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Continued Fractions and Sturmian Words
Discover the Power of Mathematics!

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What Are Continued Fractions?

A Simple Continued Fraction

$$3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}$$

The Basics

Some Terminology

- Each a_n term is called the *nth* **partial quotient**.
- Evaluating the fraction up to the *nth* partial quotient (but ignoring all of the later terms) gives a rational number, $\frac{p_n}{q_n}$. $\frac{p_n}{q_n}$ is called the *nth* **convergent**.

Partial Quotients and Convergents of π

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$\frac{p_n}{q_n} = 3 = 3.00000$

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$$3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{\dots}}}}$$

$\frac{p_n}{q_n} = \frac{333}{106} = 3.141509$

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Partial Quotients and Convergents of π

$$3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}$$

$\frac{p_n}{q_n} = \frac{355}{113}$ =3.141593

Why Use Continued Fractions? Some Advantages

Quadratic Irrational Numbers

Consider $x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}}$.

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- Similarly, any irrational solution to a quadratic equation can be represented by a periodic (repeating) continued fraction.
- Compare with the decimal representation, where $\frac{1+\sqrt{5}}{2} = 1.6180339887498948482\ldots$

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Approximations using continued fractions give around twice as much accuracy as approximations using decimal numbers (for comparable sized denominators):

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But $\pi - 3.14 \approx 0.001592$
(*i.e.* three digits of accuracy)

Continued Fraction: $\pi \approx \frac{355}{113}.$
But $\pi - \frac{355}{113} \approx -0.00000027$
(*i.e.*, seven digits of accuracy)

Some More Terminology: Subwords and Sturmian Words

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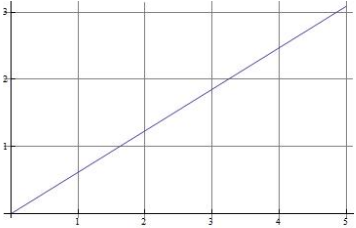
- Sturmian words are important because they are the least complicated non-periodic words. They are often studied in theoretical computer science.

Using Lines to Explain Sturmian Words

- We can also draw a line to help illustrate this word.
- Draw a line with slope m and intercept b on top of a grid. Then, start where the line intersects the y-axis and move forward.
- Just write a 0 whenever the line you drew crosses a horizontal grid line...
- ... and write a 1 whenever your line crosses a vertical grid line.
- Then a line with slope $m = \frac{-1+\sqrt{5}}{2}$ and intercept $b = 0$ would look like this:



Patterns in the Decimal Expansion

Another Clear Pattern

- Earlier we saw a number with a clear pattern in the continued fraction.
- Consider the number $0.12345678910\cdots$, which is obtained by writing a decimal point followed by each integer in binary.
- The continued fraction representation of this number is

$$\cfrac{1}{8+\cfrac{1}{9+\cfrac{1}{1+\cfrac{1}{149083+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}}}$$

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- Since this number has only 0's and 1's, let's treat it as a binary number: 0.1011010110110101101011011010...

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- Since this number has only 0's and 1's, let's treat it as a binary number: 0.10110101101101011010110110 . . .
- That number has the following continued fraction expansion:

$$\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{8+\cfrac{1}{32+\cfrac{1}{256+\cfrac{1}{8192+\cdots}}}}}}}}$$

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- As it turns out, the exponents in the partial quotients follow the Fibonacci sequence: $1, 1, 2, 3, 5, \cdots$

My Research

Questions I'm Trying to Answer

- How does changing the pattern of digits in a word affect its slope and intercept?
- How does changing the slope or intercept of a word affect its continued fraction expansion?
- Are there any other words with easy to spot patterns in both their decimal and continued fraction expansion?

Slopes and Intercepts

What Changing the Slope and Intercept Does

- The effects of some slope changes are well known.
- For example, if the intercept is 0, replacing the slope m with $1 - m$ reverses every digit; that is, all of the 1's become 0's and vice versa.
- But what happens when you change the intercept?

Slopes and Intercepts

Picking a New Intercept

- For example, let the slope $m = \frac{-1+\sqrt{5}}{2}$
- Picking an intercept $b = 0$ gives the word 1011010110110101101011
- But picking the intercept $b = \frac{1-m}{2}$ gives the word 1101011010110110101101
- There are at least two ways to explain this:

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- But picking the intercept $b = \frac{1-m}{2}$ gives the word 1101011010110110101101
- There are at least two ways to explain this:
- One way is that the change took some pairs of digits and switched them.
- Another way is that the change "rotated" either all or part of a word.

Slopes and Intercepts, Pt. 2

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- The new continued fraction is: $\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{65+\cfrac{1}{1027+\cdots}}}}}}$

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- The pattern is, of course... still waiting to be found

More Slopes and Intercepts

A Graphical Representation

- In case you were wondering, changing the intercept as we did earlier makes the word's graphical representation look like this:

- Note that the intercept change causes the lines to cross the grid lines in a slightly different order.
- This is similar to the digit switching explanation seen earlier.

Conclusion

Wrapping it up

- Numbers that have such readily apparent patterns in both their continued fraction expansion and in their decimal representation are rare.
- Because they are rare, or special, mathematicians are interested in studying numbers which do have recognizable patterns in both their decimal expansions and their continued fraction expansions.
- A better understanding of how the slope and intercept of Sturmian words relate to continued fractions may better help us predict which words have "nice" continued fraction expansions