Continued Fractions
Discover the Power of Mathematics!

Jordan Beardslee
Department of Mathematics
College of Arts & Sciences

Faculty Mentor: William Cherry
Mentor’s Department: Mathematics
Mentor’s College: Arts & Sciences

Scholars Day: April 14, 2011

1Financial support provided by a grant from the Honors College.
Rational Numbers

What is a rational number?
Rational Numbers

What is a rational number?

A number that can be expressed as a fraction with both the numerator and denominator being integers.
What is a rational number?

A number that can be expressed as a fraction with both the numerator and denominator being integers.

e.g. 1/2 or 33/77 or -44/3
What is a rational number?

A number that can be expressed as a fraction with both the numerator and denominator being integers.

e.g. 1/2 or 33/77 or -44/3

All rational numbers written as decimals will be repeating or terminating (repeating with zeros).
Rational Numbers

What is a rational number?

A number that can be expressed as a fraction with both the numerator and denominator being integers.

e.g. 1/2 or 33/77 or -44/3

All rational numbers written as decimals will be repeating or terminating (repeating with zeros).

e.g. 1/2 = .5000... and 1/3 = .3333...
Rational Numbers

What is a rational number?

A number that can be expressed as a fraction with both the numerator and denominator being integers.

e.g. $1/2$ or $33/77$ or $-44/3$

All rational numbers written as decimals will be repeating or terminating (repeating with zeros).

e.g. $1/2 = .5000...$ and $1/3 = .3333...$

But what other kinds of numbers are there!?
Irrational Numbers

Mathematicians have proved that the decimal for $\pi$ goes on forever and is never repeating (which is what makes a number irrational), therefore we cannot write it as a fraction of two integers.
Mathematicians have proved that the decimal for $\pi$ goes on forever and is never repeating (which is what makes a number irrational), therefore we cannot write it as a fraction of two integers.

But we can write it as a fraction more than two integers! Confused Yet!!?? Thats okay, but let me elighten you. Let’s take a look how we can do this!!!
The continued fraction expansion of $\pi$

$3.14159265358979323846$

$= \left[3; 7, 15, 1, 292, \ldots \right]$
The continued fraction expansion of $\pi$

$$3.14159265358979323846 = 3 + 0.14159265358979323846$$
The continued fraction expansion of $\pi$

$$3.14159265358979323846 = 3 + \frac{1}{7.062513306}$$
The continued fraction expansion of $\pi$

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{\ddots}}}}$$
The continued fraction expansion of $\pi$

$$3.14159265358979323846 = 3 + \cfrac{1}{7 + \cfrac{1}{1 + \cfrac{1}{15 + \cdots}}}$$

$$= 3 + \cfrac{1}{7 + \cfrac{1}{1 + \cfrac{1}{15.99659441}}}$$
The continued fraction expansion of $\pi$

$$3.14159265358979323846 = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \ddots}}}$$
Continued Fractions

The continued fraction expansion of \( \pi \)

\[
3.14159265358979323846 = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{1003417231}}}}
\]
Continued Fractions

The continued fraction expansion of $\pi$

$$3.14159265358979323846$$

$$= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + 0.003417231}}}$$
The continued fraction expansion of $\pi$

$$3.14159265358979323846 = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292.6345875}}}}$$
Continued Fractions

The continued fraction expansion of $\pi$

$$3.14159265358979323846 = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \cdots}}}}$$
The continued fraction expansion of $\pi$

$$3.14159265358979323846 = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \ldots}}}}$$

We write this in shorthand like this $\left[3;7,15,1,292\ldots \right]$
Continued Fractions

This way of writing a number is called a continued fraction. Furthermore, every single number can written as a continued fraction.
Continued Fractions

This way of writing a number is called a continued fraction. Furthermore, every single number can written as a continued fraction.

Continued fractions are particularly useful for approximating numbers. Anytime we see a fraction as an approximation for a number it almost always comes from the continued fraction of that number. It has been proven by Khinchin that, relative to the size of the denominator, an approximation from the continued fraction is the best approximation.

For example, if we take the first 2 pieces of the continued fraction of $\pi$ we get:

$$3 + \frac{1}{7} = \frac{22}{7}$$

$22/7$ is actually closer to $\pi$ than 3.14 (314/100), which has a denominator of 100! Talk about denominator efficiency...
This way of writing a number is called a continued fraction. Furthermore, every single number can be written as a continued fraction.

Continued fractions are particularly useful for approximating numbers. Anytime we see a fraction as an approximation for a number it almost always comes from the continued fraction of that number. It has been proven by Khinchin that, relative to the size of the denominator, an approximation from the continued fraction is the best approximation.

For example, if we take the first 2 pieces of the continued fraction of $\pi$ we get: $3 + \frac{1}{7} = \frac{21}{7} + \frac{1}{7} = \frac{22}{7}$. 
Continued Fractions

This way of writing a number is called a continued fraction. Furthermore, every single number can be written as a continued fraction.

Continued fractions are particularly useful for approximating numbers. Anytime we see a fraction as an approximation for a number it almost always comes from the continued fraction of that number. It has been proven by Khinchin that, relative to the size of the denominator, an approximation from the continued fraction is the best approximation.

For example, if we take the first 2 pieces of the continued fraction of $\pi$ we get: $3 + \frac{1}{7} = \frac{21}{7} + \frac{1}{7} = \frac{22}{7}$.

$\frac{22}{7}$ is actually closer to $\pi$ than 3.14 (314/100), which has a denominator of 100! Talk about denominator efficiency...
More Useful Facts About Continued Fractions

\[ \sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \ddots}}} } \]

Interestingly, every square root of an integer will have a nice, predictable, repeating continued fraction... Isn't Math Beautiful!

Lets look at another irrational number, \( \sqrt{2} \).
More Useful Facts About Continued Fractions

Lets look at another irrational number, $\sqrt{2}$.

$\sqrt{2} = 1.4142135623...$
More Useful Facts About Continued Fractions

Let's look at another irrational number, $\sqrt{2}$.

$\sqrt{2} = 1.4142135623...$

$$= 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2...}}}$$
More Useful Facts About Continued Fractions

Let's look at another irrational number, $\sqrt{2}$.

$$\sqrt{2} = 1.4142135623...$$

$$= 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2...}}}}$$

$$= [1; 2, 2, 2, 2...]$$
More Useful Facts About Continued Fractions

Let's look at another irrational number, \( \sqrt{2} \).

\[
\sqrt{2} = 1.4142135623...
\]

\[
= 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}}}
\]

\[
= [1; 2, 2, 2, 2, 2, \ldots]
\]

Interestingly, every square root of an integer will have a nice, predictable, repeating continued fraction...Isn't Math Beautiful!
My Interest in Continued Fractions

We learned that the square roots of intergers have nice continued fractions, but their decimals are unpredictable.
We learned that the square roots of integers have nice continued fractions, but their decimals are unpredictable.

It turns out that most irrational numbers with nice decimals (e.g. .9090090009000...) will have unpredictable continued fractions.
My Interest in Continued Fractions

We learned that the square roots of integers have nice continued fractions, but their decimals are unpredictable.

It turns out that most irrational numbers with nice decimals (e.g. .9090090009000...) will have unpredictable continued fractions.

So what do I do!? My research involvs uncovering mysteries about special numbers that have both predictable decimals and predictable continued fractions.
My Interest in Continued Fractions

We learned that the square roots of integers have nice continued fractions, but their decimals are unpredictable.

It turns out that most irrational numbers with nice decimals (e.g. .9090090009000...) will have unpredictable continued fractions.

So what do I do!? My research involves uncovering mysteries about special numbers that have both predictable decimals and predictable continued fractions.

The number that I have put most of my focus into, the rabbit number, is a binary number that is generated from a sequence closely related to the Fibonacci numbers.
My Interest in Continued Fractions

We learned that the square roots of integers have nice continued fractions, but their decimals are unpredictable.

It turns out that most irrational numbers with nice decimals (e.g. .9090090009000...) will have unpredictable continued fractions.

So what do I do!? My research involves uncovering mysteries about special numbers that have both predictable decimals and predictable continued fractions.

The number that I have put most of my focus into, the rabbit number, is a binary number that is generated from a sequence closely related to the Fibonacci numbers.

Before we look at the rabbit number lets look at Fibonacci numbers.
The first two Fibonacci numbers are defined by being $F_0 = 0$ and $F_1 = 1$. 
The first two Fibonacci numbers are defined by being $F_0 = 0$ and $F_1 = 1$.

The next number is the sum of the two numbers before it.
Fibonacci Numbers

The first two Fibonacci numbers are defined by being $F_0 = 0$ and $F_1 = 1$.

The next number is the sum of the two numbers before it.

So, $F_2 = F_0 + F_1 = 0 + 1 = 1$
Fibonacci Numbers

The first two Fibonacci numbers are defined by being $F_0 = 0$ and $F_1 = 1$.

The next number is the sum of the two numbers before it.

So, $F_2 = F_0 + F_1 = 0 + 1 = 1$

Lets do one more, $F_3 = F_1 + F_2 = 1 + 1 = 2$
The first two Fibonacci numbers are defined by being $F_0 = 0$ and $F_1 = 1$.

The next number is the sum of the two numbers before it.

So, $F_2 = F_0 + F_1 = 0 + 1 = 1$

Lets do one more, $F_3 = F_1 + F_2 = 1 + 1 = 2$

If we did this forever we would get the sequence:
The first two Fibonacci numbers are defined by being $F_0 = 0$ and $F_1 = 1$.

The next number is the sum of the two numbers before it.

So, $F_2 = F_0 + F_1 = 0 + 1 = 1$

Lets do one more, $F_3 = F_1 + F_2 = 1 + 1 = 2$

If we did this forever we would get the sequence:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377...
Fibonacci Numbers in Nature
RABBITS!

For this exercise I will call a "1" an adult rabbit and a "0" a baby rabbit.

![Rabbit Diagram]

Numbers
Continued Fractions
My Interest
Fibonacci
Example
Rabbit Number
Other Bases
RABBITS!

For this exercise I will call a "1" an adult rabbit and a "0" a baby rabbit.
Also note that adults can have babies, and babies cannot have babies.
For this exercise I will call a "1" an adult rabbit and a "0" a baby rabbit. Also note that adults can have babies, and babies cannot have babies.

I will start with one adult rabbit:

1 (1 rabbit total)
RABBITS!

For this exercise I will call a "1" an adult rabbit and a "0" a baby rabbit.
Also note that adults can have babies, and babies cannot have babies.

I will start with one adult rabbit:
1 (1 rabbit total)
My adult then has a baby, so I have a baby and an adult:
10 (2 rabbits total)
For this excercise I will call a "1" an adult rabbit and a "0" a baby rabbit.
Also note that adults can have babies, and babies cannot have babies.

I will start with one adult rabbit:
1 (1 rabbit total)
My adult then has a baby, so I have a baby and an adult:
10 (2 rabbits total)
My Adult has a baby, and my baby grows up, so I have an adult with a baby and another adult:
10 1 (3 rabbits total)

Hmmmm... 1 2 3 5 8 .... I know a saw that somewhere!
Fibonacci Numbers = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377...
For this exercise I will call a "1" an adult rabbit and a "0" a baby rabbit. Also note that adults can have babies, and babies cannot have babies.

I will start with one adult rabbit:
1 (1 rabbit total)

My adult then has a baby, so I have a baby and an adult:
10 (2 rabbits total)

My Adult has a baby, and my baby grows up, so I have an adult with a baby and another adult:
10 1 (3 rabbits total)

What would happen next?
RABBITS!

For this exercise I will call a "1" an adult rabbit and a "0" a baby rabbit.
Also note that adults can have babies, and babies cannot have babies.

I will start with one adult rabbit:
1 (1 rabbit total)
My adult then has a baby, so I have a baby and an adult:
10 (2 rabbits total)
My adult has a baby, and my baby grows up, so I have an adult with a baby and another adult:
10 1 (3 rabbits total)
What would happen next?
10 1 10 (5 rabbits total)
RABBITS!

For this exercise I will call a "1" an adult rabbit and a "0" a baby rabbit.
Also note that adults can have babies, and babies cannot have babies.

I will start with one adult rabbit:
1 (1 rabbit total)
My adult then has a baby, so I have a baby and an adult:
10 (2 rabbits total)
My adult has a baby, and my baby grows up, so I have an adult with a baby and another adult:
10 1 (3 rabbits total)
What would happen next?
10 1 10 (5 rabbits total)
10 1 10 10 1 (8 rabbits total)
RABBITS!

For this exercise I will call a "1" an adult rabbit and a "0" a baby rabbit.
Also note that adults can have babies, and babies cannot have babies.

I will start with one adult rabbit:
1 (1 rabbit total)
My adult then has a baby, so I have a baby and an adult:
10 (2 rabbits total)
My Adult has a baby, and my baby grows up, so I have an adult with a baby and another adult:
10 1 (3 rabbits total)
What would happen next?
10 1 10 (5 rabbits total)
10 1 10 10 1 (8 rabbits total)
Hmmmm... 1 2 3 5 8 .... I know a saw that somewhere!
RABBITS!

For this exercise I will call a "1" an adult rabbit and a "0" a baby rabbit.
Also note that adults can have babies, and babies cannot have babies.

I will start with one adult rabbit:
1 (1 rabbit total)
My adult then has a baby, so I have a baby and an adult:
10 (2 rabbits total)
My Adult has a baby, and my baby grows up, so I have an adult with a baby and another adult:
10 1 (3 rabbits total)
What would happen next?
10 1 10 (5 rabbits total)
10 1 10 10 1 (8 rabbits total)
Hmmmm... 1 2 3 5 8 .... I know a saw that somewhere!
Fibonacci Numbers = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377...
In the last example we saw a sequence of 1’s and 0’s where $1 \rightarrow 10$ and $0 \rightarrow 1$ and we did it over and over again.
The Rabbit Number

In the last example we saw a sequence of 1’s and 0’s where $1 \rightarrow 10$ and $0 \rightarrow 1$ and we did it over and over again. This is what we call a substitution sequence. And this specific substitution actually yields the rabbit sequence.
The Rabbit Number

In the last example we saw a sequence of 1’s and 0’s where 1→10 and 0→1 and we did it over and over again. This is what we call a substitution sequence. And this specific substitution actually yields the rabbit sequence.
Looking at it again
1
10
101
10110
10110101...
The Rabbit Number

In the last example we saw a sequence of 1’s and 0’s where $1 \rightarrow 10$ and $0 \rightarrow 1$ and we did it over and over again. This is what we call a substitution sequence. And this specific substitution actually yields the rabbit sequence.

Looking at it again:
1
10
101
10110
10110101...

We notice that the sequence is filled with Fibonacci properties, inside and out!
The Rabbit Number

In the last example we saw a sequence of 1's and 0's where 1→10 and 0→1 and we did it over and over again. This is what we call a substitution sequence. And this specific substitution actually yields the rabbit sequence. Looking at it again:

1
10
101
10110
10110101...

We notice that the sequence is filled with Fibonacci properties, inside and out!

To get the rabbit number we simply turn this sequence into a binary "decimal" to get” .10110101₂...(which means base 2)
The Rabbit Number

In the last example we saw a sequence of 1’s and 0’s where 1→10 and 0→1 and we did it over and over again. This is what we call a substitution sequence. And this specific substitution actually yields the rabbit sequence.

Looking at it again:

1
10
101
10110
10110101...

We notice that the sequence is filled with Fibonacci properties, inside and out!

To get the rabbit number we simply turn this sequence into a binary ”decimal” to get” .101101012… (which means base 2)

\[
\frac{1}{2^1} + \frac{0}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{0}{2^5} + \frac{1}{2^6} \ldots
\]
The continued fraction for this number is:

\[
\frac{1}{2 + \frac{1}{2 + \frac{1}{4 + \frac{1}{8 + \frac{1}{32 + \frac{1}{256 \ldots}}}}}}
\]

Or: \([ 0; 2, 2, 4, 8, 32, 256, \ldots ]\)
Continued Fraction for the Rabbit Number

The continued fraction for this number is:

\[
\frac{1}{2 + \frac{1}{2 + \frac{1}{4 + \frac{1}{8 + \frac{1}{32 + \frac{1}{256} + \ldots}}}}}
\]

Or: \([ 0; 2, 2, 4, 8, 32, 256, \ldots ]\)

Notice that you can get the next number by multiplying the previous two together.
The continued fraction for this number is:

$$
2 + \frac{1}{2 + \frac{1}{4 + \frac{1}{8 + \frac{1}{32 + \frac{1}{256\ldots}}}}}
$$

Or: $[0; 2, 2, 4, 8, 32, 256, \ldots]$  
Notice that you can get the next number by multiplying the previous two together.  
More specifically, the $n^{th}$ number of this continued fraction is $2^{F_n}$!
Continued Fraction for the Rabbit Number

The continued fraction for this number is:

\[
\frac{1}{2 + \frac{1}{2 + \frac{1}{4 + \frac{1}{8 + \frac{1}{32 + \frac{1}{256} + \ldots}}}}}
\]

Or: \([ 0; 2, 2, 4, 8, 32, 256, \ldots ]\)

Notice that you can get the next number by multiplying the previous two together.

More specifically, the \(n^{th}\) number of this continued fraction is \(2^{F_n}\)!

E.g. the 6th number will be \(2^{F_6} = 2^8 = 256\)
Continued Fraction for the Rabbit Number

The continued fraction for this number is:

\[ \frac{1}{2 + \frac{1}{2 + \frac{1}{4 + \frac{1}{8 + \frac{1}{32 + \frac{1}{256} \ldots}}}}} \]

Or: \[ [0; 2, 2, 4, 8, 32, 256, \ldots] \]

Notice that you can get the next number by multiplying the previous two together.

More specifically, the \( n^{th} \) number of this continued fraction is \( 2^{F_n} \! \). E.g. the 6th number will be \( 2^{F_6} = 2^8 = 256 \). Therefore we know precisely the decimal expansion and the continued fraction of this number!
The Number in Other Bases

This number in other bases yields a continued fraction that looks more complicated and it may appear like there is no pattern.
The Number in Other Bases

This number in other bases yields a continued fraction that looks more complicated and it may appear like there is no pattern.

I wrote a program in Mathematica 8 to show us what the continued fraction would look like for any given base.
The Number in Other Bases

This number in other bases yields a continued fraction that looks more complicated and it may appear like there is no pattern.

I wrote a program in Mathematica 8 to show us what the continued fraction would look like for any given base.

The majority of my research was done for this number in base 3, so let's take a look at what the continued fraction might look like!!!

\[ \left[0; 2, 1, 1, 1, 1, 18, 13, 1, 1, 121, 13122, 797161, 1, 1, 5230176601, 33354363399333138... \right] \]

There doesn't seem to be a noticeable pattern, but why???
The Number in Other Bases

This number in other bases yields a continued fraction that looks more complicated and it may appear like there is no pattern.

I wrote a program in Mathematica 8 to show us what the continued fraction would look like for any given base.

The majority of my research was done for this number in base 3, so let's take a look at what the continued fraction might look like!!!

\[\left[0; 2, 1, 1, 1, 18, 13, 1, 1, 121, 13122, 797161, 1, 1, 5230176601, 33354363399333138\ldots\right]\]
The Number in Other Bases

This number in other bases yields a continued fraction that looks more complicated and it may appear like there is no pattern.

I wrote a program in Mathematica 8 to show us what the continued fraction would look like for any given base.

The majority of my research was done for this number in base 3, so let's take a look at what the continued fraction might look like!!!

\[0; 2, 1, 1, 1, 18, 13, 1, 1, 121, 13122, 797161, 1, 1, 5230176601, 33354363399333138...\]

There doesn’t seem to be a noticeable pattern, but why???
Exploring Base 3

The reason is because we are trying to use base 10 fractions with a base 3 number!!

[0; 2, 1, 1, 1, 1, 18, 13, 1, 1, 121, 13122, 797161, 1, 1, 5230176601, 33354363399333138...]

note: base 10 uses digits 0-9, base 3 uses digits 0-2.
Exploring Base 3

The reason is because we are trying to use base 10 fractions with a base 3 number!!

\[0; 2, 1, 1, 1, 1, 18, 13, 1, 1, 121, 13122, 797161, 1, 1, 5230176601, 33354363399333138...\]

note: base 10 uses digits 0-9, base 3 uses digits 0-2.

Let’s convert these numbers to base 3 and see what we get...
The reason is because we are trying to use base 10 fractions with a base 3 number!!

\[0; 2, 1, 1, 1, 1, 18, 13, 1, 1, 121, 13122, 797161, 1, 1, 5230176601, 33354363399333138...\]

note: base 10 uses digits 0-9, base 3 uses digits 0-2.

Let’s convert these numbers to base 3 and see what we get...

\[0; 2, 1, 1, 1, 1, 200, 111, 1, 1, 11111, 200000000, 11111111111111111111, 1, 1, 1111111111111111111111111, 20000000000000000000000000000000000,...\]
Exploring Base 3

The reason is because we are trying to use base 10 fractions with a base 3 number!!

\[0; 2, 1, 1, 1, 1, 18, 13, 1, 1, 121, 13122, 797161, 1, 1, 5230176601, 33354363399333138...\]

note: base 10 uses digits 0-9, base 3 uses digits 0-2.

Let’s convert these numbers to base 3 and see what we get...

\[0; 2, 1, 1, 1, 1, 200, 111, 1, 1, 11111, 200000000, 111111111111111111111, 2000000000000000000000...\]

Looks a bit more predictable, doesn’t it?!
Formulating Base 3

\[ [0; 2, 1, 1, 1, 1, 200, 111, 1, 1, 11111, 200000000, 1111111111111111111, 1, 1, 111111111111111111111, 20000000000000000000000000000000000, \ldots] \]
Formulating Base 3

\[0; 2, 1, 1, 1, 1, 200, 111, 1, 1, 11111, 200000000, 1111111111111, 1, 1, 111111111111111111, 200000000000000000000000000000000, \ldots\]

If you look closely at this continued fraction you will see a pattern cycle in groups of 5. This is why I had to write 5 different formulas, depending if the piece was first, second, third, fourth, or fifth in the cycle.
Formulating Base 3

\[0; 2, 1, 1, 1, 1, 200, 111, 1, 1, 11111, 200000000, 11111111111111111111111, 1, 1, 1111111111111111111111111, 200000000000000000000000000000000000000000,...\]

If you look closely at this continued fraction you will see pattern cycle in groups of 5. This is why I had to write 5 different formulas, depending if the piece was first, second, third, fourth, or fifth in the cycle.
Formulating Base 3

[0; 2, 1, 1, 1, 1, 200, 111, 1, 1, 11111, 200000000, 1111111111111111111, 1, 1, 1111111111111111111111, 200000000000000000000000000000000, ...]

If you look closely at this continued fraction you will see pattern cycle in groups of 5. This is why I had to write 5 different formulas, depending if the piece was first, second, third, fourth, or fifth in the cycle.
[0; 2, 1, 1, 1, 1, 200, 111, 1, 1, 11111, 200000000, 111111111111, 1, 1, 11111111111111111111, 2000000000000000000000000000000000,...]

If you look closely at this continued fraction you will see pattern cycle in groups of 5. This is why I had to write 5 different formulas, depending if the piece was first, second, third, fourth, or fifth in the cycle.
Formulating Base 3

\[
[0; 2, 1, 1, 1, 1, 200, 111, 1, 1, \textcolor{red}{11111}, 200000000, \\
11111111111111111111111, 1, 1, \textcolor{red}{1111111111111111111111111}, \\
20000000000000000000000000000000000,...]
\]

If you look closely at this continued fraction you will see pattern cycle in groups of 5. This is why I had to write 5 different formulas, depending if the piece was first, second, third, fourth, or fifth in the cycle.
Formulating Base 3

\[ [0; 2, 1, 1, 1, 1, 200, 111, 1, 1, 11111, 200000000, 111111111111111111111, 1, 1, 111111111111111111111111111111111, 2000000000000000000000000000000000000, ...] \]

If you look closely at this continued fraction you will see pattern cycle in groups of 5. This is why I had to write 5 different formulas, depending if the piece was first, second, third, fourth, or fifth in the cycle.

For example, every fifth number in the continued fraction can be expressed as:

\[ a_n = \frac{1}{2} \left( 3^{\frac{F_3}{5}} n^{-1} - 1 \right) \]
Formulating Base 3

\[ [0; 2, 1, 1, 1, 1, 200, 111, 1, 1, 11111, 200000000, 111111111111, 1, 1, 111111111111111111111, 20000000000000000000000000000000000,...] \]

If you look closely at this continued fraction you will see pattern cycle in groups of 5. This is why I had to write 5 different formulas, depending if the piece was first, second, third, fourth, or fifth in the cycle.

For example, every fifth number in the continued fraction can be expressed as:

\[ a_n = \frac{1}{2} \left( 3 \frac{F_3}{5}^{n-1} - 1 \right) \]

Notice that we can see Fibonacci numbers everywhere:
Formulating Base 3

\[ [0; 2, \textbf{1}, 1, 1, \textbf{1}, 200, \textbf{111}, 1, 1, \textbf{11111}, 200000000, \\
\textbf{1111111111111}, 1, 1, \textbf{111111111111111111111}, 2000000000000000000000000000000,...] \]

If you look closely at this continued fraction you will see pattern cycle in groups of 5. This is why I had to write 5 different formulas, depending if the piece was first, second, third, fourth, or fifth in the cycle.

For example, every fifth number in the continued fraction can be expressed as:

\[
a_n = \frac{1}{2} \left( 3^{\frac{3}{5} n - 1} - 1 \right)
\]

Notice that we can see Fibonacci numbers everywhere:

- These have a Fibonacci number of 1’s.
Formulating Base 3

\[ [0; 2, 1, 1, 1, 1, 200, 111, 1, 1, 11111, 200000000, 111111111111111111111111, 20000000000000000000000000000000000, \ldots] \]

If you look closely at this continued fraction you will see pattern cycle in groups of 5. This is why I had to write 5 different formulas, depending if the piece was first, second, third, fourth, or fifth in the cycle.

For example, every fifth number in the continued fraction can be expressed as:

\[
a_n = \frac{1}{2} \left( 3^{\frac{3}{5}n-1} - 1 \right)
\]

Notice that we can see Fibonacci numbers everywhere:

- These have a Fibonacci number of 1’s.
- These have a Fibonacci number of 0’s.
Wrapping up

The four other formulae needed to formulate the entire continued fraction are each different, however, similar to the one previously shown.
Wrapping up

The four other formulae needed to formulate the entire continued fraction are each different, however, similar to the one previously shown.

During this research project I was hoping to find a stronger formula that could yield the continued fraction for this number in any given base, i.e. the base is part of the formula. However, among further research I found that representations became intensely more difficult as the base went up.
Wrapping up

The four other formulae needed to formulate the entire continued fraction are each different, however, similar to the one previously shown.

During this research project I was hoping to find a stronger formula that could yield the continued fraction for this number in any given base, i.e. the base is part of the formula. However, among further research I found that representations became intensely more difficult as the base went up.

For the remainder of the semester I will try to find special cases of bases and try to make a correlation between certain bases and recognizable patterns.
Wrapping up

The four other formulae needed to formulate the entire continued fraction are each different, however, similar to the one previously shown.

During this research project I was hoping to find a stronger formula that could yield the continued fraction for this number in any given base, i.e. the base is part of the formula. However, among further research I found that representations became intensely more difficult as the base went up.

For the remainder of the semester I will try to find special cases of bases and try to make a correlation between certain bases and recognizable patterns.

Thank you again to the honors college and this remarkable opportunity in mathematics research.