

# Exact Relativistic Ideal Hydrodynamical Solutions in (1+3)D with Longitudinal and Transverse Flows

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A new method for solving relativistic ideal hydrodynamics in (1+3)D is developed. Longitudinal and transverse radial flows are explicitly embedded into the ansatz for velocity field and the hydrodynamic equations are reduced to a single equation for the transverse velocity field only, which is analytically more tractable as compared with the full hydrodynamic equations. As an application we use the method to find analytically all possible solutions whose transverse velocity fields have power dependence on proper time and transverse radius. Possible application to the Relativistic Heavy Ion Collisions and possible generalizations of the method are discussed.

PACS numbers: 47.75.+f, 25.75.-q, 24.10.Nz

## I. INTRODUCTION

Relativistic hydrodynamics has wide applications in a variety of physical phenomena, ranging from the largest scales such as in cosmology and astrophysics[1] to the smallest scales such as in the relativistic nuclear collisions[2, 3]. For an introduction on the general formalism, see e.g. [4–7].

Recently there has been a remarkably successful application of Relativistic Ideal Hydrodynamics (RIHD) to the description of the space-time evolution of the hot dense QCD matter created in the Relativistic Heavy Ion Collider (RHIC) experiments. In the collisions of two relativistically moving heavy nuclei, a lot of energy is deposited in a small volume which soon creates an equilibrated system of high energy density with special initial geometry: extremely thin in the beam direction  $\hat{z}$  while in the transverse plane  $\hat{x} - \hat{y}$  it is of the size of the nuclei. The space-time evolution at RHIC is characterized by fast longitudinal expansion (longitudinal flow) and strong transverse expansion (radial and elliptic flow). In non-central collisions the created matter on the transverse plane  $\hat{x} - \hat{y}$  is initially anisotropic: such initial spatial anisotropy leads to different pressure gradients and thus different accelerations of the flow along different azimuthal directions. The resulting anisotropic transverse flow velocity eventually translates into the anisotropic azimuthal distribution of the final particle yield which is represented by the experimental observable called elliptic flow  $v_2$  — one of the milestone measurements at RHIC [8]. The RIHD model calculations[9–11] (and more recently its extension to include viscous corrections[12]), performed with realistic initial conditions and Equation of State (E.o.S) for RHIC, are able to reproduce the elliptic flow data at low-to-intermediate transverse momenta for almost all particle species and for various centralities,

beam energies and colliding nuclei. These achievements of RIHD have been the basis for the RHIC discovery that the matter being created is a strongly-coupled nearly-perfect fluid[13–16] with extremely short mean free path. It has been suggested [17–21] that the microscopic origin could be due to the strong scattering via Lorentz force between the electric and magnetic degrees of freedom co-existing in the created matter, with the magnetic ones ultimately connected with the mechanism of QCD de-confinement transition.

The great success of RIHD at RHIC has also inspired considerable interest in the formal aspects of relativistic hydrodynamics, particularly in analytical solutions of the RIHD equations with an emphasis on possible application to RHIC, see e.g. [22–29]. The idea to use exact simple RIHD solutions to describe the multi-particle production in high energy collisions dated back to Landau and Khalatnikov [30]. An important solution came from Hwa and Bjorken’s works[31, 32], i.e. the rapidity boost invariant (1+1)D solution which is widely used to describe the longitudinal expansion at RHIC. Many of the above mentioned recent works [23–26] concentrate on finding (1+1)D solutions that give an alternative description of the longitudinal expansion and a more realistic (non-boost-invariant) multiplicity distribution over rapidity.

Despite the progress in solving RIHD in (1+1)D, it is quite difficult to solve them in higher dimensions. To develop methods and find solutions in a realistic (1+3)D setting with potential application for RHIC remains an attracting but demanding task. In this work, we will develop a new method to find solutions in (1+3)D with both longitudinal and transverse flows. In Section III, we will show how the method can reduce the hydrodynamics equations to a single constraint equation for the transverse velocity field only. Using the derived equation, we will find all solutions with power-law dependence on proper time and transverse radius in Section IV. The physical relevance of our results to RHIC and possible generalizations will be discussed in Section V. We will also include a brief introduction of RIHD in Section II

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and an illustration of the method in (1+1)D in the Appendices B and C.

## II. REVIEW OF RELATIVISTIC IDEAL HYDRODYNAMICS IN (1+3)D

The hydrodynamics equations in general are simply the conservation laws for energy and momenta, i.e.

$$T^{mn}{}_{;n} = 0 \quad , \quad (1)$$

with  $m, n$  running over  $3 + 1$  space-time indices. Following usual convention (in e.g. [4, 5]), the subscript “;  $n$ ” denotes the covariant derivative  $D_n$  while a subscript “,  $n$ ” is for ordinary derivative  $\partial_n$ . Below we will introduce curved coordinates in order to simplify the hydrodynamics equations. Therefore, we use the general form for the hydrodynamics equations which involves covariant derivatives, see e.g. [4, 5]. Throughout this paper we discuss only hydrodynamics without any conserved charge, leaving the situation with conserved currents for further investigation.

For relativistic ideal hydrodynamics, the stress tensor is given by

$$T^{mn} = (\epsilon + p) u^m u^n - p g^{mn} \quad , \quad (2)$$

with  $\epsilon, p$  the energy density and pressure defined in the flowing matter’s local rest frame (L.R.F) which by definition are Lorentz scalars. The flow field  $u^m(x)$  is constrained by  $u^m \cdot u_m = 1$ . In the usual  $(t, \vec{x})$  coordinates one can express  $u^m(x)$  as  $\gamma(1, \vec{v})$  with  $\gamma = 1/\sqrt{1 - \vec{v}^2}$  and  $\vec{v} = d\vec{x}/dt$ .

We further need to specify an equation of state (E.o.S) relating the energy density  $\epsilon$  and the pressure  $p$  of the underlying fluid. Here we employ a simple, linear E.o.S, which is typically used in analytic studies of RIHD [23–26, 31, 32]:

$$p = \nu(\epsilon + p) \quad . \quad (3)$$

The above means  $\epsilon = \frac{1-\nu}{\nu} p$  implying a speed of sound  $c_s = \sqrt{\frac{\partial p}{\partial \epsilon}} = \sqrt{\frac{\nu}{1-\nu}}$ , and, in order to assure  $c_s \leq 1$ , we require  $0 < \nu \leq 1/2$ . We note that a solution obtained with the above E.o.S (3) remains valid in the presence of a nonzero bag constant  $B$ : in such case, the R.H.S. of Eq.(3) i.e.  $\epsilon + p$  remains unchanged with  $p \rightarrow p - B$  and  $\epsilon \rightarrow \epsilon + B$ , and thus Eq.(3) is modified only by adding a constant to its L.H.S.

The hydrodynamics equations together with the E.o.S thus form a complete set of 5 equations for the 5 field variables:  $\epsilon(x), p(x)$  and the three independent components of  $u^m(x)$ .

### A. Hydro Equations in Curved Coordinates

When formulating hydrodynamics for application to e.g. the relativistic heavy ion collisions, it is often useful

to use alternative coordinate systems which are curved. For our purpose of studying the (1+3)D solutions with longitudinal and transverse flow, we will use a coordinate system of  $(\tau, \eta, \rho, \phi)$ : i.e. the proper time, the spatial (longitudinal) rapidity, the transverse radius and the azimuthal angle. They are related to the usual  $(t, x, y, z)$  in the following way:

$$\begin{aligned} \tau &= \sqrt{t^2 - z^2}, \quad \eta = \frac{1}{2} \ln \frac{t+z}{t-z}, \\ \rho &= \sqrt{x^2 + y^2}, \quad \phi = \frac{1}{2i} \ln \frac{x+y \cdot i}{x-y \cdot i}, \end{aligned} \quad (4)$$

and inversely

$$\begin{aligned} t &= \tau \cosh \eta, \quad z = \tau \sinh \eta, \\ x &= \rho \cos \phi, \quad y = \rho \sin \phi. \end{aligned} \quad (5)$$

The velocity field  $u^m$  in these coordinates is related to  $u^\mu = \gamma(1, \vec{v})$  in flat coordinates  $(t, \vec{x})$  via

$$\begin{aligned} u^\tau &= \gamma(\cosh \eta - v_z \sinh \eta), \quad u^\eta = \frac{\gamma}{\tau}(v_z \cosh \eta - \sinh \eta), \\ u^\rho &= \gamma(v_x \cos \phi + v_y \sin \phi), \quad u^\phi = \frac{\gamma}{\rho}(v_y \cos \phi - v_x \sin \phi). \end{aligned} \quad (6)$$

The metric tensor associated with the  $(\tau, \eta, \rho, \phi)$  coordinates is

$$\begin{aligned} g_{mn} &= \text{Diag}(1, -\tau^2, -1, -\rho^2), \\ g^{mn} &= \text{Diag}(1, -\frac{1}{\tau^2}, -1, -\frac{1}{\rho^2}). \end{aligned} \quad (7)$$

For the covariant derivatives we will need the Affine connections  $\Gamma_{mn}^j = g^{jk} \Gamma_{kmn} = g^{jk} \frac{1}{2}(g_{km, n} + g_{kn, m} - g_{mn, k})$ . In our case, the non-vanishing connections are:

$$\begin{aligned} \Gamma_{\eta\eta}^\tau &= \tau, \quad \Gamma_{\eta\tau}^\eta = \Gamma_{\tau\eta}^\eta = \frac{1}{\tau}, \\ \Gamma_{\phi\phi}^\rho &= -\rho, \quad \Gamma_{\rho\phi}^\phi = \Gamma_{\phi\rho}^\phi = \frac{1}{\rho}. \end{aligned} \quad (8)$$

We also give the explicit form of covariant derivatives in the present coordinates for an arbitrary contra-variant-vector  $A^k$  (i.e. with upper indices  $k = \tau, \eta, \rho, \phi$ ):

$$\begin{aligned} A^k{}_{; \tau} &= A^k{}_{, \tau} + \Gamma_{\tau i}^k A^i = A^k{}_{, \tau} + \frac{1}{\tau} \delta_\eta^k A^\eta, \\ A^k{}_{; \eta} &= A^k{}_{, \eta} + \Gamma_{\eta i}^k A^i = A^k{}_{, \eta} + \tau \delta_\tau^k A^\tau + \frac{1}{\tau} \delta_\eta^k A^\tau, \\ A^k{}_{; \rho} &= A^k{}_{, \rho} + \Gamma_{\rho i}^k A^i = A^k{}_{, \rho} + \frac{1}{\rho} \delta_\phi^k A^\phi, \\ A^k{}_{; \phi} &= A^k{}_{, \phi} + \Gamma_{\phi i}^k A^i = A^k{}_{, \phi} - \rho \delta_\rho^k A^\rho + \frac{1}{\rho} \delta_\phi^k A^\rho. \end{aligned} \quad (9)$$

Inserting the above Eqs.(2,6,7) into the general hydrodynamics equations (1) and making use of Eq.(8,9), one obtains the hydrodynamics equations explicitly in the curved coordinates  $(\tau, \eta, \rho, \phi)$ .

## B. Some Known Simple Exact Solutions

We now recall some known simple exact solutions that are pertinent for our approach.

One of the most famous examples is the so-called Hwa-Bjorken solution [31, 32] which is essentially the Hubble expansion in (1+1)D. The pressure and velocity fields of this solution are given by

$$\begin{aligned} p_{Bj.} &= \frac{\text{constant}}{\tau^{1/(1-\nu)}}, \\ u_{Bj.} &= (1, 0, 0, 0). \end{aligned} \quad (10)$$

It is more transparent to look at the components of  $\vec{v}$  in flat coordinates, which are simply  $v_z = \tanh \eta = \frac{z}{t}$ ,  $v_x = v_y = 0$ .

A generalization of the Hwa-Bjorken solution to radial Hubble flow in (1+3)D is straightforward. The pressure and velocity fields are

$$\begin{aligned} p_{Hu.} &= \frac{\text{constant}}{(\tau^2 - \rho^2)^{\frac{3}{2(1-\nu)}}}, \\ u_{Hu.} &= \gamma \left( \frac{1}{\cosh \eta}, 0, \frac{\rho}{\tau}, 0 \right), \\ \gamma &= \frac{\cosh \eta}{\sqrt{1 - (\rho/\tau)^2 \cosh^2 \eta}}. \end{aligned} \quad (11)$$

with the domain of validity being  $\rho < \tau$ . In flat coordinates the velocity fields are given in simple form:  $\vec{v} = \vec{x}/t = (x/t, y/t, z/t)$ .

Further generalization of the spherically symmetric Hubble flow to arbitrary (1+d)D has been done in [23]. An ellipsoidally expanding hydrodynamical solution has also been discussed (see the first paper in [22]). There have also been study of adding radial flow to a longitudinal Bjorken profile, see e.g. [33].

## III. THE NEW REDUCTION METHOD

In this section, we use a new reduction method to find solutions for (1+3)D RIHD equations. The general idea is to first embed known solutions in lower dimensions which automatically solve 2 out of the total of 4-component hydrodynamics equations, and then reduce the remaining 2 equations into a single equation for the velocity field only. As usual, one starts with a certain ansatz for the flow velocity field: in our case we will use an ansatz with built-in longitudinal and transverse radial flow, aiming at possible application for RHIC. It would be even more interesting to include *transverse elliptic flow* which requires a suitable curved coordinates (like certain hyperbolic coordinates) other than the one used here. However generally in those cases, more Affine connections are non-vanishing, which makes the reduction method discussed below much more involved: we will leave this for future investigation.

## A. Including Longitudinal and Transverse Flow

We first embed the boost-invariant longitudinal flow as many numerical hydrodynamics calculations do, which is a suitable approach for RHIC related phenomenology. To do that, we simply set  $v_z = z/t = \tanh \eta$ , i.e.  $u^\eta = 0$ .

Next we include the transverse radial flow which is isotropic in the transverse plane. Radial flow is substantial and important at RHIC. To do so, we introduce the radial flow field  $v_\rho$  and set the flat-coordinate transverse flow fields to be  $v_x = v_\rho \cos \phi$  and  $v_y = v_\rho \sin \phi$ , which implies for the curved coordinates  $u^\rho = \gamma v_\rho$  and  $u^\phi = 0$  with the latter meaning an axially symmetric velocity field. We note that this ansatz goes beyond a simple change to cylindrical coordinates, since we require that  $u^\phi = 0$  which considerably simplifies the hydrodynamics equations.

To summarize, in order to describe a situation with both longitudinal flow and transverse radial flow we have made the following ansatz for the flow fields  $u^m$  in the coordinates  $(\tau, \eta, \rho, \phi)$ :

$$\begin{aligned} u^m &= \bar{\gamma} (1, 0, \bar{v}_\rho, 0), \\ \bar{v}_\rho &\equiv v_\rho \cosh \eta \quad , \quad \bar{\gamma} \equiv 1/\sqrt{1 - \bar{v}_\rho^2}. \end{aligned} \quad (12)$$

Note that we need to require  $\bar{v}_\rho \leq 1$ .

## B. The Equation for Transverse Velocity

With the flow fields given in (12), we can now explicitly express the stress tensor components. The non-vanishing ones are given below:

$$T^{\tau\tau} = \bar{\gamma}^2(\epsilon + p) - p = \left(\frac{\bar{\gamma}^2}{\nu} - 1\right)p, \quad (13)$$

$$T^{\rho\rho} = \bar{\gamma}^2 \bar{v}_\rho^2 (\epsilon + p) + p = \left(\frac{\bar{\gamma}^2 \bar{v}_\rho^2}{\nu} + 1\right)p, \quad (14)$$

$$T^{\tau\rho} = \bar{\gamma}^2 \bar{v}_\rho (\epsilon + p) = \frac{\bar{\gamma}^2 \bar{v}_\rho}{\nu} p, \quad (15)$$

$$T^{\eta\eta} = \frac{p}{\tau^2} \quad , \quad T^{\phi\phi} = \frac{p}{\rho^2}. \quad (16)$$

For the second equalities in each of the first three lines we have used the E.o.S (3) to substitute  $\epsilon + p$  by  $p/\nu$ .

With the above expressions and using (8)(9), the hydrodynamics equations (1) then become

$$T^{\tau\lambda}_{;\lambda} = T^{\tau\tau}_{;\tau} + \frac{T^{\tau\tau}}{\tau} + \frac{p}{\tau} + T^{\tau\rho}_{;\rho} + \frac{T^{\tau\rho}}{\rho} = 0, \quad (17)$$

$$T^{\eta\lambda}_{;\lambda} = \frac{1}{\tau^2} p_{;\eta} = 0, \quad (18)$$

$$T^{\rho\lambda}_{;\lambda} = T^{\rho\rho}_{;\rho} + \frac{T^{\rho\rho}}{\rho} - \frac{p}{\rho} + T^{\tau\rho}_{;\tau} + \frac{T^{\tau\rho}}{\tau} = 0, \quad (19)$$

$$T^{\phi\lambda}_{;\lambda} = \frac{1}{\rho^2} p_{;\phi} = 0. \quad (20)$$

The two equations involving derivatives over  $\eta$  and  $\phi$  are trivially solved by setting  $p(x) = p(\tau, \rho)$  (and the same for energy density  $\epsilon(\tau, \rho)$  due to the E.o.S) and accordingly  $\bar{v}_\rho(x) = \bar{v}_\rho(\tau, \rho)$ . We note, that the simple form of Eqs.(18,20), are a direct consequence of the vanishing components  $u^\eta = u^\phi = 0$  in the flow field ansatz, Eq.(12).

Finally we introduce a combined field variable  $\mathcal{K}$  defined as

$$\mathcal{K} \equiv \frac{T^{\tau\tau} + p}{(\rho\tau)} = \frac{\bar{\gamma}^2 p}{\nu\rho\tau} \rightarrow p = \frac{\nu\rho\tau}{\bar{\gamma}^2} \mathcal{K}. \quad (21)$$

We then substitute the pressure  $p$  in the equations (17) and (19) and obtain two equations for the fields  $\mathcal{K}$  and  $\bar{v}_\rho$ , which can be expressed as

$$D_a \cdot \mathcal{K}_{,\tau} + D_b \cdot \mathcal{K}_{,\rho} = D_1 \cdot \mathcal{K}, \quad (22)$$

$$D_b \cdot \mathcal{K}_{,\tau} + D_c \cdot \mathcal{K}_{,\rho} = D_2 \cdot \mathcal{K}. \quad (23)$$

The coefficients  $D_a, D_b, D_c, D_1, D_2$  are given by:

$$\begin{aligned} D_a &= (1 - \nu) + \nu\bar{v}_\rho^2, \\ D_b &= \bar{v}_\rho, \\ D_c &= \nu + (1 - \nu)\bar{v}_\rho^2, \\ D_1 &= -2\nu\bar{v}_\rho\bar{v}_{\rho,\tau} - \nu(1 - \bar{v}_\rho^2)/\tau - \bar{v}_{\rho,\rho}, \\ D_2 &= -2(1 - \nu)\bar{v}_\rho\bar{v}_{\rho,\rho} + \nu(1 - \bar{v}_\rho^2)/\rho - \bar{v}_{\rho,\tau}. \end{aligned} \quad (24)$$

From (22)(23) we obtain

$$\begin{aligned} \frac{\mathcal{K}_{,\tau}}{\mathcal{K}} &= (\ln \mathcal{K})_{,\tau} = \frac{D_c D_1 - D_b D_2}{D_a D_c - D_b^2} \equiv \mathcal{F}[\tau, \rho], \\ \frac{\mathcal{K}_{,\rho}}{\mathcal{K}} &= (\ln \mathcal{K})_{,\rho} = \frac{D_a D_2 - D_b D_1}{D_a D_c - D_b^2} \equiv \mathcal{G}[\tau, \rho]. \end{aligned} \quad (25)$$

with the functions  $\mathcal{F}, \mathcal{G}$  given by

$$\begin{aligned} \mathcal{F}[\bar{v}_\rho(\tau, \rho)] &= \frac{1}{\nu(1 - \nu)(1 - \bar{v}_\rho^2)^2} \times \\ &\left\{ [(1 - \nu)\bar{v}_\rho^2 - \nu]\bar{v}_{\rho,\rho} + \right. \\ &\left. [(1 - 2\nu^2) + 2\nu(\nu - 1)\bar{v}_\rho^2]\bar{v}_\rho\bar{v}_{\rho,\tau} \right. \\ &\left. - \nu[1 - \bar{v}_\rho^2][\nu + (1 - \nu)\bar{v}_\rho^2]/\tau - \nu\bar{v}_\rho[1 - \bar{v}_\rho^2]/\rho \right\}, \end{aligned} \quad (26)$$

$$\begin{aligned} \mathcal{G}[\bar{v}_\rho(\tau, \rho)] &= \frac{1}{\nu(1 - \nu)(1 - \bar{v}_\rho^2)^2} \times \\ &\left\{ [(\nu\bar{v}_\rho^2 + (\nu - 1))\bar{v}_{\rho,\tau} + \right. \\ &\left. [(-1 + 4\nu - 2\nu^2) + 2\nu(\nu - 1)\bar{v}_\rho^2]\bar{v}_\rho\bar{v}_{\rho,\rho} \right. \\ &\left. + \nu\bar{v}_\rho[1 - \bar{v}_\rho^2]/\tau + \nu[1 - \bar{v}_\rho^2][(1 - \nu) + \nu\bar{v}_\rho^2]/\rho \right\}. \end{aligned} \quad (27)$$

In (25), the function  $\ln \mathcal{K}$  depends (via  $\bar{v}_\rho$ ) on two variables  $\tau$  and  $\rho$ , and we have two equations for the two

first order derivatives  $\frac{\partial \ln \mathcal{K}}{\partial \tau}$  and  $\frac{\partial \ln \mathcal{K}}{\partial \rho}$ . For  $\ln \mathcal{K}$  as a single function of two variables  $\tau, \rho$ , the two equations can be consistent only if the following constraint on second order derivatives are satisfied  $\frac{\partial^2 \ln \mathcal{K}}{\partial \tau \partial \rho} = \frac{\partial^2 \ln \mathcal{K}}{\partial \rho \partial \tau}$ , i.e.

$$\frac{\partial}{\partial \rho} \mathcal{F}[\bar{v}_\rho(\tau, \rho)] - \frac{\partial}{\partial \tau} \mathcal{G}[\bar{v}_\rho(\tau, \rho)] = 0. \quad (28)$$

Thus we only need to solve the above single equation for the velocity field  $\bar{v}_\rho(\tau, \rho)$ . Since  $\mathcal{F}, \mathcal{G}$  already involve the first derivatives of  $\bar{v}_\rho, \tau$  and  $\bar{v}_\rho, \rho$ , the reduced velocity equation 28 is a second-order partial differential equation for the velocity field. As a minor caveat, the method applies to the case/region in which  $\ln \mathcal{K}$  is at least second-order differentiable. This reduction method can be demonstrated in the more explicit case of (1+1)D hydrodynamics, see Appendices B and C.

Given the above constraints, we can then solve from (25) the matter field  $S$  directly

$$\mathcal{K} = \mathcal{K}_0 \cdot e^{\left[ \int_{\tau_0}^{\tau} d\tau' \mathcal{F}[\tau', \rho] + \int_{\rho_0}^{\rho} d\rho' \mathcal{G}[\tau, \rho'] \right]}, \quad (29)$$

with  $\mathcal{K}_0$  being the value at arbitrary reference point  $\tau_0, \rho_0$ .

Finally let us summarize our approach: after including into the flow field ansatz the physically desired longitudinal and transverse flows, we have reduced the hydrodynamic equations into a single equation (28) involving *ONLY* the transverse velocity field  $\bar{v}_\rho$ , and any solution to this equation automatically leads to the pressure field which together with the velocity field forms a solution to the original hydrodynamics equations:

$$p = \text{constant} \times \frac{\rho\tau}{\bar{\gamma}^2} \times e^{\left[ \int^{\tau} d\tau' \mathcal{F}[\tau', \rho] + \int^{\rho} d\rho' \mathcal{G}[\tau, \rho'] \right]}. \quad (30)$$

### C. Examination of the Method

We now examine the correctness of the reduced equation (28) and the solution (30), using the two known simple analytic solutions (10) and (11) as both of them are certain special cases of our embedding with longitudinal and transverse radial flows.

For the 1-D Bjorken expansion, we have  $\bar{v}_{\rho B_j} = 0$  which leads to

$$\mathcal{F}_{B_j} = \frac{\nu}{\nu - 1} \frac{1}{\tau}, \quad \mathcal{G}_{B_j} = \frac{1}{\rho}. \quad (31)$$

One can easily verify that the above  $\mathcal{F}_{B_j}, \mathcal{G}_{B_j}$  satisfy the reduced equation (28). Furthermore by inserting  $\mathcal{F}_{B_j}, \mathcal{G}_{B_j}$  into the solution (30) one finds exactly the pressure in (10).

For the 3-D Hubble expansion, we have  $\bar{v}_{\rho H_u} = \rho/\tau$  which leads to

$$\begin{aligned} \mathcal{F}_{H_u} &= \frac{3}{\tau} + \frac{\nu - 5/2}{1 - \nu} \frac{2\tau}{\tau^2 - \rho^2}, \\ \mathcal{G}_{H_u} &= \frac{1}{\rho} + \frac{\nu - 5/2}{1 - \nu} \frac{-2\rho}{\tau^2 - \rho^2}. \end{aligned} \quad (32)$$

Again it can be easily shown that the above  $\mathcal{F}_{Hu.}, \mathcal{G}_{Hu.}$  satisfy the reduced equation (28). Furthermore by inserting  $\mathcal{F}_{Hu.}, \mathcal{G}_{Hu.}$  into the solution (30) one finds exactly the pressure in (11).

#### IV. APPLICATION OF THE METHOD

As an example for an application of the embedding-reduction method in the previous section, we show how to find all possible solutions with the following ansatz for the radial velocity field:

$$\bar{v}_\rho = A \cdot \tau^B \cdot \rho^C, \quad (33)$$

with  $A, B, C$  arbitrary real numbers. We note that the two known exact solutions we mentioned are special cases of the above form: the 1-D Bjorken expansion corresponds to  $A = 0$  while the 3-D Hubble expansion corresponds to  $A = 1, B = -1, C = 1$ . The velocity field (33), when put into (26)(27), gives the following

$$\begin{aligned} \mathcal{F}[\tau, \rho] &= \frac{1-2B}{\tau} - \frac{B+(1-4B)\nu+2B\nu^2}{\nu(1-\nu)} \cdot \frac{1}{\tau[1-\bar{v}_\rho^2]} \\ &- \frac{C+(1-C)\nu}{\nu(1-\nu)} \cdot \frac{\bar{v}_\rho}{\rho[1-\bar{v}_\rho^2]} + \frac{B(1-2\nu)}{\nu(1-\nu)} \cdot \frac{1}{\tau[1-\bar{v}_\rho^2]} \\ &+ \frac{C(1-2\nu)}{\nu(1-\nu)} \cdot \frac{\bar{v}_\rho}{\rho[1-\bar{v}_\rho^2]^2}, \end{aligned} \quad (34)$$

$$\begin{aligned} \mathcal{G}[\tau, \rho] &= \frac{-2C-\nu/(1-\nu)}{\rho} - \frac{2C\nu^2-\nu-C}{\nu(1-\nu)} \cdot \frac{1}{\rho[1-\bar{v}_\rho^2]} \\ &- \frac{B-1}{1-\nu} \cdot \frac{\bar{v}_\rho}{\tau[1-\bar{v}_\rho^2]} + \frac{C(2\nu-1)}{\nu(1-\nu)} \cdot \frac{1}{\rho[1-\bar{v}_\rho^2]^2} \\ &+ \frac{B(2\nu-1)}{\nu(1-\nu)} \cdot \frac{\bar{v}_\rho}{\tau[1-\bar{v}_\rho^2]^2}. \end{aligned} \quad (35)$$

We have used  $\bar{v}_{\rho, \rho} = \bar{v}_\rho \cdot C/\rho$  and  $\bar{v}_{\rho, \tau} = \bar{v}_\rho \cdot B/\tau$ . As a check of the above result, one can verify that by setting  $A = 0$  they reduce to (31) while by setting  $A = 1, B = -1, C = 1$  they reduce to (32), as they should.

By inserting (34)(35) into equation (28), one obtains a rather complicated constraint equation for the constants  $A, B, C$ . However after a lengthy calculations, all possible combinations of  $A, B, C$  solving the equation can actually be exhausted. Leaving the detailed (and technical) derivations to the Appendix A, we only list the final results here:

- **Solution-I:**  $A = 0$  with  $0 < \nu \leq \frac{1}{2}$  and  $|z| < t$  (1-D Bjorken) — see (10);
- **Solution-II:**  $A = 1, B = -1, C = 1$  with  $0 < \nu \leq \frac{1}{2}$  and  $\sqrt{x^2 + y^2 + z^2} < t$  (3-D Hubble) — see (11);

- **Solution-III:**  $A = 1, B = 1, C = -1$  with  $0 < \nu \leq \frac{1}{2}$ ,  $|z| < t$  and  $\sqrt{x^2 + y^2 + z^2} > t$  — the solutions are

$$\begin{aligned} v_x &= \frac{x}{t} \cdot \frac{t^2 - z^2}{x^2 + y^2}, \quad v_y = \frac{y}{t} \cdot \frac{t^2 - z^2}{x^2 + y^2}, \quad v_z = \frac{z}{t} \\ p &= \frac{\text{constant}}{(\tau\rho)^{1/(1-\nu)}(\rho^2 - \tau^2)^{(1-3\nu)/(2\nu-2\nu^2)}}; \end{aligned} \quad (36)$$

- **Solution-IV:**  $A = 1, B = 1/3, C = -1/3$  with  $\nu = 1/4$ ,  $|z| < t$  and  $\sqrt{x^2 + y^2 + z^2} > t$  — the solutions are

$$\begin{aligned} v_x &= \frac{x}{t} \cdot \left(\frac{t^2 - z^2}{x^2 + y^2}\right)^{2/3}, \quad v_y = \frac{y}{t} \cdot \left(\frac{t^2 - z^2}{x^2 + y^2}\right)^{2/3}, \quad v_z = \frac{z}{t} \\ p &= \text{constant} \times \frac{(\rho^{2/3} - \tau^{2/3})^{2/3}}{(\rho\tau)^{4/3}}; \end{aligned} \quad (37)$$

- **Solution-V:**  $A = -1, B = -1, C = 1$  with  $\nu = 1/2$  and  $\sqrt{x^2 + y^2 + z^2} < t$  — the solutions are

$$\begin{aligned} v_x &= \frac{-x}{t}, \quad v_y = \frac{-y}{t}, \quad v_z = \frac{z}{t} \\ p &= \text{constant} \times (\tau^2 - \rho^2). \end{aligned} \quad (38)$$

It can be verified that these solutions obtained by the method introduced here are indeed solutions of the original hydrodynamics equations (1). One should notice the different applicable kinematic regions in each of the above solutions which comes from the constraint that the flow velocity shall be less than the speed of light. For a detailed discussion about solutions in different regions with respect to kinematic light cone, see e.g. the Appendices of [23].

We notice that all the solutions (except the trivial Solution-I with  $A = 0$ ) satisfy two features (1)  $B = -C$  and (2)  $|A| = 1$ . The first feature may be due to dimensional reasons. The second feature,  $|A| = 1$ , may be heuristically understood in the following way. We first consider the case  $B = -C < 0$ , i.e.  $\bar{v}_\rho = A(\rho/\tau)^C$  with  $C > 0$ : in this case the solution exists in the region  $\rho < \tau \cdot |A|^{-1/C}$ , and in particular  $\rho = 0$  for  $\tau = 0$ . Thus for any  $\tau > 0$ , the flow front which travels with the speed of light,  $|\bar{v}_\rho| = 1$ , is located at  $\rho = \tau$  and hence,  $|A| = 1$ . Next we consider the case  $B = -C > 0$ , i.e.  $\bar{v}_\rho = A(\tau/\rho)^B$  with  $B > 0$ : in this case the solution exists in the region  $\rho > \tau \cdot A^{1/B}$  with  $A > 0$ , separated from an empty region by the boundary at  $\rho = \tau \cdot A^{1/B}$ . At this boundary, the flow velocity approaches the speed of light  $\bar{v}_\rho \rightarrow 1$  which enforces the matter density to drop to zero in order to avoid an infinite  $T^{mn}$  (due to the  $\gamma$ -factor in Eq.(2)). We imagine that at time  $\tau = 0$  the matter fills the whole space and then starts to flow outward, thus the boundary also moves outward from the origin with the speed of light  $\bar{v}_\rho = 1$ . This again implies the boundary should lie at  $\rho = \tau$  requiring  $A = 1$ .

The above example of the proposed embedding-reduction method demonstrates the advantage of analytical solutions. Not only could we find some solutions

of the specific type (33) but we actually were able to *exhaust* all solutions of this type. This also implies that for parametrization of flow velocity field, like e.g. in the blast wave model for RHIC fireball, there are only very limited choices for the flow profile ansatz.

## V. SUMMARY AND DISCUSSION

In summary, a general framework for the analytical treatment of RIHD equations has been developed. The method features a separation of longitudinal and transverse expansions, as inspired by RHIC phenomenology. After the separation, the longitudinal and transverse radial flows are embedded utilizing lower-dimensional solutions. The remaining equations are found to be reducible to a single constraint equation for transverse radial flow velocity field only, which can be solved completely for a certain ansatz for the velocity field. All solutions with power-law dependence on proper time and transverse radius have been found.

We now discuss various possible extensions of the present approach.

*Nontrivial longitudinal embedding:* In the current work the longitudinal flow is embedded with the Hwa-Bjorken solution. It would be very interesting to try embedding the newly found (1+1)D solutions in e.g.[23–26] with more realistic longitudinal expansion for RHIC which would be useful for studying elliptic flow in the forward/backward rapidity and their correlation[34].

*Solutions with non-power-law transverse expansion:* It would also be interesting to test more nontrivial ansatz for the embedded transverse flow. For example we know from numerical calculations of radial flow [35] in central collisions at RHIC that the radial velocity field may be parameterized as  $v_\rho \approx f(\tau)r/\tau$  with  $f(\tau \rightarrow 0) \rightarrow 0$  and  $f(\tau \gg 1) \rightarrow 1$ . Such parametrization can be cast into the derived velocity equation (28) to find possible solutions.

*Small deformation and elliptic flow:* The analytic treatment of transverse elliptic flow is difficult. One approximate method may be to introduce a parametrically small deformation of the matter field (with a certain eccentricity parameter  $\epsilon_2$ ) on top of an exact solution with transverse radial flow and using linearized hydrodynamics equations to investigate possible universal relations between the finally developed velocity field anisotropy  $v_2$  and the initial  $\epsilon_2$  [36].

*Transverse elliptic flow embedding:* Another possibility to seek exact solutions with transverse elliptic flow is to use instead of  $(\rho, \phi)$  certain hyperbolic coordinates which by definition incorporate elliptic anisotropy, see Appendix of [37] for an example of such curved coordinates which may be used to develop a similar embedding-reduction procedure describing transverse elliptic flow. Another possibility will be combining certain conformal transformations with hydrodynamics equations to degrade the elliptic geometry back to a spherical one.

*2D Hubble embedding:* In all the previously discussed, we have chosen to embed (1+1)D Hubble flow for the  $(t, z) \rightarrow (\tau, \eta)$  part, due to an emphasis on RHIC evolution. Theoretically, one can also embed a (1+2)D Hubble flow for the  $(t, x, y) \rightarrow (t, \rho, \phi) \rightarrow (\tau_\rho, \eta_\rho, \phi)$  part, and can eventually reduce the equations to a velocity equation with two variables  $(\tau_\rho, z)$  in exactly the same manner as before.

## Acknowledgments

The work is supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Divisions of Nuclear Physics, of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231. JL is grateful to Edward Shuryak for very helpful discussions.

## Appendix A

In this Appendix we give the detailed derivations leading to the solutions in Section IV with the transverse velocity ansatz (33).

We first evaluate the derivatives  $\frac{\partial \mathcal{F}}{\partial \rho}$  and  $\frac{\partial \mathcal{G}}{\partial \tau}$  with  $\mathcal{F}, \mathcal{G}$  given in (34)(35). Again we will make use of  $\bar{v}_{\rho, \rho} = \bar{v}_\rho \cdot C/\rho$  and  $\bar{v}_{\rho, \tau} = \bar{v}_\rho \cdot B/\tau$  for the velocity ansatz (33).

The result for  $\frac{\partial \mathcal{F}}{\partial \rho}$  is

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial \rho} &= \frac{\bar{v}_\rho}{\nu(1-\nu)\rho^2\tau^2(1-\bar{v}_\rho^2)^3} \times \left\{ f_1 \rho \tau \bar{v}_\rho (1-\bar{v}_\rho^2) \right. \\ &\quad + f_2 \tau^2 [(1+C)\bar{v}_\rho^2 + (C-1)] (1-\bar{v}_\rho^2) \\ &\quad \left. + f_3 \tau \rho \bar{v}_\rho + f_4 \tau^2 [(3C+1)\bar{v}_\rho^2 + (C-1)] \right\} \\ &= \frac{\bar{v}_\rho}{\nu(1-\nu)\rho^2\tau^2(1-\bar{v}_\rho^2)^3} \times \left\{ [-f_2(C+1)] \tau^2 \bar{v}_\rho^4 \right. \\ &\quad + [-f_1] \rho \tau \bar{v}_\rho^3 + [2f_2 + f_4(3C+1)] \tau^2 \bar{v}_\rho^2 \\ &\quad \left. + [f_1 + f_3] \rho \tau \bar{v}_\rho + [(f_2 + f_4)(C-1)] \tau^2 \right\}, \quad (\text{A1}) \end{aligned}$$

with coefficients  $f_{1,2,3,4}$  given by

$$\begin{aligned} f_1 &= -[B + (1-4B)\nu + 2B\nu^2] \times (2C), \\ f_2 &= -[C + (1-C)\nu], \\ f_3 &= B \times (4C) \times (1-2\nu), \\ f_4 &= C \times (1-2\nu). \end{aligned} \quad (\text{A2})$$

The result for  $\frac{\partial \mathcal{G}}{\partial \tau}$  is

$$\begin{aligned} \frac{\partial \mathcal{G}}{\partial \tau} &= \frac{\bar{v}_\rho}{\nu(1-\nu)\rho^2\tau^2(1-\bar{v}_\rho^2)^3} \times \left\{ g_1\rho\tau\bar{v}_\rho(1-\bar{v}_\rho^2) \right. \\ &\quad + g_2\rho^2[(1+B)\bar{v}_\rho^2+(B-1)](1-\bar{v}_\rho^2) \\ &\quad \left. + g_3\tau\rho\bar{v}_\rho + g_4\rho^2[(3B+1)\bar{v}_\rho^2+(B-1)] \right\} \\ &= \frac{\bar{v}_\rho}{\nu(1-\nu)\rho^2\tau^2(1-\bar{v}_\rho^2)^3} \times \left\{ [-g_2(B+1)]\rho^2\bar{v}_\rho^4 \right. \\ &\quad + [-g_1]\rho\tau\bar{v}_\rho^3 + [2g_2+g_4(3B+1)]\rho^2\bar{v}_\rho^2 \\ &\quad \left. + [g_1+g_3]\rho\tau\bar{v}_\rho + [(g_2+g_4)(B-1)]\rho^2 \right\}, \quad (\text{A3}) \end{aligned}$$

with coefficients  $g_{1,2,3,4}$  given by

$$\begin{aligned} g_1 &= -[-C-\nu+2C\nu^2] \times (2B), \\ g_2 &= -[(B-1)\nu], \\ g_3 &= (4B) \times C \times (2\nu-1), \\ g_4 &= B \times (2\nu-1). \end{aligned} \quad (\text{A4})$$

Now combining the results into (28) we obtain the following (with  $\bar{v}_\rho = A\tau^B\rho^C$  substituted in)

$$\frac{\partial \mathcal{F}}{\partial \rho} - \frac{\partial \mathcal{G}}{\partial \tau} = \frac{\bar{v}_\rho}{\nu(1-\nu)\rho^2\tau^2(1-\bar{v}_\rho^2)^3} \times \mathcal{I}[\tau, \rho] = 0,$$

$$\begin{aligned} \text{with } \mathcal{I}[\tau, \rho] &= [-f_2(C+1)A^4]\tau^{4B+2}\rho^{4C} \\ &\quad + [g_2(B+1)A^4]\tau^{4B}\rho^{4C+2} \\ &\quad + [(g_1-f_1)A^3]\tau^{3B+1}\rho^{3C+1} \\ &\quad + [(2f_2+f_4(3C+1))A^2]\tau^{2B+2}\rho^{2C} \\ &\quad + [(-2g_2-g_4(3B+1))A^2]\tau^{2B}\rho^{2C+2} \\ &\quad + [(f_1+f_3-g_1-g_3)A]\tau^{B+1}\rho^{C+1} \\ &\quad + [(f_2+f_4)(C-1)]\tau^2 \\ &\quad + [-(g_2+g_4)(B-1)]\rho^2. \end{aligned} \quad (\text{A5})$$

Clearly the solutions are

$$\bar{v}_\rho = 0, \quad (\text{A6})$$

or

$$\mathcal{I}[\tau, \rho] = 0. \quad (\text{A7})$$

The former solution is just the Hwa-Bjorken one. Any solution with non-vanishing transverse velocity then has to satisfy the latter condition Eq.(A7), which we now focus on. Again one can test the correctness of the above equation by using the 3D Hubble ( $A=1, B=-1, C=1$ ) solution.

In the Eq.(A7), terms with various powers of  $\tau, \rho$  (and only power terms) appear in  $\mathcal{I}[\tau, \rho]$ : to make all of them, either mutually cancel (among terms with exactly the same  $\tau, \rho$  powers) or vanish by respective coefficients, to eventually zero is quite nontrivial. A thorough sorting of the sequences of  $\tau, \rho$  powers can exhaust all possibilities to satisfy the algebraic equation (A5).

To see how this actually works, we give one concrete example. Let's consider the case when  $B > 0$  and  $C \neq 0$ : this implies that for the exponents of  $\tau$  we have  $4B+2 > 4B > 2B, 4B+2 > 3B+1 > B+1, 4B+2 > 2B+2 > 2B, 2B+2 > 2$  and  $2B+2 > B+1$ . So the term  $[-f_2(C+1)A^4]\tau^{4B+2}\rho^{4C}$  in Eq.(A5) can NOT be cancelled by any other one and has to vanish by itself: this leads to

$$f_2(C+1)A^4 = 0, \quad (\text{A8})$$

which in turn gives three possibilities  $f_2 = 0$  or  $C = -1$  or  $A = 0$ . In this example we follow  $C = -1$  (other choices lead to other solutions). With  $C = -1$  we notice again that the term  $[-(g_2+g_4)(B-1)]\rho^2$  can NOT be cancelled by any other remaining terms and thus shall vanish by itself: this leads to

$$-(g_2+g_4)(B-1) = 0, \quad (\text{A9})$$

which again has two possibilities  $B = 1$  or  $g_2+g_4 = 0$ . Now we choose to follow  $B = 1$ : with this choice the remaining terms are significantly simplified and finally lead to two equations about the coefficients:

$$\begin{aligned} 2g_2A^4 + (g_1-f_1)A^3 + 2(f_2-f_4)A^2 &= 0, \\ -2(g_2+2g_4)A^2 + (f_1+f_3-g_1-g_3)A \\ -2(f_2+f_4) &= 0 \end{aligned} \quad (\text{A10})$$

It can then be verified that the only solution is  $A = 1$  for arbitrary  $\nu$ .

Of course there are many but finite number of combinations that one can follow to check one by one. Note not all possibilities appearing initially can finally lead to a solution: there are only four variables  $A, B, C, \nu$  and in most cases it turns out contradiction occurs at the end which means no solution. After a tedious examination we have found all possible solutions as listed in Section IV, and there is no more solution of the power law ansatz type as in (33).

## Appendix B

In this Appendix we use (1+1)D ideal relativistic hydrodynamics to demonstrate the reduction method in a more explicit manner. The hydrodynamics equations are (in (t,z) coordinates)

$$\frac{[\partial_t + v\partial_z]\epsilon}{\epsilon + p} = -\partial_z v - \gamma_v^2[\partial_t + v\partial_z]\left(\frac{v^2}{2}\right), \quad (\text{B1})$$

$$\gamma_v^2[\partial_t + v\partial_z]v = -\frac{\partial_z p}{\epsilon + p} - \frac{v\partial_t p}{\epsilon + p}. \quad (\text{B2})$$

In the above  $v$  is the spatial velocity  $dz/dt$  and  $\gamma_v \equiv 1/\sqrt{1-v^2}$ . The energy density  $\epsilon$  and pressure  $p$  shall be related by the E.o.S, which we use in a slightly different way. We introduce the enthalpy density  $w = \epsilon + p$  and

the speed of sound  $c_s \equiv \sqrt{\partial p / \partial \epsilon}$  (which can be deduced from E.o.S), and use the following relations

$$d\epsilon = \frac{1}{1 + c_s^2} dw, \quad dp = \frac{c_s^2}{1 + c_s^2} dw \quad (\text{B3})$$

to re-write the hydrodynamics equations into:

$$\partial_t[\ln(w)] + v \partial_z[\ln(w)] = -\frac{\gamma_v^2}{1 - \xi} [\partial_x v + v \partial_t v], \quad (\text{B4})$$

$$v \partial_t[\ln(w)] + \partial_z[\ln(w)] = -\frac{\gamma_v^2}{\xi} [v \partial_x v + \partial_t v], \quad (\text{B5})$$

with  $\xi \equiv c_s^2 / (1 + c_s^2)$ . From these two equations we can obtain  $\partial_t[\ln(w)]$  and  $\partial_z[\ln(w)]$ :

$$\begin{aligned} \partial_t[\ln(w)] &= \mathcal{X}[v, \partial_t v, \partial_z v] = \mathcal{X}[t, z] \\ &= \frac{-\gamma_v^4}{\xi(1 - \xi)} \left\{ [\xi - (1 - \xi)v^2] \partial_z v + [2\xi - 1] v \partial_t v \right\}, \end{aligned} \quad (\text{B6})$$

$$\begin{aligned} \partial_z[\ln(w)] &= \mathcal{Y}[v, \partial_t v, \partial_z v] = \mathcal{Y}[t, z] \\ &= \frac{-\gamma_v^4}{\xi(1 - \xi)} \left\{ [(1 - \xi) - \xi v^2] \partial_t v + [1 - 2\xi] v \partial_z v \right\}. \end{aligned} \quad (\text{B7})$$

The necessary and sufficient conditions for the above set of equations to be soluble is the following:

$$\partial_z \mathcal{X}[t, z] - \partial_t \mathcal{Y}[t, z] = 0 \quad (\text{B8})$$

Thus we have reduced the original hydrodynamics equations into a single but second order differential equation for the velocity field  $v(t, z)$  only. With the above satisfied, the matter field is given by

$$w(t, z) = w_0 \cdot e^{\int_{t_0}^t dt' \mathcal{X}[t', z] + \int_{z_0}^z dz' \mathcal{Y}[t, z']}, \quad (\text{B9})$$

with  $w_0$  its value at arbitrary reference point  $(t_0, z_0)$ .

In the case of a linear E.o.S as the one in (3), the speed of sound  $c_s$  and thus  $\xi$  are constants independent of  $\epsilon$  or  $p$ , and we can further simplify the reduced equation (B8) for velocity field into the following:

$$\begin{aligned} &[(1 - \xi) - \xi v^2](1 - v^2)(\partial_t^2 v) \\ &+ [(2 - 3\xi) - \xi v^2](2v)(\partial_t v)^2 \\ &- [\xi - (1 - \xi)v^2](1 - v^2)(\partial_z^2 v) \\ &- [(3\xi - 1) + (\xi - 1)v^2](2v)(\partial_z v)^2 \\ &+ 2(1 - 2\xi)(1 - v^2)v(\partial_t \partial_z v) \\ &+ 2(1 - 2\xi)(1 + 3v^2)(\partial_t v)(\partial_z v) = 0. \end{aligned} \quad (\text{B10})$$

A similar scheme can be carried out for curved coordinates like  $(\tau, \eta)$  in a straightforward way. We notice a similar implementation using light-cone variables  $z_{\pm} = t \pm z$  in [24].

We emphasize that while the above procedure seems somewhat trivial in (1+1)D, its realization is much more nontrivial and involved in (1+3)D. We also point out that

the reduced equation (B8) for the velocity field (or the simplified one in case of linear E.o.S) has to be satisfied by *all* solutions to the (1+1)D hydrodynamics equations. In Appendix C we give a nontrivial and involved example from the recently found Nagy-Csörgő-Csanád (NCC) solutions[23] (which also include (1+1)D Hwa-Bjorken as a special case) to show the correctness and usefulness of the derived velocity equation.

## Appendix C

In this Appendix we show that the NCC family of analytic solutions in [23] for 1-D ideal hydrodynamics equations with a linear E.o.S can also be deduced by subjecting their velocity field ansatz to the reduced equations (B10) we derived. With the resulting velocity field we also show the matter field of NCC solutions is indeed given by (B9).

The velocity field ansatz of NCC solutions is the following (for inside-light-cone region, i.e.  $|z| < |t|$ ):

$$v = \tanh[\lambda \eta] = \frac{(t+z)^\lambda - (t-z)^\lambda}{(t+z)^\lambda + (t-z)^\lambda}, \quad \eta = \frac{1}{2} \ln \left[ \frac{t+z}{t-z} \right], \quad (\text{C1})$$

with  $\lambda$  some constant. With the above, we obtain the following relations for the derivatives:

$$\begin{aligned} \partial_t v &= \lambda(1 - v^2) \partial_t \eta, \quad \partial_z v = \lambda(1 - v^2) \partial_z \eta, \\ \partial_t^2 v &= \lambda(1 - v^2) [(\partial_t^2 \eta) - 2\lambda v (\partial_t \eta)^2], \\ \partial_z^2 v &= \lambda(1 - v^2) [(\partial_z^2 \eta) - 2\lambda v (\partial_z \eta)^2], \\ \partial_t \partial_z v &= \lambda(1 - v^2) [(\partial_t \partial_z \eta) - 2\lambda v (\partial_t \eta)(\partial_z \eta)]. \end{aligned} \quad (\text{C2})$$

Substituting the above derivatives into our velocity equation (B10), we obtain the following:

$$\begin{aligned} 0 &= \lambda(1 - v^2)^2 \times \\ &\left\{ [(1 - \xi) - \xi v^2](\partial_t^2 \eta) + [-\xi + (1 - \xi)v^2](\partial_z^2 \eta) \right. \\ &+ 2(1 - 2\xi)v(\partial_t \partial_z \eta) + 2(1 - 2\xi)\lambda(1 + v^2)(\partial_t \eta)(\partial_z \eta) \\ &\left. + 2(1 - 2\xi)\lambda v [(\partial_t \eta)^2 + (\partial_z \eta)^2] \right\}. \end{aligned} \quad (\text{C3})$$

After evaluating the derivatives of  $\eta$  in the above, we obtain:

$$\frac{2\lambda(1 - v^2)^2}{(t^2 - z^2)^2} \cdot (1 - 2\xi) \cdot (1 - \lambda) \cdot [(tz)v^2 - (t^2 + z^2)v + (tz)] = 0. \quad (\text{C4})$$

We find three classes of solutions:

- $\xi = 1/2$  with arbitrary  $\lambda$ ;
- $\lambda = 1$  with arbitrary  $\xi$  (which is nothing but the Hwa-Bjorken solution);



- $[(tz)v^2 - (t^2 + z^2)v + (tz)] = 0$  which yields only one causal solution in the forward light-cone with  $v = z/t$ , but this is just the  $\lambda = 1$  solution.

These cover the (1+1)D NCC solutions found in [23]. Note their parameter  $\kappa$  from E.o.S  $\epsilon = \kappa p$  is related to our E.o.S parameter  $\xi \equiv c_s^2/(1 + c_s^2)$  by  $\kappa = (1 - \xi)/\xi$ . It should be mentioned that the last Appendix of [23] gives the general solution of the (1+1)D relativistic hydrodynamical solutions for  $\xi = 1/2$  with arbitrary initial conditions in the forward light-cone, and it also gives various spherically symmetric solutions in arbitrary number of spatial dimensions: these aspects are not discussed in the present paper.

Next we examine the matter field corresponding to the velocity field solutions:

- for  $\xi = 1/2$  case: we have  $\mathcal{X} = (-\lambda)[2t/(t^2 - z^2)]$

and  $\mathcal{Y} = (-\lambda)[-2z/(t^2 - z^2)]$ , which via our equation (B9) gives  $w = w_0 \left[ \frac{t_0^2 - z_0^2}{t^2 - z^2} \right]^\lambda$ ;

- for  $\lambda = 1$  case: we have  $v = z/t$  and thus  $\mathcal{X} = \frac{-1}{2(\xi-1)}[2t/(t^2 - z^2)]$  and  $\mathcal{Y} = \frac{-1}{2(\xi-1)}[-2z/(t^2 - z^2)]$ , which via our equation (B9) gives  $w = w_0 \left[ \frac{t_0^2 - z_0^2}{t^2 - z^2} \right]^{1/(2-2\xi)}$ .

The two cases can be combined into a single form:

$$w = w_0 \times \left[ \frac{t_0^2 - z_0^2}{t^2 - z^2} \right]^{\frac{\lambda}{2(1-\xi)}} \quad (C5)$$

which is the same as obtained in [23].

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