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Strong Mobility in Weakly Disordered Systems

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We study transport of interacting particles in a weakly disordered medium. Our one-dimensional system includes two features: (i) disorder: the hopping rate governing the movement of a particle between two neighboring lattice sites is inhomogeneous, and (ii) hard core interaction: the maximum occupancy at each site is one particle. We find that during a substantial period, the mean-square-displacement of a particle, σ , grows super-diffusively with time t , $\sigma \sim (\epsilon t)^{2/3}$, where ϵ is the disorder strength, while in contrast, the particle displacement is sub-diffusive, $\sigma \sim t^{1/4}$, without disorder. Thus, we conclude that disorder dramatically enhances mobility. We explain this effect using scaling arguments, and verify the theoretical predictions through numerical simulations. The simulations also show that disorder generally leads to stronger mobility.

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Disorder is a ubiquitous feature of physical systems and furthermore, inhomogeneities and impurities are widely used to control properties of matter. Some of the most fascinating collective phenomena in contemporary physics including localization [1–3], slow relaxation, glassiness [4, 5], and frustration [6] are unique consequences of disorder.

While the effects of disorder on a noninteracting particle is well-understood, the consequences of disorder on strongly interacting, correlated particles remain an open question [7, 8]. In a quantum system, an isolated particle is localized by disorder, although recent results show that localization is destroyed when there are two interacting particles [9]. Hence, disorder and particle interactions compete. We investigate this interplay between disorder and particle interaction in a classical system where inhomogeneities are known to essentially trap particles and substantially slow down their movement, and find that, as in the quantum case, disorder has opposite effects on non-interacting and on interacting particles [10]. Our main result is that disorder speeds up the motion of interacting particles whereas disorder slows down the movement of non-interacting particles.

We generalize a canonical, well-studied, problem to elucidate the interplay between disorder and interactions. Our system is an unbounded one-dimensional lattice whose sites may be either occupied by a single particle or vacant. Initially, the lattice is occupied at random by identical particles with concentration c . Each particle may hop from an occupied site into a neighboring vacant site and this diffusion process is governed by the following rates: $p_+(i)$ is the hopping rate from site i to site $i+1$, and similarly, $p_-(i)$ is the hopping rate from site i to site $i-1$. While the total hopping rate is uniform, and is set to one without loss of generality, $p_-(i) + p_+(i) = 1$, the lattice is inhomogeneous. At every site there is, with equal probabilities, a bias to the right, $p_+(i) = 1/2 + \epsilon$, or a bias to the left, $p_+(i) = 1/2 - \epsilon$, as illustrated in figure 1. The

FIG. 1: Illustration of the disordered interacting particle system. The arrows indicate the bias at each site, the circles indicate vacant sites, and the bullets indicate occupied sites.

parameter $0 \leq \epsilon \leq 1/2$ quantifies the strength of the disorder. The disorder is therefore quenched, random, and uniform in strength. Moreover, since every lattice site accommodates a single particle, the particles interact via hard core repulsion.

Our focus is transport in this disordered, interacting particle, system. Since there is no overall bias in either direction, the displacement of a particle with respect to its initial position, x , does not change with time on average, $\langle x \rangle = 0$. We ask the most elementary question: how does the mean-square-displacement, σ , defined by $\sigma^2 = \langle x^2 \rangle$, evolve? We address this question via a scaling analysis of weakly disordered systems, $\epsilon \ll 1$, and numerical simulations with general disorder strengths.

Early Times. When disorder is weak, $\epsilon \ll 1$, there is an initial period during which particles do not experience the disorder, and thus, move at random, $p_+ = p_- = 1/2$. Hence, disorder is irrelevant and the behavior is dominated by particle interactions. Without disorder, the hard core repulsion causes a dramatic change in mobility: whereas an isolated particle moves diffusively, $\sigma \sim t^{1/2}$, the mean-square displacement of an interacting particle grows sub-diffusively with time [11–16]

$$\sigma \sim t^{1/4}. \quad (1)$$

Therefore, the movement of a particle is severely hindered by the presence of other particles. We demonstrate this remarkable collective behavior by considering extremely dense systems [12]. In the limit $c \rightarrow 1$, there are large clusters of occupied sites that are separated by isolated vacancies. Particles move by exchanging their position

with neighboring vacancies. A particle that, up to time t , exchanges position with a total of $N(t)$ vacancies of which $N_+(t)$ were initially located to its right and $N_-(t)$ were initially located to its left, with $N(t) = N_+(t) + N_-(t)$, has the following displacement, $x(t) = N_+(t) - N_-(t)$. First, since the vacancies are randomly distributed in the initial configuration and there is no bias in either the left or the right direction, the excess of vacancies in one direction can be deduced from the law of large numbers, $|N_+ - N_-| \sim N^{1/2}$, and consequently, $x \sim N^{1/2}$. Second, since the vacancies are sparse, they are essentially non-interacting and as a result each vacancy performs an ordinary random walk. Thus, vacancies that were initially located at distance proportional to the diffusive length scale $t^{1/2}$ from a particle may exchange position with it, and therefore, $N \sim (1 - c)t^{1/2}$. This scaling law, combining with $x \sim N^{1/2}$ yields the sub-diffusive scaling law (1). We note that while our argument is based on densely packed systems, the behavior (1) is established for arbitrary concentrations [11, 12]. We also comment that this suppressed diffusion is a direct and powerful consequence of the hard core interactions.

Intermediate Times. Eventually, the disorder becomes relevant, and the biased hopping rates do affect the particle displacement. Although there is no global bias in the hopping rates, there certainly are local biases, as illustrated in figure 1 where sites with negative bias are in the majority. We expect that, at least at intermediate times scales, or equivalently, intermediate length scales, these local biases lead to directed motion.

To quantify how such local biases affect particle mobility, we consider a particle that visits σ distinct sites of which n_+ have a positive bias and n_- have a negative bias, with $\sigma = n_+ + n_-$. Since the disorder is uncorrelated, the difference between the number of positive and negative sites, $\Delta = |n_+ - n_-|$, grows diffusively with the total number of visited sites, $\Delta \sim \sigma^{1/2}$. The excess of sites biased in one direction leads to a drift in this preferred direction with the small velocity $v \sim \epsilon \Delta / \sigma$ or $v \sim \epsilon \sigma^{-1/2}$. Furthermore, the ballistic length scale $x \sim vt$ gives an estimate for the displacement, $x \sim (\epsilon t) \sigma^{-1/2}$. We now impose self-consistency: this length scale must match the total number of sites visited, $x \sim \sigma$, and hence,

$$\sigma \sim \epsilon t \sigma^{-1/2}. \quad (2)$$

We thus arrive at our main result: the mean-square-displacement becomes super-diffusive because of the disorder,

$$\sigma \sim (\epsilon t)^{2/3}. \quad (3)$$

Of course, this length scale ultimately exceeds the suppressed diffusion length scale (1). We conclude that the inhomogeneous hopping rates generate a stochastic local velocity field, and as a result, there are local drifts that significantly enhance the mobility of the particles.

Late Times. To understand the behavior at late times, we recall that the displacement of a non-interacting particle in a random disorder is logarithmically slow,

$$x \sim \epsilon^{-2} (\ln t)^2. \quad (4)$$

We briefly explain the mechanism underlying this behavior. At sufficiently large length scales, the random disorder generates a potential well that confines the particle. The depth of this potential well is the sum of all the biases in a given range, $U(x) = \sum_{i=1}^x |p_+(i) - p_-(i)|$, and therefore, the depth of the well grows diffusively with the lateral extension, $U \sim \epsilon \sqrt{x}$. This stochastic well constitutes a barrier that the particle must overcome, and since the time to escape out of this barrier grows exponentially with the depth of the well, $t \sim \exp(U) \sim \exp(\epsilon \sqrt{x})$, the displacement is logarithmic as in (4).

We argue that the slow mobility (4) also characterizes the asymptotic behavior of our disordered interacting particle system. First, the confining potential well remains the same even when there are multiple particles. Second, the probability that a given particle escapes the well is exponentially small, and therefore, only mildly affected by the presence of other particles. We can envision a scenario where particles are stuck in a local minimum of the potential and escape the barrier one at a time. Clearly, such an escape process is governed by the very same exponential escape time that characterizes an isolated, non-interacting particle. We conclude that at late times, interacting particles in a random disorder also follow the logarithmic displacement law (4). Particle interactions become irrelevant and the behavior is governed by disorder alone.

The three time regimes. By combining the early (1), intermediate (3), and late (4) time behaviors, we conclude that the mobility of a given a given particle exhibits three distinct regimes of behavior as follows (see also figure 2),

$$\sigma \sim \begin{cases} t^{1/4} & t \ll \epsilon^{-8/5}, \\ (\epsilon t)^{2/3} & \epsilon^{-8/5} \ll t \ll \epsilon^{-4}, \\ \epsilon^{-2} (\ln t)^2 & \epsilon^{-4} \ll t. \end{cases} \quad (5)$$

The time and length scales that characterize the crossover points can be obtained by matching the two corresponding behaviors. The transition from the early regime into the intermediate regime occurs at time $t \sim \epsilon^{-8/5}$ and length $\sigma \sim \epsilon^{-2/5}$, while the transition from the intermediate domain into the late domain occurs at time $t \sim \epsilon^{-4}$ and length $\sigma \sim \epsilon^{-2}$, as shown in figure 2. In this study, we ignore the possible logarithmic corrections associated with the latter crossover [17].

Let us recap the three regimes of behavior. At the early stages, particle interactions dominate over disorder, and due to the hard core repulsion, the motion of particles is sub-diffusive. In the intermediate regime, disorder and

FIG. 2: The three regimes of behavior (5). The displacement σ is plotted versus time t using a double logarithmic scale.

interaction are both relevant. The particles stream following the stochastic local velocity field and the result is a strong, super-diffusive transport. At late times, disorder dominates and interactions become irrelevant. Particles are trapped by a stochastic potential well and the displacement is logarithmically slow because the escape time is exponentially large.

As further support of the scaling behavior above, we can show that the stochastic potential well plays no role in the intermediate regime. Clearly, since the overall hopping rate equals one, the time scale characterizing the movement between neighboring sites is also of order one. The time to escape out of a well grows exponentially with the depth of the well, $t \sim \exp(U)$, but this time scale becomes appreciable only when the depth of the potential well is large, $\epsilon\sqrt{x} \gg 1$, or equivalently, when the displacement becomes sufficiently large, $x \gg \epsilon^{-2}$. Indeed, this length scale is realized only at the late time regime, as shown in figure 2. Therefore, the potential well is negligible throughout the intermediate regime.

Let us now consider the effect of disorder on a non-interacting particle. In the absence of disorder, $\epsilon = 0$, the particle displacement is unhindered and thus, purely diffusive, $\sigma \sim t^{1/2}$. In weak disorder, $\epsilon \ll 1$, an isolated particle undergoes ordinary diffusion at early times, but is later slowed down considerably according to (4). Hence, there are two distinct regimes of behavior when interactions are absent

$$\sigma \sim \begin{cases} t^{1/2} & t \ll \epsilon^{-4}, \\ \epsilon^{-2}(\ln t)^2 & \epsilon^{-4} \ll t. \end{cases} \quad (6)$$

We note that the crossover time scale matches the upper time scale in (5). Thus, in the absence of particle interactions, disorder slows the particles down.

Surprisingly, disorder has the opposite effect on an interacting particle system. Because of disorder, particles undergo fast, super-diffusive motion as in (3) over a sub-

FIG. 3: The early and intermediate behaviors with a weak disorder, $\epsilon = 10^{-2}$. Shown are the mean-square-displacement σ versus time t (bullets), as well as two reference lines with slopes $1/4$ and $2/3$, respectively.

stantial time range. Remarkably, this mobility enhancement effect becomes more pronounced as the disorder weakens because the time and length scales characterizing the crossover points diverge as the disorder strength vanishes! We conclude in the presence of particle interactions, disorder speeds the particles up.

Numerical simulations. We performed extensive Monte Carlo simulations to test the scaling predictions. The simulation is a straightforward implementation of the transport process. Initially, N particles randomly occupy the sites of a one-dimensional lattice of size L , and the initial concentration equals c . Each lattice site has a bias in the positive or the negative direction as $p_+ = 1/2 + \epsilon$ or $p_- = 1/2 - \epsilon$ with equal probabilities. The dynamics are asynchronous. In an elementary step, a randomly chosen particle hops to the right with probability p_+ or to the left with probability p_- , and this hop is successful only if the neighboring site is vacant. Subsequently, time is augmented by the inverse

FIG. 4: The late time behavior for $\epsilon = 10^{-1}$. Shown is the mean-square-displacement σ versus time t for non-interacting particles (squares) and for interacting particles (bullets).

FIG. 5: The behavior for moderate disorders. Shown is the mean-square-displacement σ versus time t without disorder ($\epsilon = 0$, diamonds) and at moderate disorder values of $\epsilon = 0.1$ (bullets), 0.2 (squares), 0.3 (down-triangle), and 0.4 (up-triangle).

number of particles, $t \rightarrow t + 1/N$. This elementary step is repeated indefinitely. We present results of simulations with $L = 2 \times 10^5$ and $c = 1/2$.

We verified the super-diffusive behavior (3) using a very weak disorder (figure 3). Even though the super-diffusive behavior is an intermediate asymptotic, the duration of this intermediate regime grows rapidly as the disorder weakens. We performed a few additional tests: (i) we checked that the displacement σ is a function of the scaled time variable ϵt rather than t at intermediate times by using a range of weak disorders, (ii) we verified that the concentration does not play an important role using simulations with a larger concentration, $c = 3/4$, and a lower concentration, $c = 1/4$, and (iii) we used a different type of disorder with p_+ drawn from a flat distribution in the range $[1/2 : 1/2 + \epsilon]$ and recovered qualitatively similar results.

To test the behavior at late times, we also simulated a non-interacting particle system by ignoring the site occupancy restriction. These simulations show that after a very long transient period, the displacement in an interacting particle system and in a non-interacting particle system match (figure 4), thereby confirming that hard core interactions become irrelevant asymptotically, and that the behavior is governed by disorder alone.

Our scaling analysis tacitly assumes that disorder is small. A comparison of the behaviors with moderate disorders and without disorder shows that, irrespective of the disorder strength, mobility is always strengthened by disorder (figure 5). We conclude that mobility enhancement is a general effect that does not necessarily require weak disorder.

Figure 5 shows that the displacement in a homogeneous system eventually catches up with the displacement in a strongly inhomogeneous system. Indeed, the

sub-diffusive behavior (1) that characterizes a uniform system eventually exceeds the logarithmic displacement (4) in a disordered system. However, the crossover time $t \sim \epsilon^{-8}$ is astronomical at weak disorders. Therefore, disorder always generates a stronger mobility in practice. Moreover, the crossover time remains surprisingly large even at moderate and strong disorders (figure 5).

In conclusion, we studied the effect of disorder on transport in an interacting particle system. We found that whereas disorder slows down non-interacting particles, disorder speeds up interacting particles. Therefore, interaction and disorder have competing effects.

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