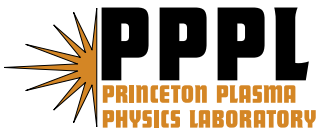

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Prepared for the U.S. Department of Energy under Contract DE-AC02-09CH11466.

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Nonlinear theoretical tools for fusion-related microturbulence: Historical evolution, and recent applications to stochastic magnetic fields, zonal-flow dynamics, and intermittency

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Abstract. Fusion physics poses an extremely challenging, practically complex problem that does not yield readily to simple paradigms. Nevertheless, various of the theoretical tools and conceptual advances emphasized at the KaufmanFest 2007 have motivated and/or found application to the development of fusion-related plasma turbulence theory. A brief historical commentary is given on some aspects of that specialty, with emphasis on the role (and limitations) of Hamiltonian/symplectic approaches, variational methods, oscillation-center theory, and nonlinear dynamics. It is shown how to extract a renormalized ponderomotive force from the statistical equations of plasma turbulence, and the possibility of a renormalized K - χ theorem is discussed. An unusual application of quasilinear theory to the problem of plasma equilibria in the presence of stochastic magnetic fields is described. The modern problem of zonal-flow dynamics illustrates a confluence of several techniques, including (i) the application of nonlinear-dynamics methods, especially center-manifold theory, to the problem of the transition to plasma turbulence in the face of self-generated zonal flows; and (ii) the use of Hamiltonian formalism to determine the appropriate (Casimir) invariant to be used in a novel wave-kinetic analysis of systems of interacting zonal flows and drift waves. Recent progress in the theory of intermittent chaotic statistics and the generation of coherent structures from turbulence is mentioned, and an appeal is made for some new tools to cope with these interesting and difficult problems in nonlinear plasma physics. Finally, the important influence of the intellectually stimulating research environment fostered by Prof. Allan Kaufman on the author's thinking and teaching methodology is described.

1. Introduction

This article records and expands upon the material I presented at the symposium on *Plasma Theory, Wave Kinetics, and Nonlinear Dynamics: KaufmanFest 2007*. Its goal is to discuss a very biased selection of analytical tools for the study of possibly strong plasma turbulence (especially of the kind relevant to fusion applications) in light of some of the methods pioneered and championed by Prof. Allan Kaufman and his coworkers. Those were inspirational to me early in my career and should be part of the repertoire of every plasma theorist, especially those just getting started. In particular, I will mention multiple-scale techniques [1], quasilinear theory [2, 3], oscillation-center transformations [4, 5], the K - χ theorem [6], Lie perturbation theory [5], and the Weyl calculus [7]. These are very well suited for applications to weak-turbulence theory. In most cases they have not been successfully applied to strong turbulence;

1959	Kraichnan [17]—Direct-interaction approximation	1959	
.	.	.	.
1960		1960	Balescu [25], Lenard [26]—BL collision operator
.	.	.	.
1	Wyld [20]—Diagrammatic formulation of NS turbulence	1	
.	Kraichnan [18]—Stochastic osc.; random-coupling model	.	.
2	Kubo [21]—Cumulant expansions	2	Vedenov <i>et al.</i> [9]—QLT
.	.	.	Drummond & Pines [10]—QLT
3	Lorenz [22]—Deterministic nonperiodic flow	3	
.	.	.	.
4		4	
.	.	.	.
1965		1965	Kadomtsev [11]—Monograph on <i>Plasma Turbulence</i>
.	.	.	.
6		6	Dupree [15]—Unmagnetized RBT
.	.	.	.
7	Kraichnan [23]—Dual cascade	7	Orszag & Kraichnan [19]—Vlasov DIA
.	.	.	Dupree [16]—RBT for $\mathbf{E} \times \mathbf{B}$ motions
8		8	Weinstock [27]—Turbulent diffusion in \mathbf{B}
.	.	.	.
1969	Benney & Newell [13]—Random wave closures	1969	Sagdeev and Galeev [12]—Monograph on <i>Nonlinear Plasma Theory</i>
.	Chirikov [24]—PhD thesis on stochasticity	.	.

Figure 1. Chronology of selected research papers on statistical methods [DIA (1959) to 1969]. More detailed versions (with some differences of emphasis) of the timeline figures presented here can be found in Ref. 28.

however, given the complexity of the subject, there really is no substitute for elegant and concise methods that apply in any regime whatsoever. They provide important reference results, and it is illuminating to understand when and why they fail and what generalizations or alternatives have evolved. With regard to the latter, I will mention the Martin–Siggia–Rose (MSR) formalism, statistical moment closures such as the direct-interaction approximation (DIA) and the eddy-damped quasinormal Markovian (EDQNM) theory, and the use of model nonlinear Langevin equations for estimating higher-order statistics. I will also describe some recent calculations that employ bifurcation theory, as well as a novel use of a wave kinetic equation to describe *nonlinear* statistical interactions. The latter analysis makes good use of a Hamiltonian formulation of $\mathbf{E} \times \mathbf{B}$ advection.

I cannot possibly give an adequate discussion of the evolution of plasma turbulence theory through its approximately half-century of development in the present short article. However, I do hope to provide some perspectives and references that may help the reader organize a vast amount of technical information.¹ Initial foundations were laid in the 1960’s (see the timeline in Fig. 1) with the development of quasilinear theory (QLT) [9, 10], weak-turbulence theory (WTT) (see Refs. 11–13 and the references in Ref. 14), and resonance-broadening theory (RBT) [15, 16]. For reference, Kraichnan’s pioneering DIA was published in 1959 [17] for strong incompressible Navier–Stokes turbulence and generalized in 1961 [18] to arbitrary quadratically nonlinear dynamical systems. Early plasma theorists were mostly oblivious to the DIA, and a fundamental 1967 article by Orszag and Kraichnan [19] critiquing RBT and describing the DIA for Vlasov turbulence was virtually ignored for about a decade.

The 1970’s (see Fig. 2) were a time of great excitement and advances on several fronts, including Davidson’s systematic treatise on WTT [14], Dewar’s pioneering work on oscillation-center QLT [4], and the work of Smith and Kaufman [37] on stochastic heating. In 1976

¹ Another related article with pedagogical intent is my written account of the four introductory lectures on plasma turbulence that I delivered at the 19th Canberra International Physics Summer School [8].

1970	Kuhn [29]— <i>The Structure of Scientific Revolutions</i>	1970	.
.	.	1	.
1	.	.	Kaufman [2]—Reformulation of QLT
.	.	2	Davidson [14]— <i>Methods in Nonlinear Plasma Theory</i>
2	Candlestickmaker [30]—Accurate determination of e	.	Benford & Thompson [36]—Probabilistic model of RBT
.	.	3	Dewar [4]—Oscillation center quasilinear theory
3	Martin, Siggia, & Rose [31]—Renormalized classical field theory	.	.
.	.	4	.
4	.	.	.
.	.	1975	Smith & Kaufman [37]—Stochastic acceleration by a single wave
1975	.	.	Krommes & Oberman [38]—Renormalization and convective cells
.	6 Orszag [32]—Review article on statistical closure	.	.
.	Kraichnan [33]—Theory of eddy viscosity	7	Cary & Kaufman [6]— K - χ theorem
7	.	.	Johnston —Mode coupling and K - χ theorem
.	.	.	& Kaufman [39] <i>via</i> Lie transforms
8	Treve [34]—Review: Theory of chaotic motion	.	DuBois & Espedal [40], Krommes [41]—Plasma DIA
.	.	1979	Littlejohn [42]—A guiding center Hamiltonian
1979	Rose [35]—Renormalization of many-particle systems	.	.

Figure 2. Chronology of selected research papers (1970 to 1979).

I published some work on the nonlinear theory of convective cells [38], a strong-turbulence calculation that led me to appreciate just how difficult the turbulence problem really was. In an attempt to broaden my horizons, I arranged for several lengthy summer visits to Berkeley, circa 1976–8. There I was introduced to Allan’s distinctive style and an immensely pleasurable and stimulating intellectual environment (quite different from the mission-oriented ambience of my home base at Princeton University’s Plasma Physics Laboratory). Allan, of course, was into everything. But I was also inspired by meeting (among others) John Cary (working on ponderomotive force *via* Hamiltonian–Lie methods); Shayne Johnston (a former fellow grad student at Princeton, then working with Allan on Hamiltonian methods for ponderomotive effects and nonlinear wave–particle interactions); Allan Lichtenberg (stochasticity); Robert Littlejohn (new tools for single-particle motion); Harry Mynick (beam–plasma interaction); and Gary Smith (stochastic heating). I also participated in the reading group that was dissecting Chirikov’s thesis line by line; that provided my first introduction to nonlinear dynamics. Either directly or indirectly, all of those interactions and research topics served to inspire my future research.

Perhaps the most significant direct consequence of my Berkeley interactions was the work on nonlinear Hamiltonian gyrokinetics by Dubin, Krommes, Oberman, and Lee [43], which built on the new techniques being developed by Robert Littlejohn [42]. However, although nonlinear gyrokinetics is absolutely key to the modern development of fusion theory and simulation, the gyrokinetics story has been told frequently [8] and reviewed at length [44, 45], so I shall not discuss it further in this article. Instead, I would like to focus on the significance of some of the other tools about which I learned at Berkeley in the context of the turbulence problem, and also point out some instances in which new tools have been required.

2. Tools for Turbulence

In Secs. 2.1–2.5 I shall give a general discussion of the challenges of a statistical theory of turbulence. The relationship of the abstract formalism to the Berkeley methods may not be immediately apparent. However, the patient reader will be rewarded in Sec. 2.6, where classical calculations relating to ponderomotive force will be reconsidered from the point of view of

renormalized statistical turbulence theory. Then in Sec. 2.7 I will describe an application of quasilinear theory to the practical problem of the determination of plasma equilibria in the presence of stochastic magnetic fields. That calculation ties together a variety of themes relating to statistical formalism.

2.1. The statistical turbulence problem

A fundamental signature of turbulence is its unpredictability, i.e., its extreme sensitivity to small changes in initial conditions. Nonlinear dynamics provides rigorous proofs that certain dynamical systems can contain behavior that is as random as a coin toss [46, 47]. Such observations motivate the application of statistical methods to nonlinear PDEs (although this is a definite leap of faith because, as Poincaré explained more than a century ago, regular and stochastic regions are usually intertwined in a very complicated fashion [48, 49]). For definiteness, let a deterministic equation evolve a field $\psi(t)$ from an initial condition ψ_0 . Formally, this can be done by the simple artifice of assigning a smooth probability density function (PDF) $P(\psi_0)$ to the initial conditions, thus creating a statistical ensemble. The general goal is then to calculate the fully multivariate probability density functional $P[\psi]$. From such a functional, one can in principle extract correlation functions involving n time points for arbitrary $n \geq 1$. Generalizations of such PDFs enable one to calculate n -point response functions as well.

Although this goal is easily stated, it is virtually impossible to achieve it in practice. The physical difficulty is that the various regions of integrable and chaotic behavior present in the typical nonlinear PDE cannot be distinguished by low-order statistical moments. As a (somewhat contrived) mathematical example, consider one ensemble comprising Gaussianly shaped spikes (examples of coherent structures) uniformly distributed on a large interval of width T , namely $\psi(t) = (2\pi\sigma^2)^{-1/2} \exp[-(t-t_0)^2/2\sigma^2]$ with $P(t_0) = T^{-1}$; and another ensemble whose realizations obey the stochastic oscillator equation [18]

$$\partial_t \tilde{\psi}(t) + i\tilde{\Omega}\tilde{\psi} = 0, \quad (1)$$

where tilde denotes a random variable and $\tilde{\Omega}$ is a centered Gaussian random variable with $\langle \tilde{\Omega}^2 \rangle = \beta^2$. The covariances $\langle \delta\psi(t)\delta\psi^*(t') \rangle$ (where $\delta\psi \doteq \tilde{\psi} - \langle \tilde{\psi} \rangle$ and the angle brackets denote ensemble averaging) can both be made identical to $\exp(-\frac{1}{2}\beta^2\tau^2)$ (where $\tau \doteq t-t'$ and \doteq denotes a definition) with the choices $\beta = (2\pi)^{1/2}T$ and $\sigma = (4\pi)^{-1/2}T^{-1}$ [neglecting corrections of $O(T^{-1})$].

In spite of such difficulties, the traditional analytical approach to the turbulence problem has been the construction of statistical closures based on low-order moments. That subject has been reviewed exhaustively [28, and references therein]. More recently, methods based on the construction of entire PDFs have been explored [50, 51]. Yet another approach is a direct attack using methods of nonlinear dynamics [52]. However, while the latter techniques have considerable promise for systems with small number of degrees of freedom (for some related discussion, see Sec. 3.1), it is unclear how to proceed for realistic systems with huge numbers of excited modes; even if there is a strange attractor embedded in the phase space, its dimension may be very large.

2.2. Statistical observables

The first step in developing a statistical formalism is to define the basic dependent variables. That is nontrivial. One obvious candidate is the mean field $\langle \psi \rangle$, but that frequently vanishes due to symmetry. At the level of two-point functions, in an Eulerian framework there are two natural functions: the correlation function $C(t, t') \doteq \langle \delta\psi(t)\delta\psi(t') \rangle$; and the response function $R(t; t') \doteq \langle \delta\psi(t)/\delta\hat{\eta}(t') \rangle$, where $\hat{\eta}$ is a statistically sharp source term added to the right-hand side of the dynamical equation, i.e., $\partial_t \tilde{\psi} + \dots = \hat{\eta}(t)$. That $\langle \psi \rangle$ and C are cumulants [21] is

clear; the interpretation of R as a cumulant is more subtle and interesting and follows from the work of MSR [28, 31].

Statistically averaged quantities such as $\langle\psi\rangle$, C , and R are called *statistical observables*. ($\tilde{\psi}$ itself is called the *primitive amplitude* and is assumed to obey a dynamical evolution equation that is of first order in time and possesses a polynomial, typically quadratic, nonlinearity.) In a cumulant description of turbulence, the goal is to find a system of equations closed in a small number of such low-order observables. That is difficult because it is easy to see that nonlinearity in stochastic quantities couples cumulants through all orders. This is the well-known *statistical closure problem* [53].

2.3. Resonance broadening

For an introductory discussion of some of the difficulties inherent in a formalism appropriate for strong turbulence, I shall focus on the response function R , which is a mean Green's function. In simple cases, it can be interpreted as a conditional probability. [Recall that the PDF of a random variable $\tilde{\psi}$ can be expressed as $P(\psi) = \langle\delta(\psi - \tilde{\psi})\rangle$.] Two important basic examples of response functions are (i) the single-particle propagator for free-streaming motion, and (ii) the diffusion Green's function. The former obeys

$$\partial_t g^{(0)}(\mathbf{x}, \mathbf{v}, t; \mathbf{x}', \mathbf{v}', t') + \mathbf{v} \cdot \nabla g^{(0)} = \delta(t - t')\delta(\mathbf{x} - \mathbf{x}')\delta(\mathbf{v} - \mathbf{v}'), \quad (2)$$

the solution of which is

$$g^{(0)}(\mathbf{x}, \mathbf{v}, t; \mathbf{x}', \mathbf{v}', t') = H(t - t')\delta(\mathbf{x} - \mathbf{x}' - \mathbf{v}(t - t'))\delta(\mathbf{v} - \mathbf{v}'). \quad (3)$$

[Here $H(\tau)$ is the Heaviside unit step function that reflects causality.] Nothing is random here; $g^{(0)}$ is its own mean. It is the conditional \mathbf{x} - \mathbf{v} phase-space PDF of finding the particle at (\mathbf{x}, \mathbf{v}) at time t , given that it was at $(\mathbf{x}', \mathbf{v}')$ at t' . This particular function is singular; its Fourier transform² ($\mathbf{x} - \mathbf{x}' \rightarrow \mathbf{k}$, $t - t' \rightarrow \omega$) is

$$\hat{g}^{(0)}(\mathbf{k}, \mathbf{v}, \omega; \mathbf{v}') = \frac{\delta(\mathbf{v} - \mathbf{v}')}{-i(\omega - \mathbf{k} \cdot \mathbf{v} + i\epsilon)}, \quad (4)$$

where the presence of ϵ (a positive infinitesimal) reflects causality. Note for future use that the reduced, \mathbf{v} -space PDF is $g^{(0)}(\mathbf{v}, t; \mathbf{v}', t') = \int d\mathbf{x} g^{(0)}(\mathbf{x}, \mathbf{v}, t; \mathbf{x}', \mathbf{v}', t') = H(t - t')\delta(\mathbf{v} - \mathbf{v}')$ or

$$\hat{g}^{(0)}(\mathbf{v}, \omega; \mathbf{v}') = \frac{\delta(\mathbf{v} - \mathbf{v}')}{-i(\omega + i\epsilon)}, \quad \hat{g}^{(0)}(\boldsymbol{\lambda}, \omega) = \frac{1}{-i(\omega + i\epsilon)}, \quad (5)$$

where $\boldsymbol{\lambda}$ is the Fourier wave number conjugate to $\mathbf{v} - \mathbf{v}'$.³

The diffusion Green's function is $g(\mathbf{v}, t; \mathbf{v}', t') \doteq \langle\tilde{g}(\mathbf{v}, t; \mathbf{v}', t')\rangle$, where

$$\partial_t \tilde{g}(\mathbf{v}, t; \mathbf{v}', t') + \tilde{\mathbf{a}}(t) \cdot \partial_{\mathbf{v}} \tilde{g} = \delta(t - t')\delta(\mathbf{v} - \mathbf{v}') \quad (6)$$

and $\tilde{\mathbf{a}}(t) = (q/m)\tilde{\mathbf{E}}(t)$ is Gaussian white noise characterized by $\langle\tilde{\mathbf{a}}\rangle = \mathbf{0}$ and $\langle\delta\tilde{\mathbf{a}}(t)\delta\tilde{\mathbf{a}}(t')\rangle = 2D\delta(t - t')$. Although the most conventional way of solving for g is to recognize that the average of Eq. (6) is rigorously the diffusion equation

$$\partial_t g(\mathbf{v}, t; \mathbf{v}', t') - D\nabla_{\mathbf{v}}^2 g = \delta(t - t')\delta(\mathbf{v} - \mathbf{v}'), \quad (7)$$

² A consistent set of conventions is described in Appendix A of either Ref. 28 or Ref. 45. In this paper I use the circumflex to denote the Fourier transform in at least one of the variables.

³ Translational invariance in \mathbf{v} is not a general property of velocity-dependent Green's functions. For example, $g^{(0)}(\mathbf{x}, \mathbf{v}, t; \mathbf{x}', \mathbf{v}', t')$ does not possess that property.

an alternate approach is to note that the solution of Eq. (6) is, via the method of characteristics,

$$\tilde{g}(\mathbf{v}, t; \mathbf{v}', t') = \delta \left(\mathbf{v} - \mathbf{v}' - \int_{t'}^t d\bar{t} \tilde{\mathbf{a}}(\bar{t}) \right), \quad (8)$$

the average of which can be calculated using cumulant expansions (which truncate at order 2 for Gaussian statistics). Either way, one finds

$$\hat{g}(\mathbf{v}, t; \mathbf{v}', t') = H(t - t') \exp \left(-\frac{|\mathbf{v} - \mathbf{v}'|^2}{4D(t - t')} \right), \quad (9a)$$

$$\hat{g}(\boldsymbol{\lambda}, \omega) = \frac{1}{-i(\omega + i\lambda^2 D)}. \quad (9b)$$

Equation (9b) should be compared with Eq. (5). Whereas in Eq. (5) all frequencies (with the exception of the single point at $\omega = 0$) are large with respect to ϵ , the resonance-broadened result (9b) opens up a continuous band of frequencies that are small with respect to the diffusion rate. For those frequencies, it is not valid to treat D (or $\tilde{\mathbf{a}}$) as small and endeavor to calculate its effect *via* regular perturbation theory. Such an attempt leads to singularities that grow progressively worse with order; thus, with $\eta \doteq \lambda^2 D$ denoting the resonance broadening, one has formally

$$\frac{1}{-i(\omega + i\eta)} = \frac{1}{-i\omega} \left[1 - i\frac{\eta}{\omega} - \left(\frac{\eta}{\omega}\right)^2 + \dots \right]. \quad (10)$$

Only the sum of the series through all orders reconstructs the well-behaved function (9b).

2.4. Renormalization

It is undesirable to be forced to sum such series explicitly for several reasons: (i) in realistic situations, the values of most of the terms in the infinite series are not available in any practical way; (ii) an asymptotic series, even through all orders, may not represent the function from which it is derived [54]. (The classic example is $e^{-1/\epsilon} \sim 0 + 0\epsilon + 0\epsilon^2 + \dots$.) It is better if one is never required to calculate the series in the first place. Very loosely, any technique that effects or, better, bypasses altogether the summation of an infinite series is called *renormalization*. The term arises from quantum field theory, where one is concerned with determining the physically measurable (observable) mass and charge of a particle.

Renormalization is a vast subject [55] that certainly cannot be delved deeply into here. But it is useful to mention some of the consequences, which can be surprising. In particular, renormalization can lead to *anomalous scalings* that cannot be predicted by naive dimensional analysis [56]. A model of random passive advection illustrates the basic point. Consider

$$\partial_t \tilde{\psi}(\mathbf{x}, t) + \tilde{\mathbf{V}}(t) \cdot \nabla \tilde{\psi} = 0. \quad (11)$$

Here $\tilde{\mathbf{V}}$ is a statistically specified random velocity. In reality, $\tilde{\mathbf{V}}$ would depend on both \mathbf{x} and t , but here let us ignore its \mathbf{x} dependence for simplicity. Then the spatial Fourier transform of Eq. (11) is

$$\partial_t \tilde{\psi}_{\mathbf{k}}(t) + i\tilde{\Omega}(t)\tilde{\psi}_{\mathbf{k}} = 0, \quad (12)$$

which introduces the random advection frequency $\tilde{\Omega}(t) \doteq \mathbf{k} \cdot \tilde{\mathbf{V}}(t)$. (This frequency also plays a role in the later considerations of zonal flows in Sec. 3.2.) For even greater ease of analysis, let us ignore the time dependence as well, i.e., $\tilde{\Omega}(t) \rightarrow \tilde{\Omega}$, a Gaussian random number with $\langle \tilde{\Omega} \rangle = 0$

and $\langle \tilde{\Omega}^2 \rangle = \beta^2$. Thus, we have arrived at the stochastic oscillator equation (1) (repeated here for convenience),

$$\partial_t \tilde{\psi} = -i\tilde{\Omega}\tilde{\psi}, \quad (13)$$

but now with a physical understanding of $\tilde{\Omega}$ as a measure of nonlinear advection.

Although Eq. (13) is nonlinear in stochastic quantities, it is dynamically linear (as emphasized by Kraichnan [18]); thus the exact solution is trivially $\tilde{\psi}(t) = \exp(-i\tilde{\Omega}t)\tilde{\psi}_0$ or $\tilde{\psi}(\omega) = 2\pi\tilde{\psi}_0\delta(\omega - \tilde{\Omega})$. (I shall assume that $\tilde{\psi}_0$ is a Gaussianly distributed random variable statistically independent of \tilde{V} .) Either of the t - or ω -dependent forms demonstrates conservative evolution of any single realization. Because the Gaussian PDF of $\tilde{\Omega}$ is largest (and smooth) at $\tilde{\Omega} = 0$, the form of $\tilde{\psi}(\omega)$ indicates a broad resonance at $\omega = 0$; this is characteristic of strong turbulence. Note that although each realization is oscillatory, statistical averaging leads *via* phase mixing to a decaying correlation function:

$$C(t, t') = C(\tau) \doteq \langle \delta\psi(t)\delta\psi^*(t') \rangle = \exp(-\frac{1}{2}\beta^2\tau^2)\langle |\delta\psi_0|^2 \rangle. \quad (14)$$

This phase-mixing-induced decay (closely related to Landau damping) is analogous to the effects of the “nonlinear scrambling” that occurs in dynamically nonlinear systems with sufficiently strong mixing properties [32].

Although in the present case one knows the exact solution, one does not have that luxury for realistic nonlinear PDEs, so one is led to consider approximate *equations* for relevant observables. In doing so, it turns out to be very useful to introduce the concept of a mean Green’s function. Thus, consider

$$\partial_t \tilde{\psi}(t) + i\tilde{\Omega}\tilde{\psi} = \hat{\eta}(t), \quad (15)$$

where a statistically sharp external source $\hat{\eta}$ has been added.⁴ The average response of the oscillator is characterized by the *mean Green’s function* or *infinitesimal response function*⁵

$$R(t; t') \doteq \left\langle \frac{\delta\psi(t)}{\delta\hat{\eta}(t')} \right\rangle \Big|_{\hat{\eta}=0}. \quad (16)$$

In general, one can postulate for R the formally closed *Dyson equation*

$$\partial_t R(t; t') + \int_{t'}^t d\bar{t} \Sigma(t; \bar{t}) R(\bar{t}; t') = \delta(t - t'). \quad (17)$$

Then the goal becomes the calculation of the *memory function* Σ .

Equation (17) is nonlocal in time, which is to be expected in any system that possesses a nonzero autocorrelation time. But if one wants to just capture the time scale, one may less ambitiously characterize the oscillator by the Markovian equation

$$\partial_\tau R(\tau) + \eta^{\text{nl}} R = \delta(\tau), \quad (18)$$

where $\eta^{\text{nl}} \doteq \int_0^\infty d\bar{\tau} \Sigma(\bar{\tau})$, and attempt to calculate η^{nl} .⁶ Let us begin naively and try to do that perturbatively. Now at lowest order (quasilinear level), it is easy to find that

$$\Sigma^{\text{QL}}(\tau) = \beta^2 R^{(0)}(\tau), \quad (19)$$

⁴ Do not confuse the external source $\hat{\eta}$ with the resonance broadening η . Unfortunately, there are not enough letters in the Greek alphabet. The notation $\hat{\eta}$ is used for consistency with Ref. 31. Here the circumflex does not indicate a Fourier transform, but rather a particular kind of source.

⁵ Setting $\hat{\eta}$ to 0 after the functional differentiation is unnecessary in this dynamically linear problem, but is required more generally.

⁶ The superscript nl stands for nonlinear and serves to distinguish η^{nl} from linear damping that will be added later.

where $R^{(0)}(\tau) = H(\tau)$ is Green's function for the linear dynamics. Thus the quasilinear approximation is causal but otherwise constant. This unfortunately leads to a severe pathology for the predicted damping:

$$\eta^{\text{QL}} = \int_0^\infty d\bar{\tau} \Sigma^{\text{QL}}(\bar{\tau}) = \int_0^\infty d\bar{\tau} \beta^2 \underbrace{R^{(0)}(\bar{\tau})}_1 = \begin{cases} 0 & (\beta = 0) \\ \infty & (\beta \neq 0), \end{cases} \quad (20)$$

which is a discontinuous and singular function of β . Such discontinuities and infinities are always signatures of trouble with the underlying approximations. (In this case, the difficulty is that $R^{(0)}$ does not decay in time.) The infinity arising at lowest order precludes further perturbation-theoretic development, since all higher-order terms would be built on the infinite lowest-order result and would have indeterminate value.

To circumvent this problem, one can regularize the infinity by adding some linear damping ν to Eq. (13):

$$\partial_t \tilde{\psi} + \nu \tilde{\psi} = -i\tilde{\Omega} \tilde{\psi}. \quad (21)$$

Now the linear Green's function is $R^{(0)}(\tau) = H(\tau)e^{-\nu\tau}$, so

$$\eta^{\text{QL}} = \int_0^\infty d\bar{\tau} \Sigma^{\text{QL}}(\bar{\tau}) = \beta^2/\nu, \quad (22)$$

which happily is a continuous function of β . With perturbation theory regularized, one can now proceed order by order and eventually sum through all orders. Although that process is still complicated [18], one crucial part of the result can be shown to be the replacement of $R^{(0)}$ by R , i.e.,⁷

$$\Sigma^{\text{nl}}(\tau) \approx \beta^2 R(\tau). \quad (23)$$

Although R is not yet known, it has been assumed to obey the Markovian dynamics $\partial_t R + \nu R + \eta^{\text{nl}} R = \delta(t - t')$, so one has the formal solution

$$R(\tau) = H(\tau) \exp[-(\nu + \eta^{\text{nl}})\tau]. \quad (24)$$

Upon calculating the total damping rate by integrating Eq. (23), one obtains the self-consistent equation

$$\eta^{\text{nl}} = \frac{\beta^2}{\nu + \eta^{\text{nl}}}, \quad (25)$$

which is a nonlinear algebraic equation that can be solved for η^{nl} . Although that can obviously be done exactly, it is more instructive to note that the equation remains well behaved in the limit $\nu \rightarrow 0$ (recall that ν was inserted only as an artifice to regularize an infinity), in which case one easily finds $\eta^{\text{nl}} \rightarrow \beta$. It is remarkable that this damping rate is finite in the limit $\nu \rightarrow 0$ although it was infinite in the lowest-order theory with $\nu = 0$.

It is important to note that the renormalized solution exhibits *anomalous scaling* in its dependence on the advection level β :

$$\eta^{\text{nl}} = \begin{cases} \beta^2/\nu & (\text{quasilinear theory}), \\ \beta & (\text{renormalized}). \end{cases} \quad (26)$$

Furthermore, the correct exponent of β , namely 1, cannot be predicted from dimensional analysis. Given that the problem is characterized by the two frequencies ν and β , dimensional analysis predicts merely that $\eta^{\text{nl}} \propto \nu^a \beta^b$ with $a + b = 1$. Only the renormalization, which

⁷ This is the direct-interaction approximation for the stochastic oscillator [18].

formally involves consideration of an infinite number of terms, is capable of selecting the proper exponent.⁸

A fundamental advance in the theory of renormalization was made by MSR [31], who generalized to classical statistical dynamics important generating-functional methods of quantum field theory (the Schwinger formalism [57, 58]). The basic observation is that all statistical observables depend functionally on external sources. Even in cases in which such sources are absent in the true physical situation (consider the Vlasov equation, for example), it is useful to introduce them anyway because (i) the cumulants can be shown to be functional derivatives of a certain generating functional with respect to the sources; and (ii) the induced dependencies on the sources can (often) be used to deduce, *via* a functional chain rule, functional relations among the observables themselves. Those persist even in the limit of vanishing sources. One is led (see Sec. 2.6.2) to a system of coupled Dyson equations for C and R (plus the exact equation for the mean field). Approximation comes in the truncation of functional equations for the constituents of Σ^{nl} . At lowest order, Kraichnan's DIA emerges. Of course, understanding the nature of the approximation is quite nontrivial. Martin has stressed that the MSR formalism is "not a panacea" [59]. However, the basic point remains that by proceeding in that way one is led to a self-consistent expression for the memory function. Note that giving any nonzero value at all to Σ^{nl} broadens the resonance predicted by the zeroth-order Green function and thus subsumes the effects of an infinite number of terms. That is, instead of writing $R = [(R^{(0)})^{-1} + \Sigma^{\text{nl}}]^{-1}$, which cannot be approximated as $R^{(0)}$ near $(R^{(0)})^{-1} = 0$, one considers $R^{-1} = (R^{(0)})^{-1} + \Sigma^{\text{nl}}$ and approximates Σ^{nl} . This philosophy was well explained in the clearly written and highly recommended introduction to the original paper by MSR [31].

2.5. The role of dissipation

In the previous example, dissipation was introduced only as a way of regularizing a divergent perturbation theory. But in classical physics dissipation is virtually always present on physical grounds, the basic paradigm being the Navier–Stokes equation, which contains the viscosity (momentum-diffusion) term $\mu \nabla^2$. In such situations the statistical dynamics contain a built-in (generally short-wavelength) cutoff. Thus the MSR/Dyson equations make no reference to a cutoff, and the focus of classical renormalization shifts to effects of statistical self-consistency.

It is important to distinguish dissipation of the primitive amplitude from dissipative effects that emerge after statistical averaging. The diffusion equation (7) is dissipative, yet it arises from a conservative amplitude equation. The same is true for the Σ^{nl} calculated for the renormalized $\nu = 0$ stochastic oscillator.

Unfortunately, dissipation in the primitive amplitude equation is incompatible with many of the elegant tools derived for conservative systems, including Hamiltonian formulations and construction of equilibrium ensembles using notions of Gibbsian statistical mechanics [60, and references therein]. With characteristic insight, Prof. Kaufman made a start on this problem with his 1984 introduction of *dissipative brackets* [61] (see also the related work of Morrison in Ref. 62 and these Proceedings.) That interesting formalism is well suited to discussion of Markovian kinetic equations such as the Fokker–Planck equation with the Landau collision operator. However, it has not as yet been shown to coexist gracefully with the more fundamental non-Markovian statistical formalisms. Thus, the techniques that so far have proven to be most useful for the general turbulence problem, which is typically forced and dissipative, have a different flavor than most of the basic tools developed by Kaufman and his collaborators.

⁸ One is led immediately to the proper answer by arguing that the ultimate statistics should not depend on ν at all. (Indeed, ν was absent from the original equation.) But in the Markovian perturbation theory for the example, it is impossible to ignore ν and deal with finite terms. A belief that the anomalous exponent of 1 is trivial for this problem amounts to an acceptance of the need for renormalization, which is exactly the point.

Nevertheless, some of those ideas and results are hiding in the general equations of classical statistical dynamics [31], as we will see in the next section.

2.6. Renormalized spectral balance, ponderomotive effects, and the search for a renormalized K - χ theorem

In this section I discuss how ponderomotive effects are embedded in the general renormalized equations of plasma turbulence, and I present some initial steps toward the possible derivation of a so-called *renormalized K - χ theorem*. In the interest of full disclosure, I must state at the outset that, although various nontrivial details have been worked out, true understanding has not yet been achieved and it is far from obvious that a renormalized K - χ theorem even exists.

2.6.1. The classical K - χ theorem. At the time I visited Berkeley in the middle 1970's, one of Allan's many interests was ponderomotive force, especially its systematic derivation from Hamiltonian- and Lie-theoretic perturbation methods. Dewar had already published his fundamental paper on "Oscillation center quasilinear theory" in 1973 [4] and had followed that up with his ambitious 1976 paper on renormalized perturbation theory [5], in which he gave an important operator generalization of Deprit's Lie perturbation method [63]. Shortly thereafter, Cary and Kaufman published their work on "Ponderomotive force and linear susceptibility in Vlasov plasmas" [6], in which the so-called *K - χ theorem* was first stated; a concise proof was soon published by Johnston and Kaufman [39]. That theorem predicts a specific and simple relation between (i) the second-order ponderomotive Hamiltonian $K^{(2)}$ for a particle in a slightly inhomogeneous but rapidly oscillating wave field, and (ii) the (linear, or lowest-order) susceptibility $\chi^{(0)}$. The theorem can be stated quite generally, as was elegantly demonstrated later by Kaufman [64], but I shall just discuss a simple special case for illustration. Consider a temporally sinusoidal electric field $\mathbf{E}_\omega(\mathbf{x})$. With $(\mathbf{X}, \mathbf{P}) \equiv \Gamma$ being the oscillation-center position and momentum, Cary and Kaufman quote the theorem in the form

$$K^{(2)}(\Gamma) = -\frac{1}{4\pi} \int d\mathbf{x} d\mathbf{x}' \mathbf{E}_\omega^*(\mathbf{x}) \cdot \frac{\delta\chi^{(0)}(\mathbf{x}, \mathbf{x}')}{\delta f(\Gamma)} \cdot \mathbf{E}_\omega(\mathbf{x}'), \quad (27)$$

where f is the background distribution function. For the special case of electrons with charge e and mass m in an electrostatic wave of wave number \mathbf{k} with weakly inhomogeneous field intensity $\mathcal{E}_{\mathbf{k},\omega}(\mathbf{X}) \doteq |E_{\mathbf{k},\omega}(\mathbf{X})|^2$, the result is

$$K^{(2)}(\Gamma) = \left(\frac{e^2}{m}\right) \frac{\mathcal{E}_{\mathbf{k},\omega}(\mathbf{X})}{[\omega - \mathbf{k} \cdot \mathbf{v}(\mathbf{P})]^2}. \quad (28)$$

The Hamiltonian equations of motion

$$\frac{\partial K}{\partial \mathbf{P}} = \dot{\mathbf{X}}, \quad (29a)$$

$$-\frac{\partial K}{\partial \mathbf{X}} = \dot{\mathbf{P}} \equiv \mathbf{F} \quad (29b)$$

thus identify the ponderomotive force on the oscillation center as $\mathbf{F} = -\nabla K$. Here the gradient operates on the weak inhomogeneity of the field intensity \mathcal{E} .

It is interesting to inquire how this result relates to statistical turbulence theory. That there should be a connection is obvious on physical grounds: ponderomotive force is a real physical effect that should not disappear when a single coherent wave is generalized to an inhomogeneous ensemble of turbulent waves. More abstractly, one is motivated by the observation that the *forms* of the statistical turbulence equations do not depend on the precise way in which the

averaging procedure $\langle \dots \rangle$ is implemented. Since the oscillation-center transformation, which systematically removes rapid oscillations, is a kind of averaging operation, one may expect that ponderomotive effects are included in the general statistical theory, and this will be seen to be true. On the other hand, no *statistical* averaging is done in the conventional derivations of either the ponderomotive force or the linear susceptibility, and that is a great simplification. In general, the presence of nontrivial statistical correlations at higher orders greatly complicates the general discussion, and a significant lack of complete understanding of those correlations turns out to be the sticking point in the derivation of a renormalized K - χ theorem.

2.6.2. General structure of the statistical theory of Vlasov turbulence; the renormalized dielectric function. Unfortunately, the general statistical theory of the Vlasov equation is somewhat tedious. It must, of course, be compatible with weak-turbulence theory [12] (which can be derived by a regular-perturbation-theoretic treatment of the Vlasov equation [14]), including at lowest nonlinear order wave-wave-wave and wave-wave-particle interactions. In addition, it must be renormalized in order to deal with resonance-broadening effects. Most of the details [28, 40] cannot be discussed here.

However, the general structure of the Dyson equations that couple Vlasov response and correlations is reasonably straightforward and instructive to consider. I shall first discuss the structure for a generic quadratically nonlinear equation of the form

$$\partial_t \psi(1) = L(1, 2)\psi(2) + \frac{1}{2}M(1, 2, 3)\psi(2)\psi(3), \quad (30)$$

where a numerical argument refers to a set of dependent variables such as $1 \equiv \{\mathbf{x}_1, \mathbf{v}_1, s_1, t_1\}$ (s being a species label); summation/integration over repeated arguments is assumed. The Vlasov equation ($\psi \rightarrow f$), which possesses the quadratic integro-differential nonlinearity $(q/m)\mathbf{E} \cdot \partial_{\mathbf{v}}f$, can be written in this form by an appropriate choice of mode-coupling coefficient M . To do so, I consider electrostatics for simplicity, write $\mathbf{E}(1) = \widehat{\mathbf{E}}(1, 2)f(2)$, where $\widehat{\mathbf{E}}$ is the linear operator that inverts Poisson's equation,⁹ and adopt the shorthand notation $\partial \doteq (q/m)\partial_{\mathbf{v}}$. Then

$$M_{\text{Vlasov}}(1, 2, 3) = \widehat{\mathbf{E}}(1, 2) \cdot \partial_1 \delta(1 - 3) + (2 \Leftrightarrow 3). \quad (31)$$

(Symmetrization is possible here because the electric field is allowed to self-consistently respond to changes in the distribution function. It is not permissible in a passive problem in which the field is specified; see the calculation below.)

Equation (30) generates the exact equation for the mean field

$$\partial_t \langle \psi \rangle(1) = L(1, 2)\langle \psi \rangle(2) + \frac{1}{2}M(1, 2, 3)\langle \psi \rangle(2)\langle \psi \rangle(3) + \frac{1}{2}M(1, 2, 3)C(2, 3), \quad (32)$$

where $C(1, 1') \doteq \langle \delta\psi(1)\delta\psi(1') \rangle$ and $\delta\psi \doteq \psi - \langle \psi \rangle$. From the exact equation for the fluctuation,

$$\partial_t \delta\psi = L\delta\psi + M\langle \psi \rangle\delta\psi + \frac{1}{2}M(\delta\psi\delta\psi - C), \quad (33)$$

one sees that the linear operator is modified by a mean-field term: $L \rightarrow \widehat{L} \doteq L + M\langle \psi \rangle$. [For the self-consistent Vlasov equation, the effective linear operator is $\widehat{L} = -(\mathbf{v} \cdot \nabla + \mathbf{E} \cdot \partial + \partial f \cdot \widehat{\mathbf{E}})$.] Then the Dyson equation for the two-point response function $R(1; 1')$ has the form

$$\partial_t R(1; 1') - \widehat{L}R + \Sigma(1; \bar{1})R(\bar{1}; 1') = \delta(1 - 1'). \quad (34)$$

⁹ Specifically, $\widehat{\mathbf{E}}_{\mathbf{k}, \omega}(s, \mathbf{v}; \bar{s}, \bar{\mathbf{v}}) = -(4\pi i \mathbf{k}/k^2)(\bar{n}q)_{\bar{s}}$, where \bar{n} is the mean density.

Σ is a generalized “turbulent collision operator”; it describes the effects of resonance broadening, self-consistent polarization processes, (part of) nonlinear mode coupling, *etc.* Its precise form depends on the particular statistical closure, such as the DIA; it is a nonlinear functional of C and R itself.¹⁰ It is frequently useful to think of Σ as a generalized Fokker–Planck operator that contains turbulent diffusion and drag terms [28, 65].

Response R is self-consistently coupled through Σ to fluctuations C . The Dyson equation for C is

$$\partial_t C(1, 1') - \widehat{L}C + \Sigma(1; \bar{1})C(\bar{1}, 1') = F(1, \bar{1})R(1'; \bar{1}). \quad (35)$$

Here a new function F (also functionally dependent on C and R) has been introduced. Upon observing that the operator on the left-hand side of Eq. (35) is Green’s function for Eq. (34), namely R^{-1} , the C equation can be written in the symmetrical form

$$C = RFR^T, \quad (36)$$

where T denotes transpose. This is one form of the *spectral balance equation* for turbulence. The positive-definite function F may be thought of as the covariance of a certain internal *incoherent noise* arising from the nonlinearity. That forcing competes with *coherent nonlinear damping* (the Σ term in the R equation); the resulting balance sets the fluctuation level C . In addition to the detailed review of spectral balance in Ref. 28, further pedagogical words about the spectral balance equation can be found in the author’s recent review article on gyrokinetic noise [45].¹¹

For nonlinear fluid equations (e.g., the Navier–Stokes equation or the Hasegawa–Mima equation), the Dyson equations can be studied as is. The most sophisticated yet (sometimes) workable approximation is the DIA; Markovian closures such as the EDQNM exist as well. Application to the Vlasov equation requires further discussion because of the multi-level nature of the physics: the fundamental random variable (the Klimontovich distribution function) lives in a 6D phase space, but the electric field, which defines the nonlinearity, lives in the 3D \mathbf{x} space. This leads to a decomposition of the damping operator Σ into two parts: one that applies the \widehat{E} operator; and one that operates directly on a phase-space function. I shall use the notation $\Sigma R = \Sigma^{(g)}R + (\partial\Delta\bar{f}) \cdot \widehat{E}R$. (The significance of the g superscript will become apparent momentarily.) The symbolic notation $\partial\Delta\bar{f}$ denotes a certain combination of nonlinear terms [28] whose details will not be required here. The notation has been chosen¹² such that this latter combination formally adds to the mean-field contribution $\partial f \cdot \widehat{E}R$; I write $\partial\bar{f} \doteq \partial f + \partial\Delta\bar{f}$. Thus, the R equation for electrostatic Vlasov turbulence becomes

$$(\partial_t + \mathbf{v} \cdot \nabla + \Sigma^{(g)})R(\mathbf{x}, \mathbf{v}, s, t; \mathbf{x}', \mathbf{v}', s', t') + \partial\bar{f} \cdot \widehat{E}R = \delta(t - t')\delta(\mathbf{x} - \mathbf{x}')\delta(\mathbf{v} - \mathbf{v}')\delta_{s,s'}. \quad (37)$$

The significance of this form is that it closely resembles the linearized Vlasov equation

$$(\partial_t + \mathbf{v} \cdot \nabla)\delta f + \partial f^{(0)} \cdot \delta \mathbf{E} = 0, \quad (38)$$

the differences being the addition of a resonance-broadening term $\Sigma^{(g)}$ and the generalization $f^{(0)} \rightarrow \bar{f}$. This strongly suggests that a renormalized dielectric \mathcal{D} can be defined by following the same operational procedure used for calculating the lowest-order dielectric from Eq. (38). If

¹⁰ In the general MSR formalism, a third (vertex) function Γ (actually a matrix with three independent components) is introduced and closures are characterized by the approximations made to the functional equation for Γ . It is not necessary to delve into the details [28] here.

¹¹ A forthcoming paper [66] (see Sec. 2.7 of the present paper) contains a useful appendix on spectral balance for a particular model of passive advection.

¹² The notation disguises the fact that $\partial\Delta\bar{f}$ is not actually the velocity gradient of a scalar, but rather is the divergence of a (wave-number-dependent) tensor.

g is the renormalized particle propagator, defined by omitting from Eq. (37) the self-consistent response term $\partial\bar{f} \cdot \widehat{\mathbf{E}}R$, then one has schematically

$$g^{-1}R + \partial\bar{f} \cdot \widehat{\mathbf{E}}R = 1. \quad (39)$$

(1 stands for the appropriate delta functions.) The solution of this equation is

$$\widehat{\mathbf{E}}R = \mathcal{D}^{-1}\widehat{\mathbf{E}}g, \quad (40a)$$

$$R = g - g\partial\bar{f}\mathcal{D}^{-1} \cdot \widehat{\mathbf{E}}g, \quad (40b)$$

where the *renormalized dielectric function* is

$$\mathcal{D} \doteq 1 + \widehat{\mathbf{E}}g \cdot \partial\bar{f}. \quad (41)$$

This has precisely the form of the well-known lowest-order (electrostatic) dielectric

$$\mathcal{D}^{(0)}(\mathbf{k}, \omega) = 1 + \sum_s \frac{\omega_{ps}^2}{k^2} \int d\mathbf{v} \frac{\mathbf{k} \cdot \partial_{\mathbf{v}} f^{(0)}}{\omega - \mathbf{k} \cdot \mathbf{v} + i\epsilon} \quad (42)$$

with the replacements $g^{(0)} \rightarrow g$ and $\partial f^{(0)} \rightarrow \partial\bar{f}$. Here $g^{(0)}$ is the linear particle propagator

$$g^{(0)}(\mathbf{x}, \mathbf{v}, s, t; \mathbf{x}', \mathbf{v}', s', t') = H(\tau)\delta(\boldsymbol{\rho} - \mathbf{v}\tau)\delta(\mathbf{v} - \mathbf{v}')\delta_{s,s'}, \quad (43a)$$

$$\widehat{g}_{\mathbf{k}, \omega}^{(0)}(\mathbf{v}, s; \mathbf{v}', s') = \frac{\delta(\mathbf{v} - \mathbf{v}')\delta_{s,s'}}{-i(\omega - \mathbf{k} \cdot \mathbf{v} + i\epsilon)}, \quad (43b)$$

where $\boldsymbol{\rho} \doteq \mathbf{x} - \mathbf{x}'$ and the Fourier transforms are with respect to $\boldsymbol{\rho}$ and τ .

Although this approach to the form of the nonlinear dielectric function is suggestive, it may not be apparent that the resulting form (41) is compatible with the conventional definition of a dielectric function that is made in linear response theory. The formal proof was first given by DuBois and Espedal [40]; an alternate derivation can be found in Ref. 28, Sec. 6.5.2, p. 165.

It is important to stress that the previous discussion applies to dynamically *self-consistent* problems in which the particles are themselves responsible for the fields: $\mathbf{E} = \widehat{\mathbf{E}}f$. One can also consider problems of *passive advection* (usually called *stochastic acceleration* [67] in the Vlasov context), in which the electric field is statistically specified. That has consequences for the forms of both the primitive amplitude equation and the renormalized Dyson equations. One is no longer allowed to write $\mathbf{E} = \widehat{\mathbf{E}}f$, since \mathbf{E} is imposed. This implies that the symmetrized form (30) may not be used. Because self-consistent response is forbidden, various ‘‘polarization’’ effects [28, 65] are absent. Nevertheless, a nontrivial, renormalized particle propagator g does exist in the passive problem. One can think of g as the conditional PDF of finding a test particle, moving under the influence of a random electric field, at phase-space point $\{\mathbf{x}, \mathbf{v}\}$ at time t , given that it was at $\{\mathbf{x}', \mathbf{v}'\}$ at time t' . In the next section I will show how to extract ponderomotive force from the Dyson equation for the passive g .

2.6.3. Ponderomotive force effects from the quasilinear theory of the weakly inhomogeneous particle propagator. The nonlinear dielectric might provide one piece of a renormalized K - χ theorem. Before attempting to draw any such connection, however, one must identify a statistical version of ponderomotive force. Because the self-consistent problem is quite complicated, I shall here consider stochastic acceleration of a test particle. (This is consistent with the fact that the original K - χ theorem assumed a given wave field.) Because such a test particle would respond to ponderomotive force, one should examine the nonlinear contributions to g in a

search for a statistical expression of ponderomotive effects. That is, one must study $\Sigma^{(g)}$. To illustrate the relevant issues with a minimum of complexity, I shall proceed by using the direct-interaction approximation. Of course, this omits a host of statistical effects (contained in the vertex renormalizations that are ignored in the DIA) and therefore precludes a discussion of complete generality. I shall also restrict attention to a single species.

It can be shown that in the passive DIA one finds

$$\Sigma^{(g)}(1, \bar{1}) = -\partial_1 \cdot g(1, 2) \mathcal{E}(2, 1) \cdot \partial_2 \delta(2 - \bar{1}), \quad (44)$$

where $\mathcal{E}(1, 1') \doteq \langle \delta \mathbf{E}(1) \delta \mathbf{E}(1') \rangle$. The presence of the two velocity derivatives identifies $\Sigma^{(g)}$ as a generalized diffusion operator. To get a feeling for its content, first assume that spatial homogeneity and temporal stationarity prevail (on the average), so that all two-point functions depend on only the differences of their spatial and temporal arguments. Those functions can then be represented in Fourier space; one finds¹³

$$\widehat{\Sigma}_{\mathbf{k}, \omega}^{(g)}(\mathbf{v}; \bar{\mathbf{v}}) = -\partial_1 \cdot \int \frac{d\bar{\mathbf{p}}}{(2\pi)^3} \frac{d\omega_{\bar{\mathbf{p}}}}{2\pi} \widehat{g}_{\mathbf{k}-\bar{\mathbf{p}}, \omega-\omega_{\bar{\mathbf{p}}}}(\mathbf{v}_1; \mathbf{v}_2) \mathcal{E}_{\bar{\mathbf{p}}, \omega_{\bar{\mathbf{p}}}} \cdot \partial_2 \delta(\mathbf{v}_2 - \bar{\mathbf{v}}). \quad (45)$$

This can be reduced to the familiar quasilinear diffusion operator by (i) passing to the long-wavelength limit $|\mathbf{k}| \ll |\bar{\mathbf{p}}|$ and the low-frequency limit $\omega \ll \omega_{\bar{\mathbf{p}}}$; (ii) assuming that the imposed field spectrum is wave-like: $\mathcal{E}_{\mathbf{k}, \omega} = 2\pi \mathcal{E}_{\mathbf{k}} \delta(\omega - \Omega_{\mathbf{k}})$, where $\Omega_{\mathbf{k}}$ is a specified frequency; and (iii) approximating g by the unperturbed propagator $g^{(0)}$ [which contains $\delta(\mathbf{v}_1 - \mathbf{v}_2)$]. Then

$$\Sigma^{(g)}(\mathbf{v}; \bar{\mathbf{v}}) \approx -\partial_1 \cdot \left(\int \frac{d\bar{\mathbf{p}}}{(2\pi)^3} \pi \delta(\Omega_{\bar{\mathbf{p}}} - \bar{\mathbf{p}} \cdot \mathbf{v}) \mathcal{E}_{\bar{\mathbf{p}}} \right) \cdot \partial_1 \delta(\mathbf{v}_1 - \bar{\mathbf{v}}). \quad (46)$$

The parenthesized term is the well-known quasilinear diffusion tensor $\mathbf{D}(\mathbf{v})$ (the significance of which was elaborated so well by Kaufman [2]).

Of course, a statistically homogeneous formalism cannot be expected to describe ponderomotive force; one must allow for weak inhomogeneities. The most elegant treatment of that topic involves the Weyl calculus, reviewed by McDonald [7]. To lowest order, however, the formalism is well known [68] and can be worked out straightforwardly [69]. For simplicity, ignore time dependence temporarily and write an arbitrary two-point space-dependent function A in terms of sum and difference variables: $A(\mathbf{x}, \mathbf{x}') = A(\mathbf{x} - \mathbf{x}' \mid \frac{1}{2}(\mathbf{x} + \mathbf{x}')) = A(\boldsymbol{\rho} \mid \mathbf{X})$; one may Fourier transform with respect to the difference argument. To first order in a weak inhomogeneity, the operation $\int d\bar{1} A(1, \bar{1}) B(\bar{1}, 1') \equiv (A \star B)(1, 1')$ can be written as

$$(\widehat{A \star B})_{\mathbf{k}}(\mathbf{X}) \approx \widehat{A}_{\mathbf{k}}(\mathbf{X}) \widehat{B}_{\mathbf{k}}(\mathbf{X}) + \frac{1}{2} i \{ \widehat{A}, \widehat{B} \}, \quad (47)$$

where

$$\{ \widehat{A}, \widehat{B} \} \doteq \frac{\partial \widehat{A}}{\partial \mathbf{X}} \cdot \frac{\partial \widehat{B}}{\partial \mathbf{k}} - \frac{\partial \widehat{A}}{\partial \mathbf{k}} \cdot \frac{\partial \widehat{B}}{\partial \mathbf{X}} = \frac{\partial}{\partial \mathbf{X}} \cdot \left(\widehat{A} \frac{\partial \widehat{B}}{\partial \mathbf{k}} \right) - \frac{\partial}{\partial \mathbf{k}} \cdot \left(\widehat{A} \frac{\partial \widehat{B}}{\partial \mathbf{X}} \right). \quad (48)$$

This is promising because gradients with respect to the weak inhomogeneity have appeared. When time dependence is included, the representation generalizes to $A(\mathbf{x}, t, \mathbf{x}', t') = A(\mathbf{x} - \mathbf{x}', t - t' \mid \frac{1}{2}(\mathbf{x} + \mathbf{x}'), \frac{1}{2}(t + t')) \equiv A(\boldsymbol{\rho}, \tau \mid \mathbf{X}, T)$. I shall consider only steady-state turbulence,

¹³ One is presented here with a classic collision of notation. A common convention in homogeneous statistical theory is to consider wave-number triads called $(\mathbf{k}, \mathbf{p}, \mathbf{q})$ with $\mathbf{k} + \mathbf{p} + \mathbf{q} = \mathbf{0}$. Unfortunately, use of \mathbf{p} here would preclude its use as a canonical momentum, to which I will also need to refer later. Therefore, in the wave-number context I shall use $\bar{\mathbf{p}}$ instead of \mathbf{p} ; hopefully that will not be confused with a canonically transformed momentum.

so no dependence with respect to T appears and temporal corrections to the Poisson bracket vanish. The formalism therefore remains correct if all AB products in Eqs. (47) and (48) are generalized to include a convolution in time difference τ (or, of course, simple products of $\widehat{A}_{\mathbf{k},\omega}$ and $\widehat{B}_{\mathbf{k},\omega}$). Thus, consider

$$\partial_\tau \widehat{g}_{\mathbf{k}}(\mathbf{X}, \mathbf{v}, \tau; \mathbf{v}') + i\mathbf{k} \cdot \mathbf{v} \widehat{g}_{\mathbf{k}} + \mathbf{v} \cdot \nabla \widehat{g}_{\mathbf{k}} + (\widehat{\Sigma}^{(g)} \star g)_{\mathbf{k}}(\mathbf{X}, \mathbf{v}, \tau; \mathbf{v}') = \delta(\tau) \delta(\mathbf{v} - \mathbf{v}'), \quad (49)$$

where

$$\begin{aligned} (\widehat{\Sigma}^{(g)} \star g)_{\mathbf{k}}(\mathbf{X}, \mathbf{v}, \tau; \mathbf{v}') &\approx \int_0^\tau d\bar{\tau} \left(\widehat{\Sigma}_{\mathbf{k}}^{(g)}(\mathbf{X}, \mathbf{v}, \bar{\tau}; \bar{\mathbf{v}}) \widehat{g}_{\mathbf{k}}(\mathbf{X}, \bar{\mathbf{v}}, \tau - \bar{\tau}; \mathbf{v}') \right. \\ &\quad \left. + \frac{1}{2} i \{ \widehat{\Sigma}_{\mathbf{k}}^{(g)}(\mathbf{X}, \mathbf{v}, \bar{\tau}; \bar{\mathbf{v}}), \widehat{g}_{\mathbf{k}}(\mathbf{X}, \bar{\mathbf{v}}, \tau - \bar{\tau}; \mathbf{v}') \} \right), \end{aligned} \quad (50)$$

and focus in particular on the Poisson-bracket terms. To isolate specific effects related to force, one may examine the time development of $\langle \mathbf{v} \rangle(\boldsymbol{\rho}, \tau) \doteq \int d\mathbf{v} \int d\mathbf{X} \mathbf{v} g(\boldsymbol{\rho}, \tau, \mathbf{v} \mid \mathbf{X}, \mathbf{v}')$. In linear theory (i.e., with the substitution $g \rightarrow g^{(0)}$), one would find $\langle \mathbf{v} \rangle = H(\tau) \mathbf{v}' \delta(\boldsymbol{\rho} - \mathbf{v}'\tau)$, which describes free-streaming propagation of a localized pulse. In general, one has

$$\partial_\tau \langle \widehat{\mathbf{v}} \rangle_{\mathbf{k}} + i\mathbf{k} \cdot \widehat{\mathbf{P}}_{\mathbf{k}} + \int d\mathbf{v} \int d\mathbf{X} (\widehat{\Sigma}^{(g)} \star g)_{\mathbf{k}}(\mathbf{X}, \mathbf{v}, \tau; \mathbf{v}') = \delta(\tau) \delta(\mathbf{v} - \mathbf{v}'), \quad (51)$$

where $\mathbf{P} \doteq \langle \mathbf{v} \mathbf{v} \rangle$. The last term on the left-hand side can be interpreted as $-\mathcal{F}/m$, where \mathcal{F} is the averaged effect of nonlinearity-induced forces. Eventually the limit $k \rightarrow 0$ will be taken, so the $i\mathbf{k} \cdot \mathbf{P}$ term will be irrelevant.

In interpreting Eq. (51), one must not forget that it describes the temporal development of the mean *particle* velocity. The statistical formalism is impartial; it will report correct statistical relations for any consistent set of independent variables in which the primitive amplitude dynamics are formulated. Since sophisticated variables such as oscillation-center coordinates and momenta are not known until the nonlinear dynamics have been examined, it is usually the original particle coordinates that are used.¹⁴ However, in perturbation theory the particle velocity differs on the average from the oscillation-center velocity by a second-order correction. I will return to this point after working out the predictions of Eq. (51).

Before considering the most general renormalized case (in which one must deal with the fact that g is nonlocal in velocity space due to resonance broadening), let us again make assumptions (ii) (wave spectrum) and (iii) ($g \rightarrow g^{(0)}$) above. Also consider times long compared to a characteristic autocorrelation time of the turbulence, so that a Markovian approximation is valid. That is, upon writing out just the temporal arguments of the required convolution, one has

$$\int_0^\tau d\bar{\tau} \Sigma^{(g)}(\bar{\tau}) g(\tau - \bar{\tau}) \approx \left(\int_0^\infty d\bar{\tau} \Sigma^{(g)}(\bar{\tau}) \right) g(\tau) = \widehat{\Sigma}_{\omega=0}^{(g)} g(\tau). \quad (52)$$

Having performed the appropriate coarse-graining, one may obtain a definite answer by evaluating the nonlinear terms at $\tau = 0$. The nonlinear contributions associated with weak inhomogeneity are then

$$\mathcal{F}/m \approx -\frac{1}{2} i \int d\mathbf{v} \int d\mathbf{X} \mathbf{v} \partial_{\mathbf{v}} \cdot \int \frac{d\bar{\mathbf{p}}}{(2\pi)^3} \frac{\partial}{\partial \mathbf{k}} \cdot [\widehat{g}_{\mathbf{k}-\bar{\mathbf{p}}, -\Omega_{\bar{\mathbf{p}}}^{(0)}}(\mathbf{v}) \boldsymbol{\mathcal{E}}_{\bar{\mathbf{p}}} \cdot \partial_{\mathbf{v}} \nabla \widehat{g}_{\mathbf{k}}(\tau = 0)], \quad (53)$$

where the second (conservative) form of Eq. (48) was used and the $\partial/\partial \mathbf{X} \equiv \nabla$ was integrated away; $\widehat{g}_{\mathbf{k},\omega}^{(0)}(\mathbf{v})$ denotes Eq. (43b) *sans* the delta function in velocity. Both velocity derivatives

¹⁴ But see Ref. 5 for some nontrivial generalizations.

can be integrated by parts, as can the ∇ :

$$\mathcal{F}/m = \frac{1}{2}i \int d\mathbf{v} \int d\mathbf{X} \int \frac{d\bar{\mathbf{p}}}{(2\pi)^3} \left(\nabla \cdot \frac{\partial}{\partial \mathbf{k}} [\partial_{\mathbf{v}} \hat{g}_{\mathbf{k}-\bar{\mathbf{p}}, -\Omega_{\bar{\mathbf{p}}}^{(0)}}(\mathbf{v}) \cdot \mathcal{E}_{\bar{\mathbf{p}}}] \right) \hat{g}_{\mathbf{k}}(0). \quad (54)$$

(A term in $\partial_{\mathbf{k}} \hat{g}_{\mathbf{k}}$ vanished at $\tau = 0$. The more general significance of that term will be discussed elsewhere.) One has $\partial_{\mathbf{v}} \hat{g}_{\mathbf{k}-\bar{\mathbf{p}}, -\Omega_{\bar{\mathbf{p}}}^{(0)}} = (-i)^{-1} (\mathbf{k} - \bar{\mathbf{p}}) / [-\Omega_{\bar{\mathbf{p}}} - (\mathbf{k} - \bar{\mathbf{p}}) \cdot \mathbf{v}]^2$. Now the $\partial/\partial \mathbf{k}$ acts on two \mathbf{k} dependencies: the \mathbf{k} in the numerator, and the $(\mathbf{k} - \bar{\mathbf{p}}) \cdot \mathbf{v}$ in the denominator; call those contributions \mathcal{F}_a and \mathcal{F}_b . After evaluating them, take the limit $\mathbf{k} \rightarrow \mathbf{0}$ in the coefficient of $\hat{g}_{\mathbf{k}}(0)$. (This could not be done before the differentiation with respect to \mathbf{k} was performed.) One obtains $\mathcal{F}_a = \langle \mathbf{F}_a \rangle$, where the average is with respect to $\hat{g}_{\mathbf{k}=\mathbf{0}}(\mathbf{v}, \tau = 0; \mathbf{v}')$ and

$$\mathbf{F}_a = -\nabla \left(\frac{1}{2} \frac{e^2}{m} \int \frac{d\bar{\mathbf{p}}}{(2\pi)^3} \frac{\nabla \mathcal{E}_{\bar{\mathbf{p}}}}{(\Omega_{\bar{\mathbf{p}}} - \bar{\mathbf{p}} \cdot \mathbf{v})^2} \right). \quad (55)$$

For a wave field localized to a single wave number $\bar{\mathbf{p}}_0$ (and its negative), one obtains

$$\mathbf{F}_a = -\nabla K, \quad (56)$$

with

$$K \doteq \frac{e^2}{m} \frac{\mathcal{E}_{\bar{\mathbf{p}}_0}}{(\Omega_{\bar{\mathbf{p}}_0} - \bar{\mathbf{p}}_0 \cdot \mathbf{v})^2}. \quad (57)$$

This is the standard, well-known expression for the second-order ponderomotive force on the oscillation center; cf. Eq. (28).

Remaining is \mathcal{F}_b , which will be argued to be related to the second-order difference between the particle velocity and the oscillation-center velocity. To show that, I briefly review the standard development of the Lie perturbation theory for the oscillation-center canonical transformation. Recall [70, 71] that the transformation between old coordinates z and new coordinates \bar{z} is generated by $\bar{z} = Tz$, where $T = T_0 + T_1 + T_2 + \dots$. One has

$$T_0 = 1, \quad (58a)$$

$$T_1 = -L_1, \quad (58b)$$

$$T_2 = -\frac{1}{2}L_2 + \frac{1}{2}L_1^2, \quad (58c)$$

where the L 's are defined in terms of the Poisson bracket with the generating function w : $L_n \doteq \{w_n, \cdot\}$. For example, the oscillation-center momentum \mathbf{P} is obtained from the laboratory momentum \mathbf{p} by

$$\mathbf{P} = \mathbf{p} + \mathbf{P}_1(z) + \mathbf{P}_2(z) + \dots, \quad (59)$$

where, e.g., $\mathbf{P}_n \doteq T_n \mathbf{p}$. To avoid secularities and vitiation of the perturbative expansion, the w_n 's are constructed to have zero average. Then

$$\langle \mathbf{P} \rangle = \mathbf{p} + \langle \mathbf{P}_2 \rangle, \quad (60)$$

where

$$\langle \mathbf{P}_2 \rangle = \frac{1}{2} \langle L_1^2 \rangle \mathbf{p}. \quad (61)$$

After spatial integration by parts under the average, this can be rearranged into

$$\frac{1}{2} \langle L_1^2 \rangle \mathbf{p} = \frac{1}{2} \frac{\partial}{\partial \mathbf{p}} \cdot \left\langle \frac{\partial w_1}{\partial \mathbf{x}} \frac{\partial w_1}{\partial \mathbf{x}} \right\rangle. \quad (62)$$

The right-hand side of Eq. (62) can be evaluated explicitly by recalling that

$$(\partial_t + \mathbf{v} \cdot \nabla)w_1 = -q\varphi(\mathbf{x}, t). \quad (63)$$

It is left as an exercise to complete the calculations and show that the contribution $\langle \dot{\mathbf{P}}_2 \rangle$ is precisely recovered by the \mathcal{F}_b term that was obtained above from the expansion in weak inhomogeneity.¹⁵

2.6.4. The possibility of a renormalized statistical K - χ theorem. The above manipulations can be repeated without the quasilinear approximation $g \rightarrow g^{(0)}$, and one is led thereby to DIA-renormalized expressions for the ponderomotive evolution of $\langle \mathbf{v} \rangle$ due to weak inhomogeneity.¹⁶ The ease with which those generalizations follow from the renormalized theory makes a strong argument in favor of MSR-style renormalizations; no series need to be summed through all orders. The formulas naturally resolve the resonant-particle singularity. There are likely practical consequences to such renormalizations; however, those will not be pursued here.

Now consider the possibility of a renormalized K - χ theorem. Of course, no such theorem is required in order to evaluate the renormalized ponderomotive effects as above, but if such a theorem did exist it would imply interesting consistency constraints on approximations made to the dielectric function and the turbulent collision operator. However, one should probably not compare apples (passive advection) with oranges (self-consistent dynamics). I have shown how to work out renormalized ponderomotive effects from the $\Sigma^{(g)}$ appropriate for passive advection. The dielectric \mathcal{D} , however, is relevant to self-consistent problems. For passive problems, R reduces to g and $\mathcal{D} \rightarrow 1$. That is not helpful in formulating a K - χ theorem.

Nevertheless, suppose that one persists in attempting to derive passive ponderomotive force from the expression for the renormalized dielectric. Note that the construction $g \mathcal{D}$ appears in the general forms of both \mathcal{D} and the renormalized $\Sigma^{(g)}$ operator. A reasonable first guess for a statistical K - χ theorem might then be¹⁷ schematically

$$\mathbf{F} = -\nabla(\hat{K}\mathcal{E}), \quad \hat{K} = -\frac{\delta\mathcal{D}}{\delta\bar{f}}. \quad (64)$$

However, difficulties emerge immediately. The averaging required for the renormalized dielectric is over not the mean distribution function f but rather the effective (wave-number-dependent) distribution \bar{f} . It is unclear what functional differentiation with respect to \bar{f} really means, and I have noted that the notation $\mathcal{D} \Delta \bar{f}$ is symbolic, being the velocity divergence of a tensor rather than literally the gradient of a scalar. Beyond the DIA, renormalized vertices that I have not discussed provide further essential complication. And if one contemplates deriving ponderomotive effects from the fully self-consistent problem, one must deal with the fact that a functional derivative with respect to the background distribution function (or any generalization thereof) would act on not just the explicit distribution in \mathcal{D} but also the implicit $\Sigma^{(g)}$ in g .

The most immediate unresolved issue is that the fundamental significance of the turbulent correction $\Delta \bar{f}$ is not understood even after three decades of research on MSR-level renormalized plasma turbulence theory.¹⁸ Its contributions to weak Vlasov turbulence theory are understood

¹⁵ As a hint, note that $d\mathcal{E}(\mathbf{X})/dt = \mathbf{V} \cdot \nabla \mathcal{E}$.

¹⁶ One must deal with the fact that the resonance-broadened g is nonlocal in velocity space. Thus $\int d\mathbf{v} \mathbf{v} \partial_v \cdot \hat{g}_{\mathbf{k}-\bar{\mathbf{p}}}(\mathbf{v}; \bar{\mathbf{v}}) \cdot \partial_{\bar{\mathbf{v}}} \hat{g}_{\mathbf{k}}(\bar{\mathbf{v}}; \mathbf{v}') = \int d\mathbf{v} [\partial_{\bar{\mathbf{v}}} \hat{g}_{\mathbf{k}-\bar{\mathbf{p}}}(\mathbf{v}; \bar{\mathbf{v}})] \hat{g}_{\mathbf{k}}(\bar{\mathbf{v}}; \mathbf{v}')$; the last derivative is on the second velocity argument.

¹⁷ Notice that the spatial gradient and the intermediate velocity derivative that are performed operate on the implicit $\Sigma^{(g)}$ that renormalizes g , not just merely the “obvious” dependencies.

¹⁸ This remark is slightly misleading. To the best of my knowledge, no one has been continuously working on the physics of $\Delta \bar{f}$ since the initial flurry of activity in the late 1970’s by such people as DuBois and Espedal [40] and Krommes [41, 72].

in the narrow sense that its relationships to specific terms in the third-order perturbative expansion of the Vlasov equation [28, 40, 72] have been identified. This means that in principle some of its connections to higher-order Hamiltonian/Lie perturbation theory are available; cf. the derivation of weak-turbulence effects by Johnston and Kaufman [39]. However, deep understanding of $\Delta\bar{f}$ as a “renormalization of the mean distribution function,” as this author has loosely suggested [41], is lacking [40]. The origins of $\Delta\bar{f}$ lie in statistical correlations arising in the rigorous theory of infinitesimal response, which is used to define the renormalized dielectric function. As was described in Ref. 28, Sec. 6.5.2, p. 165, in order to determine the mean response to a small external potential one must evaluate $\langle\tilde{R}\partial\tilde{f}\rangle$, where \tilde{R} is the random response function and \tilde{f} is the random distribution function. If that average would factor, one would have $\langle\tilde{R}\partial\tilde{f}\rangle\rightarrow R\partial f$, whereupon it is easy to show that one would obtain an expression for the renormalized dielectric \mathcal{D} that has exactly the same form as the linear dielectric $\mathcal{D}^{(0)}$ except that $g^{(0)}\rightarrow g$ and $f_0\rightarrow f$. Functional differentiation of that form with respect to f would at least be well defined. However, the statistical correlations between \tilde{R} and \tilde{f} are real and produce the $\Delta\bar{f}$ correction,¹⁹ which is required in order to produce the correct renormalized normal modes of the turbulent plasma and for a proper description of nonlinear mode coupling.

From the linear response theory cited above, one can show that $\Delta\bar{f}$ is related to the response function $G_{+--}(1, 1', 1'')$, which describes the response at point 1 due to infinitesimal perturbations at both $1'$ and $1''$. What is missing is a proven simple connection between G_{+--} and the renormalized ponderomotive effects in which we are interested. If there is such a connection, it will likely not be deduced by detailed examination of renormalized closures such as the DIA, as I have attempted to do above, but rather from a new understanding of the general structure of the renormalized theory. Another way of saying this is that the original $K\text{-}\chi$ theorem follows immediately [39] from a relation between the second-order wave-particle interaction energy (which involves the first-order four-current and the first-order four-potential) and the second-order oscillation-center Hamiltonian. We have as yet no analog of that elegant construction in the renormalized theory. Note that the statistical correlations responsible for $\Delta\bar{f}$ do not exist between the first-order quantities.

The preceding discussion illustrates both the strengths and the weaknesses of statistical turbulence theory. On the positive side, formal renormalization *a la* MSR seems to effortlessly effect a statistical averaging that mirrors the Hamiltonian oscillation-center theory, and it deals with renormalized propagators from the outset. It also has great generality, dealing with interactions between scales of all sizes, including comparable ones. On the negative side, that same generality makes it difficult to see the forest for the trees, i.e., to distill specific physical effects from the complicated mixture of nonlinear statistical couplings. Furthermore, the Hamiltonian properties of the nonlinear interactions (if they exist²⁰) do not appear to be explicitly used in any obvious way.

One is left with the uneasy feeling that some important conceptual understanding of the statistical theory remains to be achieved; certainly further work remains to be done before a renormalized $K\text{-}\chi$ theorem is proven, if indeed such a theorem exists at all. It is very likely that known results from Lie methodology and the other early work of Kaufman and his collaborators will provide important inspiration and guidance in any quest for deeper understanding. Unfortunately, the camps of statistical turbulence theory and of Lie perturbation theory have been only weakly coupled over the years. It would certainly be beneficial if that situation were corrected.

¹⁹ Some terms in that correction exist even for passive advection.

²⁰ Note that the general MSR methodology does not require that the nonlinear term in the primitive amplitude equation be derivable from a Hamiltonian formalism.

2.7. Application of renormalization to the theory of stochastic magnetic fields

A problem of considerable interest to magnetic fusion is the role of stochastic magnetic fields. Broken magnetic flux surfaces can lead to enhanced transport or even disruption of the discharge. The subject has a long history that cannot be reviewed here; see Ref. 73 for some of the early theoretical references.

A very recent calculation²¹ involving stochastic magnetic fields [66] marries a variety of the analytical tools for turbulence with the very practical problem of the calculation of plasma equilibria in toroidal devices. The methodology involves the extension of quasilinear theory (QLT) to a periodic geometry. In view of Prof. Kaufman's very significant work on QLT [2], especially his formulation of QLT in axisymmetric toroidal configurations [3], the calculation is quite relevant to this Symposium.

The usual MHD equilibrium condition²² $\mathbf{j} \times \mathbf{B} = \nabla P$ predicts that $\mathbf{B} \cdot \nabla P = 0$. For a system with good magnetic flux surfaces, this merely states that the pressure gradient is perpendicular to the flux surface. However, for regions with stochastic magnetic fields (of interest for tokamaks with ergodic limiters and for the edge regions of high- β stellarators), the condition is conventionally interpreted to mean that the pressure profile is flattened. Nevertheless, nonvanishing pressure gradients are observed in both experiments and simulations [75], so a more refined description is required.

Reiman *et al.* [75] observed that flattening need not occur in the presence of weak flows or an anisotropic pressure tensor. Furthermore, they outlined a hybrid analytical/numerical procedure wherein the magnetic-field configuration could be calculated without explicit reference to those effects. One begins with the quasineutrality condition $\nabla \cdot \mathbf{j} = 0$. That can be rewritten as

$$\mathbf{B} \cdot \nabla \mu = -\nabla \cdot \mathbf{j}_\perp, \quad (65)$$

where $\mu \doteq j_\parallel/B$; this is a magnetic differential equation for μ (\mathbf{j}_\perp is supposed to be given). Given μ , one can then proceed to solve Ampere's law for \mathbf{B} , for example by using the PIES code [76].

Let the magnetic field be written as $\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}$, where \mathbf{B}_0 possesses good flux surfaces and $\delta\mathbf{B}$ is a stochastic perturbation. Equation (65) then takes the form

$$\mathbf{B}_0 \cdot \nabla \mu + \delta\mathbf{B} \cdot \nabla \mu = -\nabla \cdot \mathbf{j}_\perp. \quad (66)$$

In coordinates for which the unperturbed magnetic lines are straight, one has

$$\mathbf{B}_0 \cdot \nabla = B_0^\phi \left(\frac{\partial}{\partial \phi} + \iota \frac{\partial}{\partial \theta} \right), \quad (67)$$

where θ and ϕ are the (generalized) poloidal and toroidal angles and ι is the rotational transform. With the convention $\mu \sim \exp(im\theta - in\phi)$, Fourier series analysis of Eq. (67) leads to

$$(B_0^\phi)^{-1} \mathbf{B}_0 \cdot \nabla \mu \rightarrow -i(n - \iota m) \hat{\mu}_{mn}, \quad (68)$$

demonstrating resonances at each of the rational surfaces. Those resonances are broadened by the stochastic component of the field. To treat the broadening quantitatively, one may think of $\delta\mathbf{B}$ as a random variable; then Eq. (66) becomes a multiplicatively nonlinear stochastic differential equation to which the usual statistical techniques may be applied. The problem is difficult because $\delta\mathbf{B}$ is self-consistently determined in terms of μ through Ampere's law. As a

²¹ So recent, in fact, that the research was not performed until well after the KaufmanFest.

²² There seems to be no area of plasma physics that Prof. Kaufman has not touched. One of his earliest papers dealt with stability of a "hydromagnetic" equilibrium [74].

first step, Krommes and Reiman [66] (KR) ignored that complication and treated Eq. (66) as a problem of *passive advection*. The resulting statistical closure contains the usual magnetic diffusion coefficient D_m (KR worked in the quasilinear limit), but also involves additional terms in principle because the source term $-\nabla \cdot \mathbf{j}_\perp$ is statistically correlated with $\delta\mathbf{B}$. This point can be illustrated with the aid of a model, as thoroughly discussed in Appendix A of Ref. 66. In the end, the additional terms can be argued to be small, so one is left with a resonance-broadened magnetic differential equation of the form

$$\mathbf{B}_0 \cdot \nabla \langle \mu \rangle - D_m \nabla^2 \langle \mu \rangle = -\nabla \cdot \langle \mathbf{j}_\perp \rangle. \quad (69)$$

(The notation “ $D_m \nabla^2$ ” denotes a nontrivial operator, since the magnetic coordinates appropriate to the underlying flux surfaces are not orthogonal.)

This is not the end of the story. Equation (69) must be solved subject to the periodicity constraints in a torus; that requires an interesting foray into the theory of periodic Green’s functions (reviewed in Appendix B of Ref. 66). Furthermore, Ampere’s law requires the *total* $\mu = \langle \mu \rangle + \delta\mu$, where $\delta\mu$ is the fluctuation about the mean. Of course, D_m incorporates mean-square information about $\delta\mu$, but that is not sufficient for the solution of Ampere’s law in a single stochastic realization. To address this issue, it is convenient to formulate the statistical closure in terms of a nonlinear Langevin equation. I will not describe the details here, since they are thoroughly discussed in Ref. 66 (see also Ref. 28). I merely remark that the nonlinear Langevin approach is undoubtedly the most intuitive formulation of the statistical closure problem, so it is highly recommended for the connoisseurs.

It was already recognized in Ref. 73 that the problem of stochastic magnetic fields presents an important venue for the application of statistical methods. The present calculation shows that the esoterica of renormalization can lead to quite practical, even quantitative predictions. Although this is well known in quantum electrodynamics, examples in plasma physics are few. One is therefore encouraged to pursue related avenues that connect analytical theory and experimental or simulation data. Some of those are described in the next several sections.

3. Tools for zonal-flow physics

The renormalized theory of spectral balance is quite general, but that is a weakness as well as a strength since it treats all fluctuations on a formally equal footing. For example, in homogeneous turbulence for which plane waves are sensible eigenfunctions, the theory describes coupling between all wave numbers $\mathbf{k} = (k_x, k_y, k_z)$, with no combination of Cartesian components being singled out for special treatment. Physics considerations, however, may point to a preferred role for specific \mathbf{k} ’s.

Consider drift waves, for example, which are very important in fusion-related microturbulence. The classic drift wave has the dispersion relation

$$\Omega_{\mathbf{k}} = \frac{\omega_*(\mathbf{k})}{1 + k_\perp^2 \rho_s^2}. \quad (70)$$

Here the diamagnetic frequency is $\omega_* \doteq k_y V_*$, the diamagnetic velocity is $V_* \doteq (\rho_s/L_n)c_s$, the sound speed is $c_s \doteq (ZT_e/m_i)^{1/2}$ (Z is the atomic number), the sound radius is $\rho_s \doteq c_s/\omega_{ci}$, and the density scale length is $L_n \doteq (-\partial_x \ln \langle n \rangle)^{-1}$. This wave is obtained from the assumptions of fluid ions ($\omega/k_\parallel v_{ti} \gg 1$) and adiabatic electrons ($\omega/k_\parallel v_{te} \ll 1$). The $k_\perp^2 \rho_s^2$ term describes the shielding effect of ion polarization drift. If that is ignored, then the physics of the wave²³ is that $\mathbf{E} \times \mathbf{B}$ advection rearranges the background ion density profile, creating a fluctuation in ion

²³ For further discussion of drift- and zonal-flow physics, see a set of pedagogical lectures by Krommes [8].

charge density. That fluctuation is then neutralized essentially instantaneously by electron flow along the magnetic field lines (adiabatic electron response).

Clearly adiabatic response fails when $\omega/k_{\parallel}v_{te} \geq 1$. Such is the case for *convective cells* with $k_{\parallel} = 0$. In toroidal geometry, for which magnetic field lines do not in general close on themselves but instead ergodically fill a magnetic flux surface, the condition $k_{\parallel} = 0$ requires both $k_y = 0$ and $k_z = 0$. Thus wave numbers with $\mathbf{k} = (k_x, 0, 0)$ have a special role. Note that electrostatic potentials dependent on only the x coordinate give rise to $\mathbf{E} \times \mathbf{B}$ flows dominantly in the y direction: $\mathbf{V}_E \propto \hat{\mathbf{b}} \times \nabla\varphi \approx \hat{\mathbf{z}} \times \nabla\varphi = \hat{\mathbf{y}}\varphi'(x)$. Such flows are called *zonal flows*. In general, they are sheared: $\varphi''(x) \neq 0$.

It has come to be appreciated that the physics of zonal flows is very important in the theory of tokamak microturbulence. (For a review, see Ref. 77.) From now on, I shall use $\mathbf{q} = (q_x, 0, 0)$ for the zonal-flow wave number. Zonal flows do not directly tap the free energy in the density gradient, since their diamagnetic frequency $\omega_*(\mathbf{q}) \propto q_y$ vanishes. They do not respond adiabatically, but rather exhibit fluid response. They can be driven up by nonlinear drift-wave–drift-wave interactions and can react back on the drift waves, both interactions being described by special cases of the triad relation $\mathbf{k} + \mathbf{p} + \mathbf{q} = \mathbf{0}$, where \mathbf{k} and \mathbf{p} denote drift waves. For example,

$$\underbrace{(k_x, k_y, k_z)}_{\mathbf{k}} + \underbrace{(k_x - q_x, -k_y, -k_z)}_{\mathbf{p}} + \underbrace{(q_x, 0, 0)}_{\mathbf{q}} = (0, 0, 0). \quad (71)$$

Because in general zonal flows are sheared and tend to be of long wavelength, they can shear apart drift-wave eddies, which tends to suppress the fluctuations that are directly responsible for the turbulent transport. Thus the presence of zonal flows can lead to reduced transport.

I shall discuss two calculations relating to zonal-flow physics, selected because they nicely illustrate a variety of fundamental tools. First I will remark on the problem of the transition to ion-temperature-gradient-driven (ITG) turbulence, in which the excitation of zonal flows plays a key role. That transition can be addressed by bifurcation theory, an important tool in the modern theory of nonlinear dynamics. Then I will mention facets of the calculation of the zonal-flow nonlinear growth rate, which affords an opportunity to apply multiple-scale methods, Hamiltonian methods for PDEs, and some interesting wave kinetics.

3.1. Bifurcation theory and the transition to ion-temperature-gradient-driven turbulence

I have remarked that zonal flows can shear apart drift-wave eddies, thereby suppressing them. Perhaps the most dramatic example of this effect is the so-called *Dimits shift*, which arises in the transition to ITG turbulence. In order to appreciate the Dimits shift, consider the most naive paradigm for the onset of turbulent transport. Imagine changing an order or bifurcation parameter ϵ in order to bring a system from linear stability, through marginal stability at $\epsilon = \epsilon_c$, to linear instability. (ϵ_c can be chosen to vanish.) Naive arguments might suggest that fluctuations, hence turbulent transport, should turn on as soon as ϵ exceeds ϵ_c . In the ITG problem, ϵ can be taken to be a normalized temperature gradient, $\epsilon \doteq R/L_T$ (R is the major radius of the tokamak). The conventional scenario is then sketched in Fig. 3.

However, large-scale numerical simulations due to Dimits and collaborators [78] showed that when collisions (which are weak) are neglected, the actual situation is more complicated. In fact, the transport does not turn on until ϵ exceeds some $\epsilon_* > \epsilon_c$. The difference $\Delta \doteq \epsilon_* - \epsilon_c$ is known as the *Dimits shift*. By qualitative analysis of the simulation results, the Dimits shift was understood in Ref. 78 to be due to the excitation of zonal flows, which then totally suppressed the ITG modes for $\epsilon_c < \epsilon < \epsilon_*$. Further discussion and insights were given by Rogers *et al.* [79].

However, more can be done. In the systematic theory of nonlinear dynamics, transitions at critical points such as $\epsilon = \epsilon_c$ are conventionally analyzed by means of bifurcation theory [80, 81], so it is natural to ask [82] whether the Dimits shift can also be explained in those terms. Now if

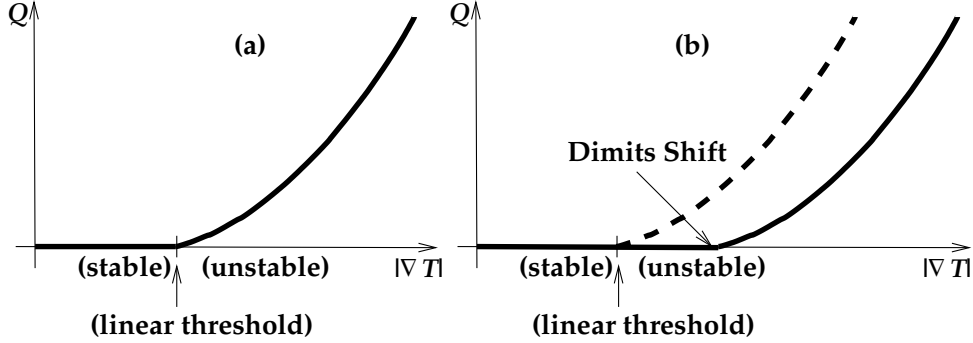


Fig. 3. Illustration of the Dimits shift. (a) Naive expectation: heat flux Q should turn on immediately after the threshold for linear instability is crossed. (b) Simulation result: turbulence and transport are suppressed in a nonzero interval (the “Dimits shift”) above threshold.

zonal flows were not involved, the transition at $\epsilon = \epsilon_c$ would be a conventional Hopf bifurcation, in which a complex-conjugate pair of eigenvalues crosses the imaginary λ axis. (Perturbations are assumed to vary as $e^{\lambda t}$.) If the bifurcation is supercritical, nonzero drift-wave activity (not necessarily turbulent) would be expected for ϵ just above ϵ_c , but that is not observed. One must rule out subcritical bifurcation, but in fact it is clear (as I will explain) that the standard theory of Hopf bifurcation does not apply when zonal flows are involved.

To gain insight into this issue, Kolesnikov and Krommes (KK) analyzed a “simple” model of ITG fluctuations. For the details, which are lengthy and nontrivial, I refer the reader to Refs. 83 and 84. Here I will just describe the qualitative situation, which I hope will argue in favor of the use of fundamental and systematic analysis tools.

The basic model consists of two nonlinear 2D PDEs that couple the vorticity²⁴ ϖ and the pressure P . The nonlinearity is due to $\mathbf{E} \times \mathbf{B}$ advection. The system supports a linear ITG instability for sufficiently large background temperature gradient. The zonal modes are assumed to be undamped in order to make contact with the collisionless simulations reported by Dimits *et al.*

KK worked in a finite-sized box, the radial width of which was intended to model the width of a radial eigenfunction in the presence of magnetic shear. They imposed standing-wave boundary conditions in the radial direction; that is the only place in which magnetic shear entered the calculation. Periodic boundary conditions were used in the poloidal direction.

By Galerkin projection, KK reduced the problem to a system of coupled ODEs. They used the minimum number of modes that leads to a well-posed problem. One satisfies the triangle condition with modes $(k_x, k_y) = (1, 1)$, $(1, -1)$, and $(2, 0)$. The latter (zonal) amplitudes are real; the others are complex. Upon recalling that one has $n = 2$ coupled fields, one can count the number of real degrees of freedom by $2n + 2n + n = 10$. Thus one deals with nonlinear dynamics in a 10D phase space.

In bifurcation calculations, progress is made by analyzing the dynamics on the *center manifold*. That is the nonlinear generalization of the center eigenspace, namely the linear subspace spanned by the eigenvectors with zero real parts at marginal stability. In a standard Hopf bifurcation, the center eigenspace would be 2D. If there are no unstable eigenvalues additional to the nearly marginal ones (in other words, the complement of the center eigenspace is a stable eigenspace), the center manifold theorem [80, 81] states that near the bifurcation

²⁴ Here the vorticity is the component of the curl of the $\mathbf{E} \times \mathbf{B}$ velocity in the direction of the magnetic field. In suitably dimensionless variables, it can be written as $\varpi = \nabla_{\perp}^2 \varphi$, where φ is the electrostatic potential. For some pedagogical discussion of the vorticity in strongly magnetized plasmas, see Ref. 85, Fig. 2.

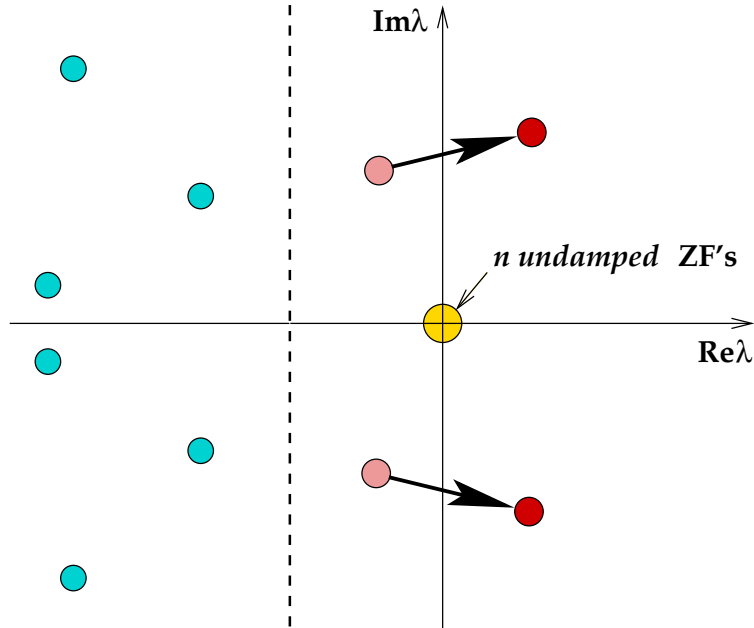


Fig. 4. (Color online.) Movement of eigenvalues as temperature gradient passes through the threshold for linear instability ($\partial_t \rightarrow \lambda$). Red: marginally stable drift waves; blue: strongly stable drift waves; yellow: undamped zonal flows.

point the nonlinear dynamics are attracted to a (curved) center manifold that is tangent to the center eigenspace at the bifurcation point. This behavior arises because the modes in the stable eigenspace are strongly damped, so rapidly relax and become adiabatically slaved to the center modes.

Thus, a standard Hopf bifurcation is associated with a 2D center manifold. The two degrees of freedom correspond to the amplitude ρ and phase θ in a complex representation $\rho e^{i\theta}$ of the reduced dynamics.

In the present model, the bifurcation is more complicated because the center eigenspace and the center manifold are 4D. This follows because each of the two (real) zonal modes that are retained contributes a linear eigenvalue of zero. Thus, the center eigenspace looks schematically like Fig. 4. If one ignores the drift-wave phase variable, the coordinate axes of the reduced phase space can be taken to be the two (real) zonal amplitudes and the drift-wave amplitude.

The ultimate goal is to understand the time-asymptotic dynamics on the center manifold. But to verify that one is on the right track, it is first useful to directly integrate the 10D system for a parameter value just slightly above marginality. A snapshot of the result is shown in Fig. 5, and a movie ([movie234.mpg](#)) is available with the online Proceedings. Those plot the dynamics versus two of the eight drift-wave degrees of freedom (on the x - y plane) and one of the zonal amplitudes (on the z axis). The phase point is projected at various time snapshots onto the horizontal and two back vertical planes; it is also drawn in three-space, with various temporal portions of the trajectory color-coded for easier visibility. It is seen that after an initial transient the trajectory settles down to a regular motion that subsequently spirals into a time-independent fixed point with zero drift-wave amplitudes but nonzero zonal amplitude, i.e., a pure zonal-flow state. The post-transient regular motion is evidence of a center manifold. The interpretation is consistent with our previous discussion: For arbitrary initial conditions that include excited drift-wave amplitudes, nonlinearity drives zonal flows that react back on the drift waves and ultimately suppress them entirely. This behavior occurs for a range of temperature gradients;

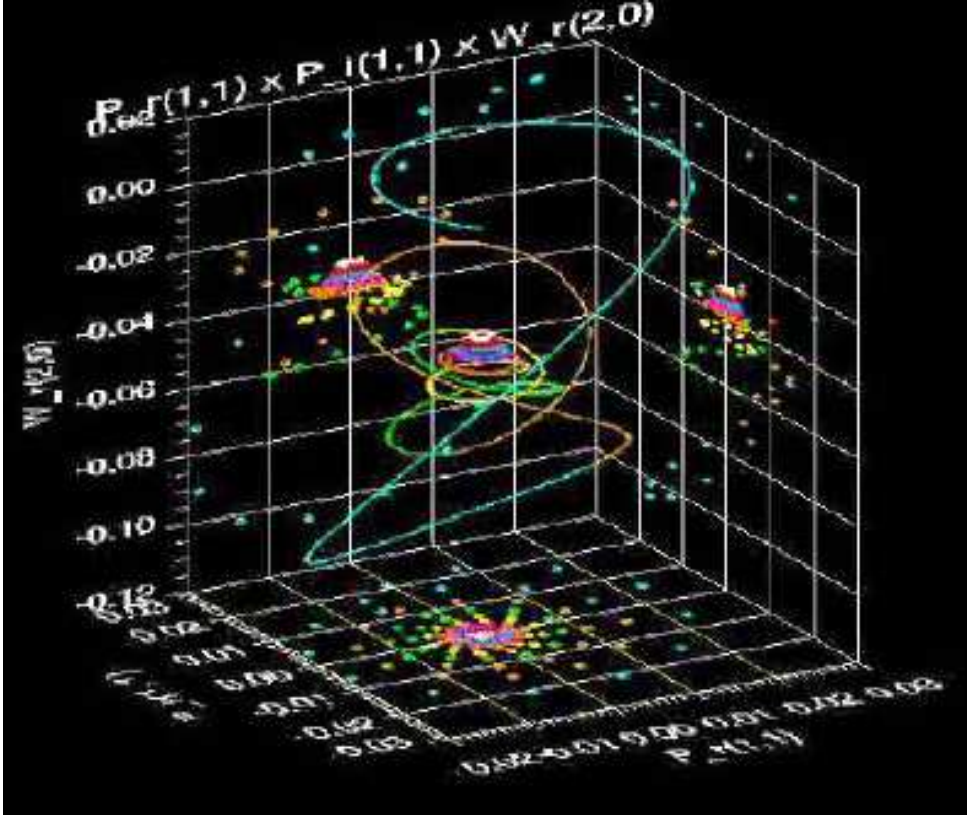


Fig. 5. (Color online.) Projection of the 10D trajectory into the three-space with axes $(x, y, z) = (\text{Im } \varpi_{1,1}, \text{Re } P_{1,1}, \varpi_{2,0})$. The time-asymptotic solution is a pure zonal-flow state.

that range defines the Dimits shift.

Analytically, one needs to (i) construct the center manifold; (ii) locate the fixed point; and (iii) discuss the stability of the fixed point and thus determine Δ , the width of the Dimits shift for the model. Construction of the center manifold is expedited by a projection method described by Kuznetsov [81], in which a contravariant coordinate system is used to locate points on the center manifold above the flat center eigenspace. (For a graph of this construction, see Ref. 84, Appendix B, Fig. 12. The projection method does not require the initial diagonalization of the linear matrix that was used in Ref. 80.) The construction is done perturbatively, thus is restricted to small amplitudes and valid only locally. Surprisingly, the location of the fixed point can be obtained analytically even though nonlinear algebraic equations need to be solved. Many details of the calculations, together with caveats²⁵ and lengthy discussion, may be found in Ref. 84.

The significance of this analysis is that it uses a systematic perturbation method to analyze a nontrivial and highly nonlinear dynamical problem. The reduction of a high-dimensional dynamics to the substantially lower-dimensional motion on the center manifold seems to be very much in the spirit of the calculations favored by Prof. Kaufman. Indeed, my initial introduction

²⁵ A significant problem of the present method is the severe Galerkin truncation. One could retain more modes, but then the center manifold becomes of higher dimensionality and difficult to analyze. Unfortunately, the predicted Dimits shift depends on the truncation. It may be that an entirely different functional analysis is required to properly calculate Δ . Certainly one desires an approach in which the influence of magnetic shear can be taken into account more precisely. There is plenty of room here for future work.

to nonlinear dynamics came *via* the detailed study of Chirikov’s thesis in the reading group in which I participated during my initial visits to Berkeley. (A subsequent additional important influence was the review article by Treve [34] on the possible applications of nonlinear dynamics ideas to controlled fusion research.) It is somewhat embarrassing that it took me almost 30 more years to accomplish a nontrivial calculation in nonlinear dynamics,²⁶ but better late than never.²⁷ Serious analytical theorists should certainly be taught the rudiments of that field.

3.2. Wave kinetics and the calculation of the zonal-flow growth rate

The bifurcation analysis emphasizes that zonal flows can suppress drift waves. However, although I have not demonstrated it here, the 10D model is not sophisticated enough to describe steady-state turbulence above the onset. There is considerable further work to do on “fundamental” studies of nonlinear dynamical behavior in fusion microturbulence. But one still has the general formalism of spectral balance in statistical closures. Here, I will discuss one aspect of that theory as it pertains to the generation of zonal flows in steady-state turbulence.

When physics is adequately described by a scalar field such as the potential φ , the general form of the spectral balance equation is Eq. (35), which I transcribe here for a Markovian, one-time description of convective cells with wave number \mathbf{q} :

$$\partial_t C_{\mathbf{q}} - 2\gamma_{\mathbf{q}}^{\text{lin}} C_{\mathbf{q}} + 2 \text{Re} \eta_{\mathbf{q}}^{\text{nl}} C_{\mathbf{q}} = 2F_{\mathbf{q}}, \quad (72)$$

which contains as usual a coherent nonlinear damping $\eta_{\mathbf{q}}^{\text{nl}}$ and an incoherent nonlinear drive $F_{\mathbf{q}}$. In principle, both of these terms contain self-interactions, i.e., triadic couplings among three convective modes satisfying $\mathbf{q} + \mathbf{q}' + \mathbf{q}'' = \mathbf{0}$, as well as cross couplings between the convective cells and other constituents of the turbulence such as drift waves. Now for nonlinearities involving $\mathbf{E} \times \mathbf{B}$ advection, the mode-coupling coefficient includes a cross product: $M_{\mathbf{k}\mathbf{p}\mathbf{q}} \propto \hat{\mathbf{z}} \cdot \mathbf{p} \times \mathbf{q}$. This implies that interactions between three zonal modes vanishes, since the wave vectors of all such modes point in the $\pm x$ direction and have vanishing cross product. There may be interactions between zonal modes and other kinds of convective cells. Here, however, I consider the important case where the zonal mode \mathbf{q} interacts with two drift waves \mathbf{k} and \mathbf{p} . Specifically, I discuss the calculation of the nonlinear zonal-flow growth rate $\gamma_{\mathbf{q}}^{\text{nl}} \doteq -\text{Re} \eta_{\mathbf{q}}^{\text{nl}}$ due to that interaction.

Such a calculation was first attempted by Diamond and coworkers [89], who contributed important insights. Their method of attack was *via* a postulated wave kinetic equation for the drift waves as modulated by the presence of a zonal flow. That captures the right idea; however, some of the details were either incorrect or unclear.

A serious attempt to understand the wave kinetic approach was mounted by Krommes and Kim [69]; subsequently, further insights were gained by Krommes and Kolesnikov [90]. Those works together provide an interesting and nontrivial tour through a variety of useful tools, including renormalized statistical theory, Hamiltonian formulation of nonlinear PDEs, multiple-scale methods, and an unusual application of wave-kinetic analysis.

The first calculation performed by Krommes and Kim was a direct derivation of $\gamma_{\mathbf{q}}^{\text{nl}}$ from the Markovian closure theory of the generalized Hasegawa–Mima equation²⁸ in the limit that $\epsilon \doteq q/k \ll 1$. In principle, this calculation is straightforward; one need “merely” write down the

²⁶ Early in my career I worked seriously on transport due to stochastic magnetic fields [73, 86], which required understanding of Chirikov’s stochasticity criterion [71] and some issues relating to Kolmogorov entropy. However, the focus there was on statistical dynamics, not nonlinear dynamics in the usual sense.

²⁷ Examples of earlier work on bifurcations in plasmas include the papers by Hinton and Horton [87] and Horton *et al.* [88], although I do not believe that the concept of a center manifold was mentioned explicitly in those works.

²⁸ In the derivation of the original Hasegawa–Mima equation, adiabatic electron response was assumed. That is incorrect for convective cells, which exhibit fluid response. In the generalized Hasegawa–Mima equation, a modified Poisson equation is used in which the electrons do not respond at all for $k_{\parallel} = 0$ modes. For further discussion, see Ref. 69.

expression for $\eta_{\mathbf{q}}^{\text{nl}}$, simplify the mode-coupling coefficients for small ϵ , and perform the indicated wave-number integrals. The rub is that those integrals, when constrained by the triad relation $\mathbf{q} + \mathbf{k} + \mathbf{p} = \mathbf{0}$, are over unusual, noncircular domains in \mathbf{k} space. Specifically, if \mathbf{q} is fixed, the tip of \mathbf{k} and the tail of \mathbf{p} are tied together, and that vertex is moved, then the vectors \mathbf{k} and \mathbf{p} sweep out areas that are not simultaneously symmetrical with respect to the same origin. Thus great care and cross checking is required, and the calculation took many months to complete.

One offshoot of this analysis was that the calculation of $\gamma_{\mathbf{q}}^{\text{nl}}$ (done there in the plasma-physics context of the anisotropic generalized Hasegawa–Mima equation) was shown to be closely related to Kraichnan’s earlier pioneering calculations of eddy viscosity for the isotropic 2D Navier–Stokes equation [33]. In fact, it was possible to formulate the plasma problem so that it embraced the Navier–Stokes equation as well, so one could take the appropriate limit. Bringing the various results into complete agreement provided an entirely nontrivial check on the calculations.

In the end, the formula for $\gamma_{\mathbf{q}}^{\text{nl}}$ is fairly simple and closely related to the result of Diamond’s heuristic wave-kinetic analysis (with some important details corrected and clarified). It is interesting to understand the underlying reasons why a wave-kinetic approach works in this context. Lest the reader argue that this is obvious, I point out that the calculations of $\gamma_{\mathbf{q}}^{\text{nl}}$ have been performed for homogeneous turbulence, for which gradients of the statistical observables with respect to space vanish; the usual wave-kinetic analysis as applied to linear wave propagation involves the Poisson bracket $\{A, B\} = \partial_{\mathbf{X}}A \cdot \partial_{\mathbf{k}}B - \partial_{\mathbf{k}}A \cdot \partial_{\mathbf{X}}B$, which vanishes identically for a homogeneous background.

The resolution of this paradox is relatively easy. In calculating $\gamma_{\mathbf{q}}$ for a specific \mathbf{q} , one is essentially considering a conditional average. A selected \mathbf{q} appears to the drift waves as a long-wavelength, inhomogeneous modulation. That modulation can be taken as the inhomogeneous background on top of which a wave-kinetic formalism can be built. This idea can be systematized by using the functional apparatus introduced by MSR, as demonstrated by Krommes and Kim.

Since the wave-kinetic formalism is being used in an unfamiliar context, one must rethink all of its assumptions very carefully. In particular, it is not immediately clear what “action” density should be evolved by the wave kinetic equation. In traditional wave theory, the appropriate action is (electrostatically) $\mathcal{J} \doteq (\partial\mathcal{D}/\partial\Omega)(E^2/8\pi)$ (Ω being the mode frequency). However, this is built from properties of the linear waves. It is unclear why linear wave physics should have anything directly to do with the calculation of $\gamma_{\mathbf{q}}^{\text{nl}}$, which follows from properties of the *nonlinear* interaction, and indeed it is inappropriate to use \mathcal{J} . Instead, one must use an invariant of the drift-wave–zonal-flow interaction.

For generalized Hasegawa–Mima dynamics, the relevant invariant was first found by Smolyakov and Diamond [91], who essentially studied the inhomogeneous interaction entirely in \mathbf{k} space. However, we know from the work of McDonald and Kaufman on the systematic analysis of weakly inhomogeneous systems with the aid of the Weyl calculus (for a review, see Ref. 7) that it is better to work in a mixed \mathbf{x} -space– \mathbf{k} -space representation. That naturally introduces Poisson brackets, the properties and interpretation of which are well known. Actually, the lowest-order version of such a calculation for weakly inhomogeneous statistical dynamics was first performed by Carnevale and Martin [68], although Krommes and Kim identified a fundamental conceptual mistake in that work. In any event, when all of the various approaches are brought into agreement, the Smolyakov–Diamond invariant is recovered. For the generalized Hasegawa–Mima equation, that is

$$\mathcal{N} = \sum_{\mathbf{k}} \mathcal{N}_{\mathbf{k}}, \quad \mathcal{N}_{\mathbf{k}} \doteq \frac{1}{2}(1 + k^2)^2 |\varphi_{\mathbf{k}}|^2. \quad (73)$$

It is important that this differs from the energy

$$\mathcal{E} = \frac{1}{2} \sum_{\mathbf{k}} (1 + k^2) |\varphi_{\mathbf{k}}|^2. \quad (74)$$

At this point, Krommes and Kim proceeded to discuss the derivation of $\gamma_{\mathbf{q}}^{\text{nl}}$ from a wave-kinetic equation that evolved $\mathcal{N}_{\mathbf{k}}$. However, the fundamental significance of $\mathcal{N}_{\mathbf{k}}$ remained unclear. Therefore, let me take a historical detour and jump ahead a few years. In the course of my work with Kolesnikov on the ITG problem, we were led to consider the generalization of the wave-kinetic approach to situations involving several coupled fields and possibly multiple nonlinear invariants. Those are complicated enough that it is essential to work with the most concise and systematic methodology available. Morrison's Hamiltonian formalism for nonlinear PDEs [92] is ideal for this purpose. For specific kinds of nonlinearity, happily including $\mathbf{E} \times \mathbf{B}$ advection, it allows one to derive the nonlinear portion of the PDE (or system of PDEs) from a particular Poisson bracket of the field with a Hamiltonian functional \mathcal{H} . Now although \mathcal{H} is conserved when all nonlinear interactions are considered, it is not conserved for a subclass of fluctuations such as short-scale drift waves because energy can be transferred to other modes. However, in a bracket formalism it is known that additional *Casimir invariants* can be conserved for an arbitrary Hamiltonian due to particular properties of the Poisson bracket. Using a covariant bracket formalism combined with multiple-scale analysis, Krommes and Kolesnikov [90] were able to show that for $\mathbf{E} \times \mathbf{B}$ advection a particular Casimir is conserved even under the interaction between short-wavelength drift waves and long-wavelength convectives cells. For the generalized Hasegawa–Mima equation, the Smolyakov–Diamond invariant is recovered. For some more general systems of PDEs, multiple Casimirs may exist.

Multiple-scale analysis leads to a wave-kinetic description of the DW–CC interaction. In the face of multiple Casimirs, that formalism is interesting and nontrivial [90], but it is sufficient here to merely consider the case of a single invariant \mathcal{N} (discussed at length in Ref. 69). Then the appropriate wave kinetic equation is

$$\partial_T \tilde{\mathcal{N}}_{\mathbf{k}} - \{\tilde{\Omega}_{\mathbf{k}}, \tilde{\mathcal{N}}_{\mathbf{k}}\} = 0, \quad (75)$$

where the Poisson bracket is defined by Eq. (48) and $\tilde{\Omega}_{\mathbf{k}}$ is an appropriate *nonlinear* advection frequency of the short scales by the long scales (*it is not the linear mode frequency of a short-scale drift wave*). A tilde emphasizes that the quantity is random (due to the stochastic nature of the zonal flows). For the generalized Hasegawa–Mima equation, one has $\tilde{\Omega}_{\mathbf{k}} = \mathbf{k} \cdot \tilde{\mathbf{V}}_{\mathbf{q}}$.

When Eq. (75) is summed over \mathbf{k} and integrated over \mathbf{X} , the Poisson bracket vanishes, confirming that \mathcal{N} is truly invariant under the interaction between the long and the short scales. Energy $\tilde{\mathcal{E}}$, however, is not invariant, since it is a different wave-number weighting of $\langle \delta\varphi^2 \rangle_{\mathbf{k}}$ than is $\tilde{\mathcal{N}}_{\mathbf{k}}$ and the Poisson bracket involves derivatives with respect to \mathbf{k} . One finds

$$\partial_T \tilde{\mathcal{E}}_{\mathbf{k}} = \{\tilde{\Omega}_{\mathbf{k}}, \tilde{\mathcal{E}}_{\mathbf{k}}\} + 2\tilde{\gamma}_{\mathbf{k}}^{(1)} \tilde{\mathcal{E}}_{\mathbf{k}}, \quad (76)$$

where, if $\tilde{\mathcal{N}}_{\mathbf{k}} = c_{\mathbf{k}} \tilde{\mathcal{E}}_{\mathbf{k}}$,

$$\tilde{\gamma}_{\mathbf{k}}^{(1)} \doteq \frac{1}{2} \{\tilde{\Omega}_{\mathbf{k}}, c_{\mathbf{k}}\}. \quad (77)$$

It is important to realize that this first-order growth rate is *not* the desired long-wavelength growth rate in the statistical spectral balance equation; note that $\tilde{\gamma}_{\mathbf{k}}^{(1)}$, being linearly proportional to the long-wavelength amplitude (buried in $\tilde{\Omega}_{\mathbf{k}}$), vanishes when averaged over an ensemble of convective cells whereas the spectral balance equation holds after all averaging has been performed. To obtain $\gamma_{\mathbf{q}}^{\text{nl}}$, one must proceed to second order. It is unnecessary to repeat the remainder of the calculation (which uses the wave kinetic equation a second time) here since the details have been described in a number of places [8, 69, 93]. (Ref. 8, in particular, makes a strong attempt to be introductory and pedagogical.) But the flavor of the calculation is clear, and one can now appreciate that systematic multiple-scale analysis and wave-kinetic formalism can be usefully applied not just to linear problems, but to nonlinear ones as well.

4. Current challenges in statistical turbulence theory:

Intermittency and coherent structures

The well-established two-point moment closures do a reasonable job of describing both fluctuations in the energy-containing part of the wave-number spectrum and the associated turbulent transport.²⁹ However, they can fail badly at describing higher-order moments [100, 101]. In this section, I introduce some of the recent issues concerning highly non-Gaussian statistics. Few, if any, questions will be resolved here; this is a fertile area for future research.

4.1. Higher-order statistics

PDFs can (usually) be described by their infinity of moments [102] or, better, cumulants [21]. A Gaussian PDF $P(x)$ is described by just two cumulants: the mean $\bar{x} \doteq \langle x \rangle$, and the variance $\sigma^2 \doteq \langle \delta x^2 \rangle$. Basic measures of non-Gaussianity include the *skewness* S and *kurtosis* K (or the flatness factor F)³⁰:

$$S \doteq \frac{\langle \delta x^3 \rangle}{\sigma^3}; \quad K \doteq F - 3, \quad \text{with } F \doteq \frac{\langle \delta x^4 \rangle}{\sigma^4}. \quad (78)$$

K is defined such that it vanishes for Gaussian statistics.

Kurtosis is a measure of the importance of large events. A PDF with large kurtosis frequently has an exponential or stretched exponential tail, i.e., more slowly decaying than a Gaussian. For example, the pure exponential PDF

$$P(x) = \frac{1}{2\sigma'} e^{-|x|/\sigma'} \quad (79)$$

has $K = 3$. Realizations sampled from such PDFs display occasional and intermittent large spikes.

That kurtosis and intermittency are linked follows from a simple example given by Frisch [103, Chap. 8]. Consider a time series $\tilde{x}(t)$ constructed by turning on a random signal $x(t)$ only a fraction λ of the time. The kurtosis of \tilde{x} is

$$K = \frac{\lambda \langle x^4 \rangle}{(\lambda \langle x^2 \rangle)^2} - 3 = \lambda^{-1} (\langle x^4 \rangle / \langle x^2 \rangle^2) - 3, \quad (80)$$

which approaches infinity as $\lambda \rightarrow 0$. [For a Gaussian signal, the formula reduces to $3(1 - \lambda)/\lambda$.]

One cannot construct PDFs by specifying S , K , and other cumulants arbitrarily. Cumulants must satisfy an infinity of *realizability constraints*, which stem from the positive-semidefiniteness of a valid PDF. The simplest one is $\langle \delta x^2 \rangle \geq 0$; another is

$$K \geq S^2 - 2. \quad (81)$$

This constraint will be important in the next section.

Various recent plasma experiments clearly display intermittent statistics. Time series taken from probes inserted in the edges of various tokamaks are routinely intermittent. Visual evidence can be found from gas-puff-imaging (GPI) data [104], which shows the intermittent formation of density *blobs* that are occasionally ejected into the scrape-off layer (SOL) and may propagate outward to the wall. A snapshot is shown in Fig. 6. Although a good deal of understanding of the

²⁹ The justification for this assertion is the extensive work done in the 1990's by Bowman [94–96] and Hu [97–99] on realizable Markovian statistical closures. That work included successful comparisons between the predictions of statistical closure theory and direct numerical simulations.

³⁰ It is unfortunate that the names of the kurtosis and flatness have not been entirely standardized in the literature. My definition of F (I follow Frisch [103, p. 41]) is often confusingly called the kurtosis, and my definition of K is sometimes called the “excess kurtosis.”

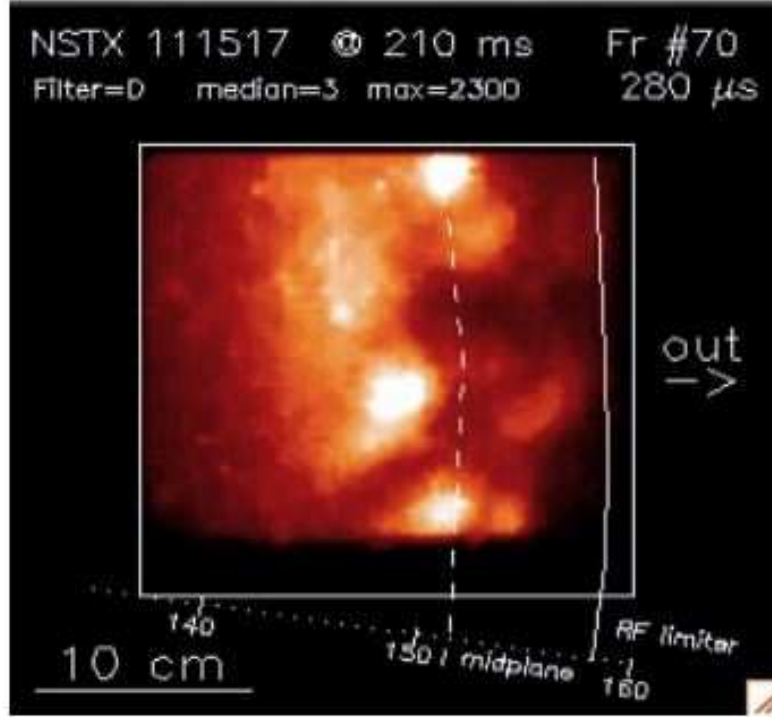


Fig. 6. (Color online.) Fluctuations in the edge of NSTX show the intermittent formation of “blobs.” From <http://www.pppl.gov/~szweben/NSTX04>.

propagation of a single blob has been developed,³¹ blob generation is much more complicated because blobs arise intermittently from a sea of turbulence. Research on blob generation is ongoing [106]; it requires a comprehension of many facets of plasma dynamics as well as statistical turbulence theory.

4.2. Physical relationships between kurtosis and skewness

A more basic physics experiment may shed light on intermittency in plasmas. In the machine TORPEX (<http://crppwww.epfl.ch/torpepx/>), skewness and kurtosis have been measured at many places in the machine in a large ensemble of shots [107]. The data shows the clear relationship

$$K = aS^2 + b, \quad (82)$$

with $a \approx 1.5$ and $b \approx -0.22$. These values are consistent with the realizability constraint (81). However, no physics basis for this relationship was offered by the original investigators.

Strikingly, essentially the same relationship, *with the same value of a* , is evinced by a global database of fluctuations in sea-surface temperature, as shown by Sura and Sardeshmukh (SS) [108]. Furthermore, those investigators advanced an analytical calculation, based on a nonlinear Langevin equation, that predicted the values $a \approx \frac{3}{2}$ and $b \approx 0$. This is clearly suggestive.

The essence of the SS model is the inclusion of a white-noise multiplicative nonlinearity in the evolution equation for the fluctuations:

$$\partial_t \delta T + \dots + \sigma \tilde{f}(t) \delta T = \tilde{f}(t), \quad (83)$$

³¹ The basic physics of blob propagation in a torus is $\mathbf{E} \times \mathbf{B}$ motion due to internal polarization arising from magnetic drifts [105].

where \tilde{f} is Gaussian white noise and σ measures the strength of the nonlinearity. Given such a stochastic differential equation [109] (to be analyzed in the Stratonovich interpretation), a Fokker–Planck equation can be deduced. The structure of that equation is simple enough that a closed hierarchy of analytical relations can be obtained for the moments through a given order. Eliminating the variance in the equations for K and S gives rise to the relation (82).

The validity of the model even for its intended target of sea-surface temperature fluctuations may be questionable; however, its prediction of $a = \frac{3}{2}$ is obviously a nontrivial success. It is reasonable, therefore, to inquire whether a similar approach may be fruitful for the application to TORPEX. Such a calculation was first outlined in Ref. 110, and some details have been published [111]. Here I shall just mention a few of the salient issues.

4.2.1. Multiplicative white noise and Markovian Langevin equations. The most important step in the modeling is the representation of all nonlinear effects by the product $\tilde{f}\delta T$, where \tilde{f} is the external noise that models the excitation of the turbulence (in lieu of a term involving the linear growth rate). Assuming that \tilde{f} is white ensures a Markovian (time-local) description. Similar descriptions are well known in statistical closure theory in the form of *realizable Langevin representations* for closures such as the EDQNM. In those theories, the effects of the original quadratic nonlinearity are represented by a coherent nonlinear damping $\eta^{\text{nl}}\delta\psi$ and an *internal* incoherent drive $\tilde{f}^{\text{nl}} \propto \tilde{w}(t)$, where \tilde{w} is Gaussian white noise. However, that \tilde{w} is specifically *not the same* as an external stirring force \tilde{f} , so one must dig deeper to understand the connection, if any, to the SS model. The remainder of the discussion becomes too technical for inclusion here; I refer the interested reader to Ref. 111.

4.2.2. The role of linear waves. The SS model applies to a single real scalar field. In confined plasmas, however, linear waves are ubiquitous. Even a simple drift mode, linearly described by $\partial_t\varphi + i\Omega\varphi = 0$, must be represented by a 2D system of first-order ODEs that couple the real and imaginary parts of the complex wave amplitude. Not only does this increase the complexity of the analysis, the presence of a real frequency has consequences for the ultimate statistics of the associated nonlinear equation. If one follows the spirit of the SS model, it can be shown that although the basic scaling $K = aS^2 + b$ still holds, one has, in particular, $a = a(\Omega)$. For the derivation of that relation for a simple model, see Refs. 110 and 111.

4.3. What can one learn from single-point statistics?

The simplest model that includes the effects of wave propagation predicts that the variation of a with Ω is bounded and modest; the principal effect of nonzero Ω is to reduce S and K . That may account for some of the variability in the data. Presumably shots with large values of skewness are highly nonlinear and relatively insensitive to a linear wave frequency; such shots should tend to track the $a = \frac{3}{2}$ curve, as indeed they appear to.

Ultimately, what does one learn from single-point statistics? At present, the answer is unclear. The level of the current modeling, which is not derived from realistic PDEs for the plasma physics, is too crude to enable one to deduce fine points of the underlying dynamics, no matter how suggestive is the $\Omega = 0$ prediction $a \approx \frac{3}{2}$. It remains possible that a more refined theory of the PDF for density fluctuations obeying specific nonlinear PDEs can be developed. The gold standard here is the *mapping closure* technique due to Chen *et al.* [50]. Unfortunately, that theory is very difficult (see additional discussion and references in Ref. 28), particularly when applied to coupled PDEs that contain linear waves. Extensive further development is required.

In addition to the theoretical complexity of systematic PDF theories, it must be stressed that knowledge of the single-point PDF of a fundamental scalar field such as temperature is not sufficient for predictions of transport coefficients, which require cross correlations. Those

are easily available from moment closures, which however have a great deal of difficulty in describing intermittency. It may be that entirely new tools are required in order to make progress. Certainly numerical investigations will continue to play an increasingly important role in explorations of complicated nonlinear statistical dynamics, but additional analytical methods would be welcome as well. As younger generations develop those, they could do no better than to look for background and motivation to the elegant tools, and the physical insights behind them, to which the present Symposium pays tribute.

5. Summary and discussion

In the previous sections, I have tried to provide a snapshot of some of the interesting issues in modern plasma turbulence theory, emphasizing connections to, or generalizations of, some of the tools that were either developed by Prof. Kaufman and/or his students and colleagues or, in my opinion, would appeal to their instincts for systematology and elegance. Although turbulent plasmas in the laboratory are incredibly complex, the nonlinear equations that describe them are overly rich with dynamical possibilities, and many questions remain unanswered, it does appear that a predictive superstructure is beginning to appear.

The short title of this article is “Tools for fusion-related microturbulence.” Although I mentioned several fusion-related applications (toroidal plasma equilibria in stochastic magnetic fields, and the Dimits shift), fusion-specific details have been discussed here only minimally. How then can the title be justified, especially when calculations aimed specifically at practical fusion phenomena such as the Dimits shift can be done only for extremely idealized models? My answer is that the more complicated the practical details, the more the field benefits from pedagogical exemplars in the sense of Kuhn [29]. (For a short discussion of Kuhn’s paradigm concept, see Ref. 112.) Hopefully the methods and models discussed above will help the reader structure his or her understanding of the basic logical threads underlying the behavior of turbulent plasma confinement devices and will point the way to future analytic calculations that are yet more concise and elegant.

In Regis’s book *Who Got Einstein’s Office?* [113], he comments on the vision of Abraham Flexner, the self-styled educator and author of the influential 1910 Flexner Report on the state of American medical education: “[Flexner] dreamt of a hybrid university—or an institute. . . —in which faculty members and students went into virgin territories together, not as equals, perhaps, but at least as partners.” Of course, the physical realization of that dream is Princeton’s Institute for Advanced Study. But the description of Flexner’s ideal interaction between faculty members and students strikes me as perfectly applicable to the intellectual environment that Allan Kaufman fostered at Berkeley. He and his students, post-docs, and visitors together tackled many difficult and subtle problems in theoretical physics, distilling new and elegant principles and perspectives that have created recognizable order from the chaos of myriad physical effects and made it so much easier for the newcomer to comprehend the basic structure of plasma dynamics. I feel privileged to have had even a brief taste of plasma-physics research *a la* Kaufman. That Allan’s influence has not faded is evidenced by the fact that in organizing my thoughts for this Symposium I was led to new results on renormalized ponderomotive force, 30 years after the original version of the K - χ theorem was published. I suggest that an aspiring theorist could do no better than to study the body of Allan’s work; it will inspire one and lead to the truth.

Finally, I would like to note the influence that my early Berkeley experiences had on my own teaching philosophy. I never sat in Allan’s classroom, for which I am very sorry. However, his approach to physics is very compatible with my own inclination, which is to seek out overarching principles that speak to the fundamental unity of the myriad of nonlinear phenomena. One must certainly teach specific known facts, but only by embedding those into a more general logical framework can one progress to generalizations and new discoveries. In my own teaching, I try to

emphasize that framework, including insights that stem from, or were motivated by, discoveries made by Allan and his collaborators. I am sure that I do it much less well than Allan, but I am very grateful indeed for his inspiration. I am deeply honored to have been able to participate in *Plasma Theory, Wave Kinetics, and Nonlinear Dynamics: KaufmanFest 2007*.

Acknowledgments

This work was supported by U.S. Department of Energy Contract No. DE-AC02-76-CHO-3073. Some parts of this material were initially presented orally at (i) a December 8, 2004, Colloquium at Princeton University's Plasma Physics Laboratory, and (ii) the June 11-12, 2007, Symposium on Recent Advances in Plasma Physics held at Princeton University in honor of Prof. R. C. Davidson. I am grateful to R. Dewar for discussions of renormalized Lie perturbation theory; to S. Johnston for providing me with some historical background on the K - χ theorem; to A. Reiman for collaboration on the problem of equilibria in stochastic magnetic fields; and to M. Labit, P. Sura, T. Stoltzfus-Dueck, and S. Zweben for discussions of blob physics and higher-order statistics.

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