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FOR GRID GENERATION AND ADAPTATION IN LP

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Generalized Monge-Kantorovich optimization for grid generation and adaptation in L_p

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Abstract

The Monge-Kantorovich grid generation and adaptation scheme of [Delzanno et al, *J. Comput. Phys.* 227, 9841 (2008)] is generalized from a variational principle based on L_2 to a variational principle based on L_p . A generalized Monge-Ampère (MA) equation is derived and its properties are discussed. Results for $p > 1$ are obtained and compared in terms of the quality of the resulting grid. We conclude that for the grid generation application, the formulation based on L_p for p close to unity leads to serious problems associated with the boundary. Results for $1.5 \lesssim p \lesssim 2.5$ are quite good, but there is a fairly narrow range around $p = 2$ where the results are close to optimal with respect to grid distortion. Furthermore, the Newton-Krylov methods used to solve the generalized MA equation perform best for $p = 2$.

Key words: Adaptive grid generation, Monge-Ampère equation,
Monge-Kantorovich optimization, Grid tangling, equidistribution, Newton-Krylov,
multigrid preconditioning, moving meshes
PACS: 02.70.-c

1 Introduction

In a recent paper [1], the fundamentals of two-dimensional grid generation and adaptation by Monge-Kantorovich optimization were laid out for the unit square $X = [0, 1] \times [0, 1]$. (In a later work [2], this theory was extended to more general physical domains and to the unit cube in three dimensions. Monge-Kantorovich optimization was also applied to grid adaptation for time-dependent blow-up problems in Ref. [3].) In this method, two densities $\rho(\mathbf{x})$

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and $\rho'(\mathbf{x}')$ are specified on X , where $\mathbf{x} = (x, y)$ and $\mathbf{x}' = (x', y')$. A specific map $\mathbf{x} \rightarrow \mathbf{x}'$ with Jacobian equal to $\rho(\mathbf{x})/\rho'(\mathbf{x}')$ was found. If these densities are, respectively, the local errors in a numerical discretization scheme at times t and $t + \Delta t$, the method *equidistributes* the errors, and it has been argued that this minimizes the total error [4]. This method is optimal in the sense that it minimizes the L_2 norm

$$\left[\int_X \rho(\mathbf{x}) |\mathbf{x}' - \mathbf{x}|^2 d\mathbf{x} \right]^{1/2} \quad (1)$$

while preserving the Jacobian constraint

$$\rho'(\mathbf{x}') \left(\frac{\partial x'}{\partial x} \frac{\partial y'}{\partial y} - \frac{\partial x'}{\partial y} \frac{\partial y'}{\partial x} \right) = \rho(\mathbf{x}). \quad (2)$$

This minimization leads to the *Monge-Ampère (MA) equation*, a nonlinear partial differential equation for a single variable. Solutions to this equation exist and are unique, and the ellipticity of the equation has been proved [5]. In Ref. [1] it was shown that the MA equation can readily be solved by Newton-Krylov methods, and that the ellipticity property means that multigrid preconditioning can be effectively used [1,6,7].

In this paper we investigate the problem of minimizing the L_p norm

$$\left[\int_X \rho(\mathbf{x}) |\mathbf{x}' - \mathbf{x}|^p d\mathbf{x} \right]^{1/p} \quad (3)$$

with the Jacobian constraint of Eq. (2). The motivation for generalizing to $p \neq 2$ is the following. First, the L_2 cost function in Eq. (1) penalizes large values of $|\mathbf{x}' - \mathbf{x}|$ but is not sensitive to small changes; if L_p for $p < 2$ is used, increased emphasis is placed on small values of $|\mathbf{x}' - \mathbf{x}|$ and decreased emphasis on large values. The opposite is true for $p > 2$, and in fact large p leads to a 'robust' cost function which is not at all sensitive to small deviations, but puts a hard constraint on large deviations. Second, a lot of work has been done in the mathematics community with regard to $p = 1$. The interest for $p = 1$ traces back to the original work of Monge [8] and is still an active research area [9,10]. However, to the best of our knowledge, there have been no attempts to apply the $p = 1$ theory. A third motivation comes from the imaging community. Recently, some researchers have applied the (L_2) Monge-Kantorovich optimization theory to find the warping transformation between two given images [11–13]. (In this case the luminosity of the two images corresponds to the initial and final densities, ρ and ρ' .) In Ref. [13], it was recognized that L_2

theory can lead to the undesirable effect of double exposure and it was argued that $L_{1+\varepsilon}$ (with $0 < \varepsilon < 1$) could have advantages over L_2 in this regard. While the application to image warping is beyond the scope of this paper, the theory and the methods developed here could be used for imaging purposes as well.

The paper proceeds as follows. In Sec. 2 we review the L_2 theory of Ref. [1]. In Sec. 3 we present the formalism for general p , leading to a generalized form of the Monge-Ampère (MA) equation (we call this equation the L_p MA equation). We present the boundary conditions and a discussion of the limit $p \rightarrow 1$. In Sec. 4 we discuss the numerical implementation details to solve the L_p MA equation while in Sec. 5 we present results. We show properties of the grid (i. e. the mapping $\mathbf{x} \rightarrow \mathbf{x}'$) for various p and compare these results in terms of minimization of the different L_p norms and in terms of the grid distortion. Much of the difference between the results for small values of p is related to boundary conditions. In Sec. 6 we discuss a variational principle based on smoothing the singularity of L_1 and one based on minimization of global grid cell distortion. In Sec. 7 we summarize our results. In particular, we summarize the conclusions showing that results with $p \rightarrow 1$ have problems in terms of grid quality, and $p = 2$ seems best with respect to the L_p norms as well as the grid distortion. Further, the numerical approach (based on Newton-Krylov methods) is more efficient for $p = 2$. In the Appendix we discuss generalized Legendre transforms.

2 Review of the L_2 theory

Let us review the L_2 variational principle from Ref. [1]. For the map $\mathbf{x} \rightarrow \mathbf{x}'$ on a volume X (with boundary ∂X), we minimize

$$E_2 = \int_X \frac{|\mathbf{x}' - \mathbf{x}|^2}{2} \rho(\mathbf{x}) d\mathbf{x} \quad (4)$$

with the constraint of Eq. (2). Here, $\rho(\mathbf{x})$ and $\rho'(\mathbf{x}')$ are two densities and the Jacobian $J(\mathbf{x}) = [x', y']$, where $[f, g] = \epsilon_{ij}(\partial f / \partial x_i)(\partial g / \partial x_j)$ is the Poisson bracket (with summation over repeated indices), with $\epsilon_{12} = -\epsilon_{21} = 1$ and $\epsilon_{11} = \epsilon_{22} = 0$. The densities are normalized to have $\int_X \rho(\mathbf{x}) d\mathbf{x} = \int_X \rho'(\mathbf{x}') d\mathbf{x}' = 1$. That is, we take the variation of

$$F_2 = E_2 - \int_X \lambda(\mathbf{x}') [\rho'(\mathbf{x}') J(\mathbf{x}) - \rho(\mathbf{x})] d\mathbf{x}. \quad (5)$$

The variable $\lambda(\mathbf{x}')$ is a Lagrange multiplier enforcing the Jacobian constraint (2) locally. Taking the variation of F_2 with respect to $\mathbf{x}'$, we find, as in Ref. [1]

$$\mathbf{x}' = \mathbf{x} - \nabla' \lambda(\mathbf{x}'), \quad (6)$$

where $\nabla' \equiv \nabla_{\mathbf{x}'}$. As shown in Ref. [1] (see also Appendix A), this equation can be put into the form

$$\mathbf{x}' = \mathbf{x} + \nabla \Phi(\mathbf{x}), \quad (7)$$

where $\lambda(\mathbf{x}')$ and $\Phi(\mathbf{x})$ are (L_2) Legendre transforms of each other. Substituting into the Jacobian condition we find

$$\nabla^2 \Phi + H[\Phi] = \frac{\rho(\mathbf{x})}{\rho'(\mathbf{x}')} - 1, \quad (8)$$

where $H[\Phi]$ is the Hessian $H[\Phi] = \partial_{xx}\Phi\partial_{yy}\Phi - (\partial_{xy}\Phi)^2$. This is the Monge-Ampère equation, a nonlinear elliptic partial differential equation for the single variable $\Phi(\mathbf{x})$. This equation is nonlinear in two ways: through the quadratic nonlinearity in the Hessian and through the dependence $\rho'(\mathbf{x}') = \rho'(\mathbf{x} + \nabla \Phi)$. The boundary conditions used, namely $\hat{\mathbf{n}} \cdot \nabla \Phi = 0$ on ∂X , reflect the requirement that boundary points are mapped to boundary points or, in other words, that boundary points can only move along the boundary but not perpendicular to it and that the total volume is preserved. For a domain with corners such as the unit square, these boundary conditions force the corner boundary points to be fixed ($\nabla \Phi = 0$).

The solution of the MA equation is the basis for the grid generation and adaptation scheme of Refs. [1,2]. Indeed, for the grid applications on the unit square [1], there is a mapping from the logical domain Ξ , which is also the unit square, to X . For $\boldsymbol{\xi} = (\xi, \eta) \in \Xi$, the density $\rho_0(\xi, \eta)$ equals unity and the logical grid is uniform. The mapping $\boldsymbol{\xi} \rightarrow \mathbf{x}$ with $d\xi d\eta = \rho(x, y) dx dy$, defines a grid in the sense that the sets of contours $\xi(x, y) = \text{const}$ and $\eta(x, y) = \text{const}$ are grid lines. Similarly, the new grid $\boldsymbol{\xi} \rightarrow \mathbf{x}'$ is defined with $\xi(x(x', y'), y(x', y')) = \text{const}$ and $\eta(x(x', y'), y(x', y')) = \text{const}$ as grid lines.

3 Optimization theory for L_p

For reasons outlined in the Introduction, we wish to generalize the L_2 theory to L_p . Let us begin by introducing an L_p variational principle for the map

$\mathbf{x} \rightarrow \mathbf{x}'$ on X which is analogous to Eqs. (4) and (5). We take

$$E_p = \int_X \frac{|\mathbf{x}' - \mathbf{x}|^p}{p} \rho(\mathbf{x}) d\mathbf{x}, \quad (9)$$

$$F_p = E_p - \int_X \lambda(\mathbf{x}') [\rho'(\mathbf{x}') J(\mathbf{x}) - \rho(\mathbf{x})] d\mathbf{x}. \quad (10)$$

The variable $\lambda(\mathbf{x}')$ is again the local Lagrange multiplier enforcing the Jacobian constraint (2). Taking the variation of F_p with respect to $\mathbf{x}'$, we find

$$\frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^{2-p}} = -\nabla' \lambda(\mathbf{x}'). \quad (11)$$

This equation can be put into the form

$$\frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^{2-p}} = \nabla \Phi(\mathbf{x}), \quad (12)$$

where $\Phi(\mathbf{x})$ is the L_p -Legendre transform of $\lambda(\mathbf{x}')$. See Appendix A. This leads to

$$|\mathbf{x}' - \mathbf{x}| = |\nabla \Phi|^{\frac{1}{p-1}}. \quad (13)$$

Substituting, we conclude

$$\mathbf{x}' = \mathbf{x} + a(\mathbf{x}) \nabla \Phi(\mathbf{x}), \quad \text{with } a(\mathbf{x}) = |\nabla \Phi|^{\frac{2-p}{p-1}} = |\mathbf{x}' - \mathbf{x}|^{2-p}. \quad (14)$$

Defining q such that $1/p + 1/q = 1$, this equation takes the form

$$a(\mathbf{x}) = |\nabla \Phi|^{q-2}. \quad (15)$$

Clearly, Eq. (14) reduces to the L_2 form of Eq. (7) and Ref. [1] for $p = 2$. Next, consider $p = 1$. From Eq. (14) we must consider the limit $p \rightarrow 1$. Using Eq. (12) leads to

$$\mathbf{x}' = \mathbf{x} + |\mathbf{x}' - \mathbf{x}| \nabla \Phi(\mathbf{x}), \quad (16)$$

which implies the Eikonal equation $|\nabla \Phi| = 1$ for $p = 1$.

For general p , Eq. (14) is substituted into the Jacobian condition to arrive at an equation which is a generalized Monge-Ampère equation for $p \neq 2$. This equation can be expressed as

$$\frac{\partial}{\partial x} \left(a(\mathbf{x}) \frac{\partial \Phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(a(\mathbf{x}) \frac{\partial \Phi}{\partial y} \right) + \left[a(\mathbf{x}) \frac{\partial \Phi}{\partial x}, a(\mathbf{x}) \frac{\partial \Phi}{\partial y} \right] = \frac{\rho(\mathbf{x})}{\rho'(\mathbf{x}')} - 1, \quad (17)$$

where $[\cdot, \cdot]$ is again the Poisson bracket.

Equation (17) can be rearranged into a different form. If one defines

$$b(\mathbf{x}) = \frac{2-p}{p-1} |\nabla \Phi|^{\frac{4-3p}{p-1}} \quad (18)$$

so that

$$\begin{aligned} \frac{\partial a}{\partial x} &= b \left(\frac{\partial \Phi}{\partial x} \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial \Phi}{\partial y} \frac{\partial^2 \Phi}{\partial y \partial x} \right) \\ \frac{\partial a}{\partial y} &= b \left(\frac{\partial \Phi}{\partial x} \frac{\partial^2 \Phi}{\partial x \partial y} + \frac{\partial \Phi}{\partial y} \frac{\partial^2 \Phi}{\partial y^2} \right), \end{aligned} \quad (19)$$

then Eq. (17) can be rewritten as

$$\begin{aligned} &\left[a + b \left(\frac{\partial \Phi}{\partial x} \right)^2 \right] \frac{\partial^2 \Phi}{\partial x^2} + \left[a + b \left(\frac{\partial \Phi}{\partial y} \right)^2 \right] \frac{\partial^2 \Phi}{\partial y^2} + 2b \frac{\partial \Phi}{\partial x} \frac{\partial \Phi}{\partial y} \frac{\partial^2 \Phi}{\partial x \partial y} + \\ &a \left[a + b \left(\frac{\partial \Phi}{\partial x} \right)^2 + b \left(\frac{\partial \Phi}{\partial y} \right)^2 \right] \left[\frac{\partial^2 \Phi}{\partial x^2} \frac{\partial^2 \Phi}{\partial y^2} - \left(\frac{\partial^2 \Phi}{\partial x \partial y} \right)^2 \right] = \frac{\rho}{\rho'} - 1. \end{aligned} \quad (20)$$

It is now easy to see that Eq. (20) belongs to the family of MA equations and it is in fact a generalized MA equation [5]. (In the following, we will refer to this equation as the L_p MA equation.) Thus, solutions of Eq. (20) exist and are unique [5]. Notice also that for $p = 2$, we have $a = 1$ and $b = 0$ so that Eq. (20) reduces to the MA equation of Ref. [1].

Equation (17) is solved in a manner similar to the Newton-Krylov methods applied to the L_2 MA equation. There are two main differences between general p and $p = 2$. First, the relation in Eq. (15) introduces a third source of nonlinearity to the L_p Monge-Ampère equation. Second, as we discuss below, for decreasing p the domain of attraction of Newton's method shrinks, making a very accurate first guess necessary for p approaching unity.

The boundary conditions for Eq. (17) are again based on mapping boundary points to boundary points. In general $a(\mathbf{x}) \neq 0$ on ∂X (otherwise $\nabla \Phi = 0$ on

∂X for $1 < p < 2$), implying that we must require

$$\hat{\mathbf{n}} \cdot \nabla \Phi = 0 \quad (21)$$

on the boundary ∂X . The unit vector $\hat{\mathbf{n}}$ is normal to the boundary. Again, this requires $\nabla \Phi = 0$ at corners, forcing them to be fixed. Therefore, it does not appear possible to solve the $p = 1$ case (the Eikonal equation) with boundary conditions on a square as in Sec. 2. This is because corner points, where $\nabla \Phi = 0$, necessarily violate the Eikonal equation. This suggests that results with $p \rightarrow 1$ may have boundary layers especially near the corners. We will see examples of this behavior in the next section.

3.1 Ellipticity

The L_p MA equation (20) is elliptic and this can be easily proved in the following way. First of all, we consider $\Phi = \Phi_0 + \delta\Phi$ (and similarly for a and b) and linearize Eq. (20) with respect to $\delta\Phi$. The function Φ_0 defines a mapping $\mathbf{x}'_0 = \mathbf{x} + |\nabla \Phi_0|^{\frac{2-p}{p-1}} \nabla \Phi_0$. In practical applications, Φ_0 could be the solution obtained at a previous time step or the previous iterate in a Newton-Krylov cycle. In the linearization procedure, we disregard all the terms that are proportional to the first derivative of $\delta\Phi$ (these are arising from linearization of a , b and ρ') since these do not affect the definition of ellipticity. Collecting the terms proportional to the second derivatives of $\delta\Phi$, there results the operator

$$L[\delta\Phi] = A \frac{\partial^2 \delta\Phi}{\partial x^2} + C \frac{\partial^2 \delta\Phi}{\partial y^2} + 2B \frac{\partial^2 \delta\Phi}{\partial x \partial y} \quad (22)$$

where

$$\begin{aligned} A &= a_0 + b_0 \left(\frac{\partial \Phi_0}{\partial x} \right)^2 + a_0 \left[a_0 + b_0 \left(\frac{\partial \Phi_0}{\partial x} \right)^2 + b_0 \left(\frac{\partial \Phi_0}{\partial y} \right)^2 \right] \frac{\partial^2 \Phi_0}{\partial y^2} \\ C &= a_0 + b_0 \left(\frac{\partial \Phi_0}{\partial y} \right)^2 + a_0 \left[a_0 + b_0 \left(\frac{\partial \Phi_0}{\partial x} \right)^2 + b_0 \left(\frac{\partial \Phi_0}{\partial y} \right)^2 \right] \frac{\partial^2 \Phi_0}{\partial x^2} \\ B &= b_0 \frac{\partial \Phi_0}{\partial x} \frac{\partial \Phi_0}{\partial y} - a_0 \left[a_0 + b_0 \left(\frac{\partial \Phi_0}{\partial x} \right)^2 + b_0 \left(\frac{\partial \Phi_0}{\partial y} \right)^2 \right] \frac{\partial^2 \Phi_0}{\partial y \partial x}. \end{aligned} \quad (23)$$

Equation (20) is elliptic if and only if $AC - B^2 > 0$ [14]. After some algebra, it is easy to show that

$$AC - B^2 = a_0 \left[a_0 + b_0 \left(\frac{\partial \Phi_0}{\partial x} \right)^2 + b_0 \left(\frac{\partial \Phi_0}{\partial y} \right)^2 \right] \frac{\rho}{\rho'} \quad (24)$$

and that the latter quantity is indeed positive since a_0 , b_0 , ρ and ρ' are positive. As for the L_2 case, ellipticity means that multigrid preconditioning is well-suited as a preconditioner for the Krylov solver (GMRES).

4 Numerical implementation

The L_p MA equation (17) is solved by using the Newton-Krylov method. The details of the discretization are the same as in Ref. [1]. To be specific, we take $\rho(\mathbf{x}) = 1$ but allow $\rho'(\mathbf{x}') \neq 1$. This allows us to use a uniform initial grid on the physical space X . (For more general $\rho(\mathbf{x})$, we difference on the uniform logical grid, as discussed in Ref. [1].) The unknowns $\Phi_{i,j}$ are located at cell centers (i and j refer to the cell center in the x and y direction, respectively) while $a_{i+1/2,j+1/2}$ are located at the vertices. Accordingly, the Laplacian of Eq. (17) is discretized with a 5-point stencil:

$$\begin{aligned} \nabla \cdot (a \nabla \Phi) |_{i,j} \approx & a_{i+1/2,j} \frac{\Phi_{i+1,j} - \Phi_{i,j}}{\Delta x} - \left(a_{i+1/2,j} + a_{i-1/2,j} \right) \frac{\Phi_{i,j}}{\Delta x^2} + a_{i-1/2,j} \frac{\Phi_{i-1,j} - \Phi_{i,j}}{\Delta x} + \\ & a_{i,j+1/2} \frac{\Phi_{i,j+1} - \Phi_{i,j}}{\Delta y} - \left(a_{i,j+1/2} + a_{i,j-1/2} \right) \frac{\Phi_{i,j}}{\Delta y^2} + a_{i,j-1/2} \frac{\Phi_{i,j-1} - \Phi_{i,j}}{\Delta y} \end{aligned} \quad (25)$$

where

$$a_{i+1/2,j+1/2} \approx \left[\left(\frac{\Phi_{i+1,j+1} + \Phi_{i+1,j} - \Phi_{i,j-1} - \Phi_{i,j}}{2\Delta x} \right)^2 + \left(\frac{\Phi_{i+1,j+1} - \Phi_{i+1,j} + \Phi_{i,j-1} - \Phi_{i,j}}{2\Delta y} \right)^2 \right]^{\frac{2-p}{2(p-1)}}$$

and

$$a_{i+1/2,j} = \frac{a_{i+1/2,j+1/2} + a_{i+1/2,j-1/2}}{2} \quad (27)$$

(and similarly). The quantities Δx and Δy represent the grid spacing in the x and y direction for the uniform initial grid. The Hessian term of Eq. (17) is discretized by a 9-point stencil, where we compose two first-order derivatives as follows. First, we define first-order derivatives at vertices ($i \pm 1/2, j \pm 1/2$) as:

$$\left. \frac{\partial \Phi}{\partial x} \right|_{i+1/2, j+1/2} \approx \frac{\Phi_{i+1, j} + \Phi_{i+1, j+1} - \Phi_{i, j} - \Phi_{i, j+1}}{2\Delta x}, \quad (28)$$

$$\left. \frac{\partial \Phi}{\partial y} \right|_{i+1/2, j+1/2} \approx \frac{\Phi_{i, j+1} + \Phi_{i+1, j+1} - \Phi_{i, j} - \Phi_{i+1, j}}{2\Delta y}. \quad (29)$$

These are introduced in similarly defined first-order derivatives at cell centers (i, j) (found by replacing $i \rightarrow i - 1/2$ and $j \rightarrow j - 1/2$ in the expressions above), to obtain the 9-point stencil discretization sought.

The boundary conditions (21) are implemented via ghost cells [1]. If we consider for instance the $x = 1$ boundary, where $\partial \Phi / \partial x = 0$, we have $\Phi_{n_x+1, j} = \Phi_{n_x, j}$, second order accurate because this segment of the boundary is at $x_{n_x+1/2} = 1$. (We use a similar formulation for the other boundary segments). For the corner ghost cells, we impose

$$\Phi_{n_x+1, n_y+1} = \Phi_{n_x+1, n_y} = \Phi_{n_x, n_y+1} = \Phi_{n_x, n_y}$$

at the top-right corner and similarly for the other corners. Here n_x and n_y label the number of grid cells in the x and y directions.

The details of the Newton-Krylov solver and its implementation are discussed at length in Ref. [1] and are not repeated here. We use a standard, non-linear Newton-Krylov solver [15], where we set the absolute error tolerance $\tau_a = 10^{-4}$ and the relative error tolerance $\tau_r = 10^{-4}$. The inner iterations are performed using the GMRES scheme, without restarting. The forcing parameter η_k is determined by the modified Eisenstat-Walker formula [15] and we start with $\eta_0 = 0.9$. We do not use a preconditioner for the GMRES iterations and we do not report any performance comparison among the different L_p methods. Rather, we focus the comparison on the quality of the resulting grids. The work to find an effective preconditioner (such as multigrid) will be justified once the best among the different L_p methods is identified.

We compare the different L_p methods in terms of quality of the grid generated. Our first measure is the finite difference approximation to square of the norm associated with E_p of Eq. (9), namely

$$\mathcal{N}(p) = \left\{ \sum_{i,j} \frac{\rho_{i,j} \left[(x'_{i,j} - x_{i,j})^2 + (y'_{i,j} - y_{i,j})^2 \right]^{p/2}}{n_x n_y} \right\}^{2/p}. \quad (30)$$

We also compute a finite difference approximation to the mean distortion of the grid cells [1]

$$\mathcal{D} = \sum_{i,j} \frac{\rho_{i,j} (g_{11} + g_{22})_{i,j}}{n_x n_y}, \quad (31)$$

where $g_{11} + g_{22}$ is the trace of the covariant metric tensor $g_{kl} = (\partial x'_m / \partial x_k)(\partial x'_m / \partial x_l)$ (repeated indices implying summation) or $g = \mathcal{J}^T \mathcal{J}$, where $\mathcal{J}$ is the Jacobi matrix, $\mathcal{J}_{mk} = \partial x'_m / \partial x_k$.

We also define another norm that is used in evaluating the quality of the grids for the smoothed L_1 case discussed in subsection 6.1:

$$\widehat{\mathcal{N}}(\epsilon) = \left\{ \sum_{i,j} \rho_{i,j} \frac{\sqrt{\epsilon^2 + (x'_{i,j} - x_{i,j})^2 + (y'_{i,j} - y_{i,j})^2} - \epsilon}{n_x n_y} \right\}^2. \quad (32)$$

5 L_p results

We first choose the example

$$\rho(\mathbf{x}) = 1; \quad \rho'(\mathbf{x}') = \frac{C}{1 + 16[(x' - 0.7)^2 + (y' - 0.9)^2]} \quad (33)$$

on the unit square, where C is a normalization constant determined by $\int_X \rho'(\mathbf{x}') d\mathbf{x}' = 1$. Figure 1 shows contours of ρ' . This is a quite challenging case since the ratio of the maximum to the minimum of ρ' is about 22. We solve the L_p MA equation (17) for several values of p in the interval $1 < p \leq 2.5$. The case $p = 1$ has been widely studied in the mathematics literature [9,10]. As we have discussed in the previous section, we obtain results by a limiting process as $p \rightarrow 1$. The smallest p we consider is $p = 1.01$, and convergence is difficult to obtain (the domain of attraction of the Newton method is small) for values of p near unity. We found it necessary to decrease p in small steps and, while decreasing p , use the latter solution as initial guess for the new p .

The grid formed using the densities in Eq. (33) is shown in Fig. 2 for various values of p having $1 < p \leq 2.5$. The grid lines shown are contours of the functions $x(x', y')$ and $y(x', y')$. The bolder lines are the contours $x = 0.25$, $x = 0.75$, $y = 0.25$, and $y = 0.75$. The grid lines appear to become more distorted as p decreases, and this distortion appears to be related to the behavior near the boundary, particularly near the corner $x' = y' = 0$. To understand this, note first that for $p = 2$ the contours $x = \text{const}$ appear to be orthogonal to

Fig. 1. Density $\rho'(x', y')$ of Eq. (33).

the boundary segments $y = 0$ and $y = 1$. Similarly, the contours $x = \text{const}$ appear to be orthogonal to the boundary segments $y = 0$ and $y = 1$. At $x' = 1$, this is expressed as $\partial y / \partial x' = 0$. In fact, this property follows directly from Eq. (7): at $x' = 1$, for example, we find $y' = y + \partial \Phi / \partial y$, so that $\partial y' / \partial x = \partial^2 \Phi / \partial x \partial y$. Since the boundary conditions on the right side are $x' = x$ or $\partial \Phi / \partial x = 0$, we conclude $\partial^2 \Phi / \partial x \partial y = 0$. This in turn implies $\partial y / \partial x' = 0$. For other values of p , this orthogonality property breaks down. Following the same chain of reasoning, we are led by Eq. (14) to $y' = y + a(\mathbf{x}) \partial \Phi / \partial y$ and $\partial y' / \partial x = (\partial / \partial x)(a(\mathbf{x}) \partial \Phi / \partial y)$, from which the above conclusion cannot be reached if $\partial a(\mathbf{x}) / \partial x \neq 0$. Indeed, the contours are far from orthogonal to the boundary, especially near the corner $x' = y' = 0$ and for smaller p . For $p = 1.01$ the contours seem severely stretched and 'creased' from this corner toward the peak in ρ' near $x' = 0.7$, $y' = 0.9$.

The function a of Eq. (15) is shown in perspective view in Fig. 3 for the case with $p = 1.01$. We see that $\partial a(\mathbf{x}) / \partial x \neq 0$ on the left side ($x = 0$) so that, according to the arguments in the previous paragraph, the contours $y = \text{const}$ are not orthogonal to the boundary there. It is interesting to notice that a is close to zero for $x = 0$ and $y = 0$ but not for $x = 1$ and $y = 1$. This is a consequence of the fact that, for $x = 0$ and $y = 0$, $|\nabla \Phi| < 1$ (see Fig. 4) and $a(\mathbf{x})$ goes to zero when $|\nabla \Phi|$ is elevated to the hundredth power (since $p = 1.01$). At $x = 1$ and $y = 1$, on the other hand, $|\nabla \Phi|$ is very slightly below unity and therefore $a = |\nabla \Phi|^{q-2} = |\nabla \Phi|^{99}$ is still not close to zero. Notice also that when $a \approx 0$, $\mathbf{x}'$ is close to $\mathbf{x}$. In the present example, the boundary points at $x = 0$ and $y = 0$ remain uniformly spaced (since the initial grid is uniform) and this contributes to the stretching of the cells near the origin for small p . Also, the crease connecting the origin with the peak of the density ρ'

Fig. 2. Contours $x(x', y') = \text{const}$ and $y(x', y') = \text{const}$ for cases with $p = 2, 2.5, 1.5, 1.25, 1.1, 1.05, 1.02, 1.01$ (64×64 cells). The bold contours have $x(x', y') = 0.25, 0.75$ and $y(x', y') = 0.25, 0.75$. Note the stretching and creasing for $p = 1.01$ between the origin and the peak of ρ' at $x' = 0.7, y' = 0.9$.

Fig. 3. The function $a(\mathbf{x}) = |\nabla\Phi|^{q-2}$ in perspective view for $p = 1.01$ ($1/p + 1/q = 1$; $q = 101$). Note that $a \approx 0$ for $x = 0$ and for $y = 0$ but not for $x = 1$ or $y = 1$. Also, the crease connected to the origin is plainly visible.

at $x' = 0.7$ and $y' = 0.9$ is clearly visible.

More insight can be gained by looking at the functions $|\nabla\Phi|^2$ in Fig. 4 for the same values of p as in Fig. 2. As p increases, this quantity becomes flatter in the interior, indeed showing that Φ becomes a solution to the Eikonal equation as $p \rightarrow 1$. However, the convergence to a solution of the Eikonal equation is far from uniform, with $|\nabla\Phi|^2$ far below unity near the boundary. This is a consequence of the fact that the corner points are fixed and these points create a boundary layer in the solution.

The results of the comparison among different L_p methods in terms of grid quality are presented in Table 1. Each row corresponds to a solution computed from the L_p MA equation (17) with $p = p_c$. Each column corresponds to measurements of the square of the norm $\mathcal{N}(p_m)$ for the same p_m . Because the solutions are minima of E_p over all $\mathbf{x}'(\mathbf{x})$ satisfying the Jacobian condition (2) and the boundary conditions (21), the minimum of the norm in each column is obtained on the diagonal $p_c = p_m$. For each column, the data is normalized to its minimum value at $p_c = p_m$ and unity is subtracted, i. e. the table entry is

$$\mathcal{M}(p_m) \equiv \mathcal{N}(p_m)/\mathcal{N}(p_c) - 1. \quad (34)$$

In the rightmost column we show the mean distortion $\mathcal{D}$ of Eq. (31). For each row (each value of p_c used to solve the equations) the mean distortion is normalized to the value of $\mathcal{D}$ obtained by the so-called *optimal distortion method* (ODM), for which $\mathcal{D}$ is minimized, and again unity is subtracted. The

Fig. 4. The quantity $|\nabla \Phi|^2$ for the same values of p as in Fig. 2.

ODM is a recently developed method presented in Ref. [1] and is based on the minimization of global grid cell distortion while enforcing equidistribution. The ODM equations and the boundary conditions used here (which differ slightly from those used in Ref. [1]) are briefly reviewed in subsection 6.2.

In order to determine the relative quality of the different L_p methods, we perform cross-comparisons of the results in Table 1. To illustrate, let us compare the columns with $p_m = 1.01$ and with $p_m = 2$. The case $p_c = 1.01$ generates a grid for which $\mathcal{M}(p_m = 2) = 0.083$. On the other hand, the grid obtained with $p_c = 2$ gives $\mathcal{M}(1.01) = 0.041$. That is, the norm $\mathcal{N}(p_m = 1.01)$ is fairly close to minimal if $p_c = 2$ is used, but $\mathcal{M}(p_m = 2)$, the deviation of the norm $\mathcal{N}(p_m = 2)$, is about a factor of two higher than when $p_c = 1.01$ is used. These results, which are representative of other pairs from Table 1, suggest that computing with larger p_c is better, but all values above $p_c = 1.5$ are fairly good. If we continue along these lines and compare the $p_m = 2$ and $p_m = 2.5$ columns, one can see that $p_c = 2.5$ gives $\mathcal{M}(2)$ which is 0.19% larger than with $p_c = 2$, while $p_c = 2$ gives $\mathcal{M}(2.5)$ which is 0.22% larger than with $p_c = 2.5$. Based solely on these considerations one might conclude that $p_c = 2.5$ is slightly better than $p_c = 2$.

We can gain further insight by focusing on the global grid cell distortion $\mathcal{D}$, of Eq. (31), in the last column of Table 1. The minimum of $\mathcal{D}$ is necessarily obtained by the ODM method (discussed in subsection 6.2), but $p_c = 2$ has the smallest value of $\mathcal{D}$ among the different p_c considered in Table 1, +0.14% above the ODM method. The actual minimum of $\mathcal{D}$ over p_c is +0.06% above the ODM results, and is achieved for $p \sim 1.9$. However, the value for $p_c = 2$ is only slightly larger than that for $p_c = 1.9$. Further, all values in the range $1.5 < p_c < 2.5$ have fairly good global grid cell distortion: $\mathcal{D}$ is within +1.6% relative to ODM. On the other hand, one can see that as p approaches unity the global cell distortion becomes very large: $\mathcal{D}$ is a whopping +25% above the ODM result for $p_c = 1.01$.

Based on the quality measures just discussed, we reach the main conclusion of the paper: $p_c = 2$ is the best among the different L_p methods. Furthermore, $p_c \rightarrow 1$ seems to be the worst. In addition, the L_2 MA equation is 'less' nonlinear and thus easier to solve. The L_2 MA equation is elliptic so that modern multigrid solvers can be effectively used as a preconditioner for the Newton-Krylov methods.

6 Other variational principles

In this section we discuss other variational principles that are useful in the comparison with the L_p results discussed above. These are a variation principle

$\downarrow p_c \setminus p_m \rightarrow$	1	1.01	1.02	1.05	1.1	1.25	1.5	1.75	2	2.25	2.5	Distortion
1.01	0.0000	0	0.0000	0.0004	0.0018	0.0103	0.0320	0.0572	0.0833	0.1093	0.1348	0.2501
1.02	0.0001	0.0000	0	0.0002	0.0013	0.0091	0.0295	0.0534	0.0782	0.1029	0.1270	0.2367
1.05	0.0006	0.0004	0.0002	0	0.0005	0.0063	0.0236	0.0444	0.0661	0.0878	0.1089	0.2130
1.1	0.0021	0.0017	0.0013	0.0005	0	0.0031	0.0162	0.0330	0.0510	0.0690	0.0867	0.1687
1.25	0.0092	0.0084	0.0076	0.0054	0.0028	0	0.0043	0.0132	0.0238	0.0351	0.0464	0.0716
1.5	0.0223	0.0210	0.0198	0.0164	0.0118	0.0036	0	0.0020	0.0063	0.0118	0.0178	0.0158
1.75	0.0334	0.0319	0.0304	0.0264	0.0208	0.0096	0.0017	0	0.0011	0.0035	0.0068	0.0018
2	0.0425	0.0408	0.0392	0.0348	0.0285	0.0155	0.0050	0.0010	0	0.0006	0.0022	0.0014
2.25	0.0499	0.0482	0.0464	0.0417	0.0349	0.0207	0.0085	0.0029	0.0006	0	0.0004	0.0069
2.5	0.0561	0.0542	0.0525	0.0475	0.0404	0.0254	0.0118	0.0052	0.0019	0.0004	0	0.0154

Table 1. Deviations of values of the norm $\hat{\mathcal{N}}(p_m)$ of Eq. (30) and the mean distortion $\mathcal{D}$ of Eq. (31). Each row corresponds to a value of $p = p_c$ used in the computation of the grid, and the columns correspond to the values of $p = p_m$ for the measured norm. All the norms are shown relative to the values on the diagonal ($p_c = p_m$) and unity is subtracted, i. e. $\mathcal{M}(p_m)$ of Eq. (34) is tabulated. Note that the values on the diagonal are minimal, as expected. Note also that for a pair of values $p_1 < p_2$, $\mathcal{M}(p_m = p_1)$ using $p_c = p_2$ is less than $\mathcal{M}(p_m = p_2)$ using $p_c = p_1$, indicating that computing with larger p_c is better than computing with smaller p_c . For instance, for $p_c = 1.01$ and $p_m = 2$ we have $\mathcal{M}(p_m = 2) = 0.0833$, whereas for $p_c = 2$ and $p_m = 1.01$ we have $\mathcal{M}(p_m = 2) = 0.0408$. The last column contains the mean distortion $\mathcal{D}$ of Eq. (31), relative to the value of $\mathcal{D}$ obtained by the ODM method of subsection 6.2, again with unity subtracted. The minimum of the values shown is at $p_c = 2$.

based on smoothing the singularity of L_1 and one based on the minimization of global grid cell distortion.

6.1 Smoothed L_1 results

The L_1 case discussed above has a singularity in $|\mathbf{x}' - \mathbf{x}|$ at $\mathbf{x}' = \mathbf{x}$. It is possible to regularize this singularity in several ways. One possibility is to consider L_p in the limit $p \rightarrow 1$, as we have done in Sec. 5. Another possibility, which we consider in this subsection, is to consider $\sqrt{\varepsilon^2 + (\mathbf{x}' - \mathbf{x})^2}$ with $\varepsilon \ll 1$. The latter has the advantage that the behavior remains that of L_1 for $(\mathbf{x}' - \mathbf{x})^2 \gg \varepsilon^2$, but behaves as L_2 for $(\mathbf{x}' - \mathbf{x})^2 \ll \varepsilon^2$. We refer to this case as the 'smoothed L_1 ', and start by defining the norm

$$E_1(\varepsilon) = \int_{\mathbf{x}} \sqrt{\varepsilon^2 + (\mathbf{x}' - \mathbf{x})^2} \rho(\mathbf{x}) d\mathbf{x} - \varepsilon. \quad (35)$$

Now consider

$$F_1(\varepsilon) = \int_{\mathbf{x}} \sqrt{\varepsilon^2 + (\mathbf{x}' - \mathbf{x})^2} \rho(\mathbf{x}) d\mathbf{x} - \int_{\mathbf{x}} \lambda(\mathbf{x}') [\rho'(\mathbf{x}') J(\mathbf{x}) - \rho(\mathbf{x})] d\mathbf{x}. \quad (36)$$

Taking the variation with respect to $\mathbf{x}'$, we find

$$\frac{\mathbf{x}' - \mathbf{x}}{\sqrt{\varepsilon^2 + (\mathbf{x}' - \mathbf{x})^2}} = -\nabla' \lambda(\mathbf{x}'). \quad (37)$$

The arguments in Appendix A show that this equation can be put into the form

$$\frac{\mathbf{x}' - \mathbf{x}}{\sqrt{\varepsilon^2 + (\mathbf{x}' - \mathbf{x})^2}} = \nabla \Phi(\mathbf{x}), \quad (38)$$

where $\Phi(\mathbf{x})$ is the Legendre transform corresponding to the cost function $c(\mathbf{x}, \mathbf{x}') = \sqrt{\varepsilon^2 + (\mathbf{x}' - \mathbf{x})^2}$. See Appendix A. We conclude

$$\mathbf{x}' - \mathbf{x} = \frac{\varepsilon \nabla \Phi(\mathbf{x})}{\sqrt{1 - |\nabla \Phi|^2}}. \quad (39)$$

From this, it is easy to see that $1 - |\nabla \Phi|^2 \sim \varepsilon^2$ for small ε . That is, the Eikonal equation $|\nabla \Phi|^2 \rightarrow 1$ is obtained in the limit $\varepsilon \rightarrow 0$. Also, for $\varepsilon \rightarrow \infty$, $\Phi \sim 1/\varepsilon$ and Eq. (39) reduces to the L_2 form of Eq. (7), namely $\mathbf{x}' - \mathbf{x} = \nabla(\varepsilon \Phi(\mathbf{x}))$.

$\downarrow \varepsilon_c \setminus \varepsilon_m \rightarrow$	0.04	0.1	0.4	1	10	Distortion
0.04	0	0.0058	0.0813	0.1260	0.1401	0.1562
0.1	0.0046	0	0.0327	0.0563	0.0636	0.0556
0.4	0.0295	0.0158	0	0.0013	0.0020	0.0003
1	0.0362	0.0221	0.0011	0	0.0001	0.0010
10	0.0378	0.0236	0.0016	0.0001	0	0.0015

Table 2

Deviations of values of the norm $\widehat{\mathcal{N}}(\varepsilon_m)$ of Eq. (30) and the mean distortion $\mathcal{D}$ of Eq. (31). Each row corresponds to a value of $\varepsilon = \varepsilon_c$ used in the computation of the grid, and the columns correspond to the values of $\varepsilon = \varepsilon_m$ for the measured norm. All the norms are shown relative to the values on the diagonal ($\varepsilon_c = \varepsilon_m$) with unity subtracted. The last column contains the mean distortion $\mathcal{D}$ of Eq. (31), relative to the value of $\mathcal{D}$ obtained by the ODM method of subsection 6.2, again with unity subtracted.

Equation (39) is substituted into the Jacobian condition (2) to arrive at an equation analogous to the modified Monge-Ampère equation in Eq. (17). (The difference is that now $a(\mathbf{x}) = \varepsilon/\sqrt{1 - |\nabla\Phi|^2}$.) This equation is solved in a manner similar to that for Eq. (17). The results are contained in Table 2 which reports the quality of the grid measured by the norm $\widehat{\mathcal{N}}(\varepsilon)$ of Eq. (32) (related to $E_1(\varepsilon)$ above), and the distortion $\mathcal{D}$ of Eq. (31). The notation is analogous to that of Table 1. Each row corresponds to $\varepsilon = \varepsilon_c$ used in the computation of the grid, while each column (except the last) uses the same $\varepsilon = \varepsilon_m$ to compute the norm (32). Each column except the last is normalized to the minimum value of that column (that is, the value for $\varepsilon_c = \varepsilon_m$) and unity is subtracted. The last column reports global cell distortion $\mathcal{D}$ (Sec. 6.2) normalized to the value obtained with the ODM method, again with unity subtracted. The results of Table 2 confirm the conclusions obtained from Table 1, showing a qualitatively similar pattern to the L_p case, and also indicating that the best performance based on $\widehat{\mathcal{N}}(\varepsilon)$ is attained for large ε , i. e. as the norm approaches the L_2 norm. The minimum value for the distortion $\mathcal{D}$ occurs for $\varepsilon = 0.4$, which is still fairly large relative to typical values of $|\mathbf{x}' - \mathbf{x}| \ll 1$.

Figure 5 shows the new adapted grid for four different values of ε , $\varepsilon = 0.04, 0.1, 0.4, 1$. The results are qualitatively similar to that of Fig. 2: for $\varepsilon = 1$ the grid is very smooth, while for $\varepsilon = 0.04$ the grid cells are highly stretched and a crease connecting the origin to the peak of the density ρ' is visible. Figure 6 shows $|\nabla\Phi|^2$ for $\varepsilon = 1$ (left) and $\varepsilon = 0.04$ (right). As ε becomes smaller, $|\nabla\Phi|$ becomes flatter in the interior and evolves toward a solution of the Eikonal equation. Again the boundary conditions create boundary layers in the solution.

Fig. 5. Contours $x(x', y') = \text{const}$ and $y(x', y') = \text{const}$ for cases with $\varepsilon = 1, 0.4, 0.1, 0.04$ (64×64 cells). The bold contours have $x(x', y') = 0.25, 0.75$ and $y(x', y') = 0.25, 0.75$. Note the stretching and creasing for $\varepsilon = 0.04$ between the origin and the peak of ρ' at $x' = 0.7, y' = 0.9$.

Fig. 6. The quantity $|\nabla \Phi|^2$ for $\varepsilon = 1$ (left) and $\varepsilon = 0.04$ (right).

6.2 The optimal distortion method (ODM)

The optimal distortion method (ODM) outlined in Ref. [1] results from minimizing the mean distortion

$$\mathcal{D} = \int_X \rho(\mathbf{x}) \left(\frac{g_{11} + g_{22}}{2} \right) dx dy \quad (40)$$

with the Jacobian constraint Eq. (2). In order to eliminate the boundary terms arising from integrating by parts in the minimization procedure, we impose

$$(\mathbf{x}' - \mathbf{x}) \cdot \hat{\mathbf{n}} = 0, \quad (41)$$

together with either of the following

$$(\mathbf{x}' - \mathbf{x}) \cdot \hat{\mathbf{t}} = f(x, y) \quad \text{or} \quad (42)$$

$$\frac{\partial y'}{\partial x} = 0 \quad \text{for } x = 0 \text{ and } 1, \quad (43)$$

$$\frac{\partial x'}{\partial y} = 0 \quad \text{for } y = 0 \text{ and } 1. \quad (44)$$

In Eq. (43), f is a given function and $\hat{\mathbf{t}}$ the unit vector tangential to the boundary. Equation (41) assures that ∂X maps to itself. (It also assures that corners stay fixed.) Equation (42) prescribes a given tangential displacement of the boundary points, while Eqs. (43) and (44) imply that the curves $x(\mathbf{x}') = \text{const}$ are normal to the bottom and top $y = 0$, $y = 1$ and $y(\mathbf{x}') = \text{const}$ are normal to the sides $x = 0$, $x = 1$. (See the arguments following Eq. (33).) With these boundary conditions, the equations for the ODM method are

$$\frac{1}{\rho} \nabla \cdot (\rho \nabla \mathbf{x}') = \nabla' \mu(\mathbf{x}'). \quad (45)$$

Again, μ is a local Lagrange multiplier to enforce the Jacobian constraint (2). Equation (45) corresponds to Eqs. (31) and (32) of Ref. [1]. As discussed in Ref. [1], these equations are of order two higher than the Monge-Ampère equation, and require a specification of the normal and tangential components of $\mathbf{x}'$ on ∂X . Unlike the case of Ref. [1], in which we considered boundary conditions similar to Eqs. (41) and (42), in this paper we use boundary conditions (43)-(44). The latter seem to be better in light of the results of Ref. [1], where we showed a case where applying $f(x, y) = 0$ on ∂X led to a grid whose grid cell distortion measured by Eq. (31) was higher than that obtained by the Monge-Kantorovich approach. (We remind the reader that the ODM method

Fig. 7. Comparison of the grid obtained by the Monge-Kantorovich approach ($p = 2$, solid line) and the ODM (dashed line) method for 32×32 grid points.

provides the grid with minimal global grid cell distortion for a *given* set of boundary conditions.)

Equations (45) and (2) with the latter boundary conditions are solved by Newton-Krylov methods in a manner similar to that used to solve the L_p Monge-Ampère equation. The grid obtained by the ODM method (dashed line) is superimposed on the grid obtained by the Monge-Kantorovich approach ($p = 2$, solid line) in Fig. 7. While there are some noticeable differences between the two grids, one can indeed see that the Monge-Kantorovich approach does a comparable job in allowing small grid cell distortion.

7 Conclusions

In Refs. [1–3] L_2 Monge-Kantorovich theory was shown to be very promising for grid generation/adaptation applications. In this paper, we have extended the L_2 Monge-Kantorovich optimization theory to L_p . This extension allowed us to study the applicability of L_p to grid generation for several p in the range $1 < p \leq 2.5$.

There are several reasons for doing this. L_2 theory places bigger penalties on

large displacement of $\mathbf{x} - \mathbf{x}$ but not a high penalty on smaller displacements. For L_p with $p > 2$ this property is emphasized. On the other hand, smaller p place a relatively larger emphasis on smaller displacements. Thus, it is of interest to investigate whether different values of p could lead to better quality grids (according to some measure) than $p = 2$. Furthermore, a lot of work has been done in the mathematics community with regard to $p = 1$ but, to the best of our knowledge, not in the context of grid generation. Also, it has been argued in Ref. [13] that L_p might have applications to image warping techniques.

The extension to L_p leads to the L_p Monge-Ampère equation (17). This equation has been solved numerically with Newton-Krylov techniques for $1 < p \leq 2.5$ and the resulting grids have been compared in terms of grid quality [essentially various measures of grid point displacement, Eq. (30), and a measure of grid cell distortion, Eq. (31)] for a challenging example.

We have concluded that $p = 2$ is the best among the different L_p methods. This is because, while $p = 2$ does an excellent job at minimizing grid point displacement and gives nearly minimal grid cell distortion, it has other definite advantages: the nonlinearities in the MA equation are more manageable (require fewer Newton iterations). Indeed we have observed that the domain of convergence of Newton's method shrinks for $p \neq 2$.

On the other hand, the limit $p \rightarrow 1$ seems to be the worst among the different L_p methods. Solution in this limit give rise to very stretched cells, with the global grid cell distortion considerably higher than what is achieved for $p = 2$. This behavior is related to the boundary conditions. The case $p = 1$ naturally requires the solution of the Eikonal equation $|\nabla\Phi|^2 = 1$ and this is not consistent with having fixed corner points where $\nabla\Phi = 0$. Thus, while $p = 1$ can only be approached in a limiting sense, these corner points create boundary layers in the solution for p close to unity.

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Appendix A. Generalized Legendre transforms

The usual (L_2) Legendre transform of a convex function $F(\mathbf{x}) \rightarrow L(\mathbf{x}')$ takes the form

$$L(\mathbf{x}') = \max_{\mathbf{x}} [\mathbf{x} \cdot \mathbf{x}' - F(\mathbf{x})] \quad (46)$$

or

$$\mathbf{x}' = \nabla F(\mathbf{x}), \quad L(\mathbf{x}') = \mathbf{x}(\mathbf{x}') \cdot \mathbf{x}' - F(\mathbf{x}(\mathbf{x}')), \quad (47)$$

where $\nabla = \nabla_{\mathbf{x}}$. (By convexity, the inverse $\mathbf{x}(\mathbf{x}')$ exists.) We conclude, from $\partial L / \partial x'_i = x_i + x'_j \partial x_j / \partial x'_i - (\partial F / \partial x_j) \partial x_j / \partial x'_i = x_i$ (summation over repeated indices implied), that

$$\mathbf{x} = \nabla' L(\mathbf{x}') \quad F(\mathbf{x}) = \max_{\mathbf{x}'} [\mathbf{x} \cdot \mathbf{x}' - L(\mathbf{x}')], \quad (48)$$

where $\nabla' \equiv \nabla_{\mathbf{x}'}$. Therefore, F and L are Legendre transforms of each other.

An alternate way of writing this is

$$\lambda(\mathbf{x}') = \max_{\mathbf{x}} \left[-\frac{(\mathbf{x}' - \mathbf{x})^2}{2} - \Phi(\mathbf{x}) \right]. \quad (49)$$

Differentiating, we find

$$\mathbf{x}' = \mathbf{x} + \nabla \Phi(\mathbf{x}). \quad (50)$$

Also,

$$\lambda(\mathbf{x}') = -(\mathbf{x}' - \mathbf{x}(\mathbf{x}'))^2 / 2 - \Phi(\mathbf{x}(\mathbf{x}')) \quad (51)$$

leads to

$$\mathbf{x} = \mathbf{x}' + \nabla' \lambda(\mathbf{x}'). \quad (52)$$

Incidentally, Eqs. (50), (52) imply

$$\nabla \Phi(\mathbf{x}) + \nabla' \lambda(\mathbf{x}') = 0. \quad (53)$$

We can also write Eq. (49) as

$$\frac{\mathbf{x}'^2}{2} + \lambda(\mathbf{x}') = \max_{\mathbf{x}} \left[\mathbf{x} \cdot \mathbf{x}' - \frac{\mathbf{x}^2}{2} - \Phi(\mathbf{x}) \right], \quad (54)$$

which is equivalent to Eq. (46) if we set $F(\mathbf{x}) = \mathbf{x}^2/2 + \Phi(\mathbf{x})$, $L(\mathbf{x}') = \mathbf{x}'^2/2 + \lambda(\mathbf{x}')$.

For L_p , we consider the analog of Eq. (49),

$$\lambda(\mathbf{x}') = \max_{\mathbf{x}} \left[-\frac{|\mathbf{x}' - \mathbf{x}|^p}{p} - \Phi(\mathbf{x}) \right], \quad (55)$$

leading as above to

$$\mathbf{x}' = \mathbf{x} + |\mathbf{x}' - \mathbf{x}|^{2-p} \nabla \Phi(\mathbf{x}). \quad (56)$$

Compare this with Eq. (12). Using $\lambda(\mathbf{x}') = -|\mathbf{x}' - \mathbf{x}(\mathbf{x}')|^p/p - \Phi(\mathbf{x}(\mathbf{x}'))$, we find

$$\begin{aligned} \frac{\partial \lambda}{\partial x'_i} &= -|\mathbf{x}' - \mathbf{x}|^{p-2} \left[x'_i - x_i - x'_j \frac{\partial x_j}{\partial x'_i} + x_j \frac{\partial x_j}{\partial x'_i} \right] - \frac{\partial \Phi}{\partial x_j} \frac{\partial x_j}{\partial x'_i} \\ &= -|\mathbf{x}' - \mathbf{x}|^{2-p} (x'_i - x_i). \end{aligned}$$

This last simplification uses Eq. (56). Therefore,

$$\mathbf{x} = \mathbf{x}' + |\mathbf{x} - \mathbf{x}'|^{2-p} \nabla' \lambda(\mathbf{x}'). \quad (57)$$

The functions $\Phi(\mathbf{x})$ and $\lambda(\mathbf{x}')$ are L_p -Legendre transforms of each other. Note that Eq. (53) again holds. This is the basis for concluding Eq. (12) from Eq. (11).

Generalizing even further, assume we have a convex function $\lambda(\mathbf{x})$ and a convex cost function $c(\mathbf{x}, \mathbf{x}')$. Then consider

$$\lambda(\mathbf{x}') = \max_{\mathbf{x}} [-c(\mathbf{x}, \mathbf{x}') - \Phi(\mathbf{x})]. \quad (58)$$

We find

$$\nabla_{\mathbf{x}} c(\mathbf{x}, \mathbf{x}') + \nabla \Phi(\mathbf{x}) = 0 \quad (59)$$

and

$$\lambda(\mathbf{x}') = -c(\mathbf{x}(\mathbf{x}'), \mathbf{x}') - \Phi(\mathbf{x}(\mathbf{x}')). \quad (60)$$

Then $\partial\lambda/\partial x'_i = -\partial c/\partial x'_i - (\partial c/\partial x_j) \partial x_j/\partial x'_i - (\partial\Phi/\partial x_j) \partial x_j/\partial x'_i = -\partial c/\partial x'_i$,
or

$$\nabla_{\mathbf{x}'} c(\mathbf{x}, \mathbf{x}') + \nabla' \lambda(\mathbf{x}') = 0. \quad (61)$$

Since Eqs. (59) and (61) take the same form as Eqs. (56) and (57), we can say that Φ and λ are generalized Legendre transforms of each other. If we assume further that $c(\mathbf{x}, \mathbf{x}') = c_0(\mathbf{x} - \mathbf{x}')$, Eqs. (59) and (61) lead to Eq. (53).

For the specific case $c(\mathbf{x}, \mathbf{x}') = \sqrt{\epsilon^2 + (\mathbf{x}' - \mathbf{x})^2}$ of subsection 6.1, these relations take the form

$$\mathbf{x}' = \mathbf{x} + \sqrt{\epsilon^2 + (\mathbf{x}' - \mathbf{x})^2} \nabla \mu(\mathbf{x}) \quad (62)$$

and

$$\mathbf{x} = \mathbf{x}' + \sqrt{\epsilon^2 + (\mathbf{x} - \mathbf{x}')^2} \nabla' \lambda(\mathbf{x}'). \quad (63)$$

Of course, since $c(\mathbf{x}, \mathbf{x}') = c_0(\mathbf{x} - \mathbf{x}')$, Eq. (53) again holds.

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