Descriptive Set Theory: Why Should We Study It?

Thomas D. Gilton
Department of Mathematics
College of Arts & Sciences

Faculty Mentor: Dr. John Krueger
Mentor’s Department: Mathematics
Mentor’s College: Arts & Sciences

Scholars Day: April 14, 2011

¹Financial support provided by a grant from the Honors College.
The goal of Descriptive Set Theory (hereafter abbreviated DST) is to study definable subsets of $\mathbb{R}$. 
The goal of Descriptive Set Theory (hereafter abbreviated DST) is to study definable subsets of $\mathbb{R}$. These sets are expected to have distinguished properties, for instance, being Lesbegue measurable, or having the property of Baire.
The goal of Descriptive Set Theory (hereafter abbreviated DST) is to study definable subsets of $\mathbb{R}$. These sets are expected to have distinguished properties, for instance, being Lebesgue measurable, or having the property of Baire. DST requires background in many areas: topology, classical set theory, analysis, and recursion theory.
Set theory studies the many properties of sets; intuitively, sets are collections of mathematical objects.
More Background

Set Theory

Set theory studies the many properties of sets; intuitively, sets are collections of mathematical objects. Set theory is framed in terms of a set of axioms; classical set theory is the study of sets in light of the so-called ZFC (Zermelo-Fraenkel-Choice) axiom system.
Set Theory

Set theory studies the many properties of sets; intuitively, sets are collections of mathematical objects. Set theory is framed in terms of a set of axioms; classical set theory is the study of sets in light of the so-called ZFC (Zermelo-Fraenkel-Choice) axiom system. One aspect of my current research is concerned with learning about the various properties of sets that are germane to Descriptive Set Theory.
Recursion theory is a branch of mathematical logic that studies **computability**, that is, problems in mathematics that could be solved in a finite amount of time using a finite amount of intelligence.
Recursion Theory

Recursion theory is a branch of mathematical logic that studies **computability**, that is, problems in mathematics that could be solved in a finite amount of time using a finite amount of intelligence.

My previous research in the mathematics department was concerned with learning the basic methods of recursion theory.
Historical Origins

A Short Summary

The subject was started by the French analysts at the turn of the 20th century, most prominently Lebesgue.
The subject was started by the French analysts at the turn of the 20th century, most prominently Lebesgue. Its rapid development came to a halt in the late 1930s, primarily because it bumped against problems which were independent of classical axiomatic set theory.
Historical Origins

A Short Summary

The subject was started by the French analysts at the turn of the 20th century, most prominently Lebesgue. Its rapid development came to a halt in the late 1930s, primarily because it bumped against problems which were independent of classical axiomatic set theory.

As mentioned above, classical set theory is concerned with the ZFC axioms. It turns out that there are problems that are undecidable in this context; these are propositions that can neither be proved nor disproved using ZFC.
Historical Origins

Summary Continued

The field became very active again in the 1960s, with the introduction of strong set-theoretic hypotheses and recursion theory.
The field became very active again in the 1960s, with the introduction of strong set-theoretic hypotheses and recursion theory. These other set-theoretic hypotheses are additional axioms added to ZFC in order to prove stronger results.
The field became very active again in the 1960s, with the introduction of strong set-theoretic hypotheses and recursion theory. These other set-theoretic hypotheses are additional axioms added to ZFC in order to prove stronger results. Recursion theory was introduced because of the intimate connection between computability (the domain of recursion theory) and definability (the chief concern of Descriptive Set Theory).
Product Spaces

Let $X = X_1 \times X_2 \times \cdots \times X_n$. Members of $X$ will be finite sequences where the $k$th term in the sequence is an element of $X_k$. 
The Method of Construction

Product Spaces

Let $X = X_1 \times X_2 \times \cdots \times X_n$. Members of $X$ will be finite sequences where the $k$th term in the sequence is an element of $X_k$. We start with a fixed collection of simple subsets of $X$, called the open sets, and construct more complicated sets with various set operations.
Product Spaces

Let $X = X_1 \times X_2 \times \cdots \times X_n$. Members of $X$ will be finite sequences where the $k$th term in the sequence is an element of $X_k$. We start with a fixed collection of simple subsets of $X$, called the open sets, and construct more complicated sets with various set operations.

Let us look at an example of a set $X \subseteq A \times B$. 

The Method of Construction

Product Spaces

Let $X = X_1 \times X_2 \times \cdots \times X_n$. Members of $X$ will be finite sequences where the $k$th term in the sequence is an element of $X_k$. We start with a fixed collection of simple subsets of $X$, called the open sets, and construct more complicated sets with various set operations.

Let us look at an example of a set $X \subseteq A \times B$. $X \subseteq A \times B$ means that $X$ is a collection of ordered pairs where the first member of the pair is an element of $A$ and the second member of the pair is an element of $B$. 
The Method of Construction

An example of a product space
The Method of Construction

Projection

As an example of a set operation, suppose

\[ X \subseteq A \times B. \]

One operation on \( X \) is that of \textit{projection}; the projection of \( X \) onto \( A \) is defined as

\[ P_A(X) = \{ a \in A \mid \exists b \in B, (a, b) \in X \} . \]
The Method of Construction

Projection

As an example of a set operation, suppose

\[ X \subseteq A \times B. \]

One operation on \( X \) is that of projection; the projection of \( X \) onto \( A \) is defined as

\[ P_A(X) = \{ a \in A \mid \exists b \in B, (a, b) \in X \} . \]

On a more intuitive level, the projection of a set is its "shadow" on an axis.
The Method of Construction

Projection onto A-axis
The Method of Construction

Projection onto B-axis
Descriptive Set Theory is of great interest to many mathematicians because the sets that are studied therein are ubiquitous.
Descriptive Set Theory is of great interest to many mathematicians because the sets that are studied therein are ubiquitous. It therefore stands to reason that having a thorough knowledge of these sets would be of great benefit.
Descriptive Set Theory is of great interest to many mathematicians because the sets that are studied therein are ubiquitous. It therefore stands to reason that having a thorough knowledge of these sets would be of great benefit. This is one of the reasons Lebesgue, who was interested in analysis, thought it was of great importance to study and develop Descriptive Set Theory.
Another of the reasons that mathematicians are interested in DST is that many problems that do not have solutions in general cases, actually have solutions in the specific context of DST. We require a bit of background:

OTHER CONTENT
Another of the reasons that mathematicians are interested in DST is that many problems that do not have solutions in general cases, actually have solutions in the specific context of DST. We require a bit of background:

**Cardinality**

One of the central notions in set theory is that of **Cardinality**.
Another of the reasons that mathematicians are interested in DST is that many problems that do not have solutions in general cases, actually have solutions in the specific context of DST. We require a bit of background:

Cardinality

One of the central notions in set theory is that of **Cardinality**. Intuitively, the cardinality of a set is the **size** of the set. Two sets have the same cardinality if there is a 1-1 correspondence between the elements of the two sets.
Another of the reasons that mathematicians are interested in DST is that many problems that do not have solutions in general cases, actually have solutions in the specific context of DST. We require a bit of background:

**Cardinality**

One of the central notions in set theory is that of **Cardinality**. Intuitively, the cardinality of a set is the **size** of the set. Two sets have the same cardinality if there is a 1-1 correspondence between the elements of the two sets. For finite sets, the cardinality is just the number of members. For example, the set \( \{a, b, c\} \) has a cardinality of 3 since it has 3 members.
Cardinality

However, in the case of infinite sets, the notion of cardinality has some surprising results.
Motivation

Cardinality

However, in the case of infinite sets, the notion of cardinality has some surprising results. It can be easily shown that the set of all natural numbers \( \mathbb{N} = \{1, 2, 3, \ldots \} \) has a "smaller size" than the set of real numbers \( \mathbb{R} \).
Motivation

Cardinality

However, in the case of infinite sets, the notion of cardinality has some surprising results.

It can be easily shown that the set of all natural numbers \( \mathbb{N} = \{1, 2, 3, \ldots\} \) has a ”smaller size” than the set of real numbers \( \mathbb{R} \).

In other words, it is logically impossible to pair each natural number with a real number.
Cardinality

However, in the case of infinite sets, the notion of cardinality has some surprising results.

It can be easily shown that the set of all natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ has a ”smaller size” than the set of real numbers $\mathbb{R}$.

In other words, it is logically impossible to pair each natural number with a real number.

One natural question to ask is whether or not there are any sets that have cardinality less than $\mathbb{R}$ and greater than $\mathbb{N}$.
The Continuum Hypothesis

There are no sets with cardinality less than $\mathbb{R}$ and greater than $\mathbb{N}$.
The Continuum Hypothesis

There are no sets with cardinality less than $\mathbb{R}$ and greater than $\mathbb{N}$. It has been shown that The Continuum Hypothesis is neither provable nor disproveable in the context of classical set theory.
The Continuum Hypothesis

*There are no sets with cardinality less than \( \mathbb{R} \) and greater than \( \mathbb{N} \).*

It has been shown that The Continuum Hypothesis is neither **provable** nor **disproveable** in the context of classical set theory. However it has also been shown that certain sets in DST, called the *analytic sets*, actually satisfy the Continuum Hypothesis; in other words any analytic set has a size greater than \( \mathbb{N} \) the same size as \( \mathbb{R} \).
This project is the second part of a multi-part project. The first part of the project (Summer 2010) was concerned with learning the methods of recursion theory.
This project is the second part of a multi-part project. The first part of the project (Summer 2010) was concerned with learning the methods of recursion theory. As I mentioned before, Descriptive Set Theory combines knowledge from many diverse mathematical disciplines. Thus one of the difficulties with the subject is the amount of background material required.
This project is the second part of a multi-part project. The first part of the project (Summer 2010) was concerned with learning the methods of recursion theory.

As I mentioned before, Descriptive Set Theory combines knowledge from many diverse mathematical disciplines. Thus one of the difficulties with the subject is the amount of background material required.

I have not made significant progress on a problem but I do now have an idea of what sort of problems I can begin working on.
This project is the second part of a multi-part project. The first part of the project (Summer 2010) was concerned with learning the methods of recursion theory.

As I mentioned before, Descriptive Set Theory combines knowledge from many diverse mathematical disciplines. Thus one of the difficulties with the subject is the amount of background material required.

I have not made significant progress on a problem but I do now have an idea of what sort of problems I can begin working on.

I am going to be looking at what properties higher-order sets (specifically sets in the *Lusin Pointclass*) can be shown to have given these stronger set-theoretic hypotheses.