Title: NONLINEARITY IN MODAL AND VIBRATION TESTING

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Nonlinearity in Modal and Vibration Testing

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Abstract

This set of slides describes some aspects of nonlinear vibration analysis thru use of analytical formulas and examples from real or simulated test systems. The systems are drawn from a set of examples based on years of vibration testing experience. Both traditional and new methods are used to describe nonlinear vibration.
What is Nonlinearity?

Linear Differential Equation: \( Mx'' + Cx' + Kx = F \)

Nonlinear Differential Equation: \( Mx'' + Cx'^2 + K(x + \alpha x^3) = F \)

- Superposition Fails
- Linear Frequency Response or Modal Model Does not Accurately Predict Measured Response.
- Non-Gaussian Probability Density for Gaussian Excitation.
- Coupling Between Frequencies.
- FRF Changes as a Function of Excitation Type or Level.
Sources of Nonlinearity

- Joints (Microslip and Macroslip). Hysteresis.
- Large Deflections (Geometric Nonlinearities).
- Material Nonlinearities (Foams, Viscoelastic Materials).
- Friction.
- Loose Parts (Rattling).
- Damage (Cracks).
How Nonlinearity Effects System Response

- Time History
- Autospectrum
- Frequency Response Function.
- Coherence
Nonlinear Effects on a Sinusoidal Time History

For sinusoidal excitation harmonic distortion indicates a nonlinearity somewhere in the system.

25% Harmonic Distortion 100% Harmonic Distortion
Nonlinear Effects on a Random Time History

- Response of Linear Oscillator
  - Linear SDOF Oscillator Acceleration Response

- Response of Polynomial Oscillator
  - Polynomial SDOF Oscillator Acceleration Response

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Nonlinearity Effects on the Autospectrum

**Autospectrum of Linear Ten DOF system.**

**Autospectrum of NonLinear Ten DOF system.**

Increased High Frequency Response

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Nonlinear Effects on the Autocorrelation

Linear Amplitude

Drive Point

Location

Frequency

Linear 8 DOF System

8 DOF System, Rattling at Location 8.

Increased High Frequency Response

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Nonlinear Effects on the FRF

Linear TenDOF System

Non-Linear TenDOF System

Increased Damping
Additional Modes

Noisy FRF,
More High Frequency Response

FRF of Linear TenDOF System

FRF of Nonlinear TenDOF System
Basic Nonlinear Effects.


Autospectrum- Broadens the Autospectrum By Adding Frequencies through frequency interactions. Often visible as Increased values at frequencies above drive range.

Frequency Response Function- FRF peaks are lower, implying Increased damping. FRF is noisier.

Coherence - lower, especially at higher frequencies.
Locating Nonlinear Elements

If the nonlinear behavior generates high frequencies, these may be greater in amplitude near the location of the nonlinearity.
Functions Designed to Quantify Nonlinear Behavior

- Amplitude Probability Density.
- Higher Order Spectra.
- Time or Amplitude Dependent Models.
- Wavelets.
- State Space Models based on Nonlinear Differential Equations.

- Amplitude Probability Density.
- Bicoherence
- TriCoherence.
**Probability Density Function**

Sample Filtered Random Time History

\[ pdf(x) = \frac{\sum_{n \text{ in range } x-\Delta x \text{ to } x+\Delta x}}{N} \]

**Probability** that the magnitude of \( x \) at a randomly chosen time lies between \( x-\Delta x \) and \( x+\Delta x \) limit as \( \Delta x \to 0 \)

\[ P(x) = \int_{x-\Delta x}^{x+\Delta x} pdf(x) \, dx \]

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Probability Density

\[ P(a < X < b) = \int_{a}^{b} p(x) \, dx \]

\[ a < b \]

\( p(x) \) is the probability density of \( x \).

\[ P(-\infty < X < \infty) = \int_{-\infty}^{+\infty} p(x) \, dx = 1.0 \]

Area under the Probability Density is unity, which is just a way of saying that for sure the value of \( x \) is somewhere.
Moments of a Density

Mean

$$\bar{x} = \frac{1}{N} \sum_{j=1}^{N} x_j.$$  

Standard Deviation

$$\sigma = \sqrt{\frac{1}{N-1} \sum_{j=1}^{N} (x_j - \bar{x})^2}$$

Skewness

$$skewness = \frac{1}{N} \sum_{j=1}^{N} \left[ \frac{x_j - \bar{x}}{\sigma} \right]^3$$  

Kurtosis

$$kurtosis = \frac{1}{N} \sum_{j=1}^{N} \left[ \frac{x_j - \bar{x}}{\sigma} \right]^4 - 3$$

For Gaussian

$$\mu$$  

$$\sigma$$  

Skewness = zero  

Kurtosis = $\frac{96}{\sqrt{N}}$
Histogram Estimate
1,024 points of random Noise.

Histogram Estimate
1,048,576 points of random Noise.

\[
y = \text{randn}(1048576, 1);
\]
\[
>> [n, xout] = \text{hist}(y, 128);
\]
\[
>> \text{bar}(xout, n/1024)
\]
\[
>> \text{xlabel('MAGNITUDE')}
\]
\[
>> \text{ylabel('PDF')}
\]

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The Kernel Estimator

\[ p(x) = \frac{1}{nh} \sum_{i=1}^{n} k \left( \frac{x - X_i}{h} \right) \]

where

- \( h \) = window width.
- \( n \) = number of data points.
- \( i \) = point index
- \( X_i \) = Kernel Location
- \( x \) = data point value.

The kernel estimate at \( x \) is a scaled summation of the points within the kernel, weighted by the kernel value at the point location.

\[ \text{Kernel } K, \int_{-\infty}^{\infty} k(x) dx = 1.0 \]
Kernel Density Estimate plotted on a Logarithmic Scale for Gaussian Random Noise.

Gaussian Probability Density, Log Scale

1024 Points

1048576 Points
Estimated Probability Density, Linear Single Degree of Freedom System
Driven by Gaussian Noise

Skewness=0.0078
Var=0.017
Kurtosis=-0.1165
Variance Kd=0.035

Green is Gaussian, Red is estimated response.


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ANALOG NONLINEAR OSCILLATOR

PDF of ASCIN

MAGNITUDE

PROBABILITY DENSITY

Skewness= -0.00015 std=+/- 0.0014
Kurtosis= 1.98 std=+/- 0.119
Crest= 2.999
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PDF of ASCOUT

MAGNITUDE

PROBABILITY DENSITY

Skewness= -0.2026 std=+/- 0.051
Kurtosis= 3.09 std=+/- 0.2939
Crest= 4.61

Nonlinearity in Modal and Vibration Testing
Higher Order Spectra

Second Order Spectra

\[
\begin{align*}
H_{xx}(\omega) &= X(\omega)X^*(\omega) \quad \text{Power Spectrum} \\
H_{xy}(\omega) &= Y(\omega)X^*(\omega) \quad \text{Cross Spectrum}
\end{align*}
\]

Higher Order Spectra

\[
\begin{align*}
H(\omega_1, \omega_2, \omega_1 + \omega_2) &= X(\omega_1)X(\omega_2)X(\omega_1 + \omega_2) \quad \text{Autobispectrum} \\
Y(\omega_1, \omega_2, \cdots \omega_N, \omega_1 + \omega_2 + \cdots + \omega_N) &= X(\omega_1)X(\omega_2)X(\omega_3) \cdots X(\omega_1 + \omega_2 + \cdots + \omega_N) \quad \text{Autonspectrum}
\end{align*}
\]

Higher Order Spectra have a number of variants depending on the Combination of X (Input) and Y(response) terms on the Right hand side of the equations.
Higher Order Spectra

Bicoherence,

\[ \gamma^2(\omega_1, \omega_2, \omega_{1+2}) = \frac{\left[ E(X(\omega_1)X(\omega_2)Y(\omega_1 + \omega_2)) \right]^2}{\left[ E(X(\omega_1)X(\omega_2)) \right]^2 \left[ E(Y(\omega_1 + \omega_2)) \right]^2} \]

\[ 0.0 \leq \gamma^2(\omega_1, \omega_2, \omega_{1+2}) \leq 1.0 \]

Unity for complete quadratic dependence, zero for no quadratic dependence

Reasonably straightforward, results readily displayed in three dimensional graphics.

Tricoherence in the special case for \( \omega_1 \) coupling to \( 3\omega_1 \).

\[ \gamma^2(\omega_1, \omega_1, \omega, 3\omega_1) = \frac{\left[ E(X(\omega_1)X(\omega_1)X(\omega_1)X(3\omega_1)) \right]^2}{\left[ E(X(\omega_1)X(\omega_1)X(\omega_1)) \right]^2 \left[ E(X(3\omega_1)) \right]^2} \]

\[ 0.0 \leq \gamma^2(\omega_1, \omega_1, \omega, 3\omega_1) \leq 1.0 \]
A Simple Example of Higher Order Spectra

$S_1 = \sin(2\pi 20t)$
$S_2 = \sin(2\pi 30t)$

$S_{out} = s_1 \cdot s_2$

Nonlinearity in Modal and Vibration Testing
Bicoherence of the Product of Two Sinusoids

Peaks Correspond to Quadratic Dependence.

The 45 degree line is a line of constant \( f_1 + f_2 \).

Blue line indicates a plane of symmetry

\[
\gamma^2(\omega_1, \omega_2, \omega_{1+2}) = \frac{\left[ E(X(\omega_1)X(\omega_2)Y(\omega_1 + \omega_2)) \right]^2}{\left[ E(X(\omega_1)X(\omega_2)) \right]^2 \left[ E(Y(\omega_1 + \omega_2)) \right]^2}
\]
These levels are based on averaging out the effects of unrelated signals, more blocks of averaging means a lower level.

STATISTICAL ERROR IN THE BICOHERENCE

75 Blocks of 256 Points each.

387 Blocks of 256 Points each.

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Bicoherence, Acceleration of Mass 10
Tendof System

linear

bilinear

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Bicoherence Example, Quadratically Related Signals

T=[0:32767]/1000;
x=0.05*randn(32767,1)+(sin(2*pi*50*t).*exp(-t/10))';
y=0.07*randn(32767,1)+x.^2;
[bc,ff,rndout]=bicoc1(x,y,1000,512);
mesh(ff,bc)
Contour(ff,bc,20);
Methods Reviewed

- Autospectrum
- Frequency Response Function
- Coherence
- Amplitude Probability Density (Skewness, Kurtosis).
- Bicoherence, Tricoherence (to be illustrated).
Review Some Systems

- Linear Ten Degree of Freedom Oscillator
- Bilinear Ten Degree of Freedom Oscillator.
- Eight Degree of Freedom System (Linear and nonlinear).
Ten Degree of Freedom System

Linear Case: \[ k_{45} = k_{\text{ref}} \]

Bilinear Case: \[ x_4 - x_5 \leq 0 \]
\[ k_{45} = 4.0k_{\text{ref}} \]
\[ x_4 - x_5 > 0 \]
\[ k_{45} = 0.25 k_{\text{ref}} \]

Loss of Stiffness in Tension
Ten Degree of Freedom Linear System

Acceleration Response of Mass 10

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Tendof Linear

Frequency Response Function
Force to Mass 10

Coherence Function, Force to Mass 10.
Ten Degree of Freedom Linear System Mass 5 Response

Skewness = 0.0111
Std = 0.044

Kurtosis = 0.0477
Std = 0.044
Tendof Linear

Bicoherence, Tendof, No Bilinear Spring

Frequency F2

Frequency F1

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Imaginary Part of Ten Degree of Freedom System Transfer Functions.
R = radius.
Shorter radius
Means increased
Damping.

Θ = angle, increasing
Angle means increasing
Frequency.

Ten Dof Linear

REAL PART

IMAGINARY PART

LINEAR TEN DOF SYSTEM

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Acceleration Time History Response
10DOF Bilinear Oscillator

Acceleration Response of Mass 10, Bilinear System

Time in Seconds

Amplitude

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Tendof Bilinear

Transfer Function

Coherence

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Autospectrum, Linear and Bilinear Tendof System

Linear System

Bilinear System

Distinct hump at location 4.5

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Tendof Bilinear Bicoherence, mass 5 response
Local Mode Shapes-Linear and Bilinear Ten DOF System
Eight Degree-of-Freedom System

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R721u7a, d7a
Kurtosis as a function of Bumper Location

Bumper at 1-2, Cases 11-12

Bumper at 4-5, cases 9-10

Bumper at 7-8, Cases 7-8

No Bumper cases 1-6

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No Bumper Present

Skewness=0.121  Std=0.158

Kurtosis= 0.277    Std=0.084

Bumper Present

Skewness=0.768  Std=0.161

Kurtosis= 0.644    Std=0.112

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8DOF Mode Shape 1 vs. Time

Time in Seconds

Location

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Nonlinearity in Modal and Vibration Testing
8DOF Mode Shape 1 vs. Time Bumber

Shape 1 R721D7a

Time in Seconds

Location

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Autospectra, 8 Dof System

Nonlinear, Bumber at Location 4-5

Linear or Nonlinear?
Bumper Location?

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Frequency Response Function 8DOF System

Imaginary Part

Transfer Functions, 8DOF, R721U7A

Location 0 0 10 20 30

Frequency 0 0 10 20 30 40 50 60 70

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Comparative Frequency Response Functions, 8 DoF System.

- Bumper at 1-2
- Bumper at 4-5
- No Bumper
- Bumper at 7-8

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8DOF BUMBER Frequency Response Function vs. Time

Shape 1 R721D7a

Time

Frequency