Study of Collective Effect in Ionization Cooling

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Abstract

As a charged particle passes through a non-gaseous medium, it polarizes the medium and induces wake fields behind it. Same thing happens in ionization cooling. The interaction with wake fields perturbs the stopping power of beam particles. The perturbation strongly depends on the densities of both the incident beam and the medium. To understand this collective effect, detailed studies have been carried out. Both analytic and simulation results are obtained and compared.

INTRODUCTION

The study of the physics of a charged particle passing through a non-gaseous medium is of long history [1, 2, 3]. For a single particle, if its momentum is high enough, it will lose energy through both ionization process and density effect. The latter has been systematically studied. For a beam consisting of a large number of particles, the interaction among the beam particles should also be taken into account in order to describe the process correctly.

Especially, density effect is introduced by the polarization of the medium. The electric fields from the polarized medium molecules generate wake fields behind the incident particle, which perturbs the motion of the beam particles following. If the particle density of the beam is high enough, the wake will enhance the stopping power of the beam particles significantly.

In this article, we derive the expressions for the wake electric field introduced by a single incident charged particle and its perturbation on the stopping power. This is extended to a two-particle system and a multi-particle system with various distributions. The comparison with simulations is next demonstrated. Finally, the damping mechanism on the wake is discussed, and its effect on the stopping power enhancement is found to be important.

WAKE ELECTRIC FIELD

First, let us focus on a single particle of charge $ze$ moving with velocity $v$ in the $z$ direction. The particle is at longitudinal position $z = z_1$ at time $t = 0$. Cylindrical coordinates are used with $\vec{\rho}$ denoting the transverse directions. The scalar potential generated by both the incident particle and polarized medium in the Coulomb gauge is given by

$$\phi(\vec{r}, t) = \frac{ze}{\pi e v} \int d\omega \int \frac{kdk J_0(k\rho)}{\kappa^2 + \omega^2 / v^2} e^{i\vec{k}(z - z_1 - vt)} e^{i(k^2, \omega)}$$

(1)

where the wave number vector is denoted by $\vec{k} = (\vec{k}, k_z)$ and the frequency by $\omega$. In above, the integration over $k_z$ and $\vec{\rho}$ have already been carried out. The polarization of the medium is described by the dielectric constant, which in a dispersive medium takes the form

$$\varepsilon(k^2, \omega) = 1 - \frac{\omega^2}{\omega_p^2} \sum_j \frac{f_j}{\omega^2 - \omega_j^2 + i\omega \Gamma_j}.$$  

(2)

where $f_j$ is the fraction of bound electrons that oscillates with the bound frequency $\omega_j$ and damping rate $\Gamma_j$ with $\sum_j f_j = 1$. In above, $\omega_p = \sqrt{4\pi n_e e^2 / m_e}$ is the plasma frequency, where $m_e$ is the electron mass and $n_e$ the electron density. We make the assertion that $\omega_p$ is much larger than the bound frequencies and damping rates. Then $\Gamma_j$ can be replaced by $2\epsilon$ with $\epsilon$ being infinitesimal, leading to

$$\frac{1}{\varepsilon} = (\omega + i\epsilon)^2 - \omega_p^2 = (\omega - \omega_p + i\epsilon)(\omega + \omega_p + i\epsilon).$$

(3)

Contour integration over $\omega$ can now be performed giving

$$\phi(\vec{r}, t) = e \int \kappa^2 \frac{d\kappa J_0(\kappa \rho)}{\kappa^2 + \omega_p^2 / v^2} e^{-\kappa(z - z_1 - vt)}$$

$$+ \frac{2e\omega_p}{v^2} \int \kappa^2 \frac{d\kappa J_0(\kappa \rho)}{\kappa^2 + \omega_p^2 / v^2} \sin \frac{\omega_p}{v} (z - z_1 - vt) \theta(z_1 + vt - z).$$

(4)

The limits of integration are from $\kappa = 0$ to $2$ \n
$$\kappa = \frac{\omega_p}{v} \sqrt{x_m^2 - 1} \quad \text{with} \quad x_m = \frac{2\gamma m_e v^2}{\hbar \omega_p},$$

(5)

which corresponds to the maximal momentum transfer in a collision. In above, $\gamma = \sqrt{1 - v^2 / c^2}$, $c$ is the velocity of light, and $\hbar$ is the Planck constant. The second term in Eq. (4) is the potential coming from the polarization of the medium, and the first term is the medium-modified self-field of the incident charged particle. The longitudinal and transverse electric fields derived from the second term vanish in front of the particle and are therefore the wake fields. Behind the particle, they take the form:

$$E_z^w(\vec{r}, t) = -\frac{2e\omega_p^2}{v^2} \int \kappa^2 \frac{d\kappa J_0(\kappa \rho)}{\kappa^2 + \omega_p^2 / v^2} \cos \frac{\omega_p(z - z_1 - vt)}{v},$$

$$E_{\rho}^w(\vec{r}, t) = \frac{2e\omega_p^2}{v^2} \int \kappa^2 \frac{d\kappa J_0(\kappa \rho)}{\kappa^2 + \omega_p^2 / v^2} \sin \frac{\omega_p(z - z_1 - vt)}{v}.$$  

(6)

Evaluating at the particle location ($z = z_1 + vt, \rho = 0$), we obtain the longitudinal field on axis,

$$E_z^w = \frac{2e\omega_p^2}{v^2} \ln x_m.$$  

(7)

1We believe the bound frequencies are one order of magnitude smaller than $\omega_p$ in liquid hydrogen.

2The integration of $k = |\vec{k}|$ has the lower limit $k = \omega_p / v$ and the upper limit $k = 2\gamma m_e v / \hbar$. 

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The corresponding energy loss per unit time or stopping power is
\[
\frac{dW}{dt} = ze\nu E_z^w = -\frac{2(ze)^2\omega_p^2}{v} \ln \rho_m, \quad (8)
\]

Behind the particle, the electric wake can be very well approximated by extending the upper limit of the \( \kappa \) integrations to infinity, resulting in
\[
E_z^w = \frac{2ze\omega_p^2}{v^2} K_0 \left( \frac{\omega_p}{\nu} \right) \cos \left[ \left( \frac{z - z_1}{\nu} - t \right) \omega_p \right],
\]
\[
E_\nu^w = \frac{2ze\omega_p^2}{v^2} K_1 \left( \frac{\omega_p}{\nu} \right) \sin \left[ \left( \frac{z - z_1}{\nu} - t \right) \omega_p \right], \quad (9)
\]
with \( K_{0,1} \) the modified Bessel functions of the second kind.

The vector potential contributes only to the medium-modified self-field when bound frequencies are neglected. The electric field derived from it consists of two parts, one part cancels the medium-modified stationary self-field from the scalar potential in Eq. (4), while the other represents the modified self-field when bound frequencies are neglected. The system. The particles are denoted by 1 and 2, respectively, with \( Z_1 \) and \( Z_2 \). In the absence of the medium (\( \omega_p = 0 \)), it reduces to the familiar pan-cake self-field,
\[
E_z^s = \frac{e\gamma Z}{(\rho^2 + \gamma^2 Z^2)^{3/2}}, \quad E_\nu^s = \frac{e\gamma \rho}{(\rho^2 + \gamma^2 Z^2)^{3/2}}, \quad (12)
\]
where \( Z = z - z_1 - vt \). In the presence of the medium, the self-field decays very much faster with respect to \( Z \). For a bunch with longitudinal and transverse radii \( \gg v/\omega_p \), the self-field has almost no influence compared with the wake fields. Therefore we ignore it in the rest of our discussions.

**TWO-PARTICLE SYSTEM**

Now let us discuss the stopping power of a two-particle system. The particles are denoted by 1 and 2, respectively, with the charge density
\[
\rho(\vec{r}, t) = ze \left[ \delta(\vec{r} - \vec{r}_1 - \vec{v}_1 t) + \delta(\vec{r} - \vec{r}_2 - \vec{v}_2 t) \right]. \quad (13)
\]

The electric field at \( (\vec{r}, t) \) is
\[
\vec{E}(\vec{r}, t) = \frac{-i2ze}{4\pi^2} \int d^3k \int d\omega \frac{\vec{k}e^{-i\omega t}}{\epsilon k^2} \times
\times \left[ e^{ik\cdot(\vec{r} - \vec{r}_1)} \delta(\vec{k} \cdot \vec{v}_1 - \omega) + e^{ik\cdot(\vec{r} - \vec{r}_2)} \delta(\vec{k} \cdot \vec{v}_2 - \omega) \right]. \quad (14)
\]

The energy gained per unit time by the two particles are
\[
\frac{dW_{1,2}}{dt} = -\frac{i2(ze)^2}{4\pi^2} \int d^3k \frac{\vec{k} \cdot \vec{v}_i}{k^2 \epsilon(k^2, \vec{k} \cdot \vec{v}_{1,2})} \times
\times \left[ 1 + e^{\pm i\vec{k} \cdot (\vec{r}_i - \vec{r}_2) \pm i(k \cdot (\vec{v}_1 - \vec{v}_2) t)} \right]. \quad (15)
\]

![Figure 1: Stopping power enhancement due to collective wake effect on a two-particle system as a function of \( R = r_0\omega_p/\nu \), where \( r \) is the separation between the two particles.](image)

For the special case where \( \vec{v}_1 = \vec{v}_2 = \vec{v} \), we have the average energy loss per particle per unit time or stopping power
\[
\frac{dW}{dt} = -\frac{i(ze)^2}{2\pi^2} \int d^3k \frac{\vec{k} \cdot \vec{v}}{k^2 \epsilon(k^2, \vec{k} \cdot \vec{v})} \left[ 1 + \cos(\vec{k} \cdot \vec{r}) \right],
\]
where \( \vec{r} = \vec{r}_1 - \vec{r}_2 \). Here, only the imaginary part of \( 1/\epsilon \) contributes. After averaging over all orientations of this two-particle system, we arrive at
\[
\left\langle \frac{dW}{dt} \right\rangle_{\text{angles}} = -\frac{(ze)^2\omega_p^2}{\nu} \ln \rho_m \left[ 1 + G(R) \right], \quad (17)
\]
where the correlation function or stopping power enhancement is defined as
\[
G(R) = \frac{-\sin R \rho_m}{R \rho_m} + \frac{\sin R}{R} - Ci(R \rho_m) + Ci(R), \quad (18)
\]
and \( Ci(x) = -\int_x^\infty dy \cos y/y \) is the cosine integral. Here, the distance of separation of the two particles, \( R = r/a_i \), has been normalized to \( a_i = v/\omega_p \) or the interaction length. The correlation function [4] as depicted in Fig. 1 shows that \( G(R) \approx 1 \) at \( R = 1 \) and decreases rapidly when \( R \gg 1 \) with oscillation period equal to the plasma wavelength \( \lambda_p = 2\pi a_i \). It is interesting to point out that the averaging over all orientations cancels out all self-field contribution.

**VARIous BEAM DISTRIBUTIONS**

In this section we will discuss the stopping power enhancement for a particle in the center of a bunch as a result of the polarization wake. Various bunch distributions are used and their importance analyzed.

**Uniformly distributed sphere**

Let us start with a uniformly distributed spherical \( N \)-particle bunch of radius \( r_0 \). The extra collective stopping power \( G_t \) received by the particle at bunch center is obtained by integrating the two-particle correlation function \( G(R) \) of Eq. (18) over all the particles in the bunch. We get
\[ G_t = \frac{3N_b}{R_0^2} \int_0^{R_0} R^2G(R)dR = \frac{N_b}{\ln x_m} \frac{f(R_0) - f(R_0x_m)}{x_m}, \]  

with \( R_0 = \frac{r_0\omega_p}{v}, \) the reduced bunch radius and

\[ f(u) = \left( \frac{1}{u^4} + \frac{1}{u} \right) \sin u - \frac{\cos u}{u^2} - \text{Ci}(u). \]

Since \( x_m \) is usually a very big number, the above can be readily approximated as

\[ G_t \approx \frac{3N_b\sin R_0}{R_0^2\ln x_m}. \]

As an example, consider a \( \gamma = 2.2 \) bunch containing \( N_b = 1 \times 10^{12} \) muons going through liquid hydrogen of density \( \rho_{\mu_2} = 0.07099 \text{ g/cm}^{-3}. \) The electron density is \( n_e = \rho_{\mu_2}N_A = 4.275 \times 10^{28} \text{ m}^{-3}, \) where \( N_A \) is the Avogadro’s number. The plasma frequency is therefore \( \omega_p = 1.166 \times 10^{16} \text{ s}^{-1} \) leading to \( x_m = 2.325 \times 10^6. \) If bunch is a uniformly distributed sphere of radius \( r_0 = 1 \text{ mm}, \) \( R_0 = 4.369 \times 10^4 \text{ and the envelope of } G_t \text{ is 0.0029. However, since the bunch edge can never be made sharper than the interaction length } a_j = 2.289 \times 10^{-8} \text{ m, the rapid oscillation of } G_t \text{ with } R_0 \text{ with period } \lambda_0 \text{ implies the enhancement of stopping power from correlation is essentially zero.}

**Cylindrical bunch**

Let us consider next distributions having the separable form \( f(z, \rho) = f_z(z)f_\rho(\rho). \) One example is a bunch with uniform distribution in the transverse direction, but is tapered at both end longitudinally, or

\[ f_z(z) = A_n \left(1 - \frac{z^2}{2z^2}\right)^n, \quad f_\rho(\rho) = \frac{2\pi\rho}{\pi\rho^2}, \]

where \( A_n = \Gamma(n + \frac{2}{3})/\sqrt{\pi}\Gamma(n + 1) \) for any \( n > -1, \) \( \hat{\rho} \) is the transverse radius of the bunch, and \( \pm \hat{z} \) is the longitudinal edges of the bunch. Notice that we can no longer apply the expression of the all-orientation-averaged correlation function \( G(R) \) of Eq. (18), because the distribution is now different in the longitudinal and transverse directions. Instead, we start from Eq. (9) to compute \( G_t, \) the collective stopping power enhancement for a particle at the center of the bunch, by the integration,

\[ G_t \ln x_m = N_b \int_0^{\hat{z}} dz f_z(z) \cos \left( \frac{z}{a_j} \right) \int_0^{\hat{\rho}} d\rho f_\rho(\rho)K_0 \left( \frac{\rho}{a_j} \right), \]

and obtain easily

\[ G_t \ln x_m = \frac{\sqrt{\pi}A_n}{(\omega_p\hat{\rho}/v)^2} \left( \frac{2v}{\omega_p\hat{z}} \right)^{n+\frac{1}{2}} \text{J}_{n+\frac{1}{2}} \left( \frac{\omega_p\hat{z}}{v} \right). \]

We see clearly that there is an oscillation in the Bessel function \( \text{J}_{n+\frac{1}{2}} \) depending very sensitively on the half bunch length \( \hat{z}. \) In fact, when \( n \) is an integer, the Bessel function reduces to sine and cosine with period \( \lambda_p, \) and the result will be similar to that of the uniform distribution in a sphere discussed earlier. We learn from Eq. (24) that the longitudinal and transverse beam sizes behave very differently, and it is the longitudinal that introduces the oscillations. To avoid oscillations, we need to go to a distribution without hard longitudinal boundaries.

**Lorentzian distribution**

Let us keep the transverse distribution to be finite and uniform, but let the longitudinal distribution be

\[ f_z(z) = \frac{z_1}{\pi z^2 + z_1^2}, \]

where \( z_1 \) is the half longitudinal length at half maximum. The correlation stopping power enhancement \( G_t \) for a particle at the bunch center is found to be

\[ G_t \ln x_m = N_b e^{-z_1/a_j}/(\rho/a_j)^2. \]

If \( \hat{\rho} = z_1 = 1 \text{ mm}, \) \( \hat{\rho}/a_j = z_1/a_j = 4.37 \times 10^4. \) Then the stopping-power enhancement is \( G_t = 45.3 \times 10^{-19000} \) which is essentially zero. In order to have a more reasonable effect at the bunch intensity of \( 1 \times 10^{12}, \) we require \( \hat{\rho}/a_j = z_1/a_j = 20. \) Then \( G_t = 0.417. \) If \( \hat{\rho}/a_j = z_1/a_j = 25, \) \( G_t = 0.0018 \) instead. However, one must remember that at \( a_j = v/\omega_p = 2.289 \times 10^{-8} \text{ m, these examples correspond to } z_1 = 4.6 \times 10^{-7} \text{ and } 5.7 \times 10^{-7} \text{ m, or bunch lengths of sub-micron sizes. At this moment, the most aggressive cooling scheme proposed by Neuffer [5] is to have an eventual bunch length of 30 cm and transverse radius } r_0 = 50 \mu m, \) or a maximum muon density of \( 10^{20} \text{ m}^{-3}. \)

**Tri-Gaussian distribution**

Let

\[ f_z(z) = \frac{e^{-z^2/(2\sigma_z^2)}}{\sqrt{2\pi\sigma_z}} \quad \text{and} \quad f_\rho(\rho) = \frac{e^{-\rho^2/(2\sigma_\rho^2)}}{\sigma_\rho}. \]

The stopping power enhancement can be integrated readily first to the Whittaker’s function and then to the exponential function. However, when \( \sigma_\rho/a_j \gg 1, \) it can be neatly reduced to

\[ G_t \ln x_m = \frac{N_b e^{-z_1/a_j^2/2}}{(\sigma_\rho/a_j)^2}, \]

which is much smaller than the situation of the Lorentzian distribution. Here, we again see that the transverse and longitudinal behave very differently. Although both are Gaussian distributed, the longitudinal decay of the collective stopping power enhancement is Gaussian, while the transverse decay is \( (a_j/\sigma_\rho)^2, \) which is very much milder.

**Exponential distribution**

In order to achieve a larger stopping power enhancement, the distribution must roll off very rapidly from the center of the bunch. A possible distribution is the exponential

\[ f_z(z) = \frac{e^{-|z|/z_1}}{2z_1} \quad \text{and} \quad f_\rho(\rho) = \frac{e^{-\rho/\rho_1}}{\rho_1}. \]
When \( z_1/a_i \gg 1 \) and \( \rho_1/a_i \gg 1 \), \( G_t \) is given by

\[
G_t \ln x_m \approx \frac{\pi N_b}{8(\rho_1/a_i)(z_1/a_i)^2} \tag{30}
\]

Or \( G_t = 3.81 \times 10^{-4} \) at \( \rho_1 = z_1 = 1 \) mm. If either the bunch sizes are each reduced 5 times or the bunch intensity is increased by a factor of 125, the collective effect enhancement will become 4.8\%, and the effect will become significant.

If we apply Neuffer’s scheme [5] again and substitute transverse bunch radius \( \rho_1 = 50 \) \( \mu \)m and bunch length \( z_1 = 30 \) cm, we get a stopping power enhancement of \( G_t \approx 8.47 \times 10^{-8} \). Here, again we notice that the longitudinal bunch length is the most important factor that determines the enhancement. If the bunch length can be further reduced to 1 mm and the bunch intensity further increased, the effect can become meaningful.

**COMPARISON WITH SIMULATION**

OOPICPro [6] developed by Tech-X Corporation is able to simulate a charged particle beam passing through matter. We simulate a \( \gamma = 2.2 \) tri-Gaussian muon bunch with rms radii 1 mm traveling through a plasma medium of electron density \( n_e = 4.28 \times 10^{18} \) m\(^{-3}\) (which is very much less than that in liquid hydrogen). The peak particle beam density is \( 5 \times 10^{19} \) m\(^{-3}\), corresponding to total bunch intensity \( N_b = 0.787 \times 10^{12} \). The longitudinal wake is shown in Fig. 2. The same wake can be computed by integrating our derived wake in Eq. (9) over all the particles in the bunch in a fake liquid hydrogen medium of the same low density. The field patterns in both the longitudinal and transverse directions agree with the simulation results. For example, the oscillation wavelength is \( \lambda_p = 1.43 \) cm corresponding to the plasma frequency of \( \omega_p = 1.17 \times 10^{11} \) s\(^{-1}\). However, OOPICPro gives the peak longitudinal wake electric field \( 1.02 \times 10^8 \) V/m, while the computed one in Fig. 3 gives \( 4 \times 10^8 \) V/m. The discrepancy may come from the fact that OOPICPro treats the medium as a fully ionized plasma while we treat the medium as a liquid with polarized molecules. This difference will be examined next.

**RELAXATION AND DAMPING**

We need to address the question of relaxation or damping of the plasma oscillation to determine whether the oscillatory wake can be established and sustained.

**Cold Plasma**

In a fully ionized plasma, electrons are free to move around as a thermal gas. In the presence of the incident muon beam, the electrons are driven into oscillations about the background ions with plasma frequency. At the same time these electrons collide with the ions. If collision takes place within one period of plasma oscillation, the plasma oscillation will be perturbed. Thus collision with ions serves as a damping mechanism. The collision frequency of an electron with the ionic background is given by [7] \( \nu_c = 2.9 \times 10^{-6} n_i T_c^{3/2} \ln \Lambda \) s\(^{-1}\), where \( n_i \) is the ionic density in cm\(^{-3}\), \( \ln \Lambda \approx 10 \) is the cutoff logarithm, and \( T_c \) is the thermal temperature of the electrons in eV. For the above OOPICPro simulation, we substitute \( n_i = 4.28 \times 10^{18} \) m\(^{-3}\) and \( T_c = 1.72 \times 10^{-3} \) eV (corresponding to 20\(^\circ\)K) to obtain \( \nu_c = 1.74 \times 10^{12} \) s\(^{-1}\), which is comparable to the plasma frequency of \( \omega_p = 1.17 \times 10^{11} \) s\(^{-1}\). So the wake will be heavily perturbed. We do see some damping of the simulated wake in Fig. 2, but not as heavy as estimated. The discrepancy may come from the relatively higher temperature of the ionized electrons, which may not even be in thermal equilibrium. Since the collision frequency increases as \( n_i \) while the plasma frequency increases with \( \sqrt{n_i} \), the damping of plasma oscillation will become more severe as the plasma density increases. Such a simulation has been carried out using OOPICPro with the plasma density increased by 10\(^3\)-fold to \( n_i = 4.28 \times 10^{22} \) m\(^{-3}\) while all other parameters remain unchanged. The result in Fig. 4 shows that the on-axis longitudinal wake is damped almost immediately.
\( \frac{dW}{dx} = 1 \) and plasma density has been increased to plasma. All parameters are the same as in Fig. 2, except that the \( \sigma \) section for the hydrogen molecule in the hard-ball model is degrees of freedom considered. The typical cross section of the H molecule in the wake oscillates sinusoidally with period \( \lambda \sim v/\omega_z \). The damping of the wake also plays an important role in the wake field. Without any damping consideration, the wake oscillates sinusoidally with period \( \lambda_p = 2\pi a_j \). Since the average separation of the incident beam particles is usually much larger than the interaction length, the wake field perturbation on stopping power is negligibly small. Damping comes from two sources: one is the collision rate between absorber molecules, which is slow and insignificant, the other is the damping rates of the bound frequencies of absorber electrons. Under certain circumstances, a shorter damped wake enhances collective perturbation.

The model used in the analysis employs the dielectric constant \( \varepsilon \) in the form of Eq. (3) where bound frequencies are considered small and neglected. Further analysis should take into account of the contribution of bound frequencies to the wake and their effects on the cooling enhancement should be fully investigated.

**REFERENCES**

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