Solution to Monthly Problem #11418

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Problem 11418 asks to evaluate (for complex $|a| > 1$)

$$J := \int_{-\infty}^{\infty} \frac{t^2 \operatorname{sech}^2(t)}{a - \tanh(t)} dt.$$

A variable change of $x = \tanh(t)$ produce for $|a| > 1$ that

$$J = \int_{-1}^1 \frac{\operatorname{arctanh}^2(x)}{a - x} dx = \frac{1}{12} \ln^3\left(\frac{a+1}{a-1}\right) + \frac{\pi^2}{12} \ln\left(\frac{a+1}{a-1}\right). \quad (1)$$

The corresponding *Maple 12* code that obtained this is

```
J1:=a->int(t^2*sech(t)^2/(a-tanh(t)),t=-infinity..infinity): J1(a)
assuming a>1;
/ 2
limit|t ln(a + 1) - t ln(a + 1 + exp(2 t) a - exp(2 t))
\
/ (a - 1) exp(2 t)\ 1 / (a - 1) exp(2 t)\
- t polylog|2, - -----| + - polylog|3, - -----|
\ a + 1 / 2 \ a + 1 /
2
+ t ln(exp(2 t) + 1) + t polylog(2, -exp(2 t)) - - polylog(3, -exp(2 t)),
2
\
t = infinity, left|
/
J2:=simplify(student[changervar](x=tanh(t),J1(a),x)) assuming
a>1:J2;
```

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$$\begin{aligned}
& -\frac{1}{12}\pi^2 \ln(a-1) + \frac{1}{12}\pi^2 \ln(a+1) - \frac{1}{4}\ln(a-1)\ln(a+1) \\
& + \frac{1}{4}\ln(a-1)^2 \ln(a+1) - \frac{1}{12}\ln(a-1)^3 + \frac{1}{12}\ln(a+1)^3
\end{aligned}$$

The newer syntax of

```
\texttt{[IntegrationTools](Change(J1(a),x=tanh(t)^2)}
```

fails to produce the desired evaluation in (1). It returns the correct but less helpful polylogarithmic limit above.

The evaluation would appear to be valid except when $-1 < a < 1$.

A human proof can be obtained from (1) on using the geometric series and integrating term-by-term carefully—which is much easier once the answer is known.