<_R) \to C$ of type $R$.

Also, for $F : (\delta^3_1)^{<\omega} \to \delta^3_1$, $C \subset \delta^3_1$ club, $R$ a type-2 tree of uniform cofinalities, $S$ a $R$-system, $r$ a type-1 tree of uniform cofinalities, $(K_1, ..., K_n)$ measures on $\omega^\omega$, and $(r_1, ..., r_n)$, $(\tau_1, ..., \tau_{n+1})$ functions, we will say that $(F, R, S, r, (K_1, ..., K_n), (r_1, ..., r_n), (\tau_1, ..., \tau_{n+1}))$ witnesses Lemma 2.25 if the conclusion of Lemma 2.25 holds where, again, the $\forall_{M^\omega}[f]$ quantifier may be interpreted as $\forall f : Dom(<_R) \to C$ of type $R$.

Let $T$ be a stabilized weakly-homogeneous tree on $\omega \times (\omega \times \omega^\omega)$ projecting to a $\Sigma^4_3$ complete set and let $S_3$ be the tree on $\omega \times \delta^1_3$ coming from the Martin-Solovay construction over $T$. Fix $x \in \omega^\omega$, we provide a theory of indiscernibles for $L[S_3, x]$ (In the next section, we show that for a cone of $x$ $L[S_3, x] = L[T_3, x]$):

**Definition 2.27.** Let $\Gamma$ be a proper class of ordinals with $\min(\Gamma) > \delta^3_1$ and let $C \subset \delta^3_1$ be club. We say $(\Gamma, C)$ is an indiscernible pair in the language $L$ iff
(i): For all $\alpha_1, \ldots, \alpha_n < \delta_3^1$, for all $(\gamma_1, \ldots, \gamma_m)$, $(\gamma'_1, \ldots, \gamma'_m)$ increasing tuples from $\Gamma$, and for all $(n+m)$-ary $L$ formulas $\phi$,

$$L[S_3, x] \models \phi(\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_m) \leftrightarrow L[S_3, x] \models \phi(\alpha_1, \ldots, \alpha_n, \gamma'_1, \ldots, \gamma'_m)$$

(ii): For all type-2 trees of uniform cofinalities $R$, for all $f_1, f_2 : dom(<_R) \to C$ of type $R$, for all $R-$systems $S$, for all $\bar{\gamma} \in \Gamma^{<\omega}$, and for all $L$-formulas $\phi$,

$$L[S_3, x] \models \phi([f^S_1], \bar{\gamma}) \leftrightarrow L[S_3, x] \models \phi([f^S_2], \bar{\gamma})$$

Remark 2.28. In view of Theorem 2.21 the import of (ii) is that, if $\bar{\delta}_1, \bar{\delta}_2$ are sequences of ordinals in $C^{\uparrow \omega}$ that are built up from $C$ in the same way, then $\bar{\delta}_1$ and $\bar{\delta}_2$ satisfy the same formulas.

Remark 2.29. All of the languages, $L$, that we will consider will have an obvious interpretation in $L[S_3, x]$. Because of this, we will not refer to $L$’s interpretation. Further, for $R$ an $L$-symbol, we will make no distinction between $R$ and it’s semantic interpretation.

Definition 2.30. Let $A \subset \delta_3^1$. We say $A$ is measure invariant if for all $\delta$, if $\delta \in A^{\uparrow}$, then, in fact, for all type 1 trees $r$

$$\forall^*_{M_r} \alpha \delta(\alpha) \in A$$

Definition 2.31. Let $(\Gamma, C)$ be an indiscernible pair in $L$. We say $(\Gamma, C)$ is full (in $L$) if there is a measure invariant $A \subset \delta_3^1$ such that

$$Hull^L(C^{\uparrow \omega} \cup \Gamma) \cap \delta_3^1 = A^{\uparrow}$$

Remark 2.32. All hulls are taken in $L[S_3, x]$.

Theorem 2.33. Let $(\Gamma, C)$ be a full indiscernible pair in the language $L$ and let $\pi : hull^L(C^{\uparrow} \cup \Gamma) \cong M$ be the transitive collapse. Then $M = L[S_3, x]$.

Proof. Let $H = hull^L(C^{\uparrow} \cup \Gamma)$.
Let $A \subset \delta^1_3$ be measure invariant so that

$$\mathcal{H} \cap \delta^1_3 = A^\uparrow$$

Let $\rho : A \cong \delta^1_3$ be the collapse of $A$.

Define the map $\tau : A^\uparrow \to \delta^1_3$ by, for $\delta \in A^\uparrow$ and $r$ a type $-1$ tree,

$$\forall^*_r \alpha \tau(\delta)(\alpha) = \rho(\delta(\alpha))$$

We first show that $\tau$ is well-defined. That is, for all $\delta \in A^\uparrow$, if $r_1$ and $r_2$ are type-1 trees, and $\bar{\delta}_1, \bar{\delta}_2$ are defined by

$$\forall^*_r \alpha \bar{\delta}_i(\alpha) = \rho(\delta(\alpha))$$

then, $\bar{\delta}_1 = \bar{\delta}_2$. To see that this is the case, fix $r_1$ and $r_2$ and, toward contradiction, fix $\delta \in A^\uparrow$ least such that $\bar{\delta}_1 \neq \bar{\delta}_2$. Without loss of generality, we assume that $\bar{\delta}_1 < \bar{\delta}_2$. Let $\bar{g}_2 : \text{Dom}(M_{r_2}) \to \delta^1_3$ be such that $[\bar{g}_2] = \bar{\delta}_1$. Note that $\forall^*_{M_{r_2}} \alpha \bar{g}_2(\alpha) < \bar{\delta}_2(\alpha)$. Define $g_2 : \text{Dom}(M_{r_2}) \to A$ by $g_2(\alpha) = \rho^{-1}(\bar{g}_2(\alpha))$. Then $[g_2] < \delta$. By the measure invariance of $A$, let $g_1 : \text{Dom}(M_{r_1}) \to A$ be such that $[g_1] = [g_2]$. If we define $\bar{g}_1 : \text{Dom}(M_{r_1}) \to \delta^1_3$ by $\bar{g}_1(\alpha) = \rho(g_1(\alpha))$ then, by the minimality of $\delta$, $[\bar{g}_1] = [\bar{g}_2]$. But this is a contradiction because $[\bar{g}_1] < \bar{\delta}_1 = [\bar{g}_2]$. This shows that $\tau$ is well-defined.

Next, we show that $\tau = \pi \upharpoonright A^\uparrow$. To see this, we merely observe that $\tau$ is onto $\delta^1_3$ and is order-preserving.

Finally, to show that $M = L[S_3, x]$, let $(\bar{n}, \bar{\delta}) = ((n_0, \ldots, n_m), (\delta_0, \ldots, \delta_m)) \in \mathcal{H} \cap S_3$. We must show that $\pi((\bar{n}, \bar{\delta})) \in S_3$. But $n_i \in \omega$, so $\pi((\bar{n}, \bar{\delta})) = (\bar{n}, \pi(\bar{\delta}))$. Further, for all ordinals $\bar{\gamma}$, $(\bar{n}, \bar{\gamma}) \in S_3$ iff $\bar{\gamma}$ codes a ranking of $T_{\bar{n}}$ via the homogeneity measures on $T_{\bar{n}}$. Knowing that $\pi(\bar{\delta}) = \tau(\bar{\delta})$, it is clear that $\pi(\bar{\delta})$ codes a ranking of $T_{\bar{n}}$, so that $\pi(\bar{n}, \bar{\delta}) \in S_3$ as required.

Next, we show how to construct a full indiscernible pair. The construction is somewhat technical. To make it easier to digest, we extract two of the main pieces in the following lemmas.
Lemma 2.34. Let $\Gamma$ have property (i) of an indiscernible pair, and let $\mathcal{L}$ be a countable language. Then there is a club $C \subseteq \delta_{3}^{1}$ such that $(\Gamma, C)$ is an indiscernible pair in $\mathcal{L}$.

Proof. For every type two tree of uniform cofinalities $R$, for every $R$-system $S$, for every $\mathcal{L}$-formula $\phi$, and for all $n \in \omega$, use the weak partition property on $\delta_{3}^{1}$ to get a club $C_{R,S,\phi,n}$ such that for all $\gamma_{1} < \ldots < \gamma_{n} \in \Gamma$, and for all $f_{1}, f_{2} : dom(<_{R}) \to C_{R,S,\phi,n}$,

$$L[S_{3}, x] = \phi([f^{S}], \gamma) \leftrightarrow L[S_{3}, x] = \phi([f^{S}], \bar{\gamma})$$

Set $C = \bigcap_{R,S,\phi,n} C_{R,S,\phi,n}$. □

Lemma 2.35. Let $C \subseteq \delta_{3}^{1}$. Let $\mathcal{F}$ be a sequence of countable collections of functions from $(\delta_{3}^{1})^{<\omega} \to \delta_{3}^{1}$ such that the following properties hold:

(i): $\mathcal{F}$ is closed under composition.

(ii): For every $m$-ary function $f \in \mathcal{F}$ and for every sequence of type-1 trees $r_{1},...,r_{n}$,

there is a function $f^{r_{1},...,r_{n}} \in \mathcal{F}$ defined by

$$\forall_{M_{r_{1}}}^{\alpha_{1}} \ldots \forall_{M_{r_{n}}}^{\alpha_{n}} f^{r_{1},...,r_{n}}(\delta_{1},...,\delta_{m})(\alpha_{1},\ldots,\alpha_{n}) = f(\delta_{1}(\alpha_{1},\ldots,\alpha_{n}),...,\delta_{m}(\alpha_{1},\ldots,\alpha_{n}))$$

Let $A$ be the closure of $C^{\uparrow,\omega}$ under $\mathcal{F}$. Then $A^{\uparrow,\omega} = A$.

Proof. It is always the case that $A \subseteq A^{\uparrow,\omega}$, we show $A^{\uparrow,\omega} \subseteq A$. To this end, let $\delta \in A^{\uparrow,\omega}$, we must show $\delta \in A$. Let $r_{1},...,r_{n}$ by type-1 trees such that $\forall_{M_{r_{1}}}^{\alpha_{1}} \ldots \forall_{M_{r_{n}}}^{\alpha_{n}} \delta(\alpha_{1},\ldots,\alpha_{n}) \in A$. By countable additivity, there is a single $f \in \mathcal{F}$, and an $l \in \omega$ such that $\forall_{M_{r_{1}}}^{\alpha_{1}} \ldots \forall_{M_{r_{n}}}^{\alpha_{n}} \delta(\alpha_{1},\ldots,\alpha_{n}) \in f[C^{\uparrow,l}]$. Say $f$ is $m$-ary and, for $i < m$, define $\delta_{i}$ by

$$\forall_{M_{r_{1}}}^{\alpha_{1}} \ldots \forall_{M_{r_{n}}}^{\alpha_{n}} \delta(\alpha_{1},\ldots,\alpha_{n}) = \text{the } i\text{th coordinate of } f^{-1}(\delta(\alpha_{1},\ldots,\alpha_{n}))$$

Each $\delta_{i} \in C^{\uparrow,\omega}$ and

$$\forall_{M_{r_{1}}}^{\alpha_{1}} \ldots \forall_{M_{r_{n}}}^{\alpha_{n}} f^{r_{1},...,r_{n}}(\delta_{1},...,\delta_{m})(\alpha_{1},...,\alpha_{n}) = f(\delta_{1}(\alpha_{1},...,\alpha_{n}),...,\delta_{m}(\alpha_{1},...,\alpha_{n})) = \delta(\alpha_{1},...,\alpha_{n})$$

so that $f^{r_{1},...,r_{n}}(\delta_{1},...,\delta_{m}) = \delta$ and $\delta \in A$ as required. □
Theorem 2.36. Let $\Gamma$ be a proper class of ordinals such that $\Gamma$ has property (i) of an indiscernible pair in the language $\mathcal{L}_0 = \{\epsilon, S_3, x\}$. Then there is a club $C \subset \delta_3^1$ and a language $\mathcal{L}$ such that $(\Gamma, C)$ is a full indiscernible pair in $\mathcal{L}$.

Proof. The construction is via an $\omega$-length induction. At stage $n$, we will have a language $\mathcal{L}_n$, a club $C_n$. Set $\mathcal{L}_0 = \{\epsilon, S_3, x\} \cup \{\gamma_i\}_{i \in \omega}$ where $\{\gamma_i\}$ is to be interpreted as an increasing sequence in $\Gamma$.

In general, at stage $n$,

1: Given $\mathcal{L}_n$, let $C_n$ be so that $(\Gamma, C_n)$ is an indiscernible pair in $\mathcal{L}_n$ (using Lemma 2.34) and so that $C_n$ witnesses Lemma 2.25 with respect to $\mathcal{L}_n$.

2: For every $\mathcal{L}_n$ skolem function with codomain $\delta_1^1$, for every type-2 tree $\mathcal{R}$, for every $\mathcal{R}$-system $S$, for every type-1 tree $r$, pick $n = n(F, \mathcal{R}, S, r)$ and $(K_1, \ldots, K_n)^{F, \mathcal{R}, S, r}, (r_1, \ldots, r_n)^{F, \mathcal{R}, S, r}$, and $(\tau_1, \ldots, \tau_{n+1})^{F, \mathcal{R}, S, r}$ such that $(F, C_n, \mathcal{R}, S, r, (K_1, \ldots, K_n)^{F, \mathcal{R}, S, r}, (r_1, \ldots, r_n)^{F, \mathcal{R}, S, r}, (\tau_1, \ldots, \tau_{n+1})^{F, \mathcal{R}, S, r}$ witnesses Lemma 2.25.

Let $\mathcal{L}_{n+1} = \mathcal{L}_n \cup \{\tau : \tau \in (\tau_1, \ldots, \tau_{n+1})^{F, \mathcal{R}, S, r}\}$ for some $\mathcal{L}_n$ skolem function $F$, some type-2 tree $\mathcal{R}$, some $\mathcal{R}$-system $S$, and some type-1 tree $r$ $\cup \{F^{r_1, \ldots, r_m} : r_1, \ldots, r_n\}$ type-1 trees and $F$ a skolem function in $\mathcal{L}_n$, where $F^{r_1, \ldots, r_m}$ is defined as in Lemma 2.35.

Define $\mathcal{L} = \bigcup_n \mathcal{L}_n$ and $C = \bigcap_n C_n$. We show that $(\Gamma, C)$ is a full indiscernible pair in $\mathcal{L}$.

Let $\mathcal{H} = Hull^\mathcal{L}(C^{\uparrow \omega} \cup \Gamma)$ and set $A = \mathcal{H} \cap \delta_3^1$. First note that $A$ is the closure of $C^{\uparrow \omega}$ under $\mathcal{F} = \{\tau : \tau$ a skolem function in $\mathcal{L}$ with codomain $\delta_3^1\}$. By construction, $\mathcal{F}$ satisfies the hypotheses of Lemma 2.35, so that $A^{\uparrow \omega} = A$. Thus it is also true that $\mathcal{H} \cap \delta_3^1 = A^{\uparrow}$.

We proceed to show that $A$ is measure invariant. Let $\delta \in A^{\uparrow}$ and $r$ a type-1 tree. We must show that $\forall^*_M, \alpha \delta(\alpha) \in A$. There is some $\tilde{\eta} \in (C^{\uparrow \omega})^{< \omega}$, some $m \in \omega$, and some $\mathcal{L}_m$ skolem function $F$ so that $\delta = F(\tilde{\eta})$. There is also some type-2 tree $\mathcal{R}$, some $f_0 : Dom(<_R) \rightarrow C$ of type $\mathcal{R}$, and some $\mathcal{R}$-system $S$ so that $\tilde{\eta} = [f_0^S]$. Further, at stage $m$ in the construction, we picked some $n = n(F, \mathcal{R}, S, r)$ and $(K_1, \ldots, K_n)^{F, \mathcal{R}, S, r}, (r_1, \ldots, r_n)^{F, \mathcal{R}, S, r}, (\tau_1, \ldots, \tau_{n+1})^{F, \mathcal{R}, S, r}$.
so that

\[ \forall_{M_r \beta}^* F([^f_0^S] ) (\beta) = \tau_1( \delta_f, r_1([f_0], \beta)) \]

\[ \forall_{M_r \beta}^* \forall_{K_1 \alpha_1} r_1([f_0], \beta, \alpha_1) = \tau_2( E^{M_r, K_1}(f_0)(\beta, \alpha_1), r_2([f_0], \beta, \alpha_1) ) \]

\[ \forall_{M_r \beta}^* \forall_{K_1 \alpha_1}^* \forall_{K_2 \alpha_2} r_2([f_0], \beta, \alpha_1, \alpha_2) = \tau_3( E^{M_r, K_1, K_2}(f_0)(\beta, \alpha_1, \alpha_2), r_3([f_0], \beta, \alpha_1, \alpha_2) ) \]

... 

\[ \forall_{M_r \beta}^* \forall_{K_1 \alpha_1}^* \forall_{K_n \alpha_n} r_n([f_0], \beta, \alpha_1, ..., \alpha_n) = \tau_{n+1}( E^{M_r, K_1, ..., K_n}(f_0)(\beta, \alpha_1, ..., \alpha_n) ) \]

A is closed under \( \tau_1, ..., \tau_{n+1} \), so that

\[ \forall_{M_r \beta}^* \forall_{K_1 \alpha_1}^* ... \forall_{K_n \alpha_n} r_n([f_0], \beta, \alpha_1, ..., \alpha_n) \in A \]

which means that

\[ \forall_{M_r \beta}^* \forall_{K_1 \alpha_1}^* ... \forall_{K_{n-1} \alpha_{n-1}} r_n([f_0], \beta, \alpha_1, ..., \alpha_{n-1}) \in A^\uparrow \]

but \( A^\uparrow = A \), so , in fact,

\[ \forall_{M_r \beta}^* \forall_{K_1 \alpha_1}^* ... \forall_{K_{n-1} \alpha_{n-1}} r_n([f_0], \beta, \alpha_1, ..., \alpha_{n-1}) \in A \]

which gives

\[ \forall_{M_r \beta}^* \forall_{K_1 \alpha_1}^* ... \forall_{K_{n-1} \alpha_{n-1}} r_{n-1}([f_0], \beta, \alpha_1, ..., \alpha_{n-1}) \in A \]

We proceed in this way inductively until we have that

\[ \forall_{M_r \beta}^* F([^f_0^S] ) (\beta) \in A \]

as required \( \square \)
2.7. The Strong Partition Property on $\delta_3^1$

As in 2.4, we fix a countable language $L$ containing $\{\epsilon, S, x\} \cup \{\gamma_i\}_{i \in \omega}$ such that $L$ has countably many $n$-ary function symbols for $n \in \omega$.

Let $\{\phi_n\}_{n \in \omega}$ be the left-most branch scale on $p[S_3]$.

The following lemma and theorem can be proven analogously to the proofs given in 2.4.

**Lemma 2.37.** $\{\phi_n\}$ is a $\Pi^1_3$ scale.

**Theorem 2.38.** For all $A \subset \delta_3^1$, there is an $x \in \omega^\omega$ such that $A \in L[S_3, x]$.

Let $(C, \Gamma)$ be full indiscernible pair for $L[S_3, x]$. Then we may code the theory of $L[S_3, x]$ by a real $y$, by putting

$$(\phi, R, S) \in y \iff \forall f : \text{Dom}(<_R) \to C \text{ of type } R \ L[S_3, x] \models \phi([f^S])$$

In this situation, we say "$y$ is a sharp for $L[S_3, x]$".

Conversely, for $y$ a real, we say "$y$ looks like a sharp" if $y$ codes a consistent theory via the above coding and $y \models ZFC + V = L[S, x] + \text{Tree Projection}.$

**Definition 2.39.** Let $\mathcal{R}_1, \ldots, \mathcal{R}_n$ be type-2 trees. Let $S_1, \ldots, S_n$ be such that $S_i$ is a $\mathcal{R}_i$-system. Let $f_1, \ldots, f_n$ be such that $f_i : \text{Dom}(<_R) \to \delta_3^1$ is a function of type $\mathcal{R}_i$. We define $\mathcal{R}^{f_1, \ldots, f_n}$ to be the unique smallest type-2 tree such that there is a function $f : \text{Dom}(<_R) \to \delta_3^1$ and for all $i < n$ there is a subtree $\mathcal{R}^i \subset \mathcal{R}$ such that $[f_i] = [f | \mathcal{R}^i]$. Let $S^i$ be the $\mathcal{R}$-system that comes from $S_i$ and the isomorphism $\mathcal{R}_i \cong \mathcal{R}^i$. Finally, we let $S^{(f_1, S_1), \ldots, (f_n, S_n)} = S^1 \cdot S^2 \cdot \ldots \cdot S^n$, where the concatenation of two $\mathcal{R}$-systems is the $\mathcal{R}$-system that comes from concatenating the component functions of the $\mathcal{R}$-systems. Note, that in this situation, $[f^S] = [f_1^{S_1}] \cdot [f_2^{S_2}] \cdot \ldots \cdot [f_n^{S_n}]$.

For $y$ a real that looks like a sharp, and $\delta < \delta_3^1$, we build the model $\mathcal{M}_y^\delta$ in the following way: the universe of $\mathcal{M}_y^\delta$, $\mathcal{M}_y^\delta$, has elements of the form $< \tau, \mathcal{R}, S, f >$ where $\tau$ is an $\mathcal{L}$-function, $\mathcal{R}$ is a type-2 tree of uniform cofinalities, $S$ is a $\mathcal{R}$-system, and $f : \text{dom}(<_R) \to \delta$ is
a function of type $\mathcal{R}$. For $(\tau_1, \mathcal{R}_1, S_1, f_1), ..., (\tau_n, \mathcal{R}_n, S_n, f_n) \in M^\delta_y$ and $\phi$ an n-ary $\mathcal{L}$-formula,

\[
\mathcal{M}^\delta_y \models \phi((\tau_1, \mathcal{R}_1, S_1, f_1), ..., (\tau_n, \mathcal{R}_n, S_n, f_n)) \iff (\phi', \mathcal{R}^{f_1,...,f_n}, S^{f_1S_1},...,f_nS_n) \in y
\]

where $\phi'([f_1]^S_1] \cdots [f_n^S_n]) \iff \phi([\tau_1([f_1^S_1]), ..., \tau_n([f_n^S_n])])$.

We make some observations about $\mathcal{M}^\delta_y$.

For $x \in \omega^\omega$ and $\mathcal{R}$ a type-2 tree of uniform cofinalities, to say "$x$ codes a function $f_x : dom(<_R) \rightarrow \delta$ of type $\mathcal{R}$" is $\Delta^3_3$ as it can be defined using ordinal quantifiers over $\delta$ and our $\Delta^3_3$ coding of functions from $\omega_\omega$ to $\delta$. In addition, for $<\alpha_1, ..., \alpha_n >$ a sequence of ordinals less than $\delta^3_3$, $\mathcal{R}$ a type-tree of uniform cofinalities, and $x \in \omega^\omega$ that codes $f_x : dom(<_R) \rightarrow \delta$, to say "$[f_x] = <\alpha_1, ..., \alpha_n >$" is also $\Delta^3_3$.

Observe also, that for quadruples $<\tau_1, \mathcal{R}_1, S_1, x_1 >, ..., <\tau_n, \mathcal{R}_n, S_n, x_n >$ and $\psi$ an $\mathcal{L}$-formula, to say "$x_i$ codes a function $f_{x_i} : Dom(<_{\mathcal{R}_i}) \rightarrow \delta$ of type $\mathcal{R}_i$ and $\mathcal{M}^\delta_y \models \psi(<\tau_1, \mathcal{R}_1, S_1, f_{x_1} >, ..., <\tau_n, \mathcal{R}_n, S_n, f_{x_n} >)$" is $\Delta^3_3$ by a similar computation (here we again use that $\Delta^3_3$ is closed under $\forall^\mathcal{L}_r$).

**DEFINITION 2.40.** For $y$ a real that looks like a sharp, $\delta < \delta^3_3$, quadruples $<\tau_1, \mathcal{R}_1, S_1, x_1 >, ..., <\tau_n, \mathcal{R}_n, S_n, x_n >$ and $<\tau_{n+1}, \mathcal{R}_{n+1}, S_{n+1}, \bar{\alpha}_{n+1} >, ..., <\tau_m, \mathcal{R}_m, S_m, \bar{\alpha}_m >$, with $\bar{\alpha}_i$ a sequence of ordinals less than $\sup_{\mu}(j_\mu(\delta))$ of type $\mathcal{R}_i$, and $\psi$ a $\mathcal{L}$ formula, we write

\[
\mathcal{M}^\delta_y \models \psi(<\tau_1, \mathcal{R}_1, S_1, x_1 >, ..., <\tau_n, \mathcal{R}_n, S_n, x_n >; <\tau_{n+1}, \mathcal{R}_{n+1}, S_{n+1}, \bar{\alpha}_{n+1} >, ..., <\tau_m, \mathcal{R}_m, S_m, \bar{\alpha}_m >)
\]

if and only if

\[
\exists x_{n+1}, ..., x_m([f_{x_1}] = \bar{\alpha}_i \land \exists x_{n+1}, ..., x_m([f_{x_1}] = \bar{\alpha}_i \land
\]

\[
\mathcal{M}^\delta_y \models \psi(<\tau_1, \mathcal{R}_1, S_1, x_1 >, ..., <\tau_n, \mathcal{R}_n, S_n, x_n >; <\tau_{n+1}, \mathcal{R}_{n+1}, S_{n+1}, x_{n+1} >, ..., <\tau_m, \mathcal{R}_m, S_m, x_m >)
\]

For $y$ a real that looks like a sharp, $\delta < \delta^3_3$, $\psi$ an $\mathcal{L}$ formula and $<\tau_{n+1}, \mathcal{R}_{n+1}, S_{n+1}, \bar{\alpha}_{n+1} >, ..., <\tau_m, \mathcal{R}_m, S_m, \bar{\alpha}_m >$ as in the previous definition, the set

\[
A = \{ <\tau_1, \mathcal{R}_1, S_1, x_1 >, ..., <\tau_n, \mathcal{R}_n, S_n, x_n > : \}
\]

\[
\mathcal{M}^\delta_y \models \psi(<\tau_1, \mathcal{R}_1, S_1, x_1 >, ..., <\tau_n, \mathcal{R}_n, S_n, x_n >; <\tau_{n+1}, \mathcal{R}_{n+1}, S_{n+1}, \bar{\alpha}_{n+1} >, ..., <\tau_m, \mathcal{R}_m, S_m, \bar{\alpha}_m >)
\]
is $\Delta^1_3$. It is $\Sigma^1_3$ be definition, and can also be computed as $\Pi^1_3$ by $\langle \tau_1, R_1, S_1, x_1 >, ..., < \tau_n, R_n, S_n, x_n > \rangle \in A$ if and only if

$$\forall x_{n+1}, ..., x_m ([f_{x_i}] = \alpha_i \rightarrow M^\delta_y \models \psi(\langle \tau_1, R_1, S_1, x_1 >, ..., < \tau_n, R_n, S_n, x_n >, < \tau_{n+1}, R_{n+1}, S_{n+1}, x_{n+1} >, ..., < \tau_m, R_m, S_m, x_m >)$$

Lastly we note that, while there may be ill founded $M^\delta_y$, every $M^\delta_y$ contains an initial segment of the ordinals, which we denote by $WFP(M^\delta_y)$. We prove the following lemma:

**Lemma 2.41.** Fix $\delta, \beta < \delta^1_3$. Let $A^\alpha_\beta = \{(< \tau, R, S, x, y) : y$ looks like a sharp and $\beta \in WFP(M^\delta_y)$ and $| < \tau, R, S, f_x > |_{M^\delta_y} = \beta \}$. Then $A^\alpha_\beta \in \delta^1_3$.

**Proof.** Fix $\delta < \delta^1_3$. Our proof is by induction on $\beta$.

For $\beta = 0$, $(< \tau, R, S, x, y) \in A^\delta_0$ iff $y$ looks like a sharp and $M^\delta_y \models | < \tau, R, S, x > |_{M^\delta_y} = 0$. (Here, we are using that 0 is definable). By observations preceding the lemma, this is in $\Delta^1_3$.

For $\beta = \alpha + 1$ a successor ordinals,

$(< \tau, R, S, x, y) \in A^\delta_\beta$ iff $y$ looks like a sharp and $\exists < \tau', R', S', x' >$

$(< \tau', R', S', x', y) \in A^\delta_\alpha$ and $M^\delta_y \models | < \tau', R', S', x' > |_{M^\delta_y} = 1 = < \tau, R, S, x >$

So $A^\delta_\beta \in \Sigma^1_3$. Similarly,

$(< \tau, R, S, x, y) \in A^\delta_\beta$ iff $y$ looks like a sharp and $\forall < \tau', R', S', x' >$

$(< \tau', R', S', x', y) \in A^\delta_\alpha \rightarrow M^\delta_y \models | < \tau', R', S', x' > |_{M^\delta_y} = 1 = < \tau, R, x >$

So $A^\delta_\beta \in \Pi^1_3$, hence $\Delta^1_3$.

For $\beta$ a limit ordinal

$(< \tau, R, S, x, y) \in A^\delta_\beta$ iff $y$ looks like a sharp and

$| < \tau, R, S, x > |_{M^\delta_y} \leq \beta$ and $\neg(| < \tau, R, S, x > |_{M^\delta_y}) < \beta$

We show that both of these conjuncts are in $\Delta^1_3$.

$| < \tau, R, S, x > |_{M^\delta_y} \leq \beta \iff \forall \alpha_1, ..., \alpha_n < \sup_\mu(j_\mu(\delta)) \forall R' \forall S' \forall \tau'$
\[(\forall x' \text{ ( } x' \text{ codes a function of type } R \text{ such that } [f_{x'}] = <\alpha_1, ..., \alpha_n>) \text{ and } \]

\[\mathcal{M}_y^\delta \models <\tau', \mathcal{R}', S', f_{x'} > < <\tau, \mathcal{R}, S, x > ) \]

\[\rightarrow \exists \alpha < \beta (<\tau', \mathcal{R}', S', f_{x'} >, y) \in A_\alpha^\delta \]

So is \( \Pi_3^1 \). Similarly,

\[ | <\tau, \mathcal{R}, S, x > |_{\mathcal{M}_y^\delta} \leq \beta \leftrightarrow \forall \bar{\alpha} < sup(\mu(\delta)) \forall \mathcal{R}' \forall S' \forall \tau' \]

\[(\mathcal{M}_y^\delta \models <\tau', \mathcal{R}', S', \bar{\alpha} \geq <\tau, \mathcal{R}, S, x >) \text{ or } \]

\[(\mathcal{M}_y^\delta \models <\tau', \mathcal{R}', S', \bar{\alpha} > < <\tau, \mathcal{R}, S, x > \text{ and } \exists \alpha < \beta < \tau', \mathcal{R}', S', \bar{\alpha} >, y) \in A_\alpha^\delta \]

So is \( \Sigma_3^1 \), hence \( \Delta_3^1 \).

We finish our proof by showing \( \neg(| <\tau, \mathcal{R}, S, x > |_{\mathcal{M}_y^\delta}) \leq \beta \) is \( \Delta_3^1 \). But this is straightforward because \( \{< <\tau, \mathcal{R}, S, x >, y > : | <\tau, \mathcal{R}, S, x > |_{\mathcal{M}_y^\delta} < \beta \} = \bigcup_{\beta < \beta} A_\beta^\delta \), hence is \( \Delta_3^1 \).

\[\square\]

We now provide our coding:

**Theorem 2.42.** There is a function \( \Phi \) with domain \( \omega^\omega \) such that the following properties hold:

(i): \( \forall z \, \Phi(z) \subset \delta_3^1 \times \delta_3^1 \).

(ii): \( \forall f : \delta_3^1 \rightarrow \delta_3^1 \, \exists z \, \Phi(z) = f \).

(iii): \( \forall \delta, \beta < \delta_3^1, \text{ the set } R_{\delta, \beta} \in \Delta_3^1 \text{ where } \]

\[z \in R_{\delta, \beta} \leftrightarrow \exists! \gamma \, \Phi(z)(\delta, \gamma) \land \gamma = \beta \]

(iv): \( \forall \delta < \delta_3^1, \text{ if } A \subset R_\delta = \bigcup_\beta R_{\delta, \beta} \text{ and } A \in \Sigma_3^1, \text{ then } \)

\[\exists \beta_0 < \delta_3^1 \forall z \in A \exists \beta < \beta_0 \, z \in R_{\delta, \beta} \]
Proof. We put
\[(\delta, \beta) \in \Phi(\tau, \mathcal{R}, S, x, y) \iff x \text{ codes a function of type } \mathcal{R} \text{ into } \delta + 1,\]
and \[M^{\delta+1}_y \models (\delta, \beta) \in \tau, \mathcal{R}, S, f_x >\]

Property (i) is immediate, while property (ii) follows from Lemma 2.38.

We now show property (iii) holds. Fix \(\delta, \beta < \delta^1_3\). The set \(B = \{\tau, \mathcal{R}, S, f_x > \mid M^{\delta+1}_y \models \tau, \mathcal{R}, S, f_x >\}\) is \(\Delta^1_3\) because
\[\langle \tau, \mathcal{R}, S, x, y >, y > \in B \iff \exists \tau_1, \mathcal{R}_1, S_1, z_1 >, \tau_2, \mathcal{R}_2, S_2, z_2 > \text{ such that }\]
\[\langle \tau_1, \mathcal{R}_1, S_1, z_1 >, y > \in A^{\delta+1}_\delta \text{ and } \langle \tau_2, \mathcal{R}_2, S_2, z_2 >, y > \in A^{\delta+1}_\beta \text{ and }\]
\[M^{\delta+1}_y \models \langle \tau_1, \mathcal{R}_1, S_1, z_1 >, \tau_2, \mathcal{R}_2, S_2, z_2 > \rangle \in \tau, \mathcal{R}, S, x >\]

So \(B\) is \(\Sigma^1_3\), a similar computation shows that \(B\) is \(\Pi^1_3\).

To finish (iii), observe that \(\tau, \mathcal{R}, S, x, y > \in R^{\delta, \beta}\) if and only if
\[\langle \tau, \mathcal{R}, S, x, y > \in B \text{ and } \forall \bar{\alpha} < sup_\mu(j_\mu(\delta)) \forall \tau' \forall \mathcal{R}' \forall S' \]
\[M^{\delta+1}_y \models \tau', \mathcal{R}', S', \bar{\alpha} > \in \tau, \mathcal{R}, S, x > \rightarrow \langle \tau', \mathcal{R}', S', \bar{\alpha} > \models M^{\delta+1}_y = \beta\]

For (iv), fix \(\delta < \delta^1_3\) and \(A \subset R^{\delta}\) such that \(A \in \Sigma^1_3\). We define a prewellordering \(<\) with \(dom(<) = A\) by
\[z_1 < z_2 \iff z_1, z_2 \in A \land \exists \beta(z_1 \in R^{\delta, \beta} \land \forall \beta' \leq \beta z_2 \notin R^{\delta, \beta'})\]

While \(<\) is not a priori \(\Sigma^1_3\), we will show that it is. It then follows from \(\Sigma^1_3\) boundedness that
\[\exists \beta < \delta^1_3 \forall z \in A \exists \beta' < \beta(z \in R^{\delta, \beta'})\]

We now show that \(<\) is \(\Sigma^1_3\). The primary difficulty is that, for \(x, y > \in R^{\delta, \beta}\), if \(w\) is such that
\[M^{\delta+1}_y \models (\delta, z) \in x\]
then it is not necessarily the case that \(|w|M^{\delta+1}_y = \beta\), because it is possible that \(w \notin WFP(M^{\delta+1}_y)\). However, we are able to avoid this difficulty by computing \(<\), in the following way:
For $< x_1, y_1 >, < x_2, y_2 > \in A$,

$$< x_1, y_2 > < < x_2, y_2 > \iff \exists w_1(M^{\delta+1}_{y_1} \models (\delta, w_1) \in x \land \exists w_2(\text{ON}^{M^{\delta+1}_{y_1}} \upharpoonright w_1 \cong \text{ON}^{M^{\delta+1}_{y_2}} \upharpoonright w_2) \land M^{\delta+1}_{y_2} \models \forall \beta < w_2 \neg (\delta, \beta) \in x_2)$$

Where "$\text{ON}^{M^{\delta+1}_{y_1}} \upharpoonright w_1 \cong \text{ON}^{M^{\delta+1}_{y_2}} \upharpoonright w_2$" means that the set of ordinals of $M^{\delta+1}_{y_1}$ below $w_1$ is order isomorphic to the set of ordinals of $M^{\delta+1}_{y_2}$ below $w_2$. Note that, if $w_1 \notin \text{WFP}(M^{\delta+1}_{y_1})$ and $\text{ON}^{M^{\delta+1}_{y_1}} \upharpoonright w_1 \cong \text{ON}^{M^{\delta+1}_{y_2}} \upharpoonright w_2$, then $w_2 \notin \text{WFP}(M^{\delta+1}_{y_2})$. Thus, if $w_1 \notin \text{WFP}(M^{\delta+1}_{y_1})$, then it is impossible for the last line of the equation to hold (because $(x_2, y_2) \in R_\delta$).

To finish our computation of $<$, we show that "$\text{ON}^{M^{\delta+1}_{y_1}} \upharpoonright w_1 \cong \text{ON}^{M^{\delta+1}_{y_2}} \upharpoonright w_2$" is a $\Sigma^1_3$ computation. To see this, observe that "$\text{ON}^{M^{\delta+1}_{y_1}} \upharpoonright w_1 \cong \text{ON}^{M^{\delta+1}_{y_2}} \upharpoonright w_2$" if and only if $\exists z$ coding a function $f_z$ with $\text{dom}(f_z)$ and $\text{range}(f_z)$ consisting of tuples of the form $< \tau, R, S, \bar{\alpha} >$ with $\bar{\alpha} < \sup(\mu(\delta + 1))$ such that $f_z$ represents a bijection between $\text{ON}^{M^{\delta+1}_{y_1}} \upharpoonright w_1$ and $\text{ON}^{M^{\delta+1}_{y_2}} \upharpoonright w_2$. It is straightforward to see that this is a $\Sigma^1_3$ computation.

\[\square\]
BIBLIOGRAPHY


