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STARS IN PHOTOGRAPHIC EMULSIONS INITIATED BY DEUTERONS

PART II. THEORETICAL

By

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University of California

January 5, 1948

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Abstract

The theory of high energy nuclear stars depends on a theory of nuclear transparency and on a theory of nuclear evaporation. The transparency can be computed on the basis of a model proposed by R. Serber as soon as the interactions between the nucleons and the incident particle are known. The evaporation can be computed on the basis of the statistical model of the nucleus as soon as the nuclear entropy and binding energies of the evaporated particles are known. The calculations have been formulated with approximate values for the above interactions, entropies, and binding energies; and by means of various mathematical methods: a method of averages, a method of reaction integrals, and one using diffusion equations. Probability distributions have been obtained for the number of prongs per star, and distributions are being computed for the energy and angle of a prong. The results are in qualitative agreement with the observations on photographic emulsions described in Part I.

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PART II. THEORETICAL¹

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Any complete theory of nuclear stars would require a complete nuclear mechanics. Still, it is of interest to use what little is known of this subject and to see how far towards an interpretation of these stars it is possible to go. In what follows it is assumed that the energy of the incident particle is of the order of 100 Mev so that the excitation energy of the struck nucleus is high enough to make resonance phenomena unimportant. At these high energies, particles may conceivably be ejected from the struck nucleus by at least two processes: direct recoil, and evaporation. If the incident particle gives so much energy to a nucleon that it leaves the nucleus in a time short compared to a nuclear period ($\sim 10^{-21}$ sec), then we have a recoil star. If, on the other hand, the energy of the incident particle is evenly distributed throughout the nucleus by many collisions of the nucleons among one another, and if a particle is ejected only after many nuclear periods (Bohr process), then we have an evaporation star. When the incident energy exceeds the total binding energy of the struck nucleus, as with some cyclotron beams incident on light nuclei, then we may expect recoil stars to be important. For lower incident energies evaporation stars should predominate. It seems that cosmic ray stars are of both these types⁽²⁾. The theory of evaporation stars depends on two things:

¹Reported at meeting of the American Physical Society, January 2-3, 1948

(2) Cosmic Radiation...W. Heisenberg...p. 131

nuclear evaporation proper, and nuclear transparency, which concerns the probability that a high energy particle pass through a nucleus with the loss of only a certain fraction of its energy.

Part A

Unless E_0 , the energy of the incident particle, is large compared to its mean energy loss per collision, (~ 20 Mev), the incident particle will almost certainly be unable to penetrate through the nucleus; there will be zero transparency. But if E_0 is many times this critical value, things may be more complex. To get a rough idea of what happens, assume the stopping power of the nucleus to be like that of a Fermi gas of protons and neutrons, so that when a particle strikes it the resulting action may be considered as a succession of n-p, n-n, and p-p interactions⁽³⁾.

(3) For a discussion of the physical model here involved, see R. Serber, Phys. Rev., 72, 1115, (1947)

We represent the interaction between the incident particle and an average particle of the nucleus by

$$A e^{-\alpha r} \quad (1)$$

For α use $.90 \cdot 10^{13} \text{ cm}^{-1}$ (4)

(4) Breit, Thaxton, Eisenbud, Phys. Rev. 55, 1018 (1939)

The total cross section for nn scattering at 90 Mev has been found to be .034 barns, that for np scattering .083 barns.⁽⁵⁾ A is chosen so as to

(5) Cook, McMillan, Peterson, and Sewell...Phys. Rev. 72, 1264, (1947)

give the average cross-section of 0.58 at 90 Mev.

The Born approximation with interaction (1) gives for the differential cross section:

$$d\sigma = \frac{m^2 A^2}{4\hbar^2 k^2} \frac{4\pi k dK}{(\alpha^2 + K^2)^2} \quad (2)$$

where $K = 2k \sin \frac{\theta}{2}$ is the momentum transfer and $\frac{\theta}{2}$ the angle of scattering in the reference system in which the struck particle is at rest; and $\hbar k$ is the momentum of one of the two colliding particles in their center of gravity system.

When this result is applied to collisions within the nucleus, two complications must be considered: the exclusion principle limits possible momentum transfers, and the struck particle is not at rest. For energies of the order of 100 Mev, $k \gg \kappa$ except for improbable values of κ , so that the momentum transferred to the struck particle is almost always nearly normal to that of the incident particle. To simplify, assume it is always exactly normal. The cross-section for energy loss E is now obtained from (2) and the relation $2mE = (\hbar\vec{\kappa} + \vec{p})^2 - p^2$, where $\vec{p}$ is the momentum of the struck particle. When this cross-section is averaged over the Fermi momentum sphere of radius p_F and summed over all values of $\kappa \perp k$ such that $(\hbar\vec{\kappa} + \vec{p})^2 > p_F^2$, so that the exclusion principle is satisfied, the result is

$$d\sigma = \frac{3A^2 \pi m dE}{2^5 \epsilon_0 \epsilon_F \alpha^2} \left[\frac{(E_B - 3E)(E_F^{1/2} - \overline{E_F - E}^{1/2})}{E_B^{3/2}} + \frac{3E^2 + 2E_B + 4E\overline{E_F - E} - E_B^2}{2E_B^2} + \tan^{-1} \frac{2E_B^{1/2}(E_F^{1/2} - \overline{E_F - E}^{1/2})E\epsilon_B}{(E + E_B)^2 + 4E_B E_F^{1/2} \overline{E_F - E}^{1/2}} \right] \quad (3)$$

where $E_B = \frac{\hbar^2 \alpha^2}{2m}$ and $E_F - E$ is to be replaced by 0 when $E > E_F = \frac{1}{2} \frac{p_F^2}{m}$. A plot of this cross section, compared with that for collisions between free particles is given in Figure 1 for the case of

incident beam energy of 100 Mev. The average energy loss $\bar{\epsilon}$ per collision in nuclear matter may now be computed by numerical integration of (3). In Figure 2 it is plotted along with the corresponding quantity for free collisions.

For further work it is convenient to fit the function $d\sigma/d\epsilon$, for collision within nuclei, by one of form $\epsilon^{-2} e^{-\epsilon/\bar{\epsilon}}$. Then we have for the probability of energy loss ϵ in one collision

$$A(\epsilon) = \frac{4}{\bar{\epsilon}^2} \epsilon^{-2} e^{-\epsilon/\bar{\epsilon}}$$

where the constants are chosen to give the correct value of the average energy loss.

Given that a particle is incident on a nucleus, it will undergo say n collisions before coming to a stop or emerging from the nucleus, and will lose a quantity ϵ of energy. We wish finally to get $P(\epsilon) d\epsilon$, the probability of an energy loss ϵ in range $d\epsilon$. To get this we compute first $R(\epsilon) d\epsilon$, the probability of there being just n collisions and of losing energy ϵ in range $d\epsilon$ to the nucleus

$$R_n(\epsilon) d\epsilon = \frac{2}{R^2} \int_0^R r^n dr \int_{\epsilon_1}^{\epsilon_2} \prod_{i=1}^n (\epsilon_i q_i dx_i d\epsilon_i) e^{-\sum_{i=1}^n q_i x_i - \frac{2}{\bar{\epsilon}} \sum_{i=1}^n \epsilon_i} \left(\frac{2}{\bar{\epsilon}}\right)^{2n} \quad (4)$$

$q_i = \rho \sigma(\epsilon_i - \sum_{j=1}^{i-1} \epsilon_j)$, ρ numerical density of nucleons in nucleus.

To derive (4) consider the expression

$e^{-q_1 x_1} \tau(\epsilon_1) d\epsilon_1 e^{-q_2 x_2} \tau(\epsilon_2) d\epsilon_2 dx_1 dx_2$
 $e^{-q_1 x_1}$ is the probability of the incident particle going a distance x_1 , inside the nucleus without collision, $\tau(\epsilon_1) d\epsilon_1 dx_1$ the probability of a collision with energy transfer ϵ_1 , occurring within distance dx_1 , $e^{-q_2 x_2}$, the probability of going a further distance x_2 , without collision, after the first collision, etc. The integrations are subject to the obvious restrictions

$$\sum_{i=1}^{n+1} x_i = t, \quad \sum_{i=1}^n \epsilon_i = E$$

where $t = 2(R^2 - r^2)^{1/2}$ is the nuclear thickness at impact parameter r .

The factor $\frac{2}{R^2} r dr$ results from averaging over the projected area of the nucleus. The quantities q_i are really functions of the ϵ_i . To simplify we evaluate them at average values of the ϵ_i , that is we put

$$q_i \approx \frac{q}{1 - (i-1) \frac{E}{E_0}}$$

The integral $R_n(E)$ then breaks up into two factors, $R_n(E) = P_n Q_n(E)$

where P_n is the probability of n collisions, and $Q_n(E) dE$ the probability of energy transfer in range dE provided that just n collisions occur.

$$Q_n(E) = \left(\frac{2E}{E_0}\right)^{2n-1} \frac{e^{-\frac{2E}{E_0}}}{(2n+1)!} \frac{2}{E_0}$$

This gives a finite probability for arbitrarily large energies. To correct this write henceforth

$$Q_n(E) = 0 \text{ if } E > E_0, \text{ but add to it a term } \left(\int_{E_0}^{\infty} Q_n dE\right) \delta(E_0 - E) \equiv C_n \delta(E_0 - E)$$

To evaluate P_n write it as

$$P_n(t) = \frac{d}{dt} \int_{\sum x_i \leq t} \prod (q_i dx_i) e^{-\sum q_i x_i} e^{-\sum (q_i - a) x_i}$$

where $a = (\prod q_i)^{1/n}$ is the geometric mean of the q_i 's. Put

$$e^{-\sum (q_i - a) x_i} = 1 - \sum C_i x_i + \frac{1}{2} (\sum C_i x_i)^2 - \dots$$

$$\sum \frac{C_i}{a_i} \equiv A_n, \quad 2 \left[\sum \frac{C_i^2}{a_i^2} + \sum_{i,j} \frac{C_i C_j}{a_i a_j} \right] \equiv B_n, \dots$$

Then evaluating the last written integral and averaging over the projected area of the nucleus gives

$$P_n = \frac{2}{(2aR)^2} \left[(n+1) I(2aR, n+1) - A_n(n+2) I(2aR, n+2) + B_n(n+3) I(2aR, n+3) + \dots \right]$$

Finally, the total energy distribution is given by

$$\sum_{n=0}^N P_n Q_n(E) dE, \quad \text{where } I(u, p) \equiv \frac{1}{p!} \int_0^u e^{-u} u^p du \quad (5)$$

where N is the greatest integer in E_0/E . To normalize this distribution, we add to it a δ function at $E=0$, of amount P_0 , representing the probability that an incident particle penetrates the nucleus without any collisions, and a δ function at $E=E_0$ of amount $\sum_{n=0}^N C_n$, representing

the probability that the incident particle is itself captured by the nucleus.

The distribution (5) is plotted in Figure 3. It should be applicable to excitation by a neutron beam of 100 Mev. If (5) is folded back on itself, the result is the distribution of Figure 4. It should be applicable to excitation by a deuteron beam of 200 Mev. The discontinuities at 100 Mev in Figure 4 are necessitated by the δ functions at $E = 0$ and $E = 100$ in Figure 3. Similar graphs have been obtained for other incident energies. Reasons for the choices of mass number $A = 13$ and $A = 100$ will appear shortly.

Part B

The evaporation of individual particles from an excited nucleus is treated by Weisskopf⁽⁶⁾. His equation (3) may be put

$$W(\bar{E}, \xi) d\xi = \nu(\bar{E} - V) \times \frac{S_B(\bar{E} - E_b - \xi) - S_A(\bar{E})}{\pi^2 h^3} d\xi, \quad \nu = \frac{\sigma g m}{\pi^2 h^3} \quad (6)$$

(6) V. Weisskopf, Phys. Rev. 52, p. 296 (1937)

where W is the probability per second that a nucleus A ejects a particle with kinetic energy ξ , thus becoming a nucleus B . m, g and E_b are the mass, statistical weight and binding energy of the ejected particle, V its coulomb or "barrier" energy. σ and $\bar{E}$ are the geometrical cross-section and excitation energy of the nucleus. S_A and S_B are the entropies of the initial and final nucleus. No fully satisfactory functional form for the entropy is known. However, on the basis of a simple model we may write (7) approximately

(7) H. A. Bethe, Phys. Rev. 50, p. 332 (1936)

$$S(E) \approx \left(\frac{A}{2.2}\right)^{1/2} E^{1/2} \equiv x E^{1/2}, \quad E \sim M \times V, \quad (7)$$

where A is the mass number. Ultimately, of course, the precise form of $S(E)$ might be deduced from (6) by an experimental measurement of or some related function.

In the application of (6) to nuclear stars, various mathematical methods are available. We treat first the "method of averages", which is adequate for describing certain average properties of the many kinds of stars produced in given experimental conditions. The threshold energy of a particle of type ν (a neutron, deuteron, etc.) is $E_{\nu} + V_{\nu} = T_{\nu}$.

If the tunnel effect is neglected, then a particle of type ν cannot be emitted unless $T_{\nu} < X$. Let $\bar{\xi}_{\nu}(X)$ be the average kinetic energy of a type ν particle, easily computed from (6) and (7). Then

$\bar{E}_{\nu} = E_{\nu} + \bar{\xi}_{\nu}$ is the average energy carried away by the first star prong if it is of type ν . Let P_{ν} , proportional to $\int W_{\nu}(\xi_{\nu}) d\xi_{\nu}$, be the probability that the first emitted particle is of type ν . Then $\sum P_{\nu} \bar{E}_{\nu} = \bar{E}(X) \equiv E_1$ is the average decrease in the original excitation X caused by the first prong. The sum $\sum P_{\nu}$ taken over all possible charged evaporated particles we call f_1 , the probability that the first evaporated prong be visible. Similarly $\bar{E}(X - E_1) \equiv E_2$ is the average decrease in remaining excitation by the second prong, corresponding to an f_2 and so on. When, after j steps in the calculation $E_{\nu} > E_j > 0$, where E_{ν} is the threshold energy of the most lightly bound particle (usually the neutron); then j is the number of particles emitted on the average from the nucleus. The energy E_j will eventually be emitted as a gamma ray or electron. The average number of visible prongs per star is now

$$N = \sum_{\nu=1}^j f_{\nu} \cdot \cdot$$

In applying the above method to stars in photographic plates,
 we need the chemical composition of the emulsion. (8) Except for small

(8)

Part I, this paper.

admixtures the emulsion is composed of a heavy component, Ag and Br, of which there are $2.1 \cdot 10^{22}$ nuclei per cc; and a light component O and C and N, of which there are $2.91 \cdot 10^{22}$ nuclei per cc. The average mass numbers for heavy and light components are 95.5 and 13.65. Figure 1 shows $\bar{\xi}$ for the heavy component. For binding energies and coulomb energies, we have used, for the heavy component:

$$\left. \begin{aligned} E_n &= 8.5, E_p = 8.5, E_\alpha = 7.0, E_d = 14.2 \\ V_p &= 6.4, V_n = 12.8, V_\alpha = 6.4 \end{aligned} \right\} \quad \text{All in Mev.}$$

With these nuclear constants, the probabilities for evaporation of various particles are as shown in Figure 5. This figure somewhat exaggerates the predominance of neutron evaporation since it ignores Gamow leakage of the charged particles through their potential barriers. With the above curves as a basis, it is now just a matter of arithmetic to compute the average number of ions evaporated from a nucleus of the heavy component with given excitation, as shown in Figure 3, curve H.

The above theory applied to the light component becomes both more difficult and less valid. Probably no statistical theory holds very accurately for nuclei as light as carbon. The fluctuations in binding energies of the evaporated particles in the case of light nuclei make it difficult to choose proper average values of these constants. A carbon nucleus with an excitation of 150 Mev may be completely broken up into α particles and lighter particles, the more so since Be^8 is unstable

with respect to α emission. A different value of $\left(\frac{A}{2.2}\right)^{1/2}$ must here be used after the evaporation of each particle, else the results are qualitatively in error. The above stepwise calculation then finally yields curve L of Figure 3. In obtaining it, we have ignored the effect of the evaporation of particles larger than the α particle, even though this is energetically quite possible at 200 Mev. The effect of this on prong number is probably not great, since such particles would usually be ejected in excited states and soon themselves decompose. The effect of direct recoils has also been ignored. But the effect of this on prong number is largely counterbalanced since the average recoil particle leaves a light nucleus with a kinetic energy of the order 35 Mev⁽⁹⁾ and calculations

(9) Wilson Powell, Bull. Am. Phys. Soc. 23, 15 (1948)

show that the average evaporated particle at 200 Mev beam energy, leaves with a kinetic energy of about 20 Mev. Curve L is physically very plausible, since it says that after a 200 Mev excitation, which is almost as much as the total binding energy of a nucleus of the light component, almost all of the charged particles are separated from one another.

If the curves of Figure 3 are weighted according to the relative amounts of light and heavy component present in the emulsion, and weighted further according to the results of Part A which give probabilities of various excitations, the result is 3.4 for an average prong number corresponding to a deuteron beam of 190 Mev. At 100 Mev and 50 Mev beam energy, the average number of prongs per star turn out to be 3.1 and 2.7. These results agree better than might be expected with the corresponding experimental result of 3.0, independent of beam energy. It may be that a trend toward higher prong number with higher beam energy is masked, in the

experimental results, by the uncertainty of the photographic plates in registering protons with energy above about 10 Mev.

Summary

Present uncertainties in the interactions between high energy nuclear particles, as well as in the density of states of excited nuclei, limit our ability to construct a theory of high energy nuclear reactions. However, there seem to be no inconsistencies between present ideas on these matters and nuclear stars as observed in photographic plates.

Part C

As a second mathematical method for treating nuclear evaporation, we now describe the method of "reaction probabilities". To illustrate, let $P_{np}(X)$ be the probability that a nucleus of excitation X dissipate all this energy in the evaporation of one neutron and one proton, except possibly for a remainder less than a neutron binding energy. If, for example the nucleus is excited by deuterons then $P_{np}(X)$ is the probability of a nuclear reaction of type (d, np) , given that the deuteron produces excitation (X) . The total probability for the (d, np) reaction, given that the deuteron hits the nucleus, will be simply the average of $P_{np}(X) + P_{pn}(X)$, weighted by the probability for excitation X described in Part A.

One way in which the nucleus may undergo the reaction (d, np) is to wait for a time t_1 after excitation without evaporating anything, then to evaporate the neutron in time dt_1 , then to wait a further time t_2 without evaporating anything, then to evaporate the proton in a time dt_2 . After the proton is emitted, the remaining energy of the nucleus must be less than ϵ_n , the binding energy of the neutron, or else some further particle will eventually be emitted (ignoring the small probability of γ -ray emission), and the reaction will then not be of type (d, np) .

-14-

Well, the probability that the nucleus wait for time t_1 is $e^{-Q_1 t_1}$, where $Q_1 = Q(\bar{X})$, and $Q(\bar{X})$ is the probability per unit time that the nucleus evaporate some particle or other. More explicitly

$$Q(\bar{X}) = \sum_j \int_0^{\bar{X}-T_j} W_j(\bar{X}, \xi) d\xi$$

Here $W_j(\bar{X}, \xi) d\xi$, given in Part B, is the probability per unit time that a nucleus with excitation $\bar{X}$ evaporate a particle of type j with kinetic energy in range $d\xi$. $T_j = \epsilon_j + V_j$ is the threshold energy for a particle of type j , ϵ_j its binding energy and V_j its Coulomb barrier. Strictly speaking the sum $\sum_j$ should be over all particles with threshold energies less than $\bar{X}$. In most cases, the series converges fast enough so that a negligible error is made if the sum is over neutrons, protons, deuterons and alpha particles only. The probability that a neutron is emitted in time dt_1 with kinetic energy in range $d\xi$ is $W_n(\bar{X}, \xi) d\xi dt_1$. The probability that the nucleus just wait for a further time t_2 is $e^{-Q_1 t_1} e^{-Q_2 t_2}$ where $Q_2 = Q(\bar{X} - \epsilon_n - \xi_1)$. Finally, $W_p(\bar{X} - \epsilon_n - \xi_1, \xi_2) d\xi_2 dt_2$ is the probability for a proton to be evaporated in time dt_2 in range $d\xi_2$.

Hence:

$$P_{np}(\bar{X}) = \int e^{-Q_1 t_1} W_1 d\xi_1 dt_1 \int e^{-Q_2 t_2} W_2 d\xi_2 dt_2$$

The time integrations may be performed at once, yielding

$$P_{np}(\bar{X}) = \int \frac{W_1 W_2}{Q_1 Q_2} d\xi_1 d\xi_2$$

The energy integrations must be performed subject to the restriction

$$0 < \bar{X} - \sum_i \xi_i - \sum_i T_i < \epsilon_n$$

It is easy to generalize this result, and obtain for the probability

$P_\nu(\bar{X})$ for the emission of ν particles of given types in a given

order, an integral of form

$$P_\nu(\bar{X}) = \int \prod_{i=1}^{\nu} \left(\frac{W_i}{Q_i} d\xi_i \right)$$

From (6)

$$\sum_{i=1}^{\nu} W_i = \prod_{i=1}^{\nu} \left[\gamma_i(\xi_i - V_i) \right] e^{-S_A(\bar{X}) - S_B(\bar{X} - \sum \epsilon_i - \sum \xi_i)} \quad (9)$$

It does not seem possible to evaluate the integrals of type (9) in closed

form, primarily because of the involved dependence of the Q_i on the $\bar{\xi}_i$. To evaluate it in a series, we recall that the first particle evaporated almost always comes off with kinetic energy not far from the average kinetic energy $\bar{\xi}_1$, which is easy to compute in terms of $\bar{X}$. Similarly, the second evaporated particle almost always comes off with kinetic energy near $\bar{\xi}_2$, and so on. It is convenient to introduce the variables

$$\eta_i = \bar{\xi}_i - V_i, \dots, \eta_v = \frac{v}{v-1}(\bar{\xi}_v - V_v) \quad ; \quad \bar{\eta}_j = \frac{j}{j-1}(\bar{\xi}_j - V_j) \quad (10)$$

Then

$$Q_i = Q(\bar{X}), \dots, Q_v = Q(\bar{X} - \sum_{i=1}^{v-1} \bar{\eta}_i - \eta_{v-1})$$

And the Taylor series for $F(\eta_j) \equiv \frac{1}{\pi Q_i}$ is:

$$F(\eta_j) = F(\bar{\eta}_j) + \sum_{i=1}^{v-1} \frac{\partial F}{\partial \eta_j} (\eta_j - \bar{\eta}_j) + \dots \quad (11)$$

will converge rapidly. Substitutions from (11) in (9) now gives a series of integrals each of which is elementary.

Applying this scheme to (8) gives

$$P_{n,p}(\bar{X}) = \frac{\gamma_n \gamma_p}{Q_1(\bar{X})} e^{-S_A(\bar{X})} \int \bar{\xi}_1 (\bar{\xi}_2 - V_2) e^{S_B(\bar{X} - \bar{\xi}_1 - \bar{\xi}_2 - E_n - E_p)} d\bar{\xi}_1 d\bar{\xi}_2 \left[\frac{1}{Q_2(\bar{X} - E_n - \bar{\xi}_1)} + \left| \frac{\partial}{\partial \bar{\xi}_1} \frac{1}{Q_2(\bar{X} - E_n - \bar{\xi}_1)} \right|_{\bar{\xi}_1 = \bar{\xi}_1} (\bar{\xi}_1 - \bar{\xi}_1) + \dots \right]$$

or, in terms of the η 's:

$$P_{n,p}(\bar{X}) = \frac{\gamma_n \gamma_p}{Q_1(\bar{X})} e^{-S_A(\bar{X})} \left\{ \frac{1}{Q(\bar{X} - E_n - \bar{\eta}_1)} I_1 + \left(\frac{\partial}{\partial \eta_1} \frac{1}{Q} \right)_{\eta_1 = \bar{\eta}_1} I_2 + \dots \right\}$$

$$I_1 = \int \eta_1 (\eta_2 - \eta_1) e^{-\alpha(\bar{X} - \sum \bar{\eta}_i - \eta_2)} d\eta_1 d\eta_2 = \int_0^{\bar{X} - \sum \bar{\eta}_i} \frac{\eta_1^3}{3!} e^{-\alpha(\bar{X} - \sum \bar{\eta}_i - \eta_1)} d\eta_1$$

$$I_2 = \int \left(\frac{\eta_1^4}{3 \cdot 4} - \bar{\eta}_1 \frac{\eta_1^3}{2 \cdot 3} \right) e^{-\alpha(\bar{X} - \sum \bar{\eta}_i - \eta_1)} d\eta_1, \text{ limits as in } I_1, \alpha \equiv \left(\frac{A}{2.2} \right)^{1/2}$$

Inspection of I_1 shows that it reaches a maximum at $\bar{X} \approx \sum \bar{\eta}_i + E_n$

and then drops off rapidly. In words, any nuclear reaction is most probable when the excitation exceeds the energy threshold for the reaction by about E_n , the binding of the neutron.

To apply this method of reaction probabilities to the heavy component is tedious because of the scores of energetically possible reactions. To lighten the labor we have introduced the quantity $W_i = \sum_j W_{ij}$ where j runs over all evaporated particles except the neutron, that is, over all ions. We then compute reaction probabilities such as P_{in}, P_{ni} where neutrons and ions are distinguished but not different kinds of evaporated ions. We further lump together probabilities such as P_{in} and P_{ni} which differ only in the order in which ions and neutrons are emitted. Adding these P's in an obvious way enables us to duplicate the curve H of figure 6 to within an error of 12%. This is regarded as a check of the method of Part B.

Part D

A third mathematical method for formulating the problem of nuclear reactions is to use diffusion equations, probably the best method for computing effects dependent on many nuclear reactions; for effects dependent on only a few reactions the method of Part C seems best.

Consider a stream of protons incident on the end of a cylinder of nuclear matter of length x_0 . Let x be distance from point of entry, E the energy of an incident proton. If $P(x, E)$ is the probability at position x that the incident particle have energy below E , then

$$\frac{\partial P(x, E)}{\partial x} = \int_E^{E_0} \frac{\partial P(x, E')}{\partial E'} \int_{E'-E}^{E'} \sigma(E', \epsilon) d\epsilon \quad (12)$$

where E_0 is the maximum value of E at which $P(x, E)$ is non-zero, and $\sigma(E, \epsilon)$ is the probability per unit distance that an incident particle with energy E suffer an energy loss ϵ . This equation of course neglects

angular scattering of the incident particle. Equation (12) with boundary condition $P(0, E) = f(E)$ determines the function $P(x, E)$. Once this is known, an integration over impact parameter b with x_0 as an appropriate function of b , will furnish a transparency curve for the nucleus, similar to Figure 3.

Let $P_n(x, E)$ be the probability at distance x that the incident particle has kinetic energy below E and has collided with n nucleons. Then a second formulation of the transparency problem is furnished by writing:

$$\frac{\partial P_n(x, E)}{\partial x} = \int_0^E \frac{\partial P_{n-1}(x, E')}{\partial E'} \sigma(E', E' - E) - \int_0^E \frac{\partial P_n(x, E')}{\partial E'} dE' \int_0^{E'} \sigma(E', \epsilon) d\epsilon \quad (13)$$

The first integral on the right of (13) sums over those processes which contribute to increases in $P_n(x, E)$, the second over those processes which contribute to decreases. (13) is a set of simultaneous diffusion equations rather than a single one as given by (12). (13), by distinguishing the separate terms in the series $P(x, E) = \sum_i P_i(x, E)$ seems more detailed than needed in computing nuclear transparencies. (13) like (12) ignores angular scattering.

To treat evaporation, consider $R(t, X)$ the probability at time t after it is struck, that an excited nucleus have excitation X .

Then

$$\frac{\partial R(t, X)}{\partial t} = \int_X^{X_0} \frac{\partial R(t, X')}{\partial X'} \int_{X'-X}^{X'} W(X', \xi) d\xi \quad (14)$$

where $W(X, \xi)$ is the probability per unit time that a nucleus with excitation X lose excitation ξ through evaporation of some particle or other. W is given in Part B. Let $r(t) dt$ be the probability of emitting an ion in time dt , $R(t) = \int_0^t r(t) dt$ the probability of having emitted an

ion before time t . Then

$$\frac{\partial R(t)}{\partial t} = \int_{\tau}^{\bar{X}_0} \frac{\partial R(t, \bar{X}')}{\partial \bar{X}'} d\bar{X}' \int_{\tau}^{\bar{X}'} W_i(\bar{X}', \xi) d\xi \quad (15)$$

where $W_i(\bar{X}, \xi)$ is probability per unit time that a nucleus with excitation $\bar{X}$ lose energy ξ through ion evaporation and τ is the threshold for ion emission. $R(t)$ can be got from this equation once $R(t, \bar{X})$ is obtained from (15). And $R(\infty)$ is the observed average number of visible prongs per star.

To compute the probability for producing a star of ν prongs, where ν is not the average number, is a problem in statistical fluctuations. This is of course more complicated than finding $R(\infty)$. When a nucleus has emitted ν ions and when further its excitation is below τ , then we may speak of a completed star of ν prongs. Let $R_\nu(t)$ be the probability at time t after the nucleus is struck that such a completed star has been formed. Then

$$\frac{\partial R_\nu(t)}{\partial t} = \int_{\tau}^{\bar{X}_0 - \tau} \frac{\partial P(\nu-1, t, \bar{X}')}{\partial \bar{X}'} d\bar{X}' \int_{\bar{X}' - \tau}^{\bar{X}'} W_i(\bar{X}', \xi) d\xi \quad (16)$$

where $P(\nu, t, \bar{X})$ is probability at time t that the struck nucleus has emitted precisely ν ions and possesses an excitation less than $\bar{X}$.

To compute the observable quantity $R_\nu(\infty)$ is now just a matter of quadratures, once $P(\nu, t, \bar{X})$ is found. The equation determining this function is

$$\frac{\partial P(\nu, t, \bar{X})}{\partial t} = \int_{\bar{X}}^{\bar{X}_0} \frac{\partial P(\nu-1, t, \bar{X}')}{\partial \bar{X}'} d\bar{X}' \int_{\bar{X}' - \bar{X}}^{\bar{X}'} W_i(\bar{X}', \xi) d\xi \quad (17)$$

+ $\int_{\bar{X}}^{\bar{X}_0} \frac{\partial P(\nu, t, \bar{X}')}{\partial \bar{X}'} d\bar{X}' \int_{\bar{X}' - \bar{X}}^{\bar{X}'} W_i(\bar{X}', \xi) d\xi - \int_0^{\bar{X}} \frac{\partial P(\nu, t, \bar{X}')}{\partial \bar{X}'} d\bar{X}' \int_{\tau}^{\bar{X}'} W_i(\bar{X}', \xi) d\xi$
 (17) is a set of simultaneous integral-differential-difference equations.

We can simplify somewhat by considering P as a smooth function of a continuous variable ν and then replacing this set of equations by a

single equation. To do this write

$$\frac{\partial P_\nu}{\partial t} = \int \frac{\partial P_{\nu-1}}{\partial \bar{x}'} d\bar{x}' W_1 d\bar{s} + \int \frac{\partial P_\nu}{\partial \bar{x}'} d\bar{x}' W_2 d\bar{s} - \int \frac{\partial P_\nu}{\partial \bar{x}'} d\bar{x}' W d\bar{s}$$

$$\frac{\partial P_{\nu-1}}{\partial t} = \int \frac{\partial P_{\nu-2}}{\partial \bar{x}'} d\bar{x}' W_1 d\bar{s} + \int \frac{\partial P_{\nu-1}}{\partial \bar{x}'} d\bar{x}' W_2 d\bar{s} - \int \frac{\partial P_{\nu-1}}{\partial \bar{x}'} d\bar{x}' W d\bar{s}.$$

or subtracting:

$$\left[\frac{\partial^2 P_\mu}{\partial \mu \partial t} \right]_{\mu=\nu-\frac{1}{2}} = \left[\frac{\partial^2 P_\mu}{\partial \mu \partial \bar{x}'} \right]_{\mu=\nu-\frac{3}{2}} d\bar{x}' \int W_1 d\bar{s} + \left[\frac{\partial^2 P_\mu}{\partial \mu \partial \bar{x}'} \right]_{\mu=\nu-\frac{1}{2}} d\bar{x}' \int W_2 d\bar{s} - \left[\frac{\partial^2 P_\mu}{\partial \mu \partial \bar{x}'} \right]_{\mu=\nu-\frac{1}{2}} d\bar{x}' \int W d\bar{s}$$

or, using the relation

$$\left[\frac{\partial^2 P_\mu}{\partial \mu \partial \bar{x}'} \right]_{\mu=\nu-\frac{1}{2}} = \left[\frac{\partial^2 P_\mu}{\partial \mu \partial \bar{x}'} \right]_{\mu=\nu-\frac{3}{2}} + \left[\frac{\partial^2 P_\mu}{\partial \mu^2 \partial \bar{x}'} \right]_{\mu=\nu-\frac{1}{2}}$$

we get

$$\left[\frac{\partial^2 P_\mu}{\partial \mu \partial t} \right]_{\mu=\nu-\frac{1}{2}} = \int \frac{\partial^2 P_\mu}{\partial \mu \partial \bar{x}'} d\bar{x}' (W_1 + W_2) d\bar{s} - \left[\frac{\partial^2 P_\mu}{\partial \mu^2 \partial \bar{x}'} d\bar{x}' \int W_1 d\bar{s} - \left[\frac{\partial^2 P_\mu}{\partial \mu^2 \partial \bar{x}'} d\bar{x}' \int W d\bar{s} \right] \right]_{\mu=\nu-\frac{1}{2}}$$

Integrating with respect to μ :

$$\frac{\partial P_\mu}{\partial t} = \int \frac{\partial P_\mu}{\partial \bar{x}'} d\bar{x}' \int_{\bar{x}-\bar{x}}^{\bar{x}_0} W d\bar{s} - \left[\frac{\partial P_\mu}{\partial \mu \partial \bar{x}'} d\bar{x}' \int W_1 d\bar{s} - \int \frac{\partial P_\mu}{\partial \bar{x}'} d\bar{x}' \int_{\bar{x}_m}^{\bar{x}'} W d\bar{s} \right]$$

From the physics of the problem it is clear that $P(\infty, \bar{x}, t) = 0$

so that the constant of integration is 0. The boundary condition is

$P(\mu, \bar{x}, 0) = \dot{q}(\mu) g(\bar{x})$ where $g(\bar{x})$ is determined by solving the transparency problem.

The angular distribution of the evaporated particles has so far been neglected. These particles should emerge with spherical symmetry with respect to the parent nucleus. But since the parent nucleus is in general moving with respect to the observer when evaporation occurs, the observed angular distribution is asymmetrical. To estimate the importance of this asymmetry consider an α particle of energy $E_\alpha = \frac{1}{2} m_\alpha v_\alpha^2$ colliding inelastically with a nucleus of mass M . After collision the velocity of the nucleus will be $V = \frac{m_\alpha}{M} v_\alpha$. Let $E_i = \frac{1}{2} m_i v_i^2$, about 12 Mev, be the average kinetic energy of the evaporated ions. Then

the asymmetry will be important if $\frac{V}{v_i}$ is of the order of 1 or greater.

We have

$$\left(\frac{V}{v_i}\right)^2 = \left(\frac{m}{M}\right)^2 \left(\frac{v_x}{v_i}\right)^2 = \left(\frac{m_x}{M}\right)^2 \frac{m_i}{m_x} \frac{E_x}{E_i}$$

If $E_x = 400 \text{ MeV}$, $E_i \approx 12 \text{ MeV}$, $\frac{m_i}{m_x} \approx \frac{1}{4}$, $M = 200$, then $\frac{V}{v_i} \approx .05$.

If the foil is as light as aluminum, $M = 27$, other conditions as above, then $\frac{V}{v_i} \approx .3$ and the asymmetry may be a sizeable effect.

For a more detailed treatment of the angular distribution, let $B(t, \vec{p})$ be the probability at time t after it is struck, that the target nucleus has already emitted an ion with momentum $\vec{p}$.

$B(\infty, \vec{p})$ will be observable from experiment. It is given by

$$\frac{\partial B(t, \vec{p})}{\partial t} = \int A(t, \vec{x}, \vec{p}) U(\vec{x}, \vec{p} - \vec{p}') d\vec{p}' d\vec{x}.$$

where $A(t, \vec{x}, \vec{p}) d\vec{p} d\vec{x}$ is probability at time t that the struck nucleus have a momentum $\vec{p}$ and an excitation $\vec{x}$, and $U(\vec{x}, \vec{p}) d\vec{p}$ is probability per unit time that a nucleus with excitation $\vec{x}$ emit an ion with momentum $\vec{p}$. To determine A we have the equation:

$$\frac{\partial A(t, \vec{x}, \vec{p})}{\partial t} = \sum_j \int A(t, \vec{x}', \vec{p}') d\vec{p}' U_j(\vec{x}, \vec{p} - \vec{p}') - A(t, \vec{x}, \vec{p}) \int U(\vec{x}, \vec{p}) d\vec{p}$$

where by conservation of energy, $\vec{x}'$ is given by

$$\vec{x}' - \vec{x} = \frac{(\vec{p}' - \vec{p})^2}{2m_j} + E_j + \frac{p'^2 - p^2}{2M}$$

From the expression for U in Part B it is easy to show

$$U_j(\vec{x}, \vec{p}' - \vec{p}) = \frac{1}{4\pi} \frac{\sigma g_j}{\pi^2 h^3} e^{-S(\vec{x}) + S(\vec{x} - E_j - \frac{(\vec{p}' - \vec{p})^2}{2m_j})} \frac{(\vec{p}' - \vec{p})^2}{2m_j} - V_j$$

where m_j , g_j and E_j , V_j are mass, statistical weight, binding energy, and Coulomb energy for an evaporated particle of type j and σ the geometrical cross section for the nucleus.

To discuss the technique of solution of the above integral equation, consider an equation of the typical form:

$$\frac{\partial P(x, \epsilon)}{\partial x} = \int_{\epsilon}^{\infty} \frac{\partial P(x, \epsilon')}{\partial \epsilon'} d\epsilon' K(\epsilon', \epsilon) \quad (18)$$

Were the kernel K of form $K \left/ \frac{f(\epsilon)}{f(\epsilon')} \right/$ then a Mellin transformation of P in the variable $f(\epsilon)$ would bring about a great simplification; were it of form $K(f(\epsilon) - f(\epsilon'))$ a Laplace transformation of $P(\epsilon, \epsilon')$ in the variable $f(\epsilon)$ would make a like simplification. If $P(x, \epsilon)$ is replaced by series expansion about the point $\epsilon = \epsilon'$, then this integral equation is easily reducible to a differential equation which has constant coefficients if K is of form $K(\epsilon - \epsilon')$. If K is of more general form, these coefficients are variable so that the method is of little use. The method of iteration may be of use in the transparency problem in the case of light nuclei, where the most probable number of collisions of the incident particle is say 3 or under; and also in the evaporation problem when the bombarding energy is low enough so that the average number of evaporated particles is not more than 3. But in general it also is impracticable.

For solving (18) with the kernels of interest in the present problem, two methods look promising. The first is to put $P(x, \epsilon) = \bar{e}^{-\alpha x} P_{\alpha}(\epsilon)$ so that (18) becomes an eigen-value problem in this the eigen-value α . Replacing the integral by a sum then in effect replaces (18) by a set of algebraic linear equations whose determinant is diagonal, which makes them quite easy to solve. Finally, we seek a linear combination of solutions of form $\bar{e}^{-\alpha x} P_{\alpha}(\epsilon)$ such that boundary conditions are satisfied. The second method is to Laplace transform $P(x, \epsilon)$ with respect to x , then to solve the resulting inhomogeneous equation by the method of Enskog.⁽¹⁰⁾

(10)

Enskog, Math. Zeit. Vol. 24, 25, 31.

with a judicious choice of the set of orthogonal functions occurring in this method. Work on these methods is now in progress.

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Fig. 1. Differential cross-section for energy loss from a neutron of 100 Mev. in a collision with a nucleon.

Fig. 2. Average energy loss $\bar{E}$ per collision with a nucleon, for a neutron of energy E_0 .

12443-9

$\delta = .213$ FOR $A = 13$
.102 $A = 100$

$\delta = .060$ FOR $A = 13$
.128 $A = 100$

PROBABILITY

.012

.010

.008

.006

.004

.002

0

A=13

A=100

A=100

A=13

 $\bar{X}$, ENERGY IN MEV

Fig. 3. Probability of energy loss $\bar{X}$ per collision with a nucleus, for a neutron of energy 100 Mev.

Fig. 4. Probability of energy loss X per collision with a nucleus, for a deuteron of energy 200 Mev.

Fig. 5. Average kinetic energies $\bar{E}$ of particles evaporated from a nucleus of the heavy component, as a function of its excitation X .

Fig. 6. Relative probabilities for evaporation of various kinds of particle from a nucleus of the heavy component, as a function of its excitation X .

Fig. 7. Average number of ions evaporated from a nucleus of the light or heavy component, as a function of excitation energy X .