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The Use of Kruskal-Newton Diagrams for Differential Equations

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The use of Kruskal-Newton diagrams for differential equations

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Abstract

The method of Kruskal-Newton diagrams for the solution of differential equations with boundary layers is shown to provide rapid intuitive understanding of layer scaling and can result in the conceptual simplification of some problems. The method is illustrated using equations arising in the theory of pattern formation and in plasma physics.

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I. INTRODUCTION

Many problems in physics involve a very small parameter ϵ such as viscosity or resistivity which appears in the equations of interest multiplying a derivative of higher order than what appears in any other terms. In this case the solutions often exhibit boundary layer phenomena, *ie* the existence of narrow layers which scale as some power of ϵ, in which the solution is rapidly varying.

For equations of this type, it is often possible to construct an analytic solution order by order in the parameter ϵ. The process is simplified by using a Kruskal-Newton diagram to discover all possible cases of dominant balance for the equation. Each dominant balance gives either the exterior equation or a layer width $\delta(\epsilon)$ and the accompanying layer equation. One then finds the solutions inside and outside the layers, and then chooses constants of integration to asymptotically match the solutions to each other and to the boundary conditions. Each solution approaches y_{match} in the region which is asymptotic to both domains. Finally, a global uniform solution is obtained by writing $y_{unif} = y_{out} + y_{layer} - y_{match}$, this solution thus providing a continuous uniform approximate solution to the differential equation over the whole domain.

For a given linear homogeneous differential equation of the form

$$\sum_{q,n,m} C_{q,m,n} \epsilon^q x^m \frac{d^n y}{dx^n} = 0, \tag{1}$$

construct a Kruskal-Newton diagram with terms given by $x^p \epsilon^q$. Here we assume the layer is at $x = 0$ so count powers of x as well as powers of dx giving $p = m - n$. The function y and the numbers $C_{q,m,n}$ are all assumed to be of order one. Each term in the sum can be represented as a point in the p, q plane. All points above a given line represent terms smaller than points on the line, and all points below a line represent terms larger than those on the line.

To find all possible combinations of dominant balance for a given equation, find all possible placements of a line so that it includes two or more terms of the equation, with all other points lying above the line. Each such placement represents a potential solution to the equation whereby the dominant balance of the solution is given by the points on the line, and all points above the line are associated with terms which are small corrections to this solution. Graphically this may be understood as bringing the line up from below until it makes contact with a point, and then rotating it one way or the other until it makes contact with a second point. A line making contact with two or more points, with all remaining points of the diagram above the line, is called a support line. The scaling of x is quickly determined by the slope of the line in the diagram. If the terms defining the line are given by $x^p\epsilon^q$ and $x^r\epsilon^s$ then balancing them gives the scaling $x \sim \epsilon^{(s-q)/(p-r)}$, the power of ϵ being minus the slope of the line. Such a plot is known as a Kruskal-Newton diagram. It was first used by Newton[1] in considering infinitesimal displacements for the development of differential calculus and subsequently further developed by Kruskal[2].

A given support line, by the neglect of all terms lying above the line, produces a simplified differential equation for y,

$$Dy(x) = 0. \tag{2}$$

There are two means of improving the accuracy of this approximation if desired. Points lying above the line constitute a small correction to the solution. The lowest order solution $y_0(x)$, is given by neglecting them. Approximating the neglected terms using $y_0(x)$, one can solve the resulting inhomogeneous equation for y. A simple iteraton equation of the form

$$Dy_{n+1}(x) = f(y_n, x) \tag{3}$$

results. This iteration is rapidly convergent according to the smallness of ϵ, and also has the advantage that it does not depend on analyticity in ϵ. It is however often not useful as a means of analytically finding a higher order solution, but is extremely powerful to use numerically.

Alternatively one can perform a perturbation expansion in ϵ by writing $y = y_0(x) + \epsilon y_1(x) + ...$ and solving order by order. In general the solution is not analytic in ϵ, so this expansion is asymptotic. In practice the parameter ϵ is often so small that the lowest or first order solution is sufficient.

In section II we discuss nested boundary layers, section III inhomogeneous equations, section IV diagrams with an infinity of points, section V the phenomenon of spurious balance, section VI boundary layers in the complex plane. Section VII treats coupled equations arising in the theory of toroidally confined plasmas. Where possible approximate methods are verified with exact solutions.

II. NESTED BOUNDARY LAYERS

It can happen that an equation possesses three different possible dominant balances, giving rise to an external solution plus two layers with different scalings, one inside the other. An example of this behavior is given by the equation

$$\epsilon^3 x y'' + x^2 y' - y(2x^3 + \epsilon) = 0, \tag{4}$$

with boundary conditions $y(0) = 1, y(1) = e$. As is well known, [3, 4] the layer location is determined by the sign of the function multiplying the derivative term. Thus the layer is at the left. The Kruskal-Newton diagram for Eq. 4 is given in Fig. 1. Ordering each term as $x^p \epsilon^q$, the first term is at (-1,3), the second at (1,0), the third at (3,0) and the last at (0,1). The outer solution is given by the terms of order zero in ϵ, and there are seen to be two other possible balances. Line 2 in the graph has slope -1, giving $x \sim \epsilon$. Line 3 has slope -2, giving $x \sim \epsilon^2$. Thus line 3 gives a layer contained within the layer given by line 2.

Use the perturbation theory approach. Begin with the outer solution. Write $y = y_0 + \epsilon y_1 + \dots$, giving $x^2 y_0' - x^3 y_0 = 0$, and $x^2 y_1' - x^3 y_1 = y_0$, and upon matching the boundary condition at $x = 1$ we have the solutions

$$y_0 = e^{x^2}, \qquad y_1 = (1 - 1/x) e^{x^2}. \tag{5}$$

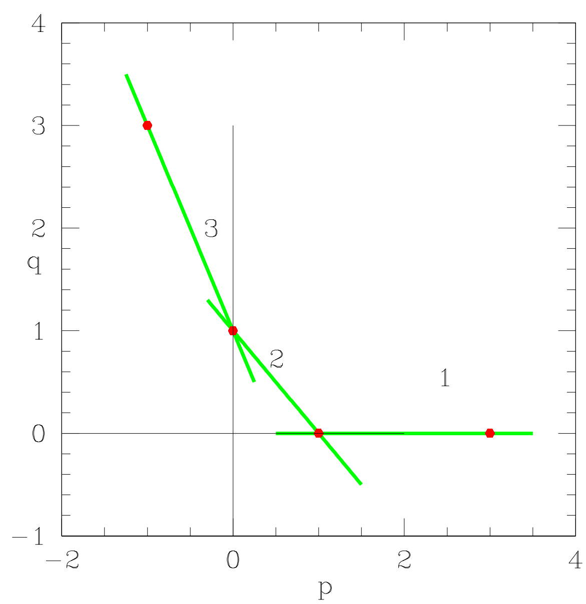

FIG. 1: Kruskal-Newton diagram for Eq. 4 with terms given by $x^p \epsilon^q$

Boundary conditions are not met at $x = 0$ by this solution, so match it to a solution in the layer given by line 2 in the Kruskal-Newton diagram. Let $x = \epsilon X$, with $y(x) = Y(X)$, giving

$$X^2 \frac{dY}{dX} - Y = -\epsilon X \frac{d^2Y}{dX^2} + 2\epsilon^2 X^3 Y. \tag{6}$$

Expanding $Y = Y_0 + \epsilon Y_1$ and matching to the outer solution we find the layer solution given by

$$Y = e^{-1/X} \left[1 + \epsilon \left(1 - \frac{2}{3X^3} + \frac{1}{4X^4} \right) \right] + O(\epsilon^2), \tag{7}$$

but it is still not possible to match the condition $y(0) = 1$. The matching solution in the domain $\epsilon << x << 1$ is given by $y_{match} = 1 + \epsilon - \epsilon/x + O(\epsilon x, x^2, \epsilon^2)$.

The second layer is given by line 3 in the Kruskal-Newton graph, with $x \sim \epsilon^2$. Writing $x = \epsilon^2 Z$, $y(x) = \mathcal{Y}(Z)$ with $\mathcal{Y} = \mathcal{Y}_0 + \epsilon \mathcal{Y}_1$ we find $\mathcal{Y}_0 = \beta_0 \sqrt{Z} I_1(2\sqrt{Z}) + \beta_1 \sqrt{Z} K_1(2\sqrt{Z})$. Since Y tends to zero for small X we must require that $\mathcal{Y}$ tend to zero for large Z. Thus we find $\beta_0 = 0$. Matching the solution to 1 at zero we find

$$\mathcal{Y}_0 = 2\sqrt{Z} K_1(2\sqrt{Z}). \tag{8}$$

Finally the uniform solution to lowest order is given by

$$y_u = y_0(x) + Y_0(X) + \mathcal{Y}_0(Z) - y_{match}, \tag{9}$$

there being no matching term coming from the domain between the two layer solutions, since each solution tends to zero exponentially in the matching region.

III. INHOMOGENEOUS EQUATIONS

Some problems involving inhomogeneous equations simplify under a rescaling of the dependent variable. Consider a differential equation of the form $\sum_{q,m,n} C_{q,m,n} \, \epsilon^q x^m y^{(n)} = x^r$. Rescale y through $y = \epsilon^{-\beta} Y$ and multiply the equation by ϵ^β, giving

$$\sum_{q,m,n} C_{q,m,n} \; \epsilon^q x^m Y^{(n)} = x^r \epsilon^\beta. \tag{10}$$

Terms involving $Y^{(n)}(x)$ are represented in a Kruskal-Newton diagram at coordinates $(p, q) = (m - n, q)$ as in the homogeneous case . The position of the inhomogeneity is at

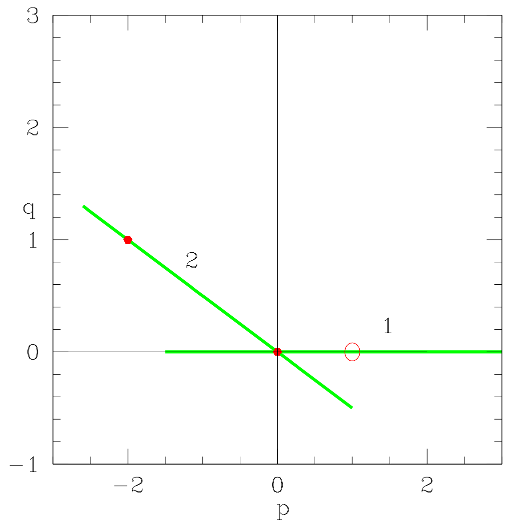

FIG. 2: Kruskal-Newton diagram for Eq. 11.

coordinates $(p, q) = (r, \beta)$, and has shifted by β. Alternatively, without multiplying through by ϵ^β it is the homogeneous terms that shift, only the relative displacement is relevant.

Draw the Kruskal-Newton diagram as in the homogeneous case. This leads to one or more support lines. Represent the inhomogeneity by a vertical line corresponding to the set of points $\{(r, \beta),\ -\infty < \beta < \infty\}$. Any vertical line must intersect all support lines, so the inhomogeneity can be included in the corresponding balances, since rescaling y corresponds to a vertical shift of all homogeneous points with respect to all inhomogeneous points. The value of β for this balance is given by the height of the intersection point.

A. A vanishing boundary layer

Consider the inhomogeneous equation

$$\epsilon y'' + xy' = x, \tag{11}$$

with boundary conditions $y(0) = A$, $y(1) = B$. The Kruskal-Newton diagram is given in Fig. 2 where the inhomogeneous term is shown with an open circle and the homogeneous terms with solid points. There is an outside solution given by line 1 and a boundary layer at $x = 0$ given by line 2 with the layer given by the scaling $x = \sqrt{\epsilon}X$. The lowest order outside solution is then $y = x + B - 1$ and the matching solution is $y_{match} = B - 1$. The interior solution is $Y = c\int_0^X e^{-t^2/2}dt + d$. Matching we find $B - 1 = c\sqrt{\pi/2} + d$ and $d = A$

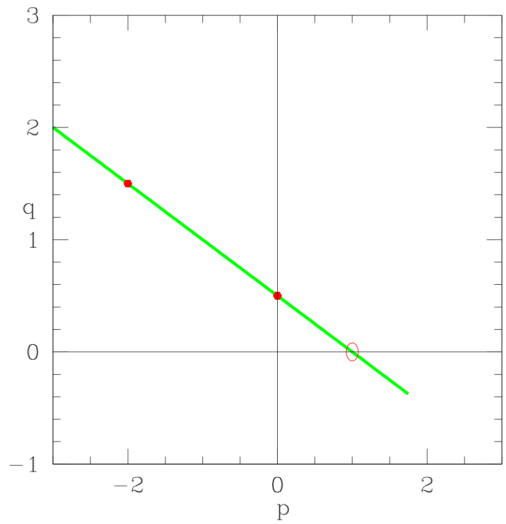

FIG. 3: the Kruskal-Newton diagram for Eq. 11 rescaled through $y = \sqrt{\epsilon}Y$.

and the uniform solution is

$$y_{unif}(x) = x + A + \sqrt{2/\pi}[B - A - 1] \int_0^{x/\sqrt{\epsilon}} e^{-t^2/2} dt. \tag{12}$$

Now consider a rescaling of the dependent variable through $y = \sqrt{\epsilon}Y$. The Kruskal-Newton diagram is then given in Fig. 3, and now it is clear that there is no layer in this problem, there is a single line in the diagram with slope $-1/2$. The rescaled equation, with $y = \sqrt{\epsilon}Y$ and $x = \sqrt{\epsilon}X$ is $Y'' + XY' = X$, a three term balance. The solution is

$$Y = a \int_0^X dte^{-t^2/2} + b + X \tag{13}$$

and matching boundary conditions we find that this is the same solution obtained above using the boundary layer formalism. This simple example demonstrates that a rescaling of the dependent variable can modify the Kruskal-Newton diagram even to the point of causing formal boundary layers to vanish or appear. Also note that the rescaling produces a diagram in which all the points are on a single support line, meaning that this solution is exact, there are no neglected terms. A simple examination of the Kruskal-Newton diagram reveals whether a rescaling of variables exists in which this is possible.

B. A $sinx$ inhomogeneity

Consider the equation

$$\epsilon\frac{d^2y}{dx^2} + x\frac{dy}{dx} = xsinx, \tag{14}$$

in the domain $(0,\pi)$, with $0 < \epsilon \ll 1$ and the boundary conditions $y(0) = y(\pi) = 1$.

The $xsinx$ term is of order x^p with $1 < p < 2$. The resulting diagram is shown in Fig. 4. Line 1 of the diagram has slope zero, giving the exterior solution, with lowest order balance given by $xdy/dx = xsinx$.

This gives upon matching to the boundary conditions at π the solution $y(x) = -cosx$. Line 2 of the diagram has slope -1/2, so $x = \sqrt{\epsilon}X$, with $y(x) = Y(X)$, giving to lowest order

$$\frac{d^2Y}{dX^2} + X\frac{dY}{dX} = 0, \tag{15}$$

with solution $Y = a\int_0^X e^{-s^2/2}ds + 1$. For large X we have $Y \to a\sqrt{\pi/2} + 1$ Matching to the exterior solutions gives $a\sqrt{\pi/2} = -2$. The lowest order uniform solution is then obtained by adding the internal and external solutions and subtracting the matching solution,

$$y_{unif}(x) = -cosx + 2 + 2\sqrt{2/\pi}\int_0^{x/\sqrt{\epsilon}} e^{-s^2/2}ds. \tag{16}$$

C. A multiple layer inhomogenous problem

Consider the equation

$$\epsilon^3 y'' - \epsilon y + x^3 y' = x, \tag{17}$$

with boundary condition $y(0) = A$ and $y(1) = B$. Figure 5 shows the Kruskal-Newton plot for this equation. The solid dots correspond to the homogeneous part and the support lines 1 to 3 are drawn neglecting the inhomogeneous terms. The dotted line represents the inhomogeneity. The intersection point with line 2 lies at height 1/2 and the dependent variable must be rescaled according to $y = \epsilon^{-\frac{1}{2}}Y$ in order to allow this balance. Otherwise the inhomogeneity would lie below line 2 and it would be a term of lower order. The other intersections lie on the p-axis giving $\beta = 0$, and there is no need to rescale.

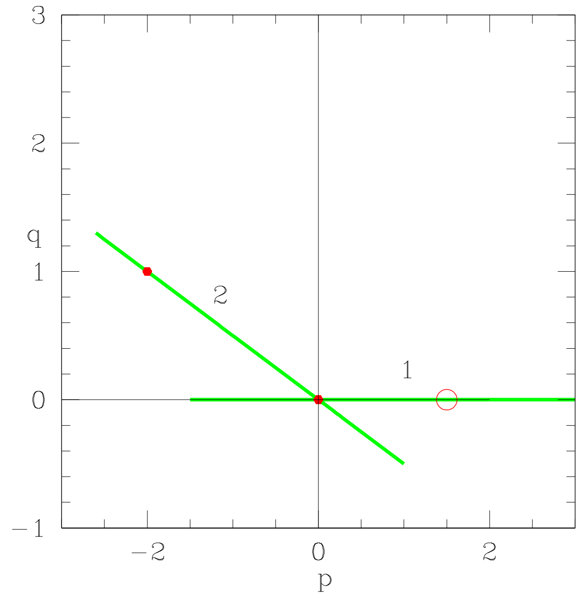

FIG. 4: Kruskal-Newton diagram for Eq. 14

Line 1 has slope zero, the dotted line intersects it, so include the inhomogeneity. The intersection lies on the p-axis, so $\beta = 0$, giving the exterior solution, with leading order balance given by $x^3 y_0'(x) = x$.

Matching the boundary condition at $x = 1$ gives

$$y \sim y_0 = -\frac{1}{x} + B + 1. \tag{18}$$

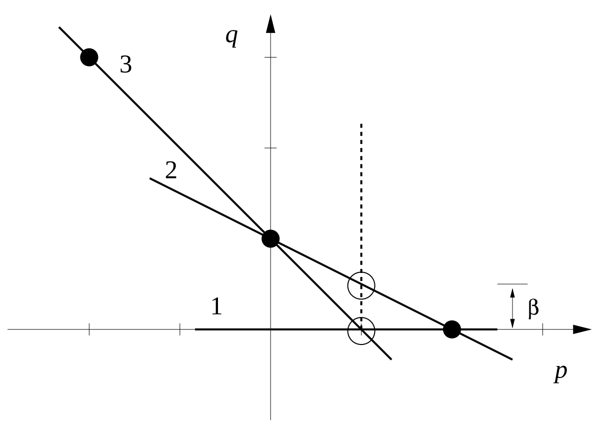

FIG. 5: The Kruskal-Newton plot for equation (17). The inhomogeneity is represented by the dotted vertical line at $p = 1$. The empty circles mark intersection points with support lines 1, 2 and 3.

Line 2 has slope $-\frac{1}{2}$, so it intersects the dotted line at height $\beta = \frac{1}{2}$, and the rescaled variables in the middle region are given by $y(x) = \epsilon^{-1/2}Y(X)$, and $x = \sqrt{\epsilon}X$. The leading order balance is given by $-Y_0(X) + X^3Y_0'(X) = X$. This equation can be solved by first changing variables to $f(s) = Y_0(\frac{1}{s})$ and then using the ansatz $f(s) = \int \sin(st)g(t)\mathrm{d}t$, leading to

$$Y_0(X) = ae^{-1/2X^2} - \int_0^\infty \sin(t/X)e^{-t^2/2}dt. \tag{19}$$

We determine the constant a by matching with the exterior solution $y_0 = \frac{1}{x} + B + 1$. The solution in the middle layer is $y = \epsilon^{-\frac{1}{2}}\left[ae^{-\frac{1}{2X^2}} - \int_0^\infty \sin\left(\frac{t}{X}\right)e^{-\frac{t^2}{2}}\mathrm{d}t\right]$. Rewritten in exterior variables this gives $y = \epsilon^{-\frac{1}{2}}\left[ae^{-\frac{\epsilon}{2x^2}} - \int_0^\infty \sin\left(\frac{\epsilon^{\frac{1}{2}}t}{x}\right)e^{-\frac{t^2}{2}}\mathrm{d}t\right]$ and expanded in the intermediate domain $\sqrt{\epsilon} \ll x \ll 1$ gives $y = \epsilon^{-\frac{1}{2}}\left[a - \frac{\epsilon^{\frac{1}{2}}}{x} + \dots\right]$. Matching these solutions we find $a = 0$ and determine the matching solution in this domain $y_{match_{12}} = -1/x$.

Line 3 has slope -1 and it intersects the dotted line at zero height. The inner variables are therefore $y(x) = \mathcal{Y}(\mathcal{X})$, and $x = \epsilon\mathcal{X}$. We find to leading order $\mathcal{Y}_0''(\mathcal{X}) - \mathcal{Y}_0(\mathcal{X}) = \mathcal{X}$, with solution

$$\mathcal{Y}_0 = a\exp(-\mathcal{X}) + b\exp(\mathcal{X}) - \mathcal{X}. \tag{20}$$

Finally, we match the inner solution with the middle solution $y \sim \epsilon^{-\frac{1}{2}}Y_0 = -\epsilon^{-\frac{1}{2}}\int_0^\infty \sin\left(\frac{t}{X}\right)e^{-\frac{t^2}{2}}\mathrm{d}t$. Expanded in the intermediate domain $\epsilon \ll x \ll \sqrt{\epsilon}$ or equivalently $\sqrt{\epsilon} \ll X \ll 1$ the contribution for $X \ll 1$ comes from the end point, giving $y \sim -\epsilon x = -\mathcal{X}$. We then find $b = 0$, and matching to the boundary condition at $x = 0$ gives $a = A$. We then find for the intermediate domain $\mathcal{X} \gg 1$, $\mathcal{Y}_0 \to y_{match_{23}} = -\mathcal{X}$.

The uniform solution is obtained by adding the outside solution to the two layer solutions and subtracting the two matching solutions,

$$y_{unif}(x) = B + 1 - \int_0^\infty \sin\left(\frac{\sqrt{\epsilon}t}{x}\right)e^{-t^2/2}\mathrm{d}t + Ae^{-x/\epsilon}. \tag{21}$$

IV. INFINITE DIAGRAMS

In the case of the presence of transcendental functions such as in example III B it can be convenient to expand these functions in infinite series, producing Kruskal-Newton diagrams

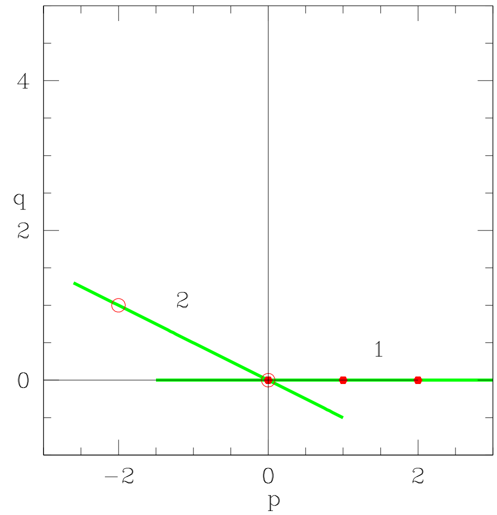

FIG. 6: Kruskal-Newton diagram for Eq. 22

with an infinity of points. Often this means that a support line contains only one term of the expansion, greatly simplifying the differential equation associated with it. If this is the case it also means that on that line the first term of the expansion is a good approximation of the function.

A. A nonpolynomial coefficient

Consider the equation

$$\epsilon y'' + 2tan(x)y' - 3y = 0, \tag{22}$$

with $y(0) = 1$, $y(\pi/2) = 1$. Expand $tan(x) = x + x^3/3 + 2x^5/15 + ...$ The Kruskal-Newton diagram is given in Fig. 6, where the solid dots correspond to $2tan(x)y'$ and the empty circles correspond to the first and third term in Eq. 22. The dots on the p-axis constitute the outer balance $2tan(x)y' \sim 3y$ with solution matching the boundary condition at $x = \pi/2$ of $y = sin(x)^{3/2}$. The matching solution for $x \to 0$ is $y_{match} = x^{3/2}$. Line 2 gives the layer equation with $x = \sqrt{\epsilon}X$ $Y'' + 2XY' - 3Y = 0$. Solutions are given by

$$Y_C = \frac{1}{i}\int_C e^{Xt+t^2/4}t^{-5/2}dt \tag{23}$$

where the contour must end at $\pm i\infty$ and can pass on either side of the cut originating at $t = 0$. If the contour is taken from $-i\infty$ to $i\infty$ to the left of the cut the large X behavior is given by a saddle point at $t = -2X$, producing asymptotic behavior of $X^{-5/2}e^{-X^2}$. A contour beginning and ending at $i\infty$ and circling the cut has asymptotic behavior of $X^{3/2}$. Thus the interior solution is given by

$$Y = \frac{a}{i}\int_{-i\infty}^{i\infty} e^{Xt+t^2/4}t^{-5/2}dt + \frac{b}{i}\oint_{i\infty}^{i\infty} e^{Xt+t^2/4}t^{-5/2}dt, \tag{24}$$

where the second contour is taken to circle the cut in a clockwise manner. Evaluating the limits for $X \to 0$ we find $Y \sim (a+b)2^{-5/4}\Gamma(-3/4)$. Evaluating the limits for $X \to \infty$ we find $Y \sim a2^{-3/2}\sqrt{\pi}X^{-5/2}e^{-X^2} + bX^{3/2}\Gamma(-3/2)$. Matching to the outside solution for $X \to \infty$ and to the boundary condition at $x = 0$ we find $b = \epsilon^{3/2}/\Gamma(-3/2)$ and $a = 2^{5/4}/\Gamma(-3/4) - \epsilon^{3/2}/\Gamma(-3/2)$. The uniform solution is then

$$y_{unif} = sin(x)^{3/2} - x^{3/2} + \frac{b}{i}\oint_{i\infty}^{i\infty} e^{xt/\sqrt{\epsilon}+t^2/4}t^{-5/2}dt + \frac{a}{i}\int_{-i\infty}^{i\infty} e^{xt/\sqrt{\epsilon}+t^2/4}t^{-5/2}dt. \tag{25}$$

B. Many dots

Consider the equation

$$\epsilon^3 y'' + tan(x)y' - [xsin(x) + \epsilon cot(x)]y = 0, \tag{26}$$

with $y(0) = A$ and $y(\pi/2) = exp(\pi/2)$. Expand $tan(x) = x + x^3/3 + 2x^5/15 + ...$, $xsin(x) = x^2 + x^4/6 + ...$, $cot(x) = 1/x - x/3 + ...$

The Kruskal-Newton diagram is given in Fig. 7 and the empty circles correspond to $xsin(x)y$. The lowest order outside solution matched to the boundary condition at $\pi/2$ is $y = e^{xsin(x)+cos(x)}$ for $x \to 0$ we have $y_{match_{12}} \sim 1 + x^2$. For the intermediate layer given by line 2, $x = \epsilon X$ and the lowest order equation is $XY' - Y/X = 0$, with solution $Y = ce^{-1/X}$, and for large X we have $Y \sim c[1 + O(\epsilon)]$ giving $c = 1$, and for small X it is exponentially small. For line 3 $x = \epsilon^2 Z$ and $\mathcal{Y}'' - \mathcal{Y}/Z = 0$, with solution, matched to the boundary condition at $Z = 0$,

$$\mathcal{Y} = A\int_0^\infty \frac{dt}{t^2}e^{-Zt-1/t} \tag{27}$$

For large Z this solution is dominated by a saddle point at $t = \sqrt{Z}$, and we have $\mathcal{Y} \sim \sqrt{\pi}Z^{3/2}e^{-Z^{3/2}}$ matching the intermediate solution to this order. The uniform solution is then

$$y_{unif}(x) = e^{xsin(x)+cos(x)} + e^{-\epsilon/x} + A\int_0^\infty \frac{dt}{t^2}e^{-xt/\epsilon^2-1/t} - 1 - x^2. \tag{28}$$

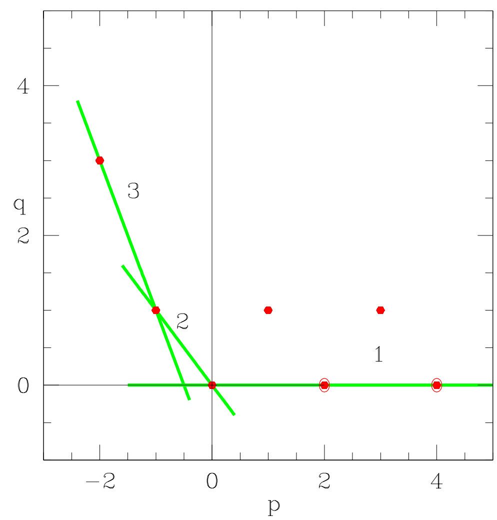

FIG. 7: Kruskal-Newton diagram for Eq. 26

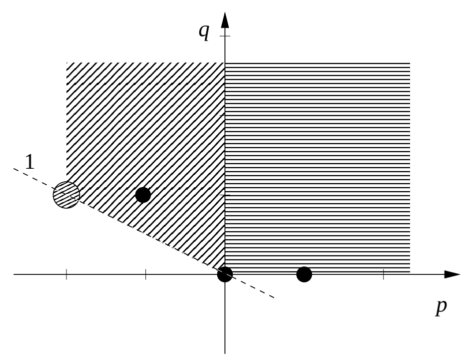

FIG. 8: Kruskal-Newton diagram for Eq. 29.

V. SPURIOUS BALANCE

It can happen that two terms lead to a formal balance between two derivatives of equal order, such as $\epsilon y'$ and xy'. Balancing these terms however cannot describe a boundary layer. A term of this nature normally only produces a small perturbation, but it is still capable of shielding otherwise important terms.

Consider the equation

$$(a\epsilon + x^2)y'' + (b\epsilon + x^2)y' = 0, \tag{29}$$

with boundary conditions $y(0) = 0$, $y(1) = 1$, a and b constants. The Kruskal-Newton diagram with both a and b of order 1 is shown in Fig. 8. Line 1 gives the spurious balance $a\epsilon y'' \sim x^2 y''$. The term $a\epsilon y''$ itself does not define a layer and contributes only little, but

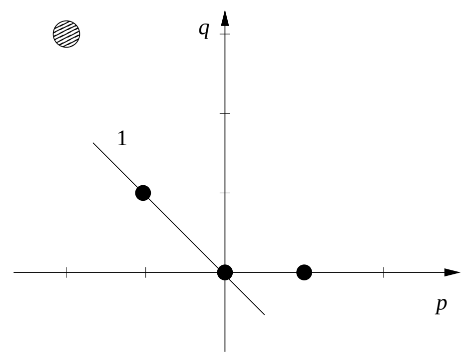

FIG. 9: Kruskal-Newton diagram for Eq. 29 with $a = 0$.

shields the term $b\epsilon y'$ represented by a dot in the hatched region.

The outside solution is

$$y(x) = \frac{e(1-1/b)}{1-e}e^{-x} + \frac{1/b - e}{1-e}, \tag{30}$$

and the solution coming from line 1, with $x = \sqrt{\epsilon}X$, matched to the boundary condition at $X = 0$ is $Y(X) = cX$. But this function cannot be matched to the exterior solution, so it is not possible to find a layer solution.

The Newton iteration method is however successful, and is worth demonstrating for this problem, since it leads to a very rapidly convergent result. Begin with the dominant balance given by line 1, giving the lowest order solution matching both boundary conditions $y_0(x) = x$. Often the lowest order solution is reasonably accurate but this linear function

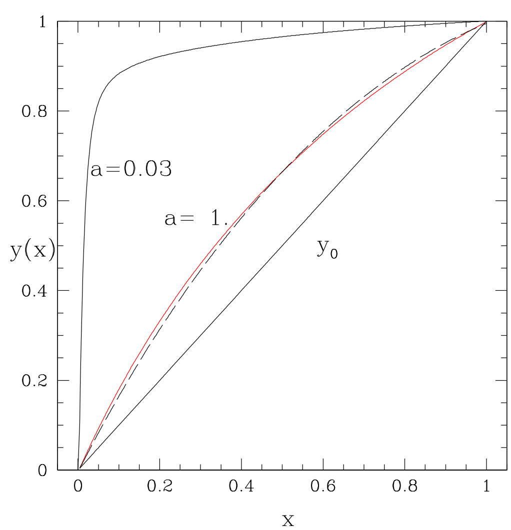

FIG. 10: Solutions of Eq. 29 with $a = 0.03$ and 1.0, with $\epsilon = .1$.

is a poor approximation to the correct solution because of the neglected terms. Performing one iteration of Eq. 3 we find the equation $y''(x) = -(b\epsilon + x^2)/(a\epsilon + x^2)$ and integration of this equation gives $y_1(x)$, a good approximation to the solution.

The situation is qualitatively different if we make a of order ϵ or smaller, as shown in Fig. 9. The previously shielded dot is now dominant in a boundary layer and cannot be neglected. For small a the layer is given by $x = \epsilon X$ with $Y(X) = (1 - e^{-bX})/b$, and the exterior solution is Eq. 30, giving a uniform solution of

$$y_{unif}(x) = \frac{1 - e^{-bx/\epsilon}}{b} + \frac{e(1 - 1/b)}{1 - e} e^{-x} + \frac{e/b + 1 - 2e}{1 - e}. \tag{31}$$

This lowest order solution is very accurate when a is of order ϵ or smaller.

Shown in Fig. 10 are numerical integrations of Eq. 29 with $y(0) = 0, y(1) = 1$ and $b = 2$, $\epsilon = 0.1$, for both $a = 1.0$ and $a = 0.03$. There is no boundary layer for the large a case, but a very pronounced layer if a is small. Shown also is the lowest order solution for large a, $y_0(x)$, using the Newton iteration method, which gives a poor approximation to the solution, and with a dotted line the solution $y_1(x)$ of the first order iteration, much more accurate.

VI. BOUNDARY LAYERS IN THE COMPLEX PLANE

Asymptotic matching in the complex plane is a strategy for calculating exponentially small terms that has been developed by Kruskal and Segur[5].

To apply the method determine the singularities of the leading-order term of the naive asymptotic expansion and find an inner equation in the vicinity. Find the exponentially small terms in the far field of the inner solution and match them with the possible exponentially small corrections to the naive expansion. Because the inner problem exhibits Stokes phenomenon in its far field[4, 6], the exponentially small corrections are only present in certain sectors. If the naive perturbation expansion contains only trivial information, the exponentially small corrections can play an important role when they control qualitative new physical phenomena or if they prevent the existence of solutions.

A. Example: Crystal growth

We consider the equation

$$\epsilon\psi''(s) + \psi(s) = \frac{1}{1+s^2}, \tag{32}$$

with boundary conditions $\psi \to 0$ as $s \to \pm\infty$. This problem is the steady state version of a problem considered by Chapman and Mortimer [7]. It is motivated by the geometrical model for crystal growth. Chapman and Mortimer used Stokes smoothing to show that no steady state solution exists although a naive perturbation expansion indicates the contrary. We reproduce this result using the Kruskal Segur method.

Expanding $\psi = \psi_0 + \epsilon\psi_1 \ldots$, we find at leading order:

$$\psi_0(s) = \frac{1}{1+s^2}. \tag{33}$$

The result satisfies the boundary conditions and it can be verified that all higher order terms do also. The leading order term has singularities at $s = \pm i$. We first study the vicinity of $s = -i$. It is useful to shift the singularity to the origin by a change of the independent variable, $s = -i + it$, $\psi(-i + it) = \phi(t)$. Equation (32) transforms to

$$-2\epsilon t\ddot{\phi}(t) + \epsilon t^2\ddot{\phi}(t) + 2t\phi(t) - t^2\phi(t) = 1 \tag{34}$$

The Kruskal-Newton diagram of this equation is shown in Fig. 11. Support line 1 marks the outer balance giving at leading order $\phi_0(t) = 1/[t(2-t)]$ in agreement with equation (33). Possible exponentially small corrections to the outer balance involve superpositions of the WKBJ solutions of the homogeneous version of equation (34)

$$\phi_\pm = e^{\pm t/\sqrt{\epsilon}}. \tag{35}$$

The inner equation is marked by support line 2. The scaling of the inner variables can be read off from the diagram: $t = \epsilon^{1/2}\tau$ and $\phi = \epsilon^{-1/2}\Phi$ and the inner equation reads

$$-2\tau\frac{d^2\Phi}{d\tau^2} + 2\tau\Phi = 1.$$

A solution of this equation is given by

$$\Phi(\tau) = \frac{1}{2}\int_C \frac{e^{-\tau z}}{1-z^2}dz. \tag{36}$$

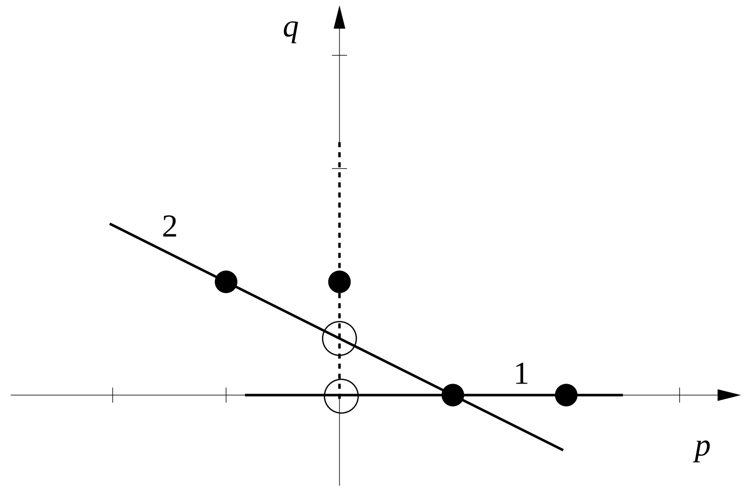

FIG. 11: The Kruskal-Newton diagram for equation (34). The dotted line represents the inhomogeneity and empty circles mark intersection points with the support lines.

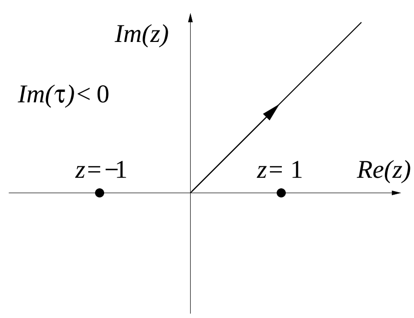

FIG. 12: The steepest decent path for the integral (36).

The path C starts at $z = 0$ and goes to $+\infty$ in the right half plane. We choose the singularity $z = 1$ to lie below the path. We can deform this contour to the steepest descent contour $arg(\tau z) = 0$ without passing through $z = 1$ provided $Im(\tau) < 0$ (Fig.12). Continuing to $Im(\tau) > 0$ the contour must be indented to include an clockwise circuit around the pole (Fig. 13). Thus, the line $Im(\tau) = 0$ is a Stokes line for $\Phi(\tau)$ and we pick up an extra pole contribution

$$\Phi_{pole} = -\pi i Res(e^{-\tau z}/(1 - z^2), z = 1) = -\frac{\pi}{2} i e^{-\tau}$$

when crossing it. The endpoint contribution $\Phi_{ep} \sim 1/2\tau$ is present everywhere.

Thus,

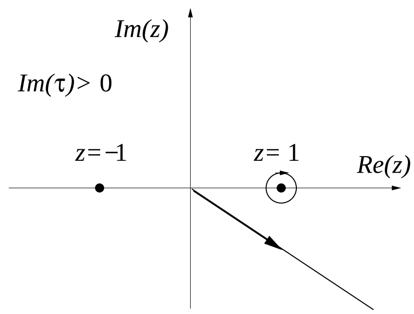

FIG. 13: The steepest decent path for the integral (36) plus a contribution from the pole.

$$\Phi(\tau) \sim \begin{cases} 1/2\tau & Im(\tau) > 0 \\ 1/2\tau - \frac{\pi}{4} i e^{-\tau} & Im(\tau) = 0 \\ 1/2\tau - \frac{\pi}{2} i e^{-\tau} & Im(\tau) < 0 \end{cases}$$

The outer solution $\phi_0(t)$ must be generalized by adding the WKBJ solutions $\phi_\pm(t)$, equation (35), as needed:

$$\phi(t) \sim \begin{cases} \phi_0(t) & Im(t) > 0 \\ \phi_0(t) + \frac{a}{2}\phi_-(t) & Im(t) = 0 \\ \phi_0(t) + a\phi_-(t) & Im(t) < 0 \end{cases}$$

The constant a can be found by matching the inner and outer expansion along the ray $Im(t) = 0$. We find

$$a = -\frac{\pi}{2} i \epsilon^{-1/2}.$$

Extrapolating back to the real s-axis we also have to take into account the contributions from the singularity at $s = i$. Since the solution $\psi(s)$ must be real on the real s-axis, this other contribution is the complex conjugate. We arrive at the uniformly valid solution for real s satisfying the boundary conditions for $s \to -\infty$:

$$\psi(s) \sim \begin{cases} 1/(1+s^2) & s < 0 \\ 1/(1+s^2) + \pi\epsilon^{-1/2} e^{-1/\sqrt{\epsilon}} sin(s/\sqrt{\epsilon}) & s > 0 \end{cases}$$

The result shows that the exact solution is not well approximated by $1/(1+s^2)$ for $s >> 1$. At large s, the exponentially small (in ϵ) correction becomes visible. The solution fails to satisfy the boundary condition as $s \to +\infty$.

To confirm the Kruskal Segur method we obtain an exact solution. Perform a Fourier transformation of Eq. 32

$$\int_{-\infty}^{\infty} dze^{-kz}[\epsilon\psi''(z) + \psi(z)] = \int_{-\infty}^{\infty} dze^{-kz}\frac{1}{1+z^2} \tag{37}$$

and write $\psi(z) = \int_{-\infty}^{\infty} dze^{ikz}\phi(k)$. Then using

$$\int_{-\infty}^{\infty} e^{-ikz}\frac{dz}{1+z^2} = \pi e^{-|k|}, \qquad \int_{-\infty}^{\infty} e^{ikz}e^{-|k|}dk = \frac{2}{1+z^2} \tag{38}$$

we find $\phi(k) = e^{-|k|}/[2(1-\epsilon k^2)]$ and an exact solution to the inhomogeneous differential equation

$$\psi(z) = \frac{1}{2}\int_{-\infty}^{\infty} e^{ikz}\frac{e^{-|k|}}{1-\epsilon k^2} = Re\int_{0}^{\infty} e^{ikz}\frac{e^{-k}}{1-\epsilon k^2}. \tag{39}$$

Now look at $z \to +\infty$. Writing the integrand as $exp(ikz - k - ln(1-\epsilon k^2))$. we find a saddle point for large $|z|$ at $k = 1/\sqrt{\epsilon} - i/z$ giving a contribution of $\psi_s(z) = cos(z/\sqrt{\epsilon})/\sqrt{2\epsilon}$. In addition there is an endpoint contribution from $k \simeq 0$, obtained to lowest order by dropping the ϵk^2 term in the denominator and using Eq. 38, giving $\psi_e(z) = 1/(1+z^2)$ so for large positive z the integral solution is asymptotic to $\psi(z) \simeq 1/(1+z^2) + cos(z/\sqrt{\epsilon})/(\sqrt{2\epsilon})$. We recognize the second term as a solution to the homogeneous equation, and thus the solution

$$\psi(z) = Re\int_{0}^{\infty} e^{ikz}\frac{e^{-k}}{1-\epsilon k^2} - \frac{cos(z/\sqrt{\epsilon})}{\sqrt{2\epsilon}}. \tag{40}$$

is asymptotic to $1/(1+z^2)$ for large positive z, and thus satisfies the requisite boundary condition.

Now continue to negative z. The integration path must be moved to the new saddle point above the real axis, as shown in Fig. 14, producing an additional circular path around the pole at $k = 1/\sqrt{\epsilon}$. Thus for negative z we have

$$\psi(z) = Re\int_{0}^{\infty} e^{ikz}\frac{e^{-k}}{1-\epsilon k^2} - \frac{cos(z/\sqrt{\epsilon})}{\sqrt{2\epsilon}} + \frac{\pi e^{-1/\sqrt{\epsilon}}}{\sqrt{\epsilon}}sin(z/\sqrt{\epsilon}). \tag{41}$$

with the integration path now passing through the upper saddle point. Evaluating again the end point and saddle point contributions we have for large negative z $\psi(z) \simeq 1/(1+z^2) + \pi e^{-1/\sqrt{\epsilon}}\sqrt{\epsilon}sin(z/\sqrt{\epsilon})/\sqrt{\epsilon}$. so it is impossible to require $\psi(z) \to 0$ for $z \to \pm\infty$.

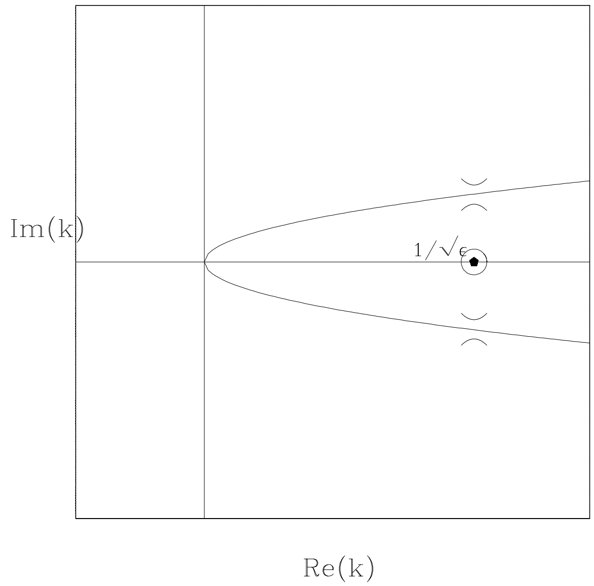

FIG. 14: Integration paths for $z > 0$ (lower) and $z < 0$ (upper) showing saddle points.

B. Example: Viscous fingering

We consider the equation

$$\epsilon\psi''(s) + (1+s^2)^2\psi(s) = 1, \tag{42}$$

with boundary condition $s\psi(s) \to 0$ as $s \to \pm\infty$. Since we expect this problem to have no solution, we also discuss the following modifications

$$\epsilon\psi''(s) + [(1+s^2)^2 + a]\psi(s) = 1, \tag{43}$$

and

$$\epsilon\psi''(s) + [(1+s^2)^2 + a + b/(1+s^2)^2]\psi(s) = 1, \tag{44}$$

where both a and b are small parameters. These equations lead to inner equations that have been discussed in the context of viscous fingering by Tanveer [8].

1. *Model equation with $a = b = 0$*

We first consider equation (42). Expanding $\psi(s) = \psi_0(s) + \epsilon\psi_1(s)\dots$ we find at leading order $\psi_0(s) = 1/((1+s^2)^2)$. This term satisfies the boundary conditions at infinity and the

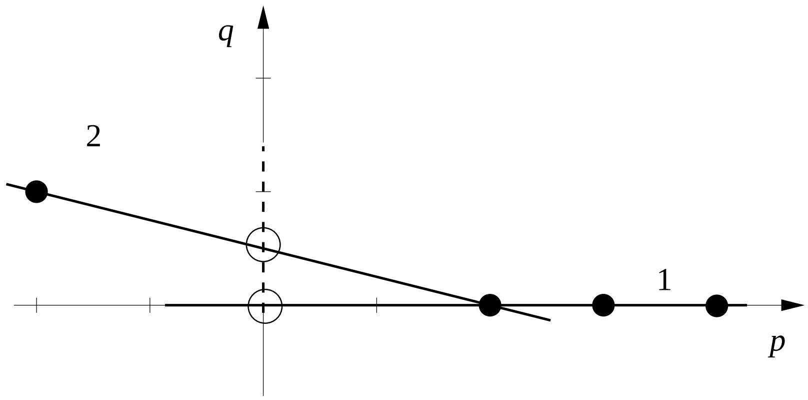

FIG. 15: The Kruskal-Newton diagram for equation (45).

higher order terms do also. The leading order term is singular at $s = \pm i$ and we study the vicinity of $s = -i$. Using shifted coordinates $s = -i + it$, $\psi(-i + it) = \phi(t)$, equation (42) transforms to

$$-\epsilon\ddot{\phi}(t) + (4t^2 - 4t^3 + t^4)\phi(t) = 1. \tag{45}$$

The Kruskal-Newton diagram for this equation is shown in Fig. 15. Line 1 gives the outer balance and we get at leading order $\phi_0(t) = 1/(t^2(2-t)^2)$. The possible exponentially small corrections are the WKBJ solutions of the homogeneous version of equation (45):

$$\phi_\pm(t) = \frac{1}{\sqrt{t(2-t)}} e^{\pm t^2(3-t)/3\sqrt{\epsilon}} \tag{46}$$

Line 2 gives the inner equation. We find from the diagram $t = \epsilon^{1/4}\tau$, $\phi = \epsilon^{-1/2}\Phi$, and

$$-\frac{d^2\Phi}{d\tau^2} + 4\tau^2\Phi = 1. \tag{47}$$

Seeking a solution that vanishes for large τ in the sector $0 < \arg(\tau) < \pi/2$ a WKBJ analysis indicates the following behavior:

$$\Phi(\tau) \sim \begin{cases} 1/4\tau^2 & 0 < arg(\tau) < \frac{\pi}{2} \\ 1/4\tau^2 + \frac{A}{2}\frac{1}{\sqrt{\tau}}e^{-\tau^2} & arg(\tau) = 0 \\ 1/4\tau^2 + A\frac{1}{\sqrt{\tau}}e^{-\tau^2} & -\frac{\pi}{2} < arg(\tau) < 0 \end{cases} \tag{48}$$

The constant A has to be determined numerically or from an exact solution of equation (47). We have to add the WKBJ solutions $\phi_\pm(t)$, equation (46), to the naive regular expansion as needed:

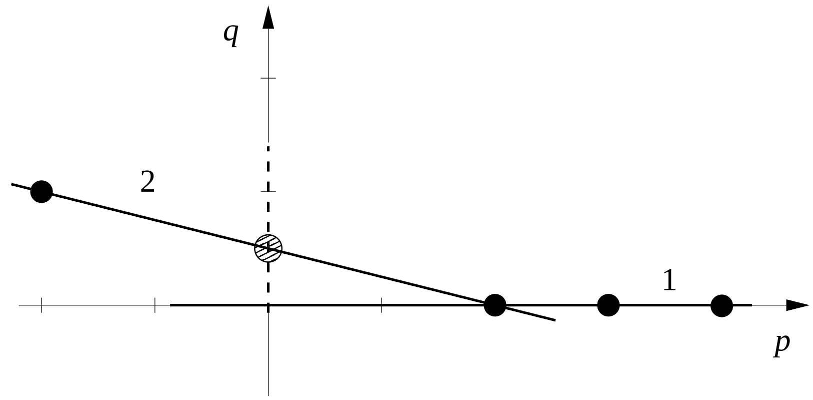

FIG. 16: Kruskal-Newton diagram for equation (49). The shaded dot represents the term $a\phi$ with a scaled to lie on support line 2.

$$\phi(t) \sim \begin{cases} 1/t^2(2-t)^2 & 0 < arg(t) < \frac{\pi}{2} \\ 1/t^2(2-t)^2 + \frac{B}{2}\phi_-(t) & arg(t) = 0 \\ 1/t^2(2-t)^2 + B\phi_-(t) & -\frac{\pi}{2} < arg(t) < 0 \end{cases}$$

The constant B can be related to A by matching along the ray $arg(t) = 0$: $B = \sqrt{2}\epsilon^{-3/8}A$.

Extrapolating back to the real s-axis and adding the complex conjugate, which is the contribution from $s = i$, we find the uniformly valid solution satisfying the boundary condition at $s = -\infty$:

$$\psi(s) \sim \begin{cases} 1/(1+s^2)^2 & s < 0 \\ 1/(1+s^2)^2 + 2^{-1/2}\epsilon^{-3/8}e^{-2/3\sqrt{\epsilon}}Acos((s^3/3+s)/\sqrt{\epsilon})/\sqrt{1+s^2} & s > 0 \end{cases}$$

Due to the presence of the exponentially small correction the solution does not satisfy the boundary condition as $s \to +\infty$.

2. *Model equation with $a \neq 0$ and $b = 0$*

Seeking a model that possesses solutions we may discuss the modified equation (43). For a small, the leading order outer solution is not modified. It is singular at $s = -i$ and we use shifted coordinates, $s = -i + it, \quad \psi(-i + it) = \phi(t)$. Equation (43) transforms to

$$-\epsilon\ddot{\phi}(t) + (4t^2 - 4t^3 + t^4 + a)\phi(t) = 1. \tag{49}$$

The Kruskal-Newton diagram for this equation is shown in Fig. 16. The parameter a has to be scaled according to $a = \alpha\epsilon^{1/2}$, with α of order unity, for the corresponding dot to lie

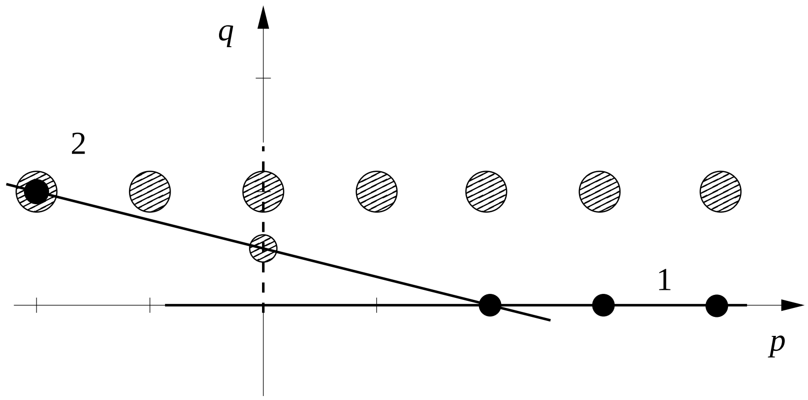

FIG. 17: Kruskal-Newton diagram for equation (51). The term multiplied by b is represented by a series of shaded dots. The parameter b is scaled, so no dot lies below support line 2.

on support line 2 and not below. The modified inner equation reads

$$-\frac{d^2\Phi}{d\tau^2} + (4\tau^2 + \alpha)\Phi = 1. \tag{50}$$

The solution of this equation for $\tau >> 1$ is still of the form (48) but with the parameter A being a function of α, $A = A(\alpha)$. If α is chosen such that $A(\alpha) = 0$, we do not pick up an exponentially small contribution when crossing the Stokes line $Im(t) = 0$ and the boundary conditions can be fulfilled. It follows from Eq. A5 that this happens for $\alpha = \alpha_n = 8n + 6$, $n = 0, 1, 2, \ldots$. This has also been shown by Tanveer[8] using a different approach.

3. *Model equation with $a \neq 0$ and $b \neq 0$*

It is interesting to study how this result is modified if the term multiplied by b in model equation (44) is present. Instead of (49) we get, using shifted coordinates

$$-\epsilon\ddot{\phi}(t) + \left[4t^2 - 4t^3 + t^4 + a + \frac{b}{t(2-t)}\right]\phi(t) = 1. \tag{51}$$

Expanding $1/t(2-t) = 1/4t^2 + 1/4t + 3/16 + \ldots$, we can draw the Kruskal-Newton diagram, see Fig. 17. No dot lies below support line 2 if we scale b according to

$$b = \beta\epsilon,$$

with β of order unity. The inner equation reads

$$-\frac{d^2\Phi}{d\tau^2} + \left(4\tau^2 + \alpha + \frac{\beta}{4\tau^2}\right)\Phi = 1. \tag{52}$$

Tanveer also studied equation (52). He found solutions for a discrete sets of eigenvalues α_n that are functions of β. This shows that the term multiplied by b in model equation (44) has an effect on the value of a even though it is numerically irrelevant on the real s-axis.

VII. COUPLED EQUATIONS

The tearing mode in a toroidally confined plasma gives an example of the use of Kruskal-Newton diagrams for two coupled second order equations. The linear evolution equations for a single helicity mode (somewhat simplified for this presentation) are given[9] by

$$\psi - x\xi = \frac{1}{\gamma\tau_R}\nabla_\perp^2\psi, \tag{53}$$

$$(\gamma\tau_A)^2\nabla_\perp^2\xi = x\left(F\psi - \nabla_\perp^2\psi\right) \tag{54}$$

where $\nabla_\perp^2 = \frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \frac{m^2}{r^2}$ and $x = r-1$. The small parameters are $\gamma\tau_A$ and $1/(\gamma\tau_R)$ and F is order one. A boundary layer occurs at $x = 0$. The outside solutions are immediately given by $\nabla_\perp^2\psi = F\psi$ and $\psi = x\xi$ and the boundary conditions then require that for small x in the case $m = 1$ ξ is order one, constant to the left of the layer, and zero to the right, whereas for $m \neq 1$ instead ψ is order one and continuous but there is a jump in its derivative across the layer $\psi'(0_+) - \psi'(0_-) = \psi\Delta'$.

First examine $m \neq 1$. Setting $\nabla_\perp^2\xi = \xi'' + x\xi'' + \xi' + R_1$, with R_1 of order 1, substituting $\nabla_\perp^2\psi$, and making use of the fact that for small x the function ψ is continuous at $x = 0$ to treat it as a constant for small x, we thus obtain a single second order differential equation for ξ

$$\psi - x\xi(x) = \frac{1}{\gamma\tau_R}\left[\frac{\gamma^2\tau_A^2(\xi''(x) + x\xi''(x) + \xi'(x) + R_1)}{x} - F\psi\right]. \tag{55}$$

The Kruskal Newton diagram is shown in Fig. 18. The dominant terms are point a, the ξ'' term, point b, the ξ term, and point c, the constant ψ appearing on the left side of the equation. Other terms lie above the line connecting points a and b. We have plotted $\gamma^2\tau_A^2/(\gamma\tau_R) \sim \epsilon^3$, but any small ordering would produce the same graph.

However, because there are two dependent variables ψ and ξ in this equation, the graph can be changed by a relative normalization of them. We now use Kruskal's asymptotic principle of maximal balance[2] according to which "the most informative ordering is that which simplifies the least, maintaining a maximal set of comparable terms"[2]. We renormalize ξ

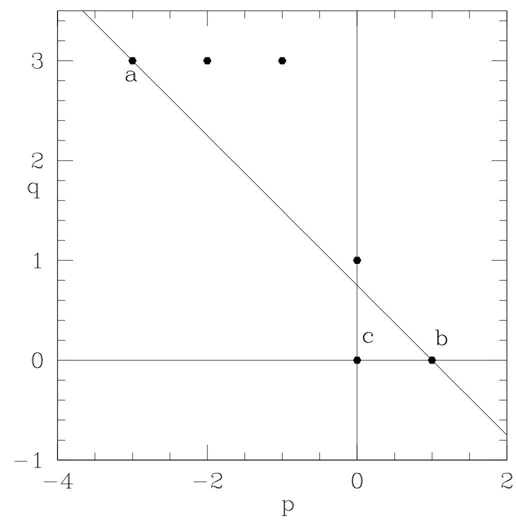

FIG. 18: Kruskal Newton diagram for the $m \geq 2$ tearing mode.

through $\xi = \psi(\gamma\tau_R/(\gamma^2\tau_A^2))^{1/4}\chi(z)$ to bring points a, b to lie on the same line as point c. The slope of the line gives the scaling of the layer as $x \sim (\gamma^2\tau_A^2/\gamma\tau_R)^{1/4}$, and we introduce the independent variable z through $x = (\gamma^2\tau_A^2/\gamma\tau_R)^{1/4}z$, and keep only the dominant terms of the Kruskal-Newton diaagram, giving

$$\chi'' - z^2\chi = z, \qquad \chi = -\frac{z}{2}\int_0^1 d\mu e^{-z^2\mu/2}(1-\mu^2)^{-1/4} \tag{56}$$

For the necessary matching to the exterior solution we have

$$\gamma^{5/4}\tau_A^{5/4}S^{3/4}\int_{-\infty}^{\infty}\frac{dz}{z}\chi'' = \Delta' \tag{57}$$

with $S = \tau_R/\tau_A \gg 1$ and $\gamma\tau_R \gg 1$ and $\gamma\tau_A \ll 1$, as assumed.

To evaluate the integral $I = \int_{-\infty}^{\infty}(dz/z)\chi''$, substitute χ and integrate over z, giving

$$I = \sqrt{\pi/2}\int_0^1 d\mu\frac{\mu^{1/2}}{(1-\mu^2)^{1/4}} = \pi\Gamma(3/4)/\Gamma(1/4) \tag{58}$$

Thus

$$\gamma\tau_A = \left(\frac{\Gamma(1/4)\Delta'}{\pi\Gamma(3/4)}\right)^{4/5}S^{-3/5}. \tag{59}$$

For the $m = 1$ mode substitute $\psi = xZ$ into Eqs. 53, 54 giving equations in the order one quantities ξ and Z

$$xZ - x\xi(x) = \frac{(2Z' + xZ'' + R_1 + xR_2)}{\gamma\tau_R},$$
$$\gamma^2\tau_A^2\xi''(x) = Fx^2Z - x^2Z'' - 2xZ' - xZ - x^2Z' - xZ \tag{60}$$

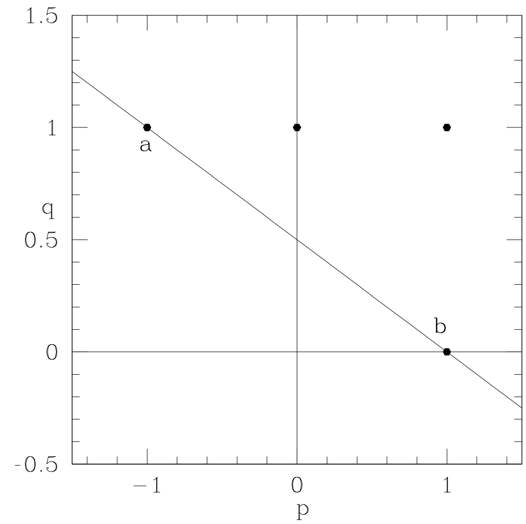

FIG. 19: Kruskal Newton diagram for the $m = 1$ tearing mode.

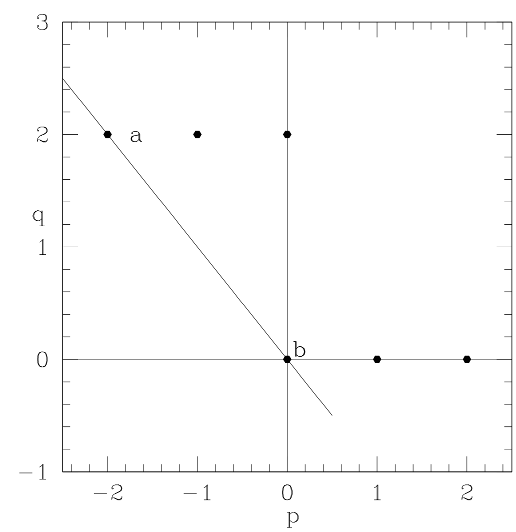

FIG. 20: Kruskal Newton diagram for the $m = 1$ tearing mode.

where R_1 and R_2 are terms of order one.

The Kruskal-Newton diagrams for these equations are shown in Figs. 19 and 20. In Fig 19 $\epsilon = 1/(\gamma\tau_R)$ and point a is the derivative terms, point b the xZ and $x\xi$ terms. This plot gives a layer scaling of $x \sim 1/\sqrt{\gamma\tau_R}$. In Fig. 20 point a is the ξ'' term, and point b the xZ' and x^2Z'' terms. This plot gives a layer scaling of $x \sim \gamma\tau_A$. In both diagrams there are other smaller terms given by points above these lines. But of course these equations are coupled and there is only one layer, and equating these scalings we find $\gamma \sim \tau_A^{-2/3}\tau_R^{-1/3}$.

Introducing the layer variable $z = (\gamma\tau_R)^{1/2}x$, and keeping only the dominant terms of the Kruskal-Newton diagram we find an exact solution which satisfies the boundary conditions $\psi \to 0, x \to +\infty$ and $\psi \to -\xi_0 x$, $x \to -\infty$

$$\xi = \frac{\xi_0}{\sqrt{2\pi}} \int_z^\infty e^{-z^2/2} dz, \qquad \psi = \frac{\xi_0}{\sqrt{2\pi}(\gamma\tau_R)^{1/2}} \left[e^{-z^2/2} - z \int_z^\infty e^{-z^2/2} dz \right] \tag{61}$$

and $\gamma = \tau_R^{-1/3} \tau_A^{-2/3}$.

VIII. SUMMARY

We illustrated the use of Kruskal-Newton diagrams for some singularly perturbed differential equations. The method is a graphical realization of Kruskal's asymptotic principle of maximal balance. It can help to decide whether a term in a differential equation plays an important role in some domain and how the size of this domain scales with the perturbation parameter. It applies to real domains as well as to regions of the complex plane for which the effect of a term is frequently counterintuitive. Examples were taken from fluid mechanics and from plasma physics.

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APPENDIX A: INTEGRAL SOLUTION FOR EQ. 50

In this appendix we give an integral representation of a particular solution of Eq. 50 and study its asymptotic behavior. Change variable to $w = \tau^2/2$ with $\Phi(\tau) = f(w)$, giving

$$2wf'' + f' - (8w + \alpha)f = -1, \tag{A1}$$

and use a Fourier-Laplace representation $f = \int e^{wt} g(t)dt$, giving a solution with

$$g(t) = A(t-2)^{-(\alpha+6)/8}(t+2)^{(\alpha-6)/8}, \qquad 2(t^2-4)ge^{wt}|_a^b = -1 \tag{A2}$$

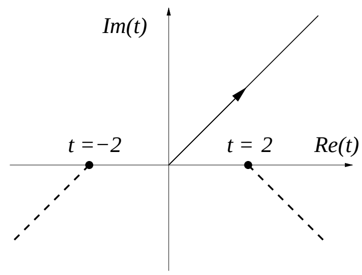

FIG. 21: The steepest descent contour for Eq. A4 for $Im(w) < 0$.

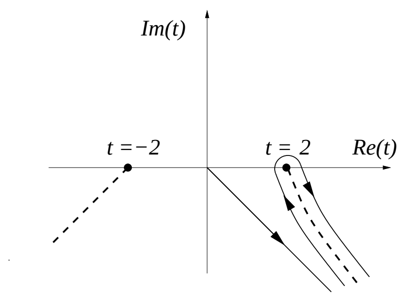

FIG. 22: The steepest descent contour for Eq. A4 for $Im(w) > 0$.

with (a, b) the contour end points. We then have the solution

$$\Phi(\tau) = A \int_{-C} e^{wt}(t-2)^{-(\alpha+6)/8}(t+2)^{(\alpha-6)/8} dt \tag{A3}$$

with $2\sqrt{2}A(-1)^{(-\alpha+2)/8} = -1$. Changing the integration variable $t \to -t$,

$$f(w) = A \int_{C} e^{-wt}(-t-2)^{-(\alpha+6)/8}(-t+2)^{(\alpha-6)/8} dt. \tag{A4}$$

The contour C starts at $t = 0$ and goes to infinity in the sector $Re(wt) > 0$. Starting in the domain $Im(w) < 0$ we choose the singularities at $t = \pm 2$ to lie below the contour. We can deform the contour to the steepest descent contour $arg(wt) = 0$ without passing through a singularity provided $Im(w) < 0$ (Fig. 21). Continuing to $Im(w) > 0$ the steepest descent contour must be deformed to go around the branch cut (Fig. 22). Thus, the line $Im(w) = 0$ is a Stokes line for $f(w)$ and we pick up an extra contribution from the cut when passing it:

$$f_{bc} \simeq \frac{i}{2\sqrt{2}} e^{-2w}(4w)^{-(\alpha+2)/8} sin\left[\pi(\alpha/8 + 1/4)\right]\Gamma(\alpha/8 + 1/4) \tag{A5}$$

The contribution $f_{ep} \simeq 1/(8w)$ from the endpoint $t \simeq 0$ is present everywhere. Thus,

$$\Phi(\tau) \simeq \begin{cases} 1/4\tau^2 & -\pi/2 < arg(\tau) < 0 \\ 1/4\tau^2 + f_{bc}/2 & arg(\tau) = 0 \\ 1/4\tau^2 + f_{bc} & 0 < arg(\tau) < \pi/2 \end{cases} . \tag{A6}$$

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