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# Transition Form Factor $\gamma\gamma^* \to \pi^0$ and QCD sum rules

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**Abstract**

We extend the QCD sum rule analysis of the form factor $F_{\gamma^*\gamma^*\to pi^0}(q_1^2, q_2^2)$ into the region of small virtuality of one of the photons: $|q_1^2| \ll 1$ GeV$^\cdot|q_2^2| \sim 1$ GeV$^2$, where one should perform more precisely an OPE to factorize large and small distance contributions. As a first step the form factor is investigated in the region of moderate virtualities: $q_1^2 \sim q_2^2 \sim -1\text{GeV}^2$ and the full $\langle 0|G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle, \langle 0|\bar{\psi}\psi|0\rangle^2$ corrections in the sum rule are obtained. It is shown that the infrared mass singularities are subtracted in the corresponding OPE for essentially nonsymmetric kinematics due to the operators of lowest two twists. On a simple scalar example the most important steps of the further calculations are demonstrated.

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# 1 Introduction

The studies of the transition form factor for the process $\gamma^*\gamma^* \to \pi^0$ when two virtual photons $\gamma^*$ produce a neutral pion is apparently the cleanest case for testing QCD predictions for elastic processes. In contrast with the pion electromagnetic form factor, the perturbative QCD subprocess $\gamma^*(q_1)+\gamma^*(q_2) \to \bar{q}(\bar{x}p)+q(xp)$ appears at the zeroth order in the QCD coupling constant $\alpha_s$, and the asymptotically leading term has no suppression. The relevant diagram resembles the handbag diagram for forward virtual Compton amplitude used in the studies of deep inelastic scattering. This gives good reasons to expect that perturbative QCD for this process may work at accessible values of spacelike photon virtualities $q_1^2 \equiv -q^2, q_2^2 \equiv -Q^2$. In the lowest order, pQCD predicts that [1]

$$F^{LO}_{\gamma^*\gamma^*\pi^0}(q^2,Q^2) = \frac{4\pi}{3}\int_0^1 \frac{\varphi_\pi(x)}{xQ^2+\bar{x}q^2}\,dx, \tag{1.1}$$

where $\varphi_\pi(x)$ is the pion distribution amplitude and $x, \bar{x} \equiv 1-x$ are the fractions of the pion light-cone momentum carried by the quarks. In the region where both photon virtualities are large: $q^2 \sim Q^2 \gtrsim 1\,GeV^2$, the pQCD predicts the overall $1/Q^2$ fall-off of the form factor, which contrasts with the naive vector meson dominance expectation $F_{\gamma^*\gamma^*\pi^0}(q^2,Q^2) \sim 1/q^2Q^2 \sim 1/Q^4$. Thus, establishing the $1/Q^2$ power law in this region is a crucial test of pQCD for this process. The study of $F_{\gamma^*\gamma^*\pi^0}(q^2,Q^2)$ over a wide range of the ratio $q^2/Q^2$ of two large photon virtualities can then provide a nontrivial information about the shape of $\varphi_\pi(x)$.

However, experimentally most favourable situation is when one of the photons is real $q^2=0$ or almost real. For experiments on $e^+e^-$-machines having one small virtuality strongly increases the cross section. It was also proposed to use real photons to study the $\pi^0$ production in $\gamma e$ collisions [2]. On fixed-target accelerators, like CEBAF, one can attempt to study the $F_{\gamma^*\gamma^*\pi^0}(q^2=0,Q^2)$ form factor through $\gamma^*\pi^* \to \gamma$ processes [3] with a photon in the final state. In the $q^2 \to 0$ limit (with $Q^2$ large), the nonperturbative information about the pion is accumulated in the same integral $I$ of $\varphi_\pi(x)/x$ that appears in the asymptotic pQCD expression for the one-gluon exchange contribution for the pion electromagnetic form factor. Hence, information extracted from the studies of $F_{\gamma^*\gamma^*\pi^0}(q^2=0,Q^2)$ can be used to settle the bounds on the pQCD hard contribution to the pion EM form factor.

Since the zeroth moment of the pion distribution function is normalized by the matrix element of the axial current (*i.e.*, by the pion decay constant $f_\pi$), the value of $I$ is sensitive to the shape of the pion distribution amplitude $\varphi_\pi(x)$. Two most popular choices include the asymptotic form $\varphi_\pi^{as}(x) = 6f_\pi x(1-x)$ [4]-[6] and the CZ model $\varphi_\pi^{CZ}(x) = 30f_\pi x(1-x)(1-2x)^2$ [7]. Using $\varphi_\pi^{CZ}(x)$ increases the integral by an extra factor of 5/3 compared to the value based on $\varphi_\pi^{as}(x)$. This observation can be used for an experimental discrimination between these two models. In fact, both the CELLO data [8] and preliminary high-$Q^2$ CLEO data [9] seem to favour the leading-order pQCD prediction with the normalization corresponding to a rather narrow distribution amplitude close to $\varphi_\pi^{as}(x)$. To perform a detailed comparison of the (future) data with theoretical predictions, one should take into account pQCD radiative corrections. These include the one-loop contributions to the hard scattering amplitude [11, 12], which, decreasing the leading-order result by about 20%, still leave a sizable gap between the predictions based on two models mentioned above. One should also add the terms generated by two-loop evolution of the pion distribution amplitude [13, 14, 15]. Originally, these corrections were found to be tiny [12]. A recent progress [16] in understanding the structure of the two-loop evolution suggests that the size of these corrections maybe somewhat larger. However, the numerical analysis of the two-loop evolution presented in ref.[17] does not indicate appreciable changes for the integral over the distribution amplitude. Hence, there are good chances that the controversial subject of the shape of $\varphi_\pi(x)$ may soon be settled experimentally.

Within the pQCD approach, the pion distribution amplitude $\varphi_\pi(x)$ is a phenomenological model function whose shape should be taken either from experiment (this was not possible so far) or calculated in some nonperturbative approach, *e.g.*, using QCD sum rules. However, applications of the QCD sum rules to nonlocal hadronic characteristics (functions), like distribution amplitudes $\varphi(x)$ are much more involved than those for the simpler classic cases [18] of hadronic masses and decay widths. The trickiest problem is that the underlying operator product expansion (OPE) for the relevant correlators has a slow convergence because some terms are parametrically enhanced, *e.g.*, by powers of $N$ for the $x^N$ moment of the distribution amplitude [7]. For this reason, one needs a very detailed information about the nonperturbative QCD vacuum to get a reliable sum rule. Such information is not available yet, and results for $\varphi(x)_\pi$ have strong model dependence[19]. In the present paper, instead of starting with the pQCD approach and taking $\varphi_\pi(x)$ from QCD sum rules, we calculate $F_{\gamma^*\gamma^*\pi^0}(q^2=0,Q^2)$ directly from a QCD sum rule for the three-point function. A serious problem for such an attempt is that one of the photons has a small virtuality and the relevant three-point amplitude is sensitive to nonperturbative long-distance QCD dynamics. For this reason, as an intermediate step we construct a QCD sum rule for a simpler kinematical situation when both photon virtualities are large. To take the $q^2 \to 0$ limit, we perform additional factorization using the methods developed in refs.[20]-[26].

The paper is organized as follows. In Section 2 we introduce our basic object: the correlator of two vector and one axial current, discuss its behaviour in different kinematical situations and interrelation between QCD sum rules and pQCD approaches. The situation when both photon virtualities are large is considered in Section 3. We construct there the QCD sum rule valid in this "large-$q^2$" kinematics and analyze its structure and some limiting cases. In Section 4, we outline specific problems one faces trying to use the operator product expansion in the limit when one of the photon virtualities is small (small-$q^2$ kinematics). The basic features of mass singularities, which appear in this limit, are illustrated using a scalar model as a toy example. The methods developed there are then used in Section 5 to construct a factorization procedure of long- and short-distance contributions in QCD. The long-distance contributions in this case have the structure of bilocal correlators which are considered in Section 6. In particular, we discuss there the continuum and $\rho$-meson contributions into the bilocal correlators. In Section 7, we discuss contact terms which appear in some bilocals. In section 8, we collect together the contributions calculated in preceding sections and write down the QCD sum rule for the $F_{\gamma^*\gamma^*\pi^0}(q^2,Q^2)$ form factor valid in small-$q^2$ kinematics. We discuss the limit $q^2 \to 0$ and present our results for

$F_{\gamma^*\gamma^*\pi^0}(q^2=0,Q^2)$. In the concluding section we summarize our findings. Some technical details of our calculations can be found in the Appendix.

# 2 Form factor $\gamma^*\gamma^* \to \pi^0$ and three-point function

## 2.1 Definitions

The form factor $F_{\gamma^*\gamma^*\to\pi^o}(q_1^2,q_2^2)$ of the $\gamma^*\gamma^* \to \pi^o$ transition is determined by the matrix element

$$4\pi \int \langle \pi, \vec{p}\ |T\{J_\mu(X)\,J_\nu(0)\}\,|0\rangle e^{-iq_1X} d^4X = \sqrt{2} i \epsilon_{\mu\nu q_1 q_2}\, F_{\gamma^*\gamma^*\to\pi^o}\left(q_1^2,q_2^2\right) \tag{2.1}$$

where $J_\mu$ is the electromagnetic current of the light quarks (divided by the electron charge):

$$J_\mu = \left(\frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d\right) , \tag{2.2}$$

$|\pi,\vec{p}\rangle$ is a one-pion state with the 4-momentum $p$ and we use throughout the convention $\epsilon_{\mu\nu\alpha\beta}q_2^\beta \equiv \epsilon_{\mu\nu\alpha q_2}$, $\epsilon_{\mu\nu\alpha\beta}q_1^\alpha q_2^\beta \equiv \epsilon_{\mu\nu q_1 q_2}$, *etc.*

To study the form factor, we should construct first a formalism in which the pion would emerge as a QCD bound state in the $\bar{q}q$ system. A possible way is to start with a three-point correlation function

$$\mathcal{F}_{\alpha\mu\nu}(q_1,q_2) = 2\pi i \int \langle 0|T\left\{j_\alpha^5(Y)\,J_\mu(X)\,J_\nu(0)\right\}|0\rangle e^{-iq_1X}\, e^{ipY} d^4X\, d^4Y\,, \tag{2.3}$$

(cf. [27]) where $p = q_1 + q_2$. In addition to the two EM currents present in eq.(2.1), the correlator (2.3) contains also the axial current $j_\alpha^5$

$$j_\alpha^5 = \left(\bar{u}\gamma_5\gamma_\alpha u\ -\ \bar{d}\gamma_5\gamma_\alpha d\right) . \tag{2.4}$$

The latter has the necessary property that its projection onto the neutral pion state is non-zero. In fact, this projection is proportional to the $\pi^- \to \mu\nu$ decay constant $f_\pi \approx 130.7\,MeV$:

$$\langle 0|j_\alpha^5(0)|\pi^0, \vec{p}\rangle\ =\ -i\,\sqrt{2}\,f_\pi p_\alpha. \tag{2.5}$$

The three-point correlator (2.3) has a richer Lorentz structure than the original amplitude (2.1), and not all the tensor structures it contains are relevant to our study of $F_{\gamma^*\gamma^*\to\pi^o}(q_1^2,q_2^2)$. Incorporating the Lorentz invariance properties of the three-point function and Bose symmetry for the virtual photons, one can write the amplitude $\mathcal{F}_{\alpha\mu\nu}$ as

$$\begin{aligned}\mathcal{F}_{\alpha\mu\nu}(q_1,q_2) &= p_\alpha\epsilon_{\mu\nu q_1 q_2}\mathcal{F}_1\left(p^2,q_1^2,q_2^2\right) + r_\alpha\epsilon_{\mu\nu q_1 q_2}\mathcal{A}_1\left(p^2,q_1^2,q_2^2\right)\\ &+ \left[\epsilon_{\alpha\mu q_1 q_2}q_{1\nu} - \epsilon_{\alpha\nu q_1 q_2}q_{2\mu}\right]\mathcal{F}_2\left(p^2,q_1^2,q_2^2\right) + \left[\epsilon_{\alpha\mu q_1 q_2}q_{2\nu} - \epsilon_{\alpha\nu q_1 q_2}q_{1\mu}\right]\mathcal{F}_3\left(p^2,q_1^2,q_2^2\right)\\ &+ \left[\epsilon_{\alpha\mu q_1 q_2}q_{1\nu} + \epsilon_{\alpha\nu q_1 q_2}q_{2\mu}\right]\mathcal{A}_2\left(p^2,q_1^2,q_2^2\right) + \left[\epsilon_{\alpha\mu q_1 q_2}q_{2\nu} + \epsilon_{\alpha\nu q_1 q_2}q_{1\mu}\right]\mathcal{A}_3\left(p^2,q_1^2,q_2^2\right)\\ &+\epsilon_{\alpha\mu\nu r}\left[\frac{p^2}{2}\mathcal{F}_4 + \frac{r^2}{2}\mathcal{F}_5 + (pr)\mathcal{A}_6\right] + \epsilon_{\alpha\mu\nu p}\left[\frac{p^2}{2}\mathcal{A}_4 + \frac{r^2}{2}\mathcal{A}_5 + (pr)\mathcal{F}_6\right],\end{aligned}$$

where $p = q_1 + q_2$, $r = q_1 - q_2$. The invariant amplitudes $\mathcal{F}, \mathcal{A}$ have the following symmetry properties: $\mathcal{F}_i(p^2, q_1^2, q_2^2) = \mathcal{F}_i(p^2, q_2^2, q_1^2)$, $\mathcal{A}_i(p^2, q_1^2, q_2^2) = -\mathcal{A}_i(p^2, q_2^2, q_1^2)$.

Utilizing the fact that, in a four-dimensional space-time, there is no antisymmetric tensor of rank 5, *i.e.*,

$$\epsilon_{\alpha\mu\nu\gamma}\, g_{\delta\epsilon} + \epsilon_{\mu\nu\gamma\delta}\, g_{\alpha\epsilon} + \epsilon_{\nu\gamma\delta\alpha}\, g_{\mu\epsilon} + \epsilon_{\gamma\delta\alpha\mu}\, g_{\nu\epsilon} + \epsilon_{\delta\alpha\mu\nu}\, g_{\gamma\epsilon} = 0\ , \tag{2.6}$$

and using the conditions $q_1^\mu \mathcal{F}_{\alpha\mu\nu} = q_2^\nu \mathcal{F}_{\alpha\mu\nu} = 0$ imposed by the EM current conservation, we get finally the expansion in terms of four invariant amplitudes:

$$\begin{aligned} \mathcal{F}_{\alpha\mu\nu}(q_1,q_2) &= \epsilon_{\mu\nu q_1 q_2}\left[p_\alpha F + r_\alpha A\right] \\ &+ \left[q_{2\nu}\epsilon_{\alpha\mu q_1 q_2} - q_{1\mu}\epsilon_{\alpha\nu q_1 q_2}\right]\widetilde{F} + \left[q_{2\nu}\epsilon_{\alpha\mu q_1 q_2} + q_{1\mu}\epsilon_{\alpha\nu q_1 q_2}\right]\widetilde{A} \\ &+ \epsilon_{\alpha\mu\nu r}\left[\frac{p^2+r^2}{4}\widetilde{F} - \frac{(pr)}{2}\widetilde{A}\right] + \epsilon_{\alpha\mu\nu p}\left[-\frac{(pr)}{2}\widetilde{F} + \frac{p^2+r^2}{4}\widetilde{A}\right]\ . \end{aligned} \tag{2.7}$$

According to eqs. (2.1), (2.5), the three-point amplitude $\mathcal{F}_{\alpha\mu\nu}(q_1,q_2)$ has a pole for $p^2 = m_\pi^2$:

$$\mathcal{F}_{\alpha\mu\nu}(q_1,q_2) = \frac{f_\pi}{p^2 - m_\pi^2} p_\alpha \epsilon_{\mu\nu q_1 q_2} F_{\gamma^*\gamma^*\to\pi^o}\left(q_1^2, q_2^2\right) + \ldots \tag{2.8}$$

Thus, the tensor structure of the pion contribution is $p_\alpha \epsilon_{\mu\nu q_1 q_2}$. In eq.(2.7), it corresponds to the invariant amplitude $F(q_1^2, q_2^2, p^2)$. In other words, $F(q_1^2, q_2^2, p^2)$ has the pole $1/(p^2 - m_\pi^2)$, and the form factor $F_{\gamma^*\gamma^*\to\pi^o}(q_1^2, q_2^2)$ can be extracted from the residue of that pole.

However, the problem is that the bound state poles are present only in full amplitudes, formally corresponding to the total sums over all orders of perturbation theory (in QCD one should not forget to add also nonperturbative contributions). Terms corresponding to any finite order do not have such poles. Fortunately, it is not always necessary to perform an explicit all-order summation (impossible in QCD) to extract information about a particular bound state.

## 2.2 Implications of the axial anomaly

Using the axial anomaly [28] relation for massless quarks

$$\partial_\alpha j_\alpha^5 = \frac{e^2}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} \tag{2.9}$$

we obtain the constraint

$$p^2 F - (q_1^2 + q_2^2)\widetilde{F} + (q_1^2 - q_2^2)(A + \widetilde{A}) = \frac{1}{\pi}. \tag{2.10}$$

For real photons ($q_1^2 = q_2^2 = 0$), this reduces to $p^2 F = \frac{1}{\pi}$, provided that $\widetilde{F}$ and $(A + \widetilde{A})$ do not have $1/q_1^2$ or $1/q_2^2$ singularities, which is true if there are no massless $\bar{q}q$ bound states in the vector channel. Hence, $F \sim 1/p^2$ for real photons, *i.e.*, $F$ really has a pole corresponding to a massless pion [29, 30]. Furthermore, the anomaly relation (2.10) fixes the value of the $F_{\gamma^*\gamma^*\to\pi^o}(q_1^2, q_2^2)$ form factor when both virtualities of the photons are zero :

$$F_{\gamma^*\gamma^*\to\pi^o}(0,0) = \frac{1}{\pi f_\pi}. \tag{2.11}$$

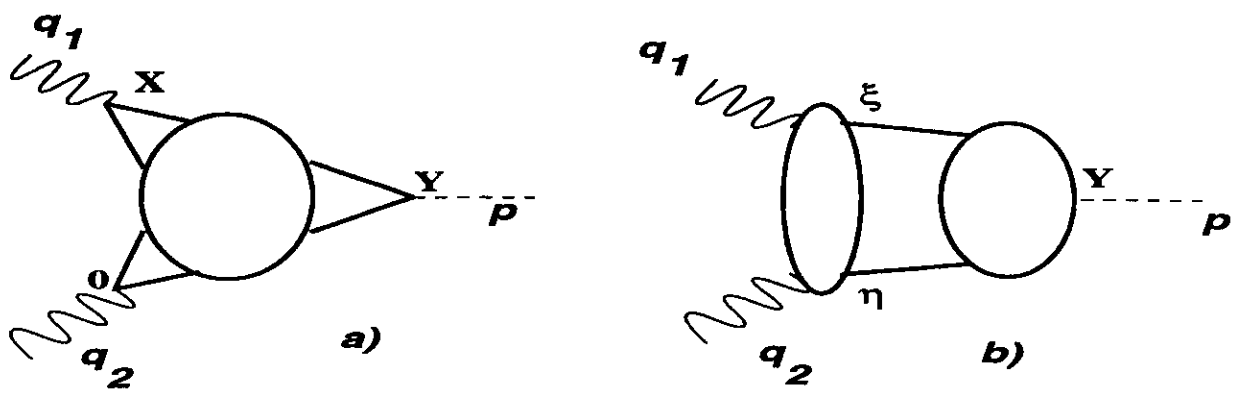


Figure 1: *a*) Three-point correlation function. *b*) Structure of factorization in the limit $Q^2, q^2 \gg |p^2|$.

It should be emphasized that the anomaly relation does not require that $F_{\gamma^*\gamma^*\to\pi^0}(0, Q^2)$ has this value for all $Q^2$: eq.(2.10) can be satisfied even if $F_{\gamma^*\gamma^*\to\pi^0}(0, Q^2)$ has a nontrivial $Q^2$-dependence.

Thus, the axial anomaly allows one to calculate exactly one particular combination of invariant amplitudes (2.10) for arbitrary values of the virtualities $q_1^2, q_2^2, p^2$. In QCD, this is a rather exceptional situation. Normally, a reliable QCD calculation is possible only in the region of large virtualities where one can incorporate the asymptotic freedom property of the theory.

## 2.3 Factorizable contributions and perturbative QCD

If both of the photon virtualities are large, the leading term of the $1/q^2$-expansion of any diagram contributing to the amplitude $F$ can be written in the factorized form (cf. [31, 32]):

$$F\left(p^2, q_1^2, q_2^2\right) \sim \int C(\xi, \eta, q_1, q_2)\, \Pi(\xi, \eta, p) d^4\xi d^4\eta\,, \tag{2.12}$$

where $C(\xi, \eta, q_1, q_2)$ is the short-distance coefficient function and $\Pi(\xi, \eta, p)$ is the long-distance factor given by a particular term of the PT expansion for the correlator of the axial current with a composite operator $\mathcal{O}(\xi, \eta) \sim \bar{q}(\xi) \dots q(\eta)$

$$\Pi(\xi, \eta, p) \sim \int \langle 0|T(\mathcal{O}(\xi, \eta) j(Y))|0\rangle\, e^{ipY} d^4Y. \tag{2.13}$$

The correlator $\Pi(\xi, \eta, p)$ is formally given by a sum over all orders of PT, and it also has a pole for $p^2 = m_\pi^2$:

$$\Pi(\xi, \eta; p) = \frac{f_\pi}{p^2 - m_\pi^2} \langle 0|\mathcal{O}(\xi, \eta)|\pi^0, \vec{p}\rangle\ . \tag{2.14}$$

Comparing eqs. (2.14) and (2.8), we conclude that the transition form factor is given by

$$F_{\gamma^*\gamma^*\to\pi^0}(q_1^2, q_2^2) = \int C(\xi, \eta, q_1, q_2) \langle 0|\mathcal{O}(\xi, \eta)|\pi^0, \vec{p}\rangle\ d^4\xi d^4\eta\ . \tag{2.15}$$

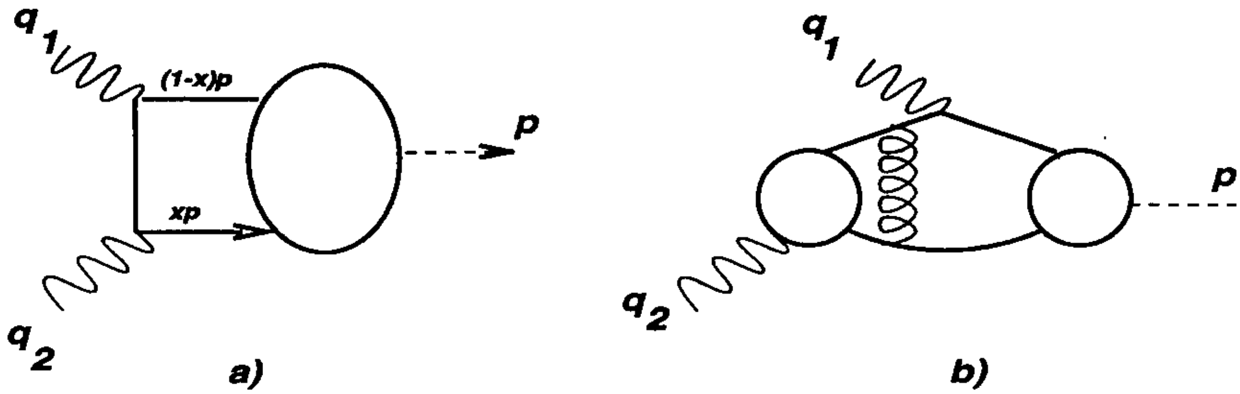


Figure 2: $a$) Lowest-order pQCD term. $b$) Hard gluon exchange diagram.

Thus, the possibility to factorize short- and long-distance contributions allows one to get an explicit expression for the form factor of a bound state, the pion, without explicitly obtaining the pion pole from a summation or a bound-state formalism.

Now, introducing the distribution amplitude $\varphi_\pi(x)$

$$\langle 0|O(\xi,\eta)|\pi^0,\vec{p}\rangle = \int_0^1 e^{ix(\xi p)+i\bar{x}(\eta p)}\varphi_\pi(x)dx \ , \tag{2.16}$$

where $\bar{x} \equiv 1-x$, we obtains the hard scattering formula

$$F_{\gamma^*\gamma^*\to\pi^0}(q_1^2,q_2^2) = \int_0^1 T(q_1,q_2;xp,\bar{x}p)\,\varphi_\pi(x)\,dx \ , \tag{2.17}$$

with $T(q_1,q_2;xp,\bar{x}p)$ being the amplitude for the subprocess $\gamma^*\gamma^* \to \bar{q}q$.

## 2.4 Perturbative QCD predictions

Calculating the subprocess amplitude in the lowest order one gets the result

$$F^{LO}(q^2,Q^2) = \frac{4\pi}{3}\int_0^1 \frac{\varphi_\pi(x)}{xq^2+\bar{x}Q^2}\,dx, \tag{2.18}$$

which was already discussed in the Introduction. For definiteness, from now on we will adhere to the convention that $q^2$ is the smaller of two virtualities: $q^2 \le Q^2$. The leading-order pQCD formula has a smooth limit when $q^2 \to 0$, predicting that the asymptotic behaviour of the $\gamma\gamma^* \to \pi^0$ form factor is [1]:

$$F^{LO}_{\gamma\gamma^*\pi^0}(Q^2) = \frac{4\pi}{3}\int_0^1 \frac{\varphi_\pi(x)}{xQ^2}dx + O(1/Q^4). \tag{2.19}$$

The $x^n$-moments of the pion distribution amplitude $\varphi_\pi(x)$ are given by the matrix elements of the twist-2 composite operators $\bar{q}\gamma_5\{\gamma^\nu D^{\nu_1}\dots D^{\nu_n}\}q$. Furthermore, the contributions to $F^{pQCD}_{\gamma\gamma^*\pi^0}(Q^2)$

from the higher twist two-body operators $\bar{q}\gamma_5\{D^{\nu_1}\dots D^{\nu_n}\}q$ and $\bar{q}\gamma_5\sigma^{\mu\nu}D^{\nu_1}\dots D^{\nu_n}q$, which are potentially dangerous due to large magnitude $O(m_\pi^2/(m_u+m_d))$ of their matrix elements [6, 33] are proportional to the light quark masses, and their net input is suppressed by $O(m_\pi^2/Q^2)$ factor for large $Q^2$. Hence, one should not expect unusually large $(1/Q^2)^N$ corrections to the leading twist term (2.19).

In the first order in $\alpha_s(Q^2)$, there is another factorizable contribution corresponding to a subprocess in which the real photon first dissociates into a quark-antiquark pair (the relevant amplitude can be called the photon distribution amplitude $\varphi_\gamma(y)$). The $\bar{q}q$-pair then interacts with the virtual photon to produce the final state quark-antiquark pair converting eventually into the pion. This contribution is analogous to the pQCD hard scattering term for the pion form factor. However, in our case, such a subprocess is realized only at the next-to-leading twist level:

$$F^{NLO}_{\gamma\gamma^*\pi^0}(Q^2) \sim \alpha_s(Q^2)\int_0^1 \frac{\varphi_\gamma(y)\varphi_\pi^P(x)}{y^2xQ^4}dx + O(1/Q^6), \tag{2.20}$$

*i.e.*, this term is suppressed both by $O(\alpha_s/\pi)$ and $O(1/Q^2)$ factors compared to the leading-order term (2.19). Perturbative QCD corrections to eq.(2.19) are known [10]-[12] and they are under control. Hence, the process $\gamma\gamma^* \to \pi^0$ provides the cleanest test of pQCD for exclusive processes.

## 2.5 Wave-function dependence and Brodsky-Lepage interpolation formula

To fix the absolute magnitude of the leading-order pQCD prediction (2.19), one should know the value of the integral

$$I = \frac{1}{f_\pi}\int_0^1 \frac{\varphi_\pi(x)}{x}dx\ , \tag{2.21}$$

which depends on the shape of the pion distribution amplitude $\varphi_\pi(x)$. In particular, using the asymptotic form[4]-[6]

$$\varphi_\pi^{as}(x) = 6f_\pi x(1-x)\,, \tag{2.22}$$

one obtains $I^{as} = 3$ resulting in the absolute prediction for the asymptotic behaviour of the $F_{\gamma\gamma^*\pi^0}(Q^2)$ form factor [1]

$$F^{as}_{\gamma\gamma^*\pi^0}(Q^2) = \frac{4\pi f_\pi}{Q^2} + O(1/Q^4). \tag{2.23}$$

On the other hand, if one uses the CZ-amplitude, the value of the integral $I$ increases: $I^{CZ} = 5$, and experimental data can, in principle, discriminate between these two possibilities. Of course, the asymptotic $1/Q^2$-dependence cannot be a true behaviour of $F_{\gamma\gamma^*\pi^0}(Q^2)$ for all $Q^2$-values: it should be somehow modified in the low-$Q^2$ region to comply with the bound on $F_{\gamma\gamma^*\pi^0}(Q^2)$ in the $Q^2 = 0$ limit imposed by the anomaly relation (2.11). Brodsky and Lepage [34] proposed the interpolation formula

$$F^{BL(int)}_{\gamma\gamma^*\pi^0}(Q^2) = \frac{1}{\pi f_\pi\left(1+\frac{Q^2}{4\pi^2 f_\pi^2}\right)} \tag{2.24}$$

which reproduces both the $Q^2 = 0$ value (2.11) and the high-$Q^2$ asymptotics (2.23) dictated by the asymptotic form of the distribution amplitude (2.22). According to refs.[8, 9], this formula agrees with the experimental data[2]. On the other hand, the curve based on the formula

$$F^{CZ(int)}_{\gamma\gamma^*\pi^0}(Q^2) = \frac{1}{\pi f_\pi \left(1 + \frac{3Q^2}{20\pi^2 f_\pi^2}\right)} \tag{2.25}$$

interpolating between the $Q^2 = 0$ value and the CZ-normalized high-$Q^2$ prediction, is far from existing data points.

## 2.6 Pion wave function and QCD sum rules

From the theoretical side, there are also doubts [19, 36] that QCD sum rules really require that the pion distribution function has the CZ shape. In particular, the QCD sum rule calculation of $\varphi_\pi(x)$ at the middle point $x = 1/2$ performed in ref. [36] produced the value $\varphi_\pi(1/2) \approx 1.2 f_\pi$, to be compared with $\varphi_\pi^{as}(1/2) = 1.5 f_\pi$ and $\varphi_\pi^{CZ}(1/2) = 0$. In ref. [19], it was pointed out that keeping in the OPE the lowest condensates only does not provide information necessary for a reliable determination of the pion distribution amplitude. This is especially clear if one writes the CZ sum rule directly for $\varphi_\pi(x)$

$$\begin{aligned} f_\pi\varphi_\pi(x) &= \frac{3M^2}{2\pi^2}(1 - e^{-s_0/M^2})x(1-x) + \frac{\alpha_s\langle GG\rangle}{24\pi M^2}[\delta(x) + \delta(1-x)] \\ &+ \frac{8}{81}\frac{\pi\alpha_s\langle\bar{q}q\rangle^2}{M^4}\{11[\delta(x) + \delta(1-x)] + 2[\delta'(x) + \delta'(1-x)]\}. \end{aligned} \tag{2.26}$$

As emphasized in [19], it is the $\delta$-function terms here which are crucial in generating a humpy form for $\varphi_\pi^{CZ}(x)$. Adding higher condensates, *e.g.*, $\langle\bar{q}D^2q\rangle$, one would get even higher derivatives of $\delta(x)$ and $\delta(1-x)$. All the subseries of such singular terms can be treated as an expansion of some finite-width functions related to nonlocal condensates. The sum rule based on a model for the nonlocal condensates consistent with the earlier estimates for $\langle\bar{q}D^2q\rangle$ [37] reduces the values of the lowest nontrivial moments of $\varphi_\pi(x)$ bringing them very close to those for the asymptotic form (2.22). As a result, the model distribution amplitude constructed in [19] gives $I \sim 3$. One can also try to construct a QCD sum rule directly for the integral $I$. However, it is clear that such a sum rule cannot be derived by a simple substitution of the original CZ sum rule into the integral (2.21), because of singularities generated by the $\delta(x)/x$ and $\delta'(x)/x$ terms. The singularities disappear if one uses the nonlocal condensates in the sum rule for $\varphi_\pi(x)$ [38], and the resulting value for $I$ is close to 3.

A more radical way is to consider the sum rule for the original amplitude as a whole, without approximating it by the first term of the pQCD expansion. As we will see, the singularities mentioned above will appear then as power $(1/q^2)^n$ infrared singularities in the relevant operator product expansion (OPE). However, these singularities are caused by a formal extension of the

[2] It also agrees with the curve obtained from a constituent quark model calculation by Ito, Buck and Gross [35].

OPE formulas from the large $q^2$ region where they are valid, into the small-$q^2$ region, where the OPE breaks down. In fact, the OPE should be modified in the small-$q^2$ region. Such a modification is equivalent to the regularization of the infrared singularities. Our goal in the present paper is to construct a QCD sum rule for the transition form factor valid in the region of small $q^2$.

## 2.7 Unfactorizable contributions and QCD sum rules

If the photon virtualities are not very large, then it is normally impossible to factorize the $p^2$-dependence of the diagrams contributing to the three-point amplitude $F(p^2, q_1^2, q_2^2)$ into separate factors one of which will produce the pion pole. We know, of course, that the full amplitude $F(p^2, q_1^2, q_2^2)$ must have the pion pole, but it is unclear how much a particular finite-order diagram contributes to such a pole.

To display the pole structure of $F(p^2, q_1^2, q_2^2)$, it is convenient to use the dispersion relation for the three-point amplitude:

$$F\left(p^2, q_1^2, q_2^2\right) = \frac{1}{\pi}\int_0^\infty \frac{\rho\left(s, q_1^2, q_2^2\right)}{s - p^2}\, ds + \text{“}subtractions\text{”}. \tag{2.27}$$

The pion contribution to the spectral density is proportional to the $F_{\gamma^*\gamma^*\to\pi^0}$ form factor:

$$\rho\left(s, q_1^2, q_2^2\right) = \pi f_\pi \delta(s - m_\pi^2) F_{\gamma^*\gamma^*\to\pi^0}\left(q_1^2, q_2^2\right) + \text{“}higher\ states\text{”}, \tag{2.28}$$

but any approximation for $F(p^2, q_1^2, q_2^2)$ obtained from a PT expansion, gives information not only about the pion, but also about the higher states. So, there are two problems. First, we should arrange a reliable PT expansion for $F(p^2, q_1^2, q_2^2)$. The only way is to take $p^2$ spacelike and large enough to produce a reasonbly converging OPE. The second problem is that when $p^2$ is large, the pion contribition does not overwhelmingly dominate the dispersion integral and, to disentangle it, one should subtract the contribution due to the higher states. Though the contribution of a state located at $s = m^2$ is suppressed by the $-p^2/(m^2 - p^2)$-factor compared to that of a state at $s = 0$, the relative suppression of higher states disappears as $-p^2$ tends to infinity.

The higher states include $A_1$ and higher pseudovector resonances which presumably become broader and broader, with their sum rapidly approaching the pQCD spectral density. So, the simplest model is to approximate all the higher states, including the $A_1$, by the spectral density $\rho^{PT}(s, q_1^2, q_2^2)$ calculated in perturbation theory:

$$\rho^{mod}\left(s, q_1^2, q_2^2\right) = \pi f_\pi \delta(s) F_{\gamma^*\gamma^*\to\pi^0}\left(q_1^2, q_2^2\right) + \theta(s - s_o)\rho^{PT}(s, q_1^2, q_2^2) \tag{2.29}$$

where the parameter $s_o$ is the effective threshold for the higher states.

The smaller $p^2$, the bigger relative contribution of the lowest state. Hence, the strategy is to take the smallest possible $p^2$ within the region where the $1/p^2$ expansion is still legitimate. In fact, it is more convenient to use the faster decreasing exponential weight $\exp[-s/M^2]$ instead of $1/(s - p^2)$. This is achieved by applying to $F(q_1^2, q_2^2, p^2)$ the SVZ-Borel transformation [18]:

$$\hat{B}(p^2 \to M^2) F(q_1^2, q_2^2, p^2) \equiv \Phi(q_1^2, q_2^2, M^2) = \frac{1}{\pi M^2}\int_0^\infty e^{-s/M^2} \rho(s, q_1^2, q_2^2)\, ds. \tag{2.30}$$

Another merit of the SVZ-Borel transformation is that using it one gets a factorially improved OPE power series: $1/(-p^2)^N \to (1/M^2)^N/(N-1)!$.

In a sense, the QCD sum rules can be treated as a method of extracting information about the lowest state from the behaviour of $F(q_1^2, q_2^2, p^2)$ in the large-$p^2$ region. To construct a QCD sum rule, one should calculate the SVZ transform $\Phi(q_1^2, q_2^2, M^2)$ as a power expansion in $1/M^2$ for large $M^2$. To this end, one should calculate first the three-point function $T(p^2, q^2, Q^2)$ as a power expansion in $1/p^2$ for large $p^2$. However, a particular form of the $1/p^2$ expansion depends on the interrelation between the values of the photon virtualities $q^2$ and $Q^2$.

The simplest case is when both virtualities are sufficiently large and similar in magnitude: $Q^2 \sim q^2 \sim -p^2 > \mu^2$, where $\mu$ is a typical hadronic scale $\mu^2 \sim 1\,GeV^2$. This case will be referred to as the "large-$q^2$" kinematics. Then all power-behaved $(1/M^2)^n$ contributions correspond to the situation when all the currents $J_\mu(X)$, $J_\nu(0)$ and $j_\alpha^5(Y)$ are close to each other, *i.e.*, all the intervals $X^2, Y^2, (X-Y)^2$ are small.

A more complicated case is when $q^2$ is small $q^2 \ll \mu^2$, while $Q^2$ is still large: $Q^2 > \mu^2$ . In this case, to be referred to as "small-$q^2$" kinematics, one should also take into account the configuration when the electromagnetic current $J_\mu(X)$ related to the $q$-photon is far away from two other currents, *i.e.*, there is a possibility of long-distance propagation in the $q$-channel. In the limit $q^2 \to 0$, such a propagation is responsible for the mass singularities ($1/q^2$, $\ln(q^2)$, *etc.*) in the Feynman diagrams contributing to the three-point amplitude.

# 3 QCD sum rules for the $F_{\gamma^*\gamma^*\pi^0}(Q^2, q^2)$ form factor in large-$q^2$ kinematics

Let us consider first the simpler case, when both $Q^2$ and $q^2$ are large. In this situation, it is sometimes convenient to introduce another set of variables: average virtuality $\tilde{Q}^2 = (Q^2+q^2)/2$ and the asymmetry parameter $\omega = (Q^2-q^2)/(Q^2+q^2)$.

## 3.1 Lowest-order perturbative term

The starting point of the operator expansion is the perturbative triangle graph (fig.3$a$). Its contribution contains almost all types of the tensor structures present in eq. (2.7). Extracting the $F$ part and using the Feynman parameterization we get:

$$F^{PT}(q^2, Q^2, p^2) = \frac{2}{\pi}\int_0^1 \frac{x_1 x_2}{[q^2 x_1 x_3 + Q^2 x_2 x_3 - p^2 x_1 x_2]}\,\delta(1-\sum_{i=1}^3 x_i)\, dx_1 dx_2 dx_3 \ . \qquad (3.1)$$

In this representation, it is straightforward to apply the SVZ-Borel transformation to get

$$\Phi^{PT}(q^2, Q^2, M^2) = \frac{2}{\pi M^2}\int_0^1 \exp\left\{-\frac{q^2 x_1 x_3 + Q^2 x_2 x_3}{x_1 x_2 M^2}\right\}\,\delta(1-\sum_{i=1}^3 x_i)\, dx_1 dx_2 dx_3 \ . \qquad (3.2)$$

Comparing this representation with the definition of the SVZ transform (2.30), we can immediately write down the formula for the perturbative spectral density $\rho^{PT}(s,q^2,Q^2)$:

$$\rho^{PT}(s,q^2,Q^2) = 2\int_0^1 \delta\left(s - \frac{q^2x_1x_3 + Q^2x_2x_3}{x_1x_2}\right)\ \delta(1-\sum_{i=1}^{3}x_i)\ dx_1dx_2dx_3. \tag{3.3}$$

Scaling the integration variables: $x_1 + x_2 = y$, $x_2 = xy$, $x_1 = (1-x)y \equiv \bar{x}y$ and taking trivial integrals over $x_3$ and $y$, we get

$$\rho^{PT}(s,q^2,Q^2) = 2\int_0^1 \frac{x\bar{x}(xQ^2+\bar{x}q^2)^2}{[sx\bar{x}+xQ^2+\bar{x}q^2]^3}\,dx\,, \tag{3.4}$$

or, in terms of $\widetilde{Q}^2$ and $\omega$:

$$\rho^{PT}(s,q^2,Q^2) = 2\int_0^1 \frac{x\bar{x}\widetilde{Q}^4(1+\omega(x-\bar{x}))^2}{[sx\bar{x}+\widetilde{Q}^2(1+\omega(x-\bar{x}))]^3}\,dx. \tag{3.5}$$

It should be noted that the variable $x$ in the integrals above is the light-cone fraction of the total pion momentum $p$ carried by one of the quarks.

A very simple result for $\rho^{quark}(s,q^2,Q^2)$ appears when $q^2=0$:

$$\rho^{quark}(s,q^2=0,\,Q^2) = \frac{Q^2}{(s+Q^2)^2}. \tag{3.6}$$

As $Q^2$ tends to zero, the spectral density $\rho^{quark}(s,q^2=0,Q^2)$ becomes narrower and higher converting into $\delta(s)$ in the $Q^2\to 0$ limit [29]:

$$\rho^{quark}(s,q^2=0,\,Q^2=0) = \delta(s). \tag{3.7}$$

Thus, the perturbative triangle diagram "tells us" that, for $Q^2=0$, two photons can produce only a single massless pseudoscalar state, and there are no other states in the spectrum of the final hadrons. As $Q^2$ increases, the spectral function broadens, *i.e.*, higher states can also be produced.

## 3.2 Gluon condensate corrections

When the Borel parameter $M^2$ (or the probing virtuality $-p^2$) decreases, both perturbative (logarithmic or $\mathcal{O}(\alpha_s)$ ) and nonperturbative (power or $\mathcal{O}(1/p^2)$) corrections come into play. As argued by SVZ [18], the power corrections proportional to quark and gluon condensates: $\langle 0|\bar{\psi}\psi|0\rangle^2$, $\langle 0|G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle$, *etc.*, are much more important than the higher order perturbative corrections. In many cases, the latter can safely be neglected.

The lowest-order diagrams proportional to the gluon condensate are shown in fig.3*b*–*g*. They take into account the fact that propagating through the QCD vacuum, quarks interact with the nonperturbative gluon fluctuations which can be treated as a background field. The most

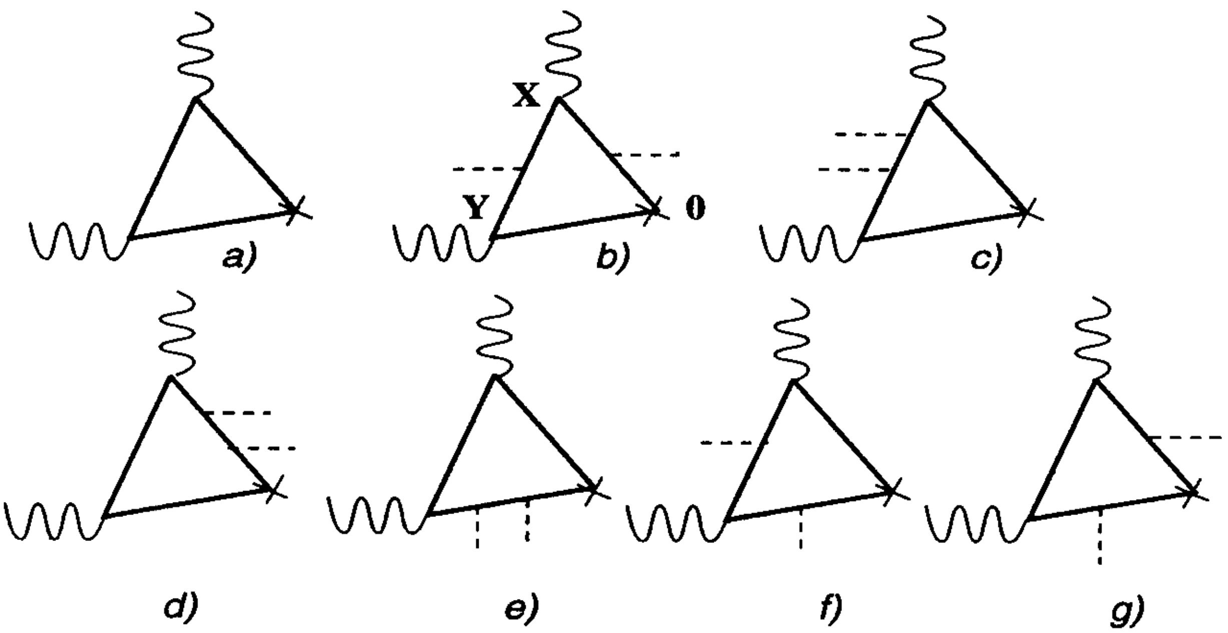


Figure 3: $a$) Lowest-order perturbative contribution. $b) - g$) Gluon condensate corrections.

straightforward way to calculate these diagrams is to take quark propagators in the coordinate representation using the Fock-Schwinger gauge $z_\mu A_\mu(z) = 0$ for the background field $A_\mu(z)$:

$$\hat{S}(X,Y) = \frac{\hat{\Delta}}{2\pi^2\Delta^4} - \frac{1}{8\pi^2}\frac{\Delta_\alpha}{\Delta^2}\tilde{G}_{\alpha\beta}(0)\gamma_\beta\gamma_5 + \left\{\frac{i}{4\pi^2}\frac{\hat{\Delta}}{\Delta^4}Y_\rho X_\mu G_{\rho\mu}(0) - \frac{1}{192\pi^2}\frac{\hat{\Delta}}{\Delta^4}(X^2Y^2 - (XY)^2)G_{\beta\chi}(0)G_{\beta\chi}(0)\right\}. \tag{3.8}$$

Here $\Delta = X - Y$ and $\tilde{G}_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\rho\sigma}G_{\rho\sigma}$. Using this expression for each quark propagator of the original triangle diagram and retaining the $O(GG)$ terms, one obtains all the diagrams of fig.3$b$–$g$. In particular, one can immediately see that contributions 3$d$ and 3$e$ vanish due to our choice of the coordinate origin at the axial current vertex (such a choice, in which the two photons are treated symmetrically, is more convenient in this calculation than that implied by our original definition (2.3)). The remaining diagrams 3$b, c, f, g$ are easily calculated in the coordinate representation. After performing the Fourier transformation to the momentum space, we extract the tensor structure corresponding to the $F$ form factor and then apply the SVZ-Borel transformation to the relevant contribution $F^{(GG)}(q^2, Q^2, p^2)$. The final result for the sum of the $O(GG)$ diagrams reads:

$$\Phi^{(GG)}(q^2, Q^2, M^2) = \frac{\pi^2}{9}\langle\frac{\alpha_s}{\pi}GG\rangle\left(\frac{1}{2M^4Q^2} + \frac{1}{2M^4q^2} - \frac{1}{M^2Q^2q^2}\right) = \frac{\pi^2}{9}\langle\frac{\alpha_s}{\pi}GG\rangle\left(\frac{1}{\tilde{Q}^2M^4} - \frac{1}{\tilde{Q}^4M^2}\right)\frac{1}{1-\omega^2}. \tag{3.9}$$

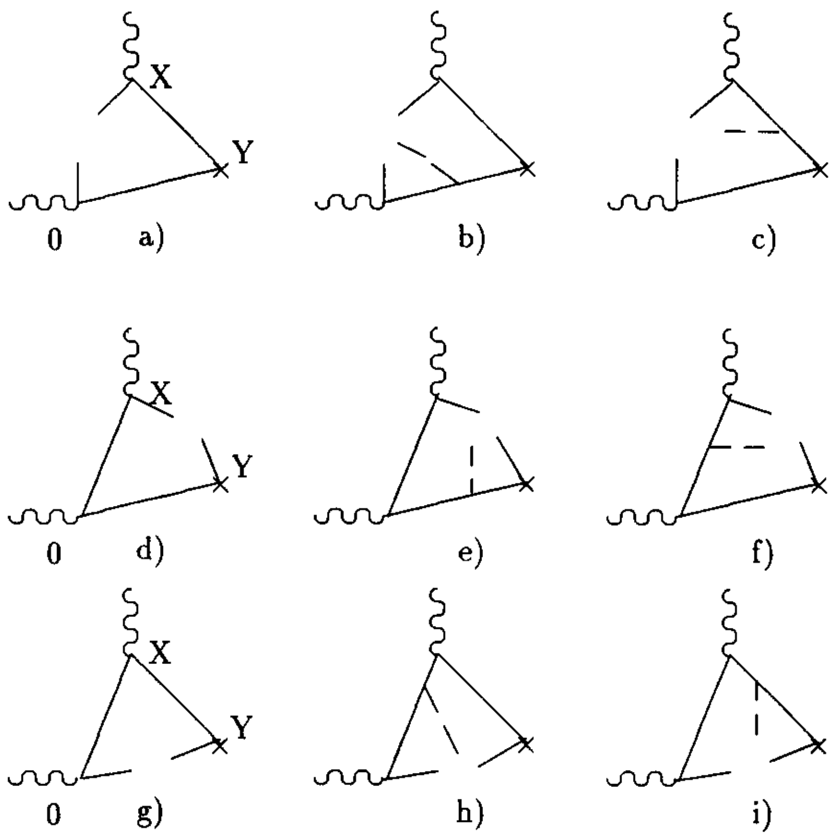


Figure 4: Quark condensate corrections involving "soft" gluons.

### 3.3 Quark condensate corrections

For massless $u$- and $d$-quarks, the quark condensate contribution starts with terms proportional to $\langle 0|\bar{q}\Gamma q\bar{q}\Gamma q|0\rangle$. Using the usual vacuum dominance hypothesis [18], these can be reduced to $\langle 0|\bar{q}q|0\rangle^2$. There are two types of diagrams producing the $\langle 0|\bar{q}q|0\rangle^2$ contibutions.

First, there are diagrams shown in fig.4$a$–$i$ (sometimes called the soft gluon diagrams) corresponding to local operators of $\bar{q}DDDq$, $\bar{q}GDq$ and $\bar{q}DGq$ type, with the covariant derivatives $DDD$, and the $GD$, $DG$ factors producing an extra $\bar{q}q$ term by equations of motion.

The diagrams shown in fig.5$a$–$r$ (and corresponding to a hard gluon exchange) give the $\langle 0|\bar{q}q|0\rangle^2$ structure directly . However, only the diagrams 5$a$–$d$ contribute to the form factor $F$. The total $O(\langle\bar{q}q\rangle^2)$ contribution to the borelized amplitude is

$$\Phi^{<\bar{q}q>^2} = \frac{64\pi^2\alpha_s\langle\bar{q}q\rangle^2}{243M^2}\left(\frac{1}{M^4}\left[\frac{Q^2}{q^4}+\frac{9}{2q^2}+\frac{9}{2Q^2}+\frac{q^2}{Q^4}\right]+\frac{9}{Q^2q^4}+\frac{9}{Q^4q^2}\right)$$
$$= \frac{64\pi^2\alpha_s\langle qq\rangle^2}{243M^2}\left(\frac{11-3\omega^2}{\tilde{Q}^2M^4}+\frac{18}{\tilde{Q}^6}\right)\frac{1}{(1-\omega^2)^2}. \qquad (3.10)$$

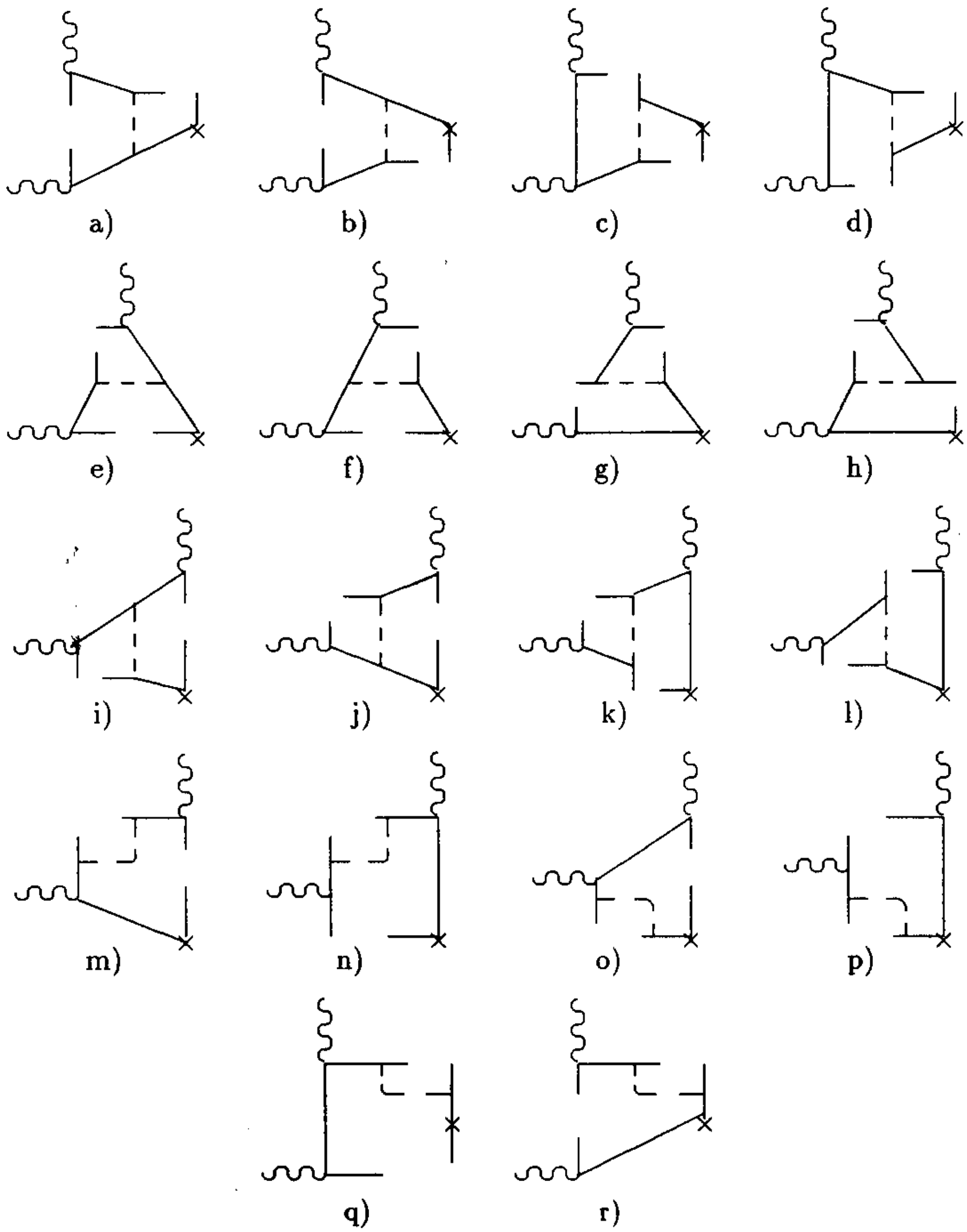


Figure 5: Quark condensate corrections involving "hard" gluons.

## 3.4 QCD sum rule

Combining now eqs.(2.29), (2.30), (3.4), (3.9) and (3.10), we obtain the QCD sum rule for the $F$ form factor valid in the region where both virtualities of the photons are large:

$$\pi f_\pi F_{\gamma^*\gamma^*\pi^0}(q^2,Q^2) = 2\int_0^{s_0} ds\, e^{-s/M^2} \int_0^1 \frac{x\bar{x}(xQ^2+\bar{x}q^2)^2}{[sx\bar{x}+xQ^2+\bar{x}q^2]^3}\, dx$$
$$+\frac{\pi^2}{9}\langle\frac{\alpha_s}{\pi}GG\rangle\left(\frac{1}{2M^2Q^2}+\frac{1}{2M^2q^2}-\frac{1}{Q^2q^2}\right)$$
$$+\frac{64}{243}\pi^3\alpha_s\langle\bar{q}q\rangle^2\left(\frac{1}{M^4}\left[\frac{Q^2}{q^4}+\frac{9}{2q^2}+\frac{9}{2Q^2}+\frac{q^2}{Q^4}\right]+\frac{9}{Q^2q^4}+\frac{9}{Q^4q^2}\right), \tag{3.11}$$

or, in terms of the variables $\widetilde{Q}^2$ and $\omega$:

$$\begin{aligned} F_{\gamma^*\gamma^*\to\pi^0}(q^2,Q^2) &= \frac{1}{\pi f_\pi}\left\{2\int_0^{s_0} ds\, e^{-s/M^2}\int_0^1 dx\, \frac{x\bar{x}\widetilde{Q}^4(1+\omega(x-\bar{x}))^2}{[sx\bar{x}+\widetilde{Q}^2(1+\omega(x-\bar{x}))]^3}\right. \\ &+ \frac{\pi^2}{9}\langle\frac{\alpha_s}{\pi}GG\rangle\left(\frac{1}{\widetilde{Q}^2M^2}-\frac{1}{\widetilde{Q}^4}\right)\frac{1}{1-\omega^2} \\ &+ \left.\frac{64}{243}\pi^3\alpha_s\langle\bar{q}q\rangle^2\left(\frac{11-3\omega^2}{\widetilde{Q}^2M^4}+\frac{18}{\widetilde{Q}^6}\right)\frac{1}{(1-\omega^2)^2}\right\} \end{aligned} \tag{3.12}$$

In particular, taking the exactly symmetric kinematics, when $\omega = 0$ and $q^2 = Q^2$, we obtain the sum rule

$$\begin{aligned} F_{\gamma^*\gamma^*\to\pi^0}(Q^2,Q^2) &= \frac{1}{\pi f_\pi}\left\{2\int_0^{s_0} ds\, e^{-s/M^2}\int_0^1 dx\, \frac{x\bar{x}\, Q^4}{[sx\bar{x}+Q^2]^3}+\right. \\ &+ \frac{\pi^2}{9}\langle\frac{\alpha_s}{\pi}GG\rangle\left(\frac{1}{Q^2M^2}-\frac{1}{Q^4}\right) \\ &+ \left.\frac{64}{243}\pi^3\alpha_s\langle\bar{q}q\rangle^2\left(\frac{11}{Q^2M^4}+\frac{18}{Q^6}\right)\right\} \end{aligned} \tag{3.13}$$

derived in ref.[27].

## 3.5 $\widetilde{Q}^2 \to \infty$ limit and transition to perturbative QCD

In the limit $\widetilde{Q}^2 \to \infty$ with $\omega$ fixed, eq.(3.12) reproduces the sum rule

$$\begin{aligned} &F_{\gamma^*\gamma^*\to\pi^0}((1-\omega)\widetilde{Q}^2,(1+\omega)\widetilde{Q}^2) \\ &= \frac{1}{f_\pi\widetilde{Q}^2}\left\{\frac{2M^2}{\pi}(1-e^{-s_0/M^2})\int_0^1\frac{x\bar{x}\,dx}{(1+\omega(x-\bar{x}))}\right. \\ &+ \left.\frac{\pi}{9M^2(1-\omega^2)}\langle\frac{\alpha_s}{\pi}GG\rangle+\frac{64\pi^2}{243M^4}\frac{(11-3\omega^2)}{(1-\omega^2)^2}\alpha_s\langle\bar{q}q\rangle^2\right\} \end{aligned} \tag{3.14}$$

obtained by Gorsky [39], who calculated the leading $1/\widetilde{Q}^2$ contribution only.

To make a connection with the perturbative QCD approach, it is instructive to rewrite eq.(3.14) as

$$
\begin{aligned}
&F_{\gamma^*\gamma^*\to\pi^0}((1-\omega)\tilde{Q}^2,(1+\omega)\tilde{Q}^2) \\
&= \frac{4\pi}{3f_\pi\tilde{Q}^2}\int_0^1 \frac{dx}{(1+\omega(x-\bar{x}))}\left\{\frac{3M^2}{2\pi^2}(1-e^{-s_0/M^2})x\bar{x}\right. \\
&\qquad + \frac{1}{24M^2}\langle\frac{\alpha_s}{\pi}GG\rangle[\delta(x)+\delta(\bar{x})] \\
&\left. + \frac{8}{81M^4}\pi\alpha_s\langle\bar{q}q\rangle^2\Big(11[\delta(x)+\delta(\bar{x})]+2[\delta'(x)+\delta'(\bar{x})]\Big)\right\}.
\end{aligned}
\tag{3.15}
$$

Note, that the expression in large curly brackets coincides with the QCD sum rule (2.26) for $f_\pi\varphi_\pi(x)$. Hence, in the large-$\tilde{Q}^2$ limit, the QCD sum rule (3.16) exactly reproduces the perturbative QCD result

$$
F_{\gamma^*\gamma^*\to\pi^0}\left((1-\omega)\tilde{Q}^2,(1+\omega)\tilde{Q}^2\right) = \frac{4\pi}{3\tilde{Q}^2}\int_0^1 \frac{\varphi_\pi(x)\,dx}{(1+\omega(x-\bar{x}))}. \tag{3.16}
$$

# 4 Operator product expansion in small-$q^2$ kinematics

In the nonsymmetric kinematics $q^2 \ll Q^2 \sim 1$ GeV$^2$, it is more convenient to use the "old" variables $q^2 = -q_1^2$ and $Q^2 = -q_2^2$ instead of $\omega$ and $\tilde{Q}^2$.

## 4.1 General features of the small-$q^2$ kinematics

Incorporating a general analysis of the asymptotic behaviour of Feynman diagrams (see Appendix A and [31, 32]), one can show that, in the small-$q^2$ kinematics, there are two different regions of integration capable of producing a contribution to the correlator (see fig.6), which behaves like an inverse power of $p^2$. The first one is the purely short-distance region corresponding to a situation when all three currents are separated by short intervals, *i.e.*, all the intervals $X^2, Y^2, (X-Y)^2$ are small. The second region corresponds to another short distance regime, in which the vertex

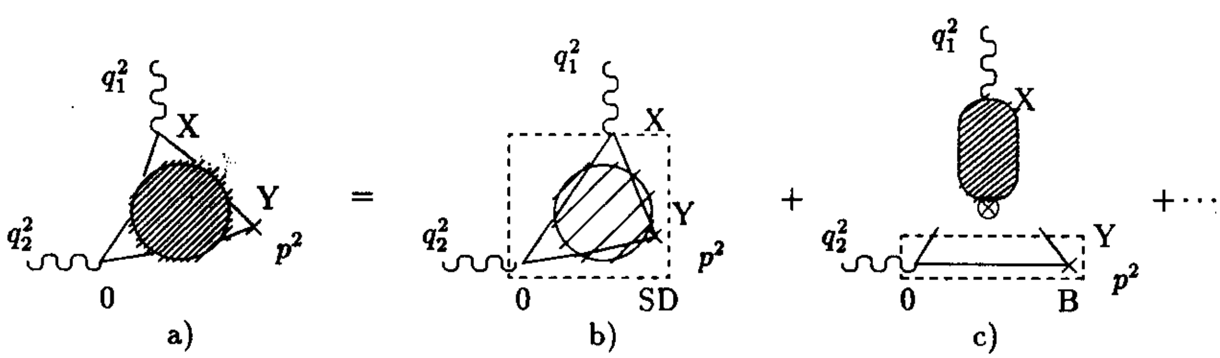


Figure 6: Configurations responsible for the power-behaved contributions.

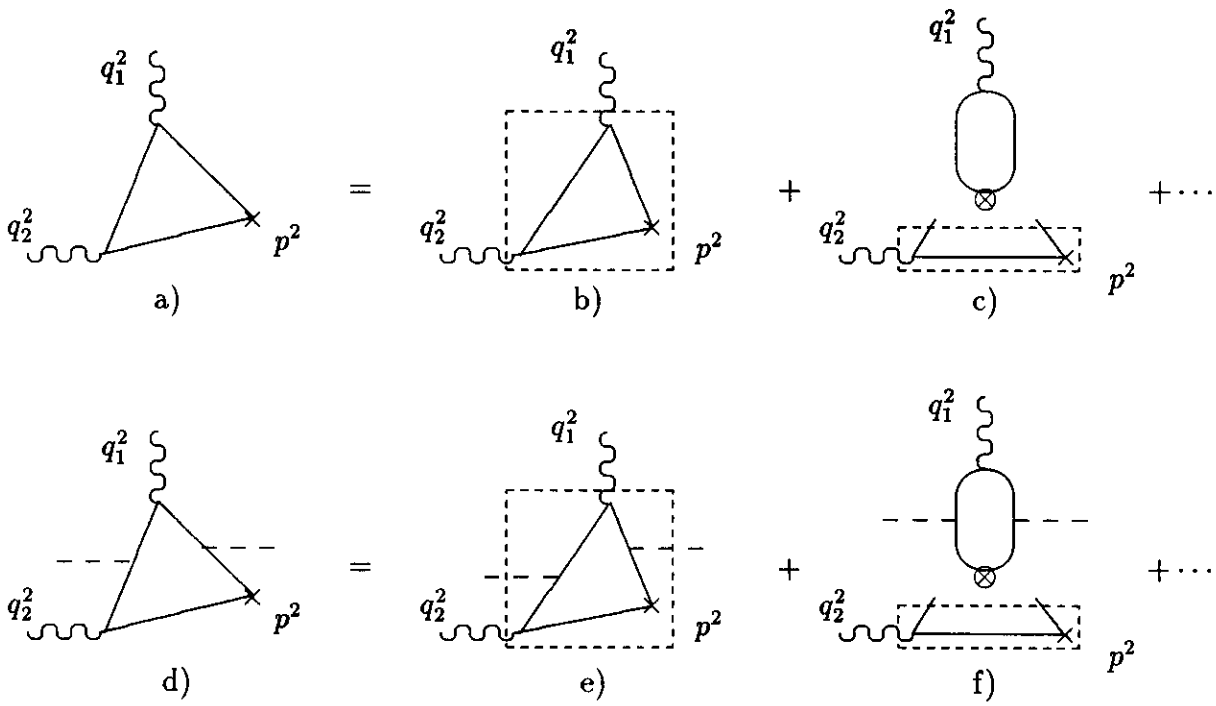


Figure 7: Separation of $SD$- and $B$-contributions for perturbative diagrams and gluon condensate corrections.

$X$ related to the small virtuality photon is separated by a long interval from two other currents (this means that $Y^2$ is small, but $X^2$ and $(X-Y)^2$ are large). This is illustrated by fig.6, where 6$a$ shows the full correlator $\mathcal{F}_{\alpha\mu\nu}(q_1, q_2)$, whereas figs.6$b, c$ represent the two possibilities of getting the power-law contributions.

In fig.6c, the long distance contribution (the dashed blob) is given by a two-point (bilocal) correlator $\Pi(q)$ of the electromagnetic current $J_\mu$ and some composite operator $\mathcal{O}$ of quark and gluon fields, the latter represented by $\otimes$. At low $q^2$, the correlator $\Pi(q)$ cannot be calculated in perturbation theory. The standard strategy is to model this nonperturbative object by the "first resonance plus continuum" ansatz, with the parameters of the spectrum determined from a QCD sum rule for such a correlator. In other words, one should calculate $\Pi(q)$ at large $q^2$ using the OPE, then extract, in a standard way, the parameters of the model spectrum in the $q$-channel and, finally, use the model spectral density in a dispersion relation for $\Pi(q)$ to obtain $\Pi(q)$ at small $q^2$.

On the diagrammatic level, the factorization procedure should be applied for each contributing diagram: the lowest-order triangle graph (Fig. 7$a, b, c$), gluon-condensate graphs (Fig. 7$d, e, f$), *etc.* In this way, each diagram is represented as the sum of its purely short-distance ($SD$) and bilocal ($B$) parts, the latter obtained through the relevant contribution into the operator product expansion for $J_\nu(Y)\, j^5_\alpha(0)$. In fact, it is more convenient to define the $SD$-part of a diagram as the difference between the original unfactorized expression and its $B$-part determined *via* the OPE.

The total SD-contribution is given by a sum of the SD-parts of all relevant diagrams:

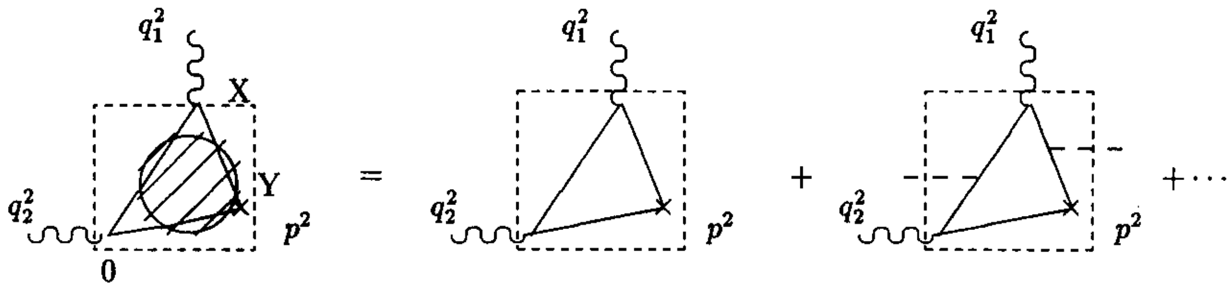


Figure 8: $SD$-contribution.

Now, substituting the $SD$-terms by the "original minus $B$-terms", we obtain the expansion shown on Fig.9. The dots there stand for the rest of the condensate diagrams and higher-order corrections.

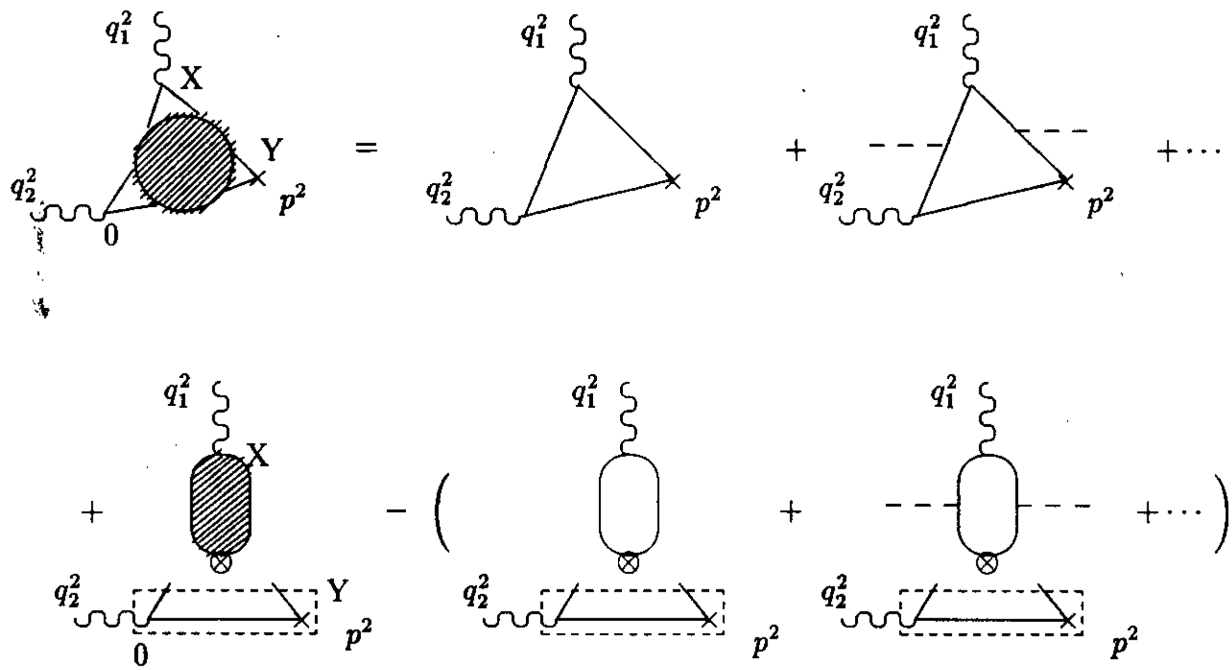


Figure 9: Structure of factorization in the small-$q^2$ kinematics.

The first row in **Fig.9** corresponds to the usual OPE for the correlator (2.3) in the large-$q^2$ kinematics ($q^2 \sim Q^2 \sim -p^2 \sim \mu^2$). The second row corresponds to additional terms which should be taken into account in the case of small-$q^2$ kinematics ($q^2 \ll Q^2 \sim -p^2 \sim \mu^2$). Note, that at large values of $q^2$, these additional terms should die out to convert the modified OPE specific to the small-$q^2$ kinematics into the standard OPE for the large-$q^2$ kinematics (cf. [24, 26]). Another statement is that, at low $q^2$, the "subtraction" terms (those in the parentheses in the second row of Fig.9) have exactly the same singularities as the corresponding terms in the first row of Fig.9 and, therefore, the total expression is regular in this kinematic limit. In Section 5, it is shown that, in this problem, the singular terms (so-called mass or infrared singularities) are generated only by the operators of two lowest twists.

## 4.2 Logarithmic singularities

According to eqs.(3.11), the condensate terms contain $1/q^2$, $1/q^4$, *etc.* singularities in the $q^2 \to 0$ limit. On the other hand, the perturbative expression, though finite as $q^2 \to 0$, contains contributions which are non-analytic at this point. To study the structure of the non-analytic terms, it is convenient to use the following method. First, let us introduce another set of integration variables in the expression for the Borel transform $\Phi^{PT}(q^2, Q^2, M^2)$ (3.2):

$$x_1 + x_3 = \lambda, \quad x_3 = y\lambda, \quad x_1 = (1-y)\lambda \equiv \bar{y}\lambda, \quad x_2 = 1 - \lambda.$$

This gives

$$\Phi^{PT}(q^2, Q^2, M^2) = \frac{2}{\pi M^2} \int_0^1 \lambda d\lambda \, dy e^{-q^2 y\lambda/M^2 \bar{\lambda}} e^{-yQ^2/\bar{y}M^2}. \tag{4.1}$$

Performing a formal $q^2$-expansion of the exponential in the integrand, we will get divergent integrals for all the coefficients of the $(q^2)^n$ expansion, except for the lowest term. To get a more sensible result, we use a continuous version of the series expansion for the exponential, *i.e.*, the Mellin representation:

$$e^{-A} = \frac{1}{2\pi i} \int_{-\delta - i\infty}^{-\delta + i\infty} A^J \, \Gamma(-J) \, dJ \; . \tag{4.2}$$

Now, the integral over $\lambda$ can be taken easily, and the next step is to calculate the $J$-integral by taking residues at $J = 0, 1, \ldots$. This gives:

$$\begin{aligned} \Phi^{PT}(q^2, Q^2, M^2) &= \frac{1}{\pi M^2} \int_0^1 dy e^{-Q^2 y/M^2 \bar{y}} \\ &\times \left\{ 1 + \frac{q^2 y}{M^2} e^{q^2 y/M^2} + \left[ 2\frac{q^2 y}{M^2} + \frac{q^4 y^2}{M^4} \right] e^{q^2 y/M^2} \ln\left(\frac{q^2 y}{M^2}\right) \right. \\ &\left. - \sum_{n=1}^{\infty} \left(\frac{q^2 y}{M^2}\right)^n \frac{\psi(n)(n+1)}{(n-1)!} \right\} \end{aligned} \tag{4.3}$$

where $\psi(z) \equiv \Gamma'(z)/\Gamma(z)$ and $\Gamma(z)$ is the Euler Gamma function.

The non-analytic terms $q^2 \ln q^2$ and $q^4 \ln q^2$ are typical examples of mass singularities [40]. They appear when a long-distance propagation of particles is possible. In our case, the mass singularities are related to the possibility to create, for massless quarks, a $q\bar{q}$-pair by a zero-virtuality photon. To apply perturbative QCD methods, one should first factorize the contributions due to short and long distance dynamics, as outlined above (see Figs.5,6).

## 4.3 Scalar model

To clarify the origin of the singularities and outline the method of their factorization, let us consider first the analogue of our form factor in a simple scalar theory $g\phi^3_{(4)}$:

$$\mathcal{F}(q_1, q_2) = \int d^4X \, d^4Y \; e^{-iq_1 X} \, e^{ipY} \langle 0 | T\left\{ j(X) j(Y) j(0) \right\} | 0 \rangle, \tag{4.4}$$

where $j(X) =: \phi(X)\phi(X):$ . The perturbative contribution and some of the power corrections are shown in Fig.10.

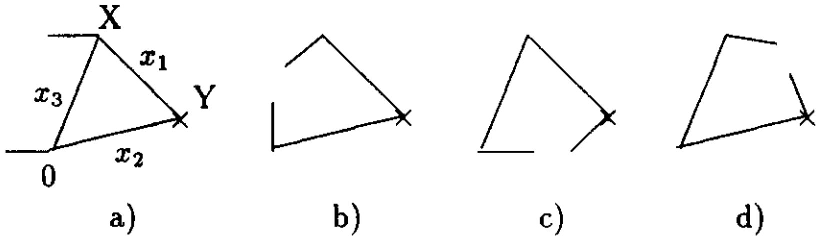


Figure 10: Lowest terms of the OPE in a scalar toy model.

Calculating the perturbative contribution (fig.10$a$) in the $\alpha$-representation or using the Feynman parameters, we get

$$\begin{aligned} \Phi_{(a)} &\equiv \hat{B}(-p^2 \to M^2)\mathcal{F}_{(a)} = \int_0^1 dx_1 dx_3 \theta(x_1+x_3<1)\frac{1}{x_2x_3}e^{-\frac{q^2x_1x_3+Q^2x_2x_3}{x_1x_2M^2}} \\ &= \int_0^1 \frac{d\lambda dy}{\bar{\lambda}\bar{y}M^2}e^{-yq^2\lambda/M^2\bar{\lambda}}e^{-yQ^2/M^2\bar{y}} \ . \end{aligned} \tag{4.5}$$

Using the Mellin representation, as described above, we obtain the expression

$$\Phi_{(a)} = \frac{1}{2\pi^2}\int_0^1 dy e^{-yQ^2/\bar{y}M^2}\frac{1}{\bar{y}M^2}\left\{-e^{yq^2/M^2}\ln\left(\frac{yq^2}{M^2}\right)+\sum_{n=0}^{\infty}\left(\frac{yq^2}{M^2}\right)^n\frac{\psi(n+1)}{n!}\right\}, \tag{4.6}$$

in which the term containing the logarithmic singularity is explicitly displayed.

The diagrams 10$b,c,d$ corresponding to power corrections give

$$\Phi_{(b)} = \hat{B}(-p^2 \to M^2)\left(-\frac{8\langle\phi^2\rangle}{q^2Q^2}\right) = 0, \quad \Phi_{(c)} = -\frac{8\langle\phi^2\rangle}{Q^2M^2}, \quad \Phi_{(d)} = -\frac{8\langle\phi^2\rangle}{q^2M^2} \ . \tag{4.7}$$

Since the diagrams 10$a,d$ contain terms singular in $q^2$, it is necessary to perform an additional factorization of short- and long-distance contributions for these diagrams.

To get the bilocal contribution for the amplitude (4.4), let us extract terms related to the simplest coefficient function proportional to the propagator $S(Y) = i/4\pi^2(Y^2 - i0)$ (see Fig.6$c$):

$$\begin{aligned} \mathcal{F}^B &= \int d^4Y\, e^{ipY}\ \frac{1}{\pi^2Y^2}\ \sum_{n=0}^{\infty}\frac{1}{n!}\ Y^{\mu_1}\dots Y^{\mu_n} \\ &\times\int d^4X\, e^{-iq_1X}\ \langle 0|T\left\{j(X)\phi(0)(\partial_{\mu_1}\dots\partial_{\mu_n})\phi(0)\right\}|0\rangle \ . \end{aligned} \tag{4.8}$$

Here, the long-distance contribution is described by the two-point correlator

$$\begin{aligned} \Pi_{\mu_1\dots\mu_n}(q_1) &\equiv \int d^4X\, e^{-iq_1X}\ \langle 0|T\left\{j(X)\phi(0)\{\partial_{\mu_1}\dots\partial_{\mu_n}\}\phi(0)\right\}|0\rangle \\ &= (-i)^n q_{1\mu_1}\dots q_{1\mu_n}\,\Pi_n(q_1^2) + \dots \ , \end{aligned} \tag{4.9}$$

where the dots denote the terms containing $g_{\mu_i\mu_j}$.

The large-$p^2$ behaviour of $\mathcal{F}^B$ is governed by the leading light-cone singularity of the integrand. To display the $Y^2$-structure of the integrand in eq.(4.9), we reexpand the composite operator over the traceless operators possessing a definite twist [32]:

$$Y^{\mu_1}\dots Y^{\mu_n}\,(\phi\partial_{\mu_1}\dots\partial_{\mu_n}\phi) = \\ = \sum_{l=0}^{[n/2]}\frac{n!(n-2l+1)}{l!(n-l+1)!}\left(\frac{Y^2}{4}\right)^l\{Y^{\mu_1}\dots Y^{\mu_{n-2l}}\}(\phi(\partial^2)^l\{\partial_{\mu_1}\dots\partial_{\mu_{n-2l}}\}\phi)\,. \tag{4.10}$$

In (4.10), we used the standard notation $\{\partial_{\mu_1}\dots\partial_{\mu_n}\}$ for a traceless group of indices $\mu_1\dots\mu_n$: $g^{\mu_i\mu_j}O_{\{\dots\mu_i\dots\mu_j\dots\}} = 0$. The leading $1/p^2$ contribution in (4.8) in the $p^2\to\infty$ limit comes from the lowest twist operators ($t = 2$ in this case). All other operators are accompanied by the factors $\sim Y^2$, $(Y^2)^2$, *etc.*, which cancel singularity of the propagator $\sim 1/Y^2$ and, hence, give no contribution in the large-$p^2$ limit.

To get the B-regime contribution for the perturbative triangle loop, one should substitute in eq. (4.8) the perturbative expression for the twist-2 part of the two-point correlator

$$\Pi_n(q_1^2)\to\Pi_n^{PT}(q^2) = \frac{1}{8\pi^2}\int_0^1 dy\, y^n\,\ln\frac{q^2y\bar{y}}{\mu^2} \tag{4.11}$$

where $\mu^2$ is the UV cut-off parameter for the composite operator $\phi\{\partial_{\mu_1}\dots\partial_{\mu_n}\}\phi$, and $q^2$ serves as a cut-off for the low-momentum region of integration. As expected, in the limit $q^2\to 0$, one obtains a mass singularity.

Substituting (4.11) into (4.8) and taking into account that $\{q_{1\mu_1}\dots q_{1\mu_n}\}$ differs from $q_{1\mu_1}\dots q_{1\mu_n}$ only in inessential terms $\sim Y^2$, it is straightforward to perform summation over $n$ to get

$$\mathcal{F}^B = \int_0^1 dy\,\ln\frac{q^2y\bar{y}}{\mu^2}\int\frac{d^4Y}{\pi^2Y^2}e^{i(pY)-iy(q_1Y)} = \int_0^1 dy\,\frac{1}{(p-yq_1)^2}\ln\frac{q^2y\bar{y}}{\mu^2}. \tag{4.12}$$

Since $(p-yq_1)^2 = p^2-2y(pq_1)+y^2q_1^2$ and $(p-q_1)^2 = -Q^2$, we can write $(p-yq_1)^2 = \bar{y}p^2+y\bar{q}^2-yQ^2$. Applying the SVZ-Borel transformation, we get

$$\Phi^{B(PT)} = -\frac{1}{2\pi^2}\int_0^1\frac{dy}{yM^2}e^{-yQ^2/\bar{y}M^2}\left\{e^{yq^2/M^2}\ln\left(\frac{y\bar{y}q^2}{\mu^2}\right)\right\}. \tag{4.13}$$

Comparing this result with the exact expression (4.6), one can observe that the non-analytic terms of two expressions coincide. Hence, their difference, *i.e.*, the coefficient function of the SD regime (see Fig.7*b*, Figs.8,9 ) does not have non-analytic terms (mass singularities) in the $q^2\to 0$ limit (cf. [40, 20]).

For the "$\langle\phi^2\rangle$-condensate" correction given by the diagram shown in fig.9*d*, the contribution corresponding to the lowest twist operators exactly reproduces the singular term (4.7) $\sim 1/q^2$. Thus, this singularity will not appear in the modified OPE suitable for the nonsymmetric kinematics $q^2 \ll Q^2$.

As we stressed above, the two-point correlator in (4.8) is responsible for the long-distance contribution $\sim 1/|q_1|$ and is not directly calculable in perturbation theory. However, we can write it through a dispersion relation

$$\Pi_n(q_1^2) = \frac{1}{\pi}\int_0^\infty \frac{\delta\Pi_n(s)}{s-q_1^2}ds + (\text{ subtractions }), \tag{4.14}$$

where $\delta\Pi_n(s) \equiv (\Pi_n(s+i0) - \Pi_n(s-i0))/2i$ is the relevant spectral density. As usual, we will model it by a "first resonance plus continuum" ansatz. A similar dispersion relation can be written for the perturbative correlator $\Pi_n^{pert}(q_1^2)$ (4.11), with the perturbative spectral density $\delta\Pi_n^{pert}(s)$ substituting the exact one. It is easy to realize that the ambiguity in the value of the $\mu^2$-parameter in eqs. (4.11),(4.13) corresponds to the ambiguity in the UV subtraction procedure for the correlators. However, the large-$s$ behavior of the exact and perturbative spectral densities is the same, and it is not necessary to specify the subtraction procedure because the correlators appear in the modified OPE only through the difference $\Pi_n(q_1^2) - \Pi_n^{pert}(q_1^2)$ (see fig.9), which is UV finite.

Writing explicitly the sum over the "hadronic " states in (4.14), we get

$$\begin{aligned} i^{n+1}\Pi_n(q_1,Y) &\equiv i^{n+1}\Pi_{\{\mu_1\ldots\mu_n\}}(q_1)\, Y^{\mu_1}\ldots Y^{\mu_n} = \\ &= \frac{f_\phi^2(Yq_1)^n\langle y^n\rangle}{m_\phi^2-q_1^2} + \frac{1}{\pi}\int_{s_\phi}^\infty ds\frac{i\delta\Pi_n^{PT}(s)(Yq_1)^n}{s-q_1^2} + (\text{ subtractions }). \end{aligned} \tag{4.15}$$

By analogy with the definition of the matrix element for the $\pi$-meson, we define $\langle 0|j(0)|\phi,p\rangle = if_\phi$. We can introduce the twist-2 distribution amplitude $\phi(y)$ by treating the matrix elements of composite operators as its moments:

$$\langle\phi,p|\phi(0)(Y\partial)^n\phi(0)|0\rangle = i^n(Yp)^n(-if_\phi)\int_0^1 y^n\varphi(y)dy. \tag{4.16}$$

We also used a convenient shorthand notation $\langle y^n\rangle \equiv \int_0^1 \phi(y)y^n dy$. The contribution of the higher excited states (continuum) in eq.(4.16) is approximated, as usual, by the perturbative spectral density $\delta\Pi_n^{PT}(s) = -(1/8\pi)\int_0^1 y^n dy$ starting from the continuum threshold $s_\phi$.

The parameters of the model spectral density in eq.(4.15), namely, the mass $m_\phi$, the residue of the first pole $f_\phi$, the moments of the distribution amplitude $\langle y^n\rangle$ and the threshold $s_\phi$ should be extracted from the auxiliary sum rule for the moments of the scalar meson distribution amplitude. In this case, the additional terms in the modified OPE will decrease with increasing $q^2$. As a result, the modified SR will reproduce the original SR for the three-point correlator valid in the region where both $q^2$ and $Q^2$ are large (see fig.9). Treating these parameters as known, we substitute (4.15) in (4.8) and define the "bilocal" contribution in the r.h.s. of the SR for the three-point correlator. As a result, all the additional terms in the OPE (see the second row of fig.9) can be written in the following form:

$$
\begin{aligned}
\Delta\Phi &\equiv \Phi^{bilocal} - \left(\Phi^B_{(a)} + \Phi^B_{(d)} + \cdots\right) \\
&= \int_0^1 \frac{1}{yM^2}\, e^{-Q^2\bar{y}/M^2y}\, e^{q^2\bar{y}/M^2} \\
&\times \left[\frac{4f_\phi^2\varphi(y)}{m_\phi^2+q^2} + \frac{1}{2\pi^2}\ln\left(\frac{q^2}{s_\phi+q^2}\right) + 8\frac{\langle\phi^2\rangle}{q^2}(\delta(y)+\delta(\bar{y}))\right].
\end{aligned}
\tag{4.17}
$$

Now we can write our final expression for the rhs of the modified sum rule:

$$
\begin{aligned}
&\Phi(q^2, Q^2, M^2) = \Phi_{(a)} + \Phi_{(b)} + \Phi_{(c)} + \Phi_{(d)} + \Delta\Phi = \\
&= \int_0^1 \frac{1}{yM^2}\, e^{-Q^2\bar{y}/M^2y} \left\{\frac{4f_\phi^2\varphi(y)}{m_\phi^2+q^2}\, e^{q^2\bar{y}/M^2} - \frac{1}{2\pi^2}\ln\frac{(s_\phi+q^2)\bar{y}}{M^2}\, e^{q^2\bar{y}/M^2} + \right. \\
&\qquad \left. + \sum_{n=0}^{\infty}\left(\frac{q^2\bar{y}}{M^2}\right)^n \frac{\psi(n+1)}{n!}\right\} - \frac{8\langle\phi^2\rangle}{Q^2M^2}.
\end{aligned}
\tag{4.18}
$$

It is straightforward to observe that this expression is well-defined for $q^2 = 0$. Below, in our analysis of the QCD sum rule for the $F_{\gamma^*\gamma^*\to\pi^0}(Q^2, q^2)$ form factor we will follow a similar strategy. Of course, instead of the toy scalar meson, the $\rho$-meson will play the role of the lowest state in the bilocal contributions.

# 5 Mass singularities in the QCD case

As discussed above, the B-regime for the correlator (2.3) corresponds to a situation when only the points $Y$ and 0 are separated by a small interval (see fig.6$b$. In simple cases, the short-distance coefficient function is given by a single quark propagator or a product of propagators. The long-distance contribution is represented by a particular two-point correlator of the electromagnetic current $J_\mu(X)$ and some composite operator (see [22, 24]).

## 5.1 Terms with single-propagator coefficient function

The simplest case is when the coefficient function is given by a single quark propagator $S(X) = \hat{X}/2\pi^2(X^2 - i0)^2$. The relevant contribution (Fig.6$b$) can be written as

$$
\begin{aligned}
\mathcal{F}^B_{\alpha\mu\nu} &= -\frac{2\pi}{3}\int d^4Y\, e^{ipY}\, \frac{Y^\beta}{2\pi^2 Y^4} \sum_{n=0} \frac{1}{n!} Y^{\mu_1}\ldots Y^{\mu_n} \\
&\times \left[\left\{ -S_{\nu\beta\alpha\sigma}\int d^4X\, e^{-iq_1X}\, \langle 0|T\{J_\mu(X)\,\bar{u}(0)(\overleftarrow{\partial}_{\mu_1}\ldots\overleftarrow{\partial}_{\mu_n})\gamma_\sigma\gamma_5 u(0)\}|0\rangle \right.\right. \\
&\qquad \left. + i\epsilon_{\nu\beta\alpha\sigma}\int d^4X\, e^{-iq_1X}\, \langle 0|T\{J_\mu(X)\,\bar{u}(0)(\overleftarrow{\partial}_{\mu_1}\ldots\overleftarrow{\partial}_{\mu_n})\gamma_\sigma u(0)\}|0\rangle \right\} \\
&+ \left\{ S_{\nu\beta\alpha\sigma}\int d^4X\, e^{-iq_1X}\, \langle 0|T\{J_\mu(X)\,\bar{u}(0)(\overrightarrow{\partial}_{\mu_1}\ldots\overrightarrow{\partial}_{\mu_n})\gamma_\sigma\gamma_5 u(0)\}|0\rangle \right. \\
&\qquad \left.\left. + i\epsilon_{\nu\beta\alpha\sigma}\int d^4X\, e^{-iq_1X}\, \langle 0|T\{J_\mu(X)\,\bar{u}(0)(\overrightarrow{\partial}_{\mu_1}\ldots\overrightarrow{\partial}_{\mu_n})\gamma_\sigma u(0)\}|0\rangle \right\}\right],
\end{aligned}
\tag{5.1}
$$

where $S_{\nu\beta\alpha\sigma} \equiv (g_{\nu\beta}\, g_{\alpha\sigma} - g_{\nu\alpha}\, g_{\beta\sigma} + g_{\nu\sigma}\, g_{\alpha\beta})$.

Let us consider the bilocal correlators with the right-sided derivatives

$$R_n^5(q_1, Y) = \int d^4X\, e^{-iq_1X} \,\langle 0|T\{J_\mu(X)\,\bar{u}(0)(Y\vec{\partial})^n\gamma_\sigma\gamma_5 u(0)\}|0\rangle,$$
$$R_n(q_1, Y) = \int d^4X\, e^{-iq_1X} \,\langle 0|T\{J_\mu(X)\,\bar{u}(0)(Y\vec{\partial})^n\gamma_\sigma u(0)\}|0\rangle. \quad (5.2)$$

The bilocals $L_n^5(q_1, Y)$ and $L_n(q_1, Y)$ with the left-sided derivatives can be treated in the same way.

For any $n$, we can expand the current with derivatives over a set of traceless operators. More precisely, one should deal with traceless combinations of the indices $\beta, \mu_1, \ldots, \mu_n$. Therefore the essential contribution to eq.(5.1) comes from two types of operators, *viz.*, the lowest twist operators which correspond to the traceless $\{\beta, \mu_1, \ldots, \mu_n\}$ combination and the next-to-leading twist operators which contain one contraction of $\sim g_{\beta\mu_i}$ or $\sim g_{\mu_i\mu_j}$ type. The operators with higher twists are accompanied by the factors $(Y^2)^2$, $(Y^2)^3$, *etc.* which cancel the singularity of the quark propagator $1/Y^4$ and, hence, do not produce mass singularities.

## 5.2 Factorization of the perturbative term

The factorization procedure for the perturbative loop is illustrated on Fig.7$a, b, c$. The diagram 7c corresponds to the expression (5.1) with the two-point correlators given by their lowest-order perturbative form:

$$\left\{ \begin{array}{c} R_n^5(q_1, Y) \\ R_n(q_1, Y) \end{array} \right\} = 12 \left[ \begin{array}{c} -i\epsilon_{\mu\alpha\sigma\beta} \\ S_{\mu\alpha\sigma\beta} \end{array} \right] \int d^D\hat{k}\, \frac{(-iYk)^n[k_\alpha k_\beta - k_\alpha q_{1\beta}]}{k^2(k-q_1)^2} \quad (5.3)$$

where $d^D\hat{k} \equiv d^Dk/(2\pi)^D$, $D = 4 - 2\varepsilon$; here and in the following we use dimensional regularization for the UV divergencies and the $\overline{MS}$ subtraction scheme.

To extract the contribution of the lowest two twists it is sufficient to keep only the terms up to $Y^2$ in the expansion of the integral (5.3) in powers of $Y^2$. Indeed, the twist-3 contribution in (5.1) is obtained by taking formally $Y^2 = 0$ in the numerator of the integrand. Terms proportional to $Y^2$ give the contribution of the twist-5 operators. Using this expansion (see Appendix D) and (5.1), it is possible to perform summation over $n$ and to integrate over $d^4Y$. After simple but lengthy calculations we find:

$$\Phi^{B(PT)}(q^2, Q^2, M^2) = \frac{2}{\pi}\int_0^1 dy e^{-Q^2y/M^2\bar{y}} \frac{1}{2M^2}$$
$$\times \left\{ e^{q^2y/M^2} \ln\frac{q^2y\bar{y}}{\mu^2}\left[2\frac{q^2y}{M^2} + \frac{q^4y^2}{M^4}\right] + 2\frac{q^2y}{M^2} - \frac{3}{2}\frac{q^4y^2}{M^4} \right\}. \quad (5.4)$$

As expected, the terms proportional to $q^2 \ln q^2$ (given by the twist-3 operators) and $q^4 \ln q^2$ (produced by the twist-5 operators), coincide with the non-analytic terms in (4.3).

## 5.3 Factorization for terms proportional to the gluonic condensate

In a similar way, we can consider the diagrams proportional to the gluon condensate (see Fig.3). In this case, it is convenient to perform factorization diagram by diagram, because different groups of diagrams correspond to different coefficient functions (CF) of the B regime. Applying the same technique that was used for the perturbative contribution, we get the following representation for Fig.3*b*, *c*:

$$\Phi_b(q^2,Q^2,M^2) = -\frac{\pi}{18M^6}\langle\frac{\alpha_s}{\pi}GG\rangle\int_0^1 dy\frac{\bar{y}}{y^2}e^{-Q^2\bar{y}/M^2y} \times\left\{\frac{M^2}{q^2\bar{y}} + e^{q^2\bar{y}/M^2}\ln\frac{q^2\bar{y}}{M^2} - \sum_{n=1}^{\infty}\left(\frac{q^2\bar{y}}{M^2}\right)^n\frac{\psi(n+1)}{n!}\right\} \tag{5.5}$$

$$\Phi_c(q^2,Q^2,M^2) = \frac{\pi}{18M^4}\langle\frac{\alpha_s}{\pi}GG\rangle\left\{\frac{1}{q^2}+\frac{1}{Q^2}\right\} -\frac{\pi}{9M^6}\langle\frac{\alpha_s}{\pi}GG\rangle\int_0^1 dye^{-Q^2\bar{y}/M^2y}\left\{\frac{M^2}{q^2y} + \left(\frac{\bar{y}}{y}-\frac{1}{y^2}\right)e^{q^2\bar{y}/M^2}\ln\frac{q^2\bar{y}}{M^2} - \left(\frac{\bar{y}}{y}-\frac{1}{y^2}\right)\sum_{n=1}^{\infty}\left(\frac{q^2\bar{y}}{M^2}\right)^n\frac{\psi(n+1)}{n!}\right\}. \tag{5.6}$$

The factorization procedure for these diagrams can be performed just like it was done in the perturbative case. The only difference is that now we should take the $\langle GG\rangle$ part of the correlator (5.2) rather than its perturbative part. For Fig.3*b*, the bilocal contribution is

$$\Phi_b^B(q^2,Q^2,M^2) = -\frac{\pi}{18}\langle\frac{\alpha_s}{\pi}GG\rangle\frac{1}{M^6}\int_0^1 dy\frac{y}{\bar{y}^2}e^{-Q^2y/\bar{y}M^2} \times\left\{\frac{M^2}{yq^2}e^{yq^2/M^2} + e^{yq^2/M^2}\ln\frac{y\bar{y}q^2}{\mu^2} + e^{yq^2/M^2}(2y-\frac{7}{6})\right\}, \tag{5.7}$$

and for Fig.3*c* we obtain

$$\Phi_c^B(q^2,Q^2,M^2) = \frac{\pi}{18}\langle\frac{\alpha_s}{\pi}GG\rangle\frac{1}{M^4}\frac{1}{q^2} -\frac{\pi}{18}\langle\frac{\alpha_s}{\pi}GG\rangle\frac{2}{M^6}\int_0^1 dye^{-yQ^2/\bar{y}M^2}\left\{\frac{M^2}{\bar{y}q^2}e^{yq^2/M^2} + \left(\frac{y}{\bar{y}}-\frac{1}{\bar{y}^2}\right)e^{yq^2/M^2}\ln\frac{y\bar{y}q^2}{\mu^2}\right\}. \tag{5.8}$$

Note, that the terms proportional to $1/q^2$ are due to the traceless combination of indices $\beta,\mu_1,\ldots,\mu_n$ in (5.1), whereas the terms proportional to $\ln q^2$ correspond to the next-to-leading twist operators.

Diagrams 3*f*, *g* can be treated in a similar way, their total contribution being

$$\Phi_{f+g}(q^2,Q^2,M^2) = -\frac{\pi}{18}\langle\frac{\alpha_s}{\pi}GG\rangle\frac{1}{M^6}\int_0^1 dye^{-yQ^2/\bar{y}M^2} \times\left\{\frac{(2\bar{y}^2+y)}{\bar{y}^2}\left(e^{yq^2/M^2}\ln\frac{yq^2}{M^2} - \sum_{n=1}^{\infty}\left(\frac{yq^2}{M^2}\right)^n\frac{\psi(n+1)}{n!}\right) -\frac{(1-2y)}{\bar{y}^2}\frac{M^2}{q^2}\right\}. \tag{5.9}$$

In this case, the short-distance part of the relevant bilocal contribution is given by a product of two quark propagators:

$$\mathcal{F}^B_{\alpha\mu\nu} = \frac{2\pi}{3}\int d^4Y\, e^{ipY}\, d^4Z\, \frac{(Y-Z)^\delta}{2\pi^2(Y-Z)^4}\, \frac{Z^\epsilon}{2\pi^2 Z^4} \tag{5.10}$$
$$\sum_{n,m=0}^{\infty} \frac{1}{n!m!}\, Y^{\mu_1}\dots Y^{\mu_n}\, Z^{\nu_1}\dots Z^{\nu_m} \int d^4X\, e^{-iq_1X}$$
$$\langle 0|T\left\{J_\mu(X)\,\bar{u}(0)(\partial_{\mu_1}\dots\partial_{\mu_n})\gamma_\nu\gamma_\delta\gamma_\gamma g(\partial_{\nu_1}\dots\partial_{\nu_m}A^b_\gamma(0))t^b\gamma_\epsilon\gamma_5\gamma_\alpha u(0)\right\}|0\rangle.$$

The $\langle GG\rangle$ contribution in (5.11) leads to the expression:

$$\Phi^B_{f+g}(q^2,Q^2,M^2) = -\frac{\pi}{18}\langle\frac{\alpha_s}{\pi}GG\rangle\frac{1}{M^6}\int_0^1 dy e^{-yQ^2/\bar{y}M^2}$$
$$\times\left\{\frac{(2\bar{y}^2+y)}{\bar{y}^2}e^{yq^2/M^2}\ln\frac{y\bar{y}q^2}{\mu^2} - \frac{(1-2y)}{\bar{y}^2}\frac{M^2}{q^2}e^{yq^2/M^2}\right.$$
$$\left. + e^{yq^2/M^2}(\dots)\right\}, \tag{5.11}$$

where $(\dots)$ denote terms regular for $q^2 = 0$.

## 5.4 Factorization for quark condensate corrections

The factorization procedure should be applied also to the diagrams with the quark condensate. The relevant contributions are analogous to the power corrections of the toy scalar model.

Consider first the diagrams with a soft gluon (Fig.7). The first three diagrams (figs.7$a, b, c$) produce a $p^2$-independent contribution into $F(p^2, q_1^2, q_2^2)$ which vanishes after the SVZ-Borel transformation in $p^2$. The last three diagrams (figs.7$g, h, i$) are regular in the $q^2 \to 0$ limit, and there is no need for an additional factorization. Finally, the contributions of the remaining three diagrams (figs.7$d, e, f$) has a manifestly factorized form (see (5.1),(5.11)), *i.e.*, they are completely absorbed by the bilocal terms in the modified OPE.

For the diagrams with a hard gluon exchange (Fig.8), the situation is very similar. The first three diagrams (figs.8$a, b, c$ ) produce a manifestly factorized contributions into $F(p^2, q_1^2, q_2^2)$. The basic difference between them is that the coefficient function of the $B$-regime is given by a product of two quark propagators while those related to the diagrams 8$b, c$ is formed by a product of two quark propagators and a gluon one. The contribution of Fig.8$d$ is regular in the $q^2 \to 0$ limit, and there is no need for an additional factorization.

As mentioned earlier, the remaining diagrams 8$e$–$r$ do not contribute to $F(p^2, q_1^2, q_2^2)$ in the large-$q^2$ regime. However, a more careful analysis (see Section 7) of the bilocal objects corresponding to some of these diagrams shows that there may appear nontrivial contributions into $F(p^2, q_1^2, q_2^2)$ due to the so-called contact terms [22]. Within the QCD sum rule method, the contact terms, in particular, play a crucial role in establishing the correct normalization for the electromagnetic form factors at zero momentum transfer [22, 24, 26].

After application of the subtraction procedure illustrated on Fig.5, all the infrared singularities of the original sum rule cancel with the corresponding singular contributions from the diagrams in which the $B$-term is extracted explicitly. In our case, it is sufficient to consider the bilocal objects corresponding to operators of the lowest two twists. The terms remaining after this subtraction are regular in the $q^2 \to 0$ limit and contribute to the modified sum rule.

# 6 Bilocal correlators

## 6.1 Bilocals related to a single-propagator coefficient function

In the simplest case the coefficient function of the $B$-regime is given by a single quark propagator $S(Y) = \hat{Y}/2\pi^2(Y^2 - i0)^2$, and the corresponding contribution into the three-point amplitude $\mathcal{F}_{\alpha\mu\nu}(q_1, p)$ can be written as

$$\mathcal{F}^B_{\alpha\mu\nu}(q_1,p) \sim \int e^{ipY} \frac{Y^\beta d^4Y}{(Y^2 - i0)^2} \int e^{-iq_1 X} \langle 0|T(J_\mu(X)(\bar{\psi}(0)\gamma_\nu\gamma_\beta\gamma_5\gamma_\alpha\psi(Y))|0\rangle d^4X \,. \tag{6.1}$$

As emphasized before, the correlators (bilocals) $\langle 0|T\{J_\mu(X)\mathcal{O}^{(i)}(0)\}|0\rangle$ of the $J_\mu(X)$-current with composite operators $\mathcal{O}^{(i)}(0)$ are sensitive to the long-distance ($\sim 1/|q_1|$) dynamics and, for this reason, they are not directly calculable in perturbation theory. The standard way out is to write them in a dispersion form and assume the simplest ansatz ("lowest resonance + continuum") for the model spectral density, with the continuum starting at some effective threshold $s_\rho$. The parameters of the model spectrum can be extracted from an auxiliary QCD sum rule. The quantum numbers of the electromagnetic current $J_\mu(X)$, which appears in all the bilocals, dictates that the lowest resonance shall be represented by the $\rho^0$-meson. In particular, for the bilocals corresponding to the-single propagator coefficient function, we can write

$$\begin{aligned} R_n(q_1,Y) &= \frac{1}{\pi}\int_0^\infty \frac{\delta R_n(s)}{s-q_1^2} ds + (\text{ subtractions }) \\ &= \text{“}\rho^o\text{-meson contribution”} + \frac{1}{\pi}\int_{s_\rho}^\infty ds \frac{\delta R_n^{PT}(s,Y)}{s-q_1^2} + (\text{ subtractions }), \end{aligned} \tag{6.2}$$

where $\delta R_n(s,Y)$ is the relevant discontinuity: $\delta R_n(s,Y) \equiv (R_n(s+i0,Y) - R_n(s-i0,Y))/2i$.

Picking out the $\rho^o$-meson term

$$\delta^{(+)}(p^2 - m_\rho^2)\, \frac{d^4p}{(2\pi)^3} \sum_{\lambda=-1}^{1} |\rho^o_\lambda; \vec{p}\rangle\, \langle\rho^o_\lambda; \vec{p}\,| \subset \hat{\mathbf{I}}, \tag{6.3}$$

in the sum over physical hadronic states (with $\lambda$ being the helicity of the $\rho^o$), we extract the $\rho^0$ contribution. As a result, we obtain a set of matrix elements which can be parameterized as

$$\begin{aligned} \langle 0|\bar{\psi}(0)\gamma^\sigma\psi(Y)|\rho^o_{\lambda=0}; \vec{p}\rangle &= ip^\sigma\, f_\rho^V\, \phi_\rho^V(Yp,\mu^2) + \cdots \\ \langle 0|\bar{\psi}(0)\gamma^\sigma\psi(Y)|\rho^o_{|\lambda|=1}; \vec{p}\rangle &= \varepsilon^\sigma_\perp\, f_\rho^V m_\rho\, \phi^V_{\rho\perp}(Yp,\mu^2) \end{aligned} \tag{6.4}$$

$$+ia_{V_1}\,p^{\sigma}\;f_{\rho}^{V}m_{\rho}(\varepsilon_{\perp}Y)\,\phi_{\rho\perp}^{V_1}(Yp,\mu^2)+\cdots \tag{6.5}$$

$$\langle 0|\bar{\psi}(0)\gamma^{\sigma}\gamma_5\psi(Y)|\rho^o_{|\lambda|=1};\vec{p}\rangle \;=\epsilon^{\sigma\alpha\beta\rho}\,\varepsilon_{\perp}^{\alpha}\,p_{\beta}\,Y_{\rho}\;f_{\rho}^{A}\,\phi_{\rho}^{A}(Yp,\mu^2)+\cdots\;. \tag{6.6}$$

Here only the twist 3 terms are written explicitly, and the dots stand for the higher twist contributions, $\varepsilon_{\sigma}$ is the polarization vector of the $\rho^o$-meson, and the helicity components have an evident interpretation in terms of the longitudinal and transverse polarizations: $\rho^o_{\lambda=0}\equiv\rho^o_L$ , $\rho^o_{|\lambda|=1}\equiv\rho^o_{\perp}$.

In a standard way, the functions $\phi_{\rho}(Yp,\mu^2)$ can be related to the usual wave functions $\varphi_{\rho}(y,\mu^2)$ describing the light-cone momentum distribution inside the $\rho$:

$$\phi_{\rho}^{(i)}(Yp,\mu^2)=\int_0^1 dy e^{-i(Yp)y}\varphi_{\rho}^{(i)}(x,\mu^2), \tag{6.7}$$

with $\mu^2$ being the renormalization parameter for the relevant composite operators. The constant $f_{\rho}^{V}$ fixing the normalization of the simplest wave function is known from previous QCD sum rule studies: $f_{\rho}^{V}\simeq 200\,\mathrm{MeV}$ [18, 41], while the constants $f_{\rho}^{A}$ and $a_{V1}$ in eqs. (6.5), (6.6) can be fixed by equations of motion (cf. [39, 36]), which form an infinite set of relations connecting the moments of different wave functions (see Appendix C).

For our purposes, it is more convenient to write down the matrix elements in a form suitable for an arbitrary polarization of the $\rho^o$-meson:

$$\begin{aligned}\langle 0|\bar{\psi}(0)\gamma_{\sigma}\psi(Y)|\rho^o_{\lambda};\vec{p}\rangle \;&=\varepsilon_{\sigma}^{(\lambda)}\;f_{\rho}^{V}m_{\rho}\;\left[\phi_{\rho\perp}^{V}(Yp,\mu^2)+C_{V5}\,Y^2\phi_{\rho\perp}^{V5}(Yp,\mu^2)+\cdots\right]\\ &+ia_{V1}\,p_{\sigma}\;f_{\rho}^{V}m_{\rho}(\varepsilon^{(\lambda)}Y)\;\left[\phi_{\rho\perp}^{V1}(Yp,\mu^2)+C_{V15}\,Y^2\phi_{\rho\perp}^{V15}(Yp,\mu^2)+\cdots\right]\\ &+f_{\rho}^{V25}\;\left(Y_{\sigma}(\varepsilon^{(\lambda)}Y)\;-\;\varepsilon_{\sigma}^{(\lambda)}Y^2/4\right)\;\phi_{\rho\perp}^{V25}(Yp,\mu^2)+\cdots\end{aligned} \tag{6.8}$$

$$\begin{aligned}\langle 0|\bar{\psi}(0)\gamma_{\sigma}\gamma_5\psi(Y)|\rho^o_{\lambda};\vec{p}\rangle \;&=\epsilon_{\sigma\alpha\beta\rho}\,\varepsilon_{\alpha}^{(\lambda)}\,p_{\beta}\,Y_{\rho}\;f_{\rho}^{A}\;\left[\phi_{\rho}^{A}(Yp,\mu^2)\right.\\ &\left.+C_{A5}\,Y^2\phi_{\rho}^{A5}(Yp,\mu^2)+\cdots\right].\end{aligned} \tag{6.9}$$

Since the C-parity of the $\rho^o$-meson is negative, its wave functions have the following properties (here and below $\bar{y}\equiv 1-y$):

$$\begin{aligned}&\varphi_{\rho\perp}^{V,V5,V25,A,A5}(y)=\varphi_{\rho\perp}^{V,V5,V25,A,A5}(\bar{y})\quad,\quad\varphi_{\rho\perp}^{V1,V15}(y)=-\varphi_{\rho\perp}^{V1,V15}(\bar{y}),\\ &\int_0^1 dy\;\varphi_{\rho\perp}^{V,V5,V25,A,A5}(y)=1\quad,\quad\int_0^1 dy\;y\,\varphi_{\rho\perp}^{V1,V15}(y)=1.\end{aligned} \tag{6.10}$$

In the relations above, we have explicitly displayed wave functions up to twist 4. Note that, for a longitudinally polarized $\rho^o$-meson,

$$\varepsilon_{\sigma}^{\lambda=0}\simeq i\,p_{\sigma}/m_{\rho}+\mathcal{O}\,(m_{\rho}/p_z)$$

as $p_z\to\infty$, and the twist-3 part in eq.(6.8) coincides with the well known definition (6.4).

Let us consider first the twist-3 $\rho^o$-meson contribution in eq. (6.2). Applying (6.3), (6.8) and (6.9), we obtain:

$$\left(\sum_{n=0}^{\infty}\frac{1}{n!}R_n(q_1,Y)\right)_{\rho^o} = \frac{(-i)(f_\rho^V m_\rho)^2}{m_\rho^2-q_1^2}\Big\{\varepsilon_\mu\varepsilon_\sigma^*\int_0^1 dy\, e^{-i(Yq_1)\bar y}\,\varphi_{\rho\perp}^V(y)$$
$$-ia_{V1}\,q_{1\sigma}\varepsilon_\mu(\varepsilon^*Y)\int_0^1 dy\, e^{-i(Yq_1)\bar y}\,\varphi_{\rho\perp}^{V1}(y)\Big\} \tag{6.11}$$

$$\left(\sum_{n=0}^{\infty}\frac{1}{n!}R_n^5(q_1,Y)\right)_{\rho^o} = \frac{(-i)(f_\rho^A f_\rho^V m_\rho)}{m_\rho^2-q_1^2}\varepsilon_\mu\varepsilon_\delta^*\,\epsilon_{\sigma\delta\beta\rho}\,q_{1\beta}Y_\rho\int_0^1 dy\; e^{-i(Yq_1)\bar y}\;\varphi_{\rho\perp}^A(y). \tag{6.12}$$

Here we use the shorthand notation:

$$\varepsilon_\mu\varepsilon_\sigma^*\equiv\sum_{\lambda=0,\pm1}\varepsilon_\mu^\lambda\varepsilon_\sigma^{\lambda*}=-g_{\mu\sigma}+\frac{q_{1\mu}q_{1\sigma}}{m_\rho^2}. \tag{6.13}$$

Substituting eqs. (6.11), (6.12) into eq. (5.1) and extracting the proper tensor structure we get:

$$F^{B(\rho)}=\frac{4\pi}{3}\frac{f_\rho^V m_\rho}{m_\rho^2-q_1^2}\int_0^1 dy\frac{1}{\tilde p^4}\left[-a_{V1}\,f_\rho^V m_\rho\varphi_{\rho\perp}^{V1}(y)-f_\rho^A\,\varphi_{\rho\perp}^A(y)(1+2\bar y)\right], \tag{6.14}$$

where

$$\tilde p^{\,2}\equiv(q_2+\bar y q_1)^2=-q_1^2\,y\bar y+q_2^2\,y+p^2\,\bar y$$

is the virtuality of the hard quark written in the "parton" form. This formula includes an extra factor of 2 which appears when one adds the contribution of the correlators $L_n$, $L_n^5$ and uses the symmetry properties of the wave functions (6.10). It should be noted that the wave functions $\varphi_{\rho\perp}^V$, $\varphi_{\rho\perp}^{V5}$ and $\varphi_{\rho\perp}^{V25}$ do not contribute to the form factor $F$ which is considered here.

Deriving, in a similar way, the twist-5 contribution and applying the SVZ-Borel transformation to the resulting amplitude, we obtain:

$$\begin{aligned}\Phi^{B(\rho)} &= \frac{4\pi}{3}\frac{f_\rho^V m_\rho}{m_\rho^2+q^2}\int_0^1 dy\frac{1}{\bar y^2M^4}\,e^{-Q^2y/M^2\bar y}\,e^{q^2y/M^2}\\ &\times\left[-a_{V1}\,f_\rho^V m_\rho\left(\varphi_{\rho\perp}^{V1}(y)-\frac{4C_{V51}}{\bar yM^2}\,\varphi_{\rho\perp}^{V51}(y)\right)\right.\\ &\left.-f_\rho^A(1+2\bar y)\left(\varphi_{\rho\perp}^A(y)-\frac{4C_{A5}}{\bar yM^2}\,\varphi_{\rho\perp}^{A5}(y)\right)\right].\end{aligned} \tag{6.15}$$

As expected, the twist-5 contribution is suppressed by one power of $1/M^2$.

Now, taking $q^2=0$ and introducing the integration variable $s=yQ^2/\bar y$, we can write this term as

$$A(Q^2)=-\frac{4\pi}{3Q^2}\frac{f_\rho^V}{m_\rho}\int_0^\infty\frac{ds}{M^4}e^{-s/M^2}\left[a_{V1}\,f_\rho^V m_\rho\left(\varphi_{\rho\perp}^{V1}(\frac{s}{s+Q^2})+\frac{4C_{V51}(s+Q^2)}{M^2Q^2}\,\varphi_{\rho\perp}^{V51}(\frac{s}{s+Q^2})\right)\right.$$
$$\left.+f_\rho^A(1+2\frac{s+Q^2}{Q^2})\left(\varphi_{\rho\perp}^A(\frac{s}{s+Q^2})+\frac{4C_{A5}(s+Q^2)}{M^2Q^2}\,\varphi_{\rho\perp}^{A5}(\frac{s}{s+Q^2})\right)\right] \tag{6.16}$$

Note, that this representation has "wrong" powers of the SVZ-Borel parameter $M^2$ compared to the canonical form

$$\Phi(M^2, Q^2) = \frac{1}{\pi M^2} \int_0^\infty e^{-s/M^2} \rho(s, Q^2)\, ds. \tag{6.17}$$

Using the transformation

$$\int_0^\infty ds\; e^{-s/M^2} \frac{g(s)}{M^2} = \int_0^\infty ds\; e^{-s/M^2} (g(0)\, \delta(s) + g'(s)), \tag{6.18}$$

$$\int_0^\infty ds\; e^{-s/M^2} \frac{g(s)}{M^4} = \int_0^\infty ds\; e^{-s/M^2} (g'(0)\, \delta(s) + g(0)\, \delta'(s) + g''(s)) \tag{6.19}$$

*etc.*, we can always cast the terms with the "wrong" powers into the canonical form (6.17).

In particular, taking the asymptotic forms (see Appendix C)

$$a_{V1} = \frac{1}{40},\ f_\rho^A = -\frac{f_\rho^V m_\rho}{4}, \varphi_{V1} = \varphi_{V1}^{as} = 60 y\bar{y}\,(2y-1),\ \varphi_A = \varphi_A^{as} = 6y\bar{y}, \tag{6.20}$$

as the simplest estimate for the twist-3 contributions, we get

$$g_{3,\rho}(s) \;=\; 8\pi^2 (f_\rho^V)^2 \frac{s\, Q^2}{(s+Q^2)^3} \tag{6.21}$$

$$\rho_3^\rho(s, Q^2) \;=\; g'_{3,\rho}(s) \;=\; 8\pi^2 (f_\rho^V)^2 \frac{(Q^4 - 2sQ^2)}{(s+Q^2)^4} \tag{6.22}$$

Note, that the $s$-integral over this spectral density is zero.

The asymptotic forms for the twist-5 distribution amplitudes can be directly extracted from the corresponding correlators:

$$\varphi_{V15} = \varphi_{V15}^{as} = 420 (y\bar{y})^2\,(2y-1), \quad \varphi_{A5} = \varphi_{A5}^{as} = 30 y^2 \bar{y}^2. \tag{6.23}$$

From the equations of motion (see (C.4), Appendix C) it follows that $C_{V15} = 5C_{A5}/7$. Taking into account this expression and eqs.(6.23), we obtain:

$$A_5(Q^2) = \frac{1}{\pi M^2} \int_0^\infty ds\, e^{-s/M^2} \frac{g_{5,\rho}(s)}{M^4}\ , \tag{6.24}$$

where

$$g_{5,\rho}(s) = -4\pi^2 (f_\rho^V)^2\, 40\, C_{A5} \frac{s^2\, Q^2}{(s+Q^2)^4}\ . \tag{6.25}$$

Hence, the relevant spectral density is

$$\rho_5^\rho(s, Q^2) = -8\pi^2 (f_\rho^V)^2\, 40\, C_{A5} Q^2 \frac{(3s^2 - 6sQ^2 + Q^4)}{(s+Q^2)^6} = g''_{5,\rho}(s). \tag{6.26}$$

## 6.2 Continuum contribution

In the basic OPE for the small-$q^2$ kinematics, one always deals with the difference between an "exact" bilocal correlator $R$ and its perturbative analog $R^{PT}$ (see Fig.7). An important observation is that the subtraction terms in the dispersion relation for $R^{PT}$ coincide with those in the dispersion relation for $R$, since the ultraviolet behaviour of these two correlators is the same because of asymptotic freedom. Hence, there is no need to explicitly specify a subtraction prescription for the correlators.

Now, incorporating our model for the bilocal correlators, in which the contribution due to higher excited states is approximated by the perturbative spectral density (see (6.2)), *i.e.*, by the continuum starting at $s_\rho$, we can easily write down an expression for the difference between the continuum contribution to $R$ and the perturbative bilocal $R^{PT}$. Then, substituting the result into the original expansion (5.1) and performing some straightforward calculations, we obtain:

$$\Phi^{B(cont)} - \Phi^{B(PT)} = \frac{1}{\pi}\int_0^1 dy \frac{1}{M^2}\, e^{-Q^2y/M^2\bar{y}}\, e^{q^2y/M^2}$$
$$\times \left[\frac{2y}{M^2}\left(q^2\ln\frac{s_\rho+q^2}{q^2} - s_\rho\right) + \frac{y^2}{M^4}\left(q^4\ln\frac{s_\rho+q^2}{q^2} - q^2 s_\rho + \frac{s_\rho^2}{2}\right)\right]. \tag{6.27}$$

The terms collected in the ( ) brackets correspond to contributions due to operators with twist 3 and 5. Note, that these terms exactly cancel the logarithmic contributions $q^2\ln q^2, q^4\ln q^2$ present in the coeficient function of the unit operator for the usual OPE valid in the large-$q^2$ kinematics. As a result, the non-analytic terms are replaced by the combinations $q^2\ln(s_\rho+q^2)$ and $q^4\ln(s_\rho+q^2)$, which are "safe" in the $q^2 \to 0$ limit. On the other hand, for large $q^2$, the usual OPE without additional terms must work, *i.e.*, the difference between "exact" bilocal term and its perturbative analogue must vanish faster than any power of $1/q^2$ in the large-$q^2$ limit. It is easy to check that the terms on the rhs of eq.(6.27) behave like $1/q^2$ when $q^2 \to \infty$. To get the total expression for the additional terms, we should add the $\rho$-contribution to eq.(6.27). The $\rho$-contribution also has the $1/q^2$-behaviour for large $q^2$. However, to produce a perfect transition to the pure SD-case, the $1/q^2$-terms must cancel. Basically, this means that using a rough model for the correlator one should not rely too heavily on the extrapolation of our result (6.27) beyond the region $q^2 \lesssim m_\rho^2$. However, we can require that the model, at least, should provide the cancellation of the $1/q^2$ terms. If we choose the asymptotic form for the lowest-twist distribution amplitudes, the cancellation of the $1/q^2$-terms produces the estimate:

$$\tilde{s}_\rho^2 = 8\pi^2(f_\rho^V)^2 m_\rho^2. \tag{6.28}$$

Our result (6.27) simplifies in the $q^2 \to 0$ limit:

$$\Phi^{B(cont)} - \Phi^{B(PT)} = \frac{1}{\pi M^2}\int_0^\infty ds\; e^{-s/M^2}\frac{\bar{y}^2}{Q^2}\left[-\frac{2y}{M^2}s_\rho + \frac{y^2}{M^4}\frac{s_\rho^2}{2}\right], \tag{6.29}$$

where $s$ again is defined by $s = yQ^2/\bar{y}$. The twist-3 contribution can be represented in the canonical form with the following spectral density:

$$\rho_3^B(s,Q^2) = -2s_\rho \frac{(Q^4 - 2sQ^2)}{(s+Q^2)^4}. \tag{6.30}$$

For the twist-5 contribution in eq.(6.29), we obtain:

$$\rho_5^B(s,Q^2) = s_\rho^2 Q^2 \frac{(3s^2 - 6sQ^2 + Q^4)}{(s+Q^2)^6} = g''_{5,\text{cont}}(s), \tag{6.31}$$

where

$$g_{5,\text{cont}}(s) = \frac{s_\rho^2}{2}\frac{s^2Q^2}{(s+Q^2)^4}. \tag{6.32}$$

Using (6.28), the $\rho$-meson contribution from (6.22) can be rewritten as

$$\rho_3^\rho(s,Q^2) \;=\; g'_{3,\rho}(s) = \frac{\widetilde{s}_\rho^2}{m_\rho^2}\,\frac{(Q^4 - 2sQ^2)}{(s+Q^2)^4}. \tag{6.33}$$

For the next-to-leading-twist contributions, similar duality arguments give the following estimates for the normalization constants of the relevant two-body distribution amplitudes:

$$\widetilde{C}_{A5} = \frac{\widetilde{s}_\rho}{60} \simeq 2.28\cdot 10^{-2}\,\text{GeV}^2\;;\quad \widetilde{C}_{V15} = \frac{\widetilde{s}_\rho}{84} \simeq 1.63\cdot 10^{-2}\,\text{GeV}^2. \tag{6.34}$$

Hence, the expression (6.26) can be written as

$$\rho_5^\rho(s,Q^2) = -2\frac{\widetilde{s}_\rho^3}{3m_\rho^2}Q^2\frac{(3s^2 - 6sQ^2 + Q^4)}{(s+Q^2)^6}. \tag{6.35}$$

If the distribution amplitudes deviate from their asymptotic forms, then the duality condition between the $\rho$-meson and the continuum should keep the form of the expressions (6.33), (6.35), but instead of $\widetilde{s}_\rho$ we should get an effective duality interval.

Note that for $f_\rho^V = 0.2\,\text{GeV}$, our estimates give $\widetilde{s}_\rho \simeq 1.37\,\text{GeV}^2$ for the duality interval. This value is in good agreement with the "canonical" one: $s_\rho^{LD} = 4\pi^2(f_\rho^V)^2 \simeq 1.58\,\text{GeV}^2$. The latter follows from the local duality considerations for the two-point correlator of two vector currents.

## 6.3 Twist-2 bilocals for two-propagator coefficient functions

Next in complexity is the contribution related to the coefficient function formed by a product of two propagators $S(Y,Z)$ and $S(Z,0)$:

$$\mathcal{F}^{B(2)}_{\alpha\mu\nu} = \frac{2\pi}{3}\int d^4Y\, e^{ipY}\, d^4Z\,\frac{(Y-Z)^\delta}{2\pi^2(Y-Z)^4}\,\frac{Z^\epsilon}{2\pi^2Z^4}\,\sum_{n,m=0}^{\infty}\frac{1}{n!m!}\,Y^{\mu_1}\ldots Y^{\mu_n}\,Z^{\nu_1}\ldots Z^{\nu_m} \tag{6.36}$$
$$\times\int d^4X\, e^{-iq_1X}\,\langle 0|T\{J_\mu(X)\,\bar u(0)(\overleftarrow{\partial}_{\mu_1}\ldots\overleftarrow{\partial}_{\mu_n})\gamma_\nu\gamma_\delta\; g_\gamma g(\overrightarrow{\partial}_{\nu_1}\ldots\overrightarrow{\partial}_{\nu_n}A^b_\gamma(0))t^b\gamma_\epsilon\gamma_5\gamma_\alpha u(0)\}|0\rangle\,.$$

Here we explicitly extracted the bilocal correlator containing a composite operator composed of two quark and one gluonic field. Note that the gluonic field $A^b_\gamma(Z)$ here may be treated as taken in the Fock-Schwinger gauge, *i.e.*, it can be substituted by

$$A^b_\gamma(Z) = Z_\varphi \int_0^1 \alpha\, G^b_{\varphi\gamma}(\alpha Z) d\alpha\,. \tag{6.37}$$

As a result, the $\rho^o$-meson contribution is determined by the following matrix elements:

$$\begin{aligned} \langle 0|\bar u(Z_1)\gamma_\beta\gamma_5 g_s G^b_{\varphi\gamma}(Z_3) t^b u(Z_2)|\rho^o_\lambda;\vec p\rangle &= p_\beta\, \epsilon_{\varphi\gamma\theta\kappa}\, p_\theta\, \varepsilon^{(\lambda)}_\kappa\; f^A_{3\rho}\, \phi^A_{3\rho}(Z_i p,\mu^2) \\ &\quad + \text{higher twist contributions} \end{aligned} \tag{6.38}$$

$$\begin{aligned} \langle 0|\bar u(Z_1)\gamma_\beta\, i g_s G^b_{\varphi\gamma}(Z_3) t^b u(Z_2)|\rho^o_\lambda;\vec p\rangle &= p_\beta \left(p_\varphi \varepsilon^{(\lambda)}_\gamma - p_\gamma \varepsilon^{(\lambda)}_\varphi\right) f^V_{3\rho}\, \phi^V_{3\rho}(Z_i p,\mu^2) \\ &\quad + \text{higher twist contributions}\,. \end{aligned} \tag{6.39}$$

In a standard way, we can introduce the momentum distribution amplitudes[3] $\varphi^{V,A}_{3\rho}(y_i)$:

$$\phi^{V,A}_{3\rho}(Z_i p,\mu^2) = \int_0^1 [dy]_3\, \varphi^{V,A}_{3\rho}(y_i)\, e^{-i\sum y_i (Z_i p)}. \tag{6.40}$$

They have the following symmetry properties:

$$\varphi^A_{3\rho}(y_1,y_2;y_3) = \varphi^A_{3\rho}(y_2,y_1;y_3)\ ,\ \ \varphi^V_{3\rho}(y_1,y_2;y_3) = -\varphi^V_{3\rho}(y_2,y_1;y_3). \tag{6.41}$$

In our definition, the normalization constants $f^A_{3\rho} = 0.6 \cdot 10^{-2}\,\mathrm{GeV}^2, f^V_{3\rho} = 0.25 \cdot 10^{-2}\,\mathrm{GeV}^2$ [41] are factored out, so that the distribution amplitudes are normalized to unity:

$$\int_0^1 [dy]_3\, \varphi^A_{3\rho}(y_i) = 1\ ,\ \ \int_0^1 [dy]_3\, (y_1 - y_2)\,\varphi^V_{3\rho}(y_i) = 1\ . \tag{6.42}$$

Following the procedure described in Sec.2.1, we find the $\rho^o$-meson contribution:

$$\begin{aligned} \Phi^{B(2)}_\rho &= \frac{8\pi}{3}\frac{f^V_\rho m_\rho}{m_\rho^2+q^2}\int_0^1 d\alpha\,\alpha\int_0^1 d\beta\int_0^1 [dy]_3\; e^{\,b/aM^2} \\ &\times \left\{ f^A_{3\rho}\,\varphi^A_{3\rho}(y_1,y_2;y_3)\left[\frac{c_1}{a^2M^4} - \frac{d_1}{2a^3M^6}\right] - f^V_{3\rho}\,\varphi^V_{3\rho}(y_1,y_2;y_3)\left[\frac{c_2}{a^2M^4} - \frac{d_2}{2a^3M^6}\right]\right\}, \end{aligned} \tag{6.43}$$

where

$$\begin{aligned} a &= \alpha\beta y_3 + y_2, \\ b &= -q^2\,(\alpha^2\beta y_3^2 + 2\alpha\beta y_2 y_3 - \alpha\beta y_3 + y_2^2 - y_2) + Q^2\,(\alpha\beta y_3 + y_2 - 1) \end{aligned} \tag{6.44}$$

and

$$\begin{aligned} c_1 &= (\alpha\beta y_3)/(\alpha\beta y_3 + y_2), \\ d_1 &= c_1\left(-q^2\,(\alpha^2\beta y_3^2 + 2\alpha y_2 y_3 + \alpha\beta y_3 + y_2^2 + y_2) + Q^2\right), \\ c_2 &= (\alpha\beta y_3 + 2\beta y_2)/(\alpha\beta y_3 + y_2), \\ d_2 &= c_1\left(q^2\,(\alpha^2\beta y_3^2 + 2\alpha\beta y_2 y_3 + \alpha\beta y_3 + 2\beta y_2^2 - y_2^2 + y_2) + Q^2\,(1-2\beta)\right). \end{aligned} \tag{6.45}$$

[3] $[dy]_3 \equiv dy_1 dy_2 dy_3\, \delta(1 - \sum_i y_i)$

The perturbative spectral density for all of these correlators is suppressed by $O(\alpha_s/\pi)$-factor, and for this reason we neglect here the contribution due to higher states.

Taking $q^2 = 0$, we get

$$a = \alpha\beta y_3 + y_2, \quad b = -Q^2(1-a) \tag{6.46}$$

and

$$\begin{aligned} c_1 &= (\alpha\beta y_3)/a, \quad d_1 = c_1 Q^2, \\ c_2 &= (\alpha\beta y_3 + 2\beta y_2)/a, \quad d_2 = c_1 Q^2 (1-2\beta). \end{aligned} \tag{6.47}$$

Introducing the variables:

$$s = Q^2(1-a)/a = Q^2 \frac{1 - y_2 - \alpha\beta y_3}{y_2 + \alpha\beta y_3}, \; v(s) = \frac{Q^2}{s+Q^2} \tag{6.48}$$

and integrating over $\beta, \alpha$, we obtain the representation

$$\Phi_\rho^{B(2)} = \frac{8\pi}{3} \frac{f_\rho^V}{m_\rho} \int_0^\infty ds \; e^{-s/M^2} \frac{(s+Q^2)}{Q^4 M^4} \times \tag{6.49}$$

$$\times \int_0^1 [dy]_3 \; \theta(y_2 \le v(s)) \; \frac{(v(s) - y_2)}{y_3^2} \times$$

$$\left\{ f_{3\rho}^A \varphi_{3\rho}^A(y_1, y_2; y_3) (y_2 + y_3 - v(s)) \left[ 1 - \frac{(s+Q^2)}{2M^2} \right] - \right.$$

$$- f_{3\rho}^V \varphi_{3\rho}^V(y_1, y_2; y_3) \left[ \left( y_2 + y_3 - v(s) - 2y_2 \ln\left( \frac{v(s) - y_2}{y_3} \right) \right) - \right.$$

$$\left. \left. - \frac{(s+Q^2)}{2M^2} \left( y_2 + y_3 - v(s) + 2(v(s) - y_2) \ln\left( \frac{v(s) - y_2}{y_3} \right) \right) \right] \right\}$$

To estimate these contributions, we use the asymptotic forms of the corresponding three-body $\rho$-meson distribution amplitudes [41]:

$$\begin{aligned} \varphi_{3A}(y_1 y_2 y_3) &\to \varphi_{3A}^{as} y_1 y_2 y_3 = 360 y_1 y_2 y_3^2, \\ \varphi_{3V}(y_1 y_2 y_3) &= \varphi_{3V}^{as}(y_1 y_2 y_3) = 7!(y_1 - y_2) y_1 y_2 y_3^2. \end{aligned} \tag{6.50}$$

## 6.4 Twist-2 bilocals for three-propagator coefficient functions

The bilocals associated with the coefficient functions given by a product of three propagators can appear in the $\langle\bar\psi\psi\rangle^2$ quark condensate diagrams of the unmodified OPE (see Figs.7$a-r$). Furthermore, it was pointed out there that for large and moderate $q_1^2$ only diagrams $7a-d$ contribute to the invariant amplitude $F$ we are interested in. So, let us consider them first. In fact, among these diagrams, only 7$b$ and 7$c$ produce bilocals with the three-propagator coefficient function. After some algebra, we obtain

$$\mathcal{F}^{B(3)}_{\alpha\mu\nu} = \frac{32\pi^2\alpha_s\langle\bar{u}u\rangle}{27} \int dY e^{ipY} \left(\frac{p_\alpha}{p^2}\right) \frac{Y_\beta}{8\pi^2Y^2} \sum_{n=0}^{\infty} \frac{1}{n!} Y^{\mu_1}\dots Y^{\mu_n}$$
$$\times \int dX e^{-iq_1X} \langle 0|T\{J_\mu(X)\,\bar{u}(0)(\partial_{\mu_1}\dots\partial_{\mu_n})\gamma_\nu\gamma_\beta\gamma_5 u(0)\}|0\rangle. \qquad (6.51)$$

Adding the charge conjugate contribution produces an extra factor of 2. The correlator which appeared in eq.6.51 can also be treated as a distribution amplitude $\varphi_{\gamma^*}(y,q^2)$ of a photon with virtuality $q_1^2 = -q^2$. The perturbative spectral density for this correlator is zero, so the simplest approximation is to model $\varphi_{\gamma^*}(y,q^2)$ by contributions from lowest resonances:

$$\varphi_{\gamma^*}(y,q^2) = \frac{m_\rho f_\rho^V f_\rho^T}{m_\rho^2+q^2}\varphi_\rho^T(y) + \frac{m_{\rho'} f_{\rho'}^V f_{\rho'}^T}{m_{\rho'}^2+q^2}\varphi_{\rho'}^T(y) + \dots\ , \qquad (6.52)$$

where the $\rho$-contribution is determined by the matrix element

$$\langle 0|\bar{\psi}(0)\sigma_{\nu\beta}\psi(Y)|\rho^o_\lambda;\vec{p}\rangle \ = i\,(\,\varepsilon_\nu^{(\lambda)}p_\beta - \varepsilon_\beta^{(\lambda)}p_\nu\,)\ f_\rho^T\ \phi_\rho^T(Yp,\mu^2) + \text{higher twists}\ . \qquad (6.53)$$

Here, as usual, $\sigma_{\nu\beta} = \frac{i}{2}[\gamma_\nu,\gamma_\beta]$, $p = q_1 + q_2$. The simplest approximation is to model the photon distribution amplitude $\varphi_{\gamma^*}(y,q^2)$ by the $\rho^o$-meson contribution alone.

Proceeding as described above, we get for the SVZ-transform $\Phi$:

$$\Phi^{\rho(3)} = -\frac{64\pi^2}{27}\frac{\alpha_s\langle\bar{u}u\rangle}{M^6}\frac{m_\rho f_\rho^V f_\rho^T}{m_\rho^2+q^2}$$
$$\iint_0^1 dy d\beta\, e^{\beta(q^2y\bar{y}-Q^2y)/((1-y\beta)M^2)}\frac{\beta\varphi_\rho^T(y)}{(1-y\beta)^3}, \qquad (6.54)$$

where $\varphi_\rho^T(y)$ is the normalized distribution amplitude (its zeroth moment equals 1). To get a correct overall normalization of the correlator for large $q^2$, we should approximate $m_\rho f_\rho^V f_\rho^T$ by $-2\langle\bar{u}u\rangle$. For $q^2 = 0$, introducing the variable $s = Q^2y\beta/(1-y\beta)$, we then get:

$$\Phi^{\rho(3)} = \frac{128\pi^2}{27}\frac{\alpha_s\langle\bar{u}u\rangle^2}{M^6Q^4m_\rho^2}\int_0^\infty e^{-s/M^2}\,g^T(s)\,ds\,, \qquad (6.55)$$

where

$$g^T(s) = s\int_{s/s+Q^2}^{1} dy\,\frac{\varphi_\rho^T(y)}{y^2}\,, \qquad (6.56)$$

If we model $\varphi_\rho^T(y)$ by its asymptotic form $\varphi_\rho^T(y) = 6y(1-y)$, this gives

$$g^T(s) = 6s\left(\ln\frac{Q^2+s}{s} - \frac{Q^2}{Q^2+s}\right)\,. \qquad (6.57)$$

Using the formulas (6.18),(6.19) we can convert the expression (6.55) into the canonical form. Note, however, that $g_T''(s)$ contains the $1/s$-singularity, and one should be careful when calculating the relevant spectral density $\rho^T(s,Q^2)$. The simplest procedure is to represent the $\ln s$ term as $\lim_{\lambda^2\to 0}\ln(s+\lambda^2)$. Then application of eq.(6.19) is straightforward, and we get

$$\Phi^T = \frac{128\pi^2}{27}\frac{\alpha_s\langle\bar{u}u\rangle^2}{M^2Q^4m_\rho^2}\lim_{\lambda^2\to 0}\left[\ln\frac{Q^2}{\lambda^2} - 2 + \int_0^\infty e^{-s/M^2}\left(\frac{s^2+3sQ^2+4Q^4}{(s+Q^2)^3} - \frac{1}{s+\lambda^2}\right)ds\right]. \quad (6.58)$$

This trick amounts to using an alternative regularized form for the $(1/s)_+$-distribution usually defined by

$$\int_0 f(s)\left(\frac{1}{s_+}\right)ds = \int_0 \frac{f(s)-f(0)}{s}ds. \quad (6.59)$$

# 7 Bilocals and contact terms

A special care must be taken about the correlators containing the Dirac operator $\gamma_\mu D^\mu$ acting on the quark field $\psi$. Since the correlator is a $T$-product of the electromagnetic current and a composite operator, applying the equation of motion one gets the $\delta^{(4)}(X)$-function, *i.e.*, the external vertices of the bilocal are contracted into a single point and it reduces to a $q^2$-independent constant.

Let us sketch a simple derivation for such terms (see, *e.g.*, [21]). Using the functional representation for the correlator

$$\langle 0|T\{\ldots\bar{\psi}(X)\hat{\nabla}\psi(0)\}|0\rangle = \int \mathcal{D}[\bar{\psi}]\,\mathcal{D}[\psi]\,\mathcal{D}[A]\,\{\ldots\bar{\psi}(X)\hat{\nabla}\psi(0)\}\,\exp\left(i\int\mathcal{L}(Z)\,d^4Z\right), \quad (7.1)$$

where $\mathcal{L}(Z) = \bar{\psi}(Z)\,i\hat{\nabla}\psi(Z) + \cdots$ , we can write

$$\hat{\nabla}\psi(0)\,\exp\left(i\int\mathcal{L}(Z)\,d^4Z\right) = -\frac{\delta}{\delta\bar{\psi}(0)}\exp\left(i\int\mathcal{L}(Z)\,d^4Z\right). \quad (7.2)$$

Integrating by parts in (7.1) results in

$$\int \mathcal{D}[\bar{\psi}]\,\mathcal{D}[\psi]\,\mathcal{D}[A]\left\{\cdots\frac{\delta\bar{\psi}(Z)}{\delta\bar{\psi}(0)}\right\}\exp\left(i\int\mathcal{L}(Z)\,d^4Z\right). \quad (7.3)$$

It is the derivative $\delta\bar{\psi}(X)/\delta\bar{\psi}(0)$ that produces the $\delta^{(4)}(X)$ term mentioned above.

The contact terms play an important role in all applications of the QCD sum rules to low-momentum behaviour of hadronic form factors. In particular, without them, it is impossible to satisfy the Ward identities fixing the pion form factor normalization at zero momentum transfer [24, 26].

## 7.1 Separating short- and long-distance contributions

Consider the hard gluon exchange diagrams shown in Figs.5$a-r$ which produce, in the $B$-regime, the bilocals associated with the three-propagator coefficient function. Take first the diagrams 5e,f. Their contributions are

$$5e \sim \frac{32i}{p^2 q_1^2 q_2^4}\, q_{1\mu}\, \epsilon_{\alpha\nu q_1 q_2} \qquad 5f \sim \frac{32i}{p^4 q_1^2 q_2^2}\, q_{1\mu}\, \epsilon_{\alpha\nu q_1 q_2}, \tag{7.4}$$

*i.e.*, they do not contribute to the invariant form factor $F$. However, as we will see below, in the B-regime, equations of motion "extract" the appropriate tensor structure, *i.e.*, these diagrams cannot be ignored. The relevant term from the three-point correlation function can be written as[4]

$$\begin{aligned} \mathcal{F}^{B,5ef}_{\alpha\mu\nu}(q_1,q_2) &= -\frac{32\pi^2}{9}\frac{\alpha_s\langle\bar{u}u\rangle}{p^2 q_2^2}\, \epsilon_{\alpha\nu q_1 q_2} \int d^4Z\, e^{ipZ}\, \frac{1}{4\pi^2 Z^2} \sum_{n=0}^{\infty} \frac{1}{n!}\, Z^{\mu_1}\dots Z^{\mu_n} \\ &\times \int dX e^{-iq_1X}\, \langle 0|T\{J_\mu(X)\,\bar{u}(0)(\partial_{\mu_1}\dots\partial_{\mu_n})u(0)\}|0\rangle . \end{aligned} \tag{7.5}$$

Extracting the bilocal term from (7.5), one should pick out the traceless combination of indices $\mu_1,\dots\mu_n$, *i.e.*, the lowest twist term which gives the leading power contribution with respect to $1/p^2; 1/q_2^2$. Introducing the notation

$$\Pi_{\mu\{\mu_1\dots\mu_n\}}(q_1) = \int dX e^{-iq_1X}\, \langle 0|T\{J_\mu(X)\,\bar{u}(0)\{\partial_{\mu_1}\dots\partial_{\mu_n}\}u(0)\}|0\rangle, \tag{7.6}$$

we can represent the correlator (7.6) in the following form:

$$\begin{aligned} \Pi_{\mu\{\mu_1\dots\mu_n\}}(q_1) &= A^{(n)}(q_1^2)\, q_{1\mu}\, \{q_{1\mu_1}\cdots q_{1\mu_n}\} + B^{(n)}(q_1^2)\, \{q_{1\mu}, q_{1\mu_1}\cdots q_{1\mu_n}\} + \\ &+ C^{(n)}(q_1^2)\, g_{\mu}\{_{\mu_1} q_{1\mu_2}\cdots q_{1\mu_n}\}, \end{aligned} \tag{7.7}$$

where $\{\dots\}$ denotes the traceless-symmetric part of a tensor. Because of the electromagnetic current conservation, we have the constraint $q_{1\mu}\, \Pi_{\mu\{\mu_1\dots\mu_n\}}(q_1) = 0$ which produces some relations between the invariant functions $A^{(n)}, B^{(n)}, C^{(n)}$. Using the formula from [42]

$$q_1^\alpha\, \{q_{1\mu_1}\cdots q_{1\mu_{n-1}}, q_\alpha\} = q_1^2\, \frac{n+1}{2n}\, \{q_{1\mu_1}\cdots q_{1\mu_{n-1}}\}, \tag{7.8}$$

we obtain

$$\left(A^{(n)} + \frac{(n+2)}{2(n+1)}\, B^{(n)}\right) q_1^2 + C^{(n)} = 0. \tag{7.9}$$

Contracting (7.6) with $g_{\mu\mu_1}$ gives

$$\Pi_{\mu\{\mu_1\dots\mu_n\}}(q_1)\, g_{\mu\mu_1} = \left(A^{(n)}\, q_1^2\, \frac{(n+1)}{2n} + C^{(n)} \left(\frac{n+1}{n}\right)^2\right) \{q_{1\mu_2}\cdots q_{1\mu_n}\}. \tag{7.10}$$

[4]The charge conjugate contribution can be trivially added.

Furthermore, applying the technique symbolized by eqs. (7.1) - (7.3) (see Appendix E) we obtain, in the leading-twist approximation:

$$\Pi_{\mu\{\mu_1\ldots\mu_n\}}(q_1)\, g_{\mu\mu_1} \simeq (-i)^{n-1}\, \{q_{1\mu_2}\ldots q_{1\mu_n}\}\times$$
$$\times\left[-2\langle\bar{u}u\rangle - \frac{1}{2}\, q_{1\epsilon} \int d^4X e^{-iq_1X}\, \langle 0|T\{J_\mu(X)\,\bar{u}(0)\{\partial_{\mu_1}\ldots\partial_{\mu_n}\}\sigma_{\mu\epsilon}u(0)\}|0\rangle\right], \qquad (7.11)$$

where the first contribution inside the brackets is just the contact term , while the second one cannot be reduced to any other contact terms. Since the relevant perturbative spectral density vanishes, the continuum contribution is not necessary, and the bilocal is modeled by the lowest resonances, *i.e.*, by the $\rho$-meson[5] in the simplest approximation. In this case, we get

$$\Pi_{\mu\{\mu_1\ldots\mu_n\}}(q_1)\, g_{\mu\mu_1} = (-i)^{n-1}\, \{q_{1\mu_2}\ldots q_{1\mu_n}\}\times$$
$$\times\left[-2\langle\bar{u}u\rangle + \frac{(f_\rho^V m_\rho) f_\rho^T}{m_\rho^2 - q_1^2}\, \frac{3}{2}\, q_1^2 \int dy\, y^{n-1}\, \varphi_\rho^T(y)\right], \qquad (7.12)$$

where $\varphi_\rho^T(y)$ is the tensor wave function of the $\rho$-meson. Note that the term containing $\varphi_\rho^T(y)$ vanishes for $q_1^2 = 0$.

Using the formulas from Appendix D, one can easily perform the necessary contractions:

$$Z_{\mu_1}\ldots Z_{\mu_n}\, \Pi_{\mu\{\mu_1\ldots\mu_n\}} = A^{(n)}\, q_{1\mu}\, \tau^n\, C_n^{(1)}(\eta) +$$
$$+\frac{B^{(n)}}{n+1}\left[-Z_\mu \frac{q_1^2}{2}\, \tau^{n-1}\, C_{n-1}^{(2)}(\eta) + q_{1\mu}\, \tau^n\, C_n^{(2)}(\eta)\right]$$
$$+\frac{C^{(n)}}{n}\left[Z_\mu\, \tau^{n-1}\, C_{n-1}^{(2)}(\eta) - q_{1\mu}\, \frac{Z^2}{2}\, \tau^{n-2}\, C_{n-2}^{(2)}(\eta)\right], \qquad (7.13)$$

where $C_n^{(\lambda)}(\eta)$ are the Gegenbauer polynomials and the notation $\eta = i\,(q_1 Z)/\sqrt{-Z^2 q_1^2}$, $\tau = -i\sqrt{-Z^2 q_1^2}/2$ is introduced.

The tensor structure ($\sim p_\alpha\, \epsilon_{\mu\nu q_1 q_2}$) we are interested in, can be only produced in (7.13) by the terms $\sim Z_\mu$. Hence, other terms can be ignored. Combining now eqs. (7.9) – (7.12), we get, modulo the next-to-leading twist contributions:

$$Z_\mu\, \tau^{n-1}\, C_{n-1}^{(2)}(\eta)\left[-\frac{q_1^2\, B^{(n)}}{2(n+1)} + \frac{C^{(n)}}{n}\right] \simeq \qquad (7.14)$$
$$\simeq\; Z_\mu(-i q_1 Z)^{n-1}\, \frac{2n^2}{(n+1)(n+2)}\left[-2\langle\bar{u}u\rangle + \frac{(f_\rho^V m_\rho) f_\rho^T}{m_\rho^2 - q_1^2}\, \frac{3}{2}\, q_1^2 \int dy\, y^{n-1}\, \varphi_\rho^T(y)\right].$$

Substituting (7.15) into (7.5), integrating over $d^4Z$ and summing over $n$ by using the generating function technique we get for the SVZ-transform of the contact terms:

$$\Phi^{5ef(C)} = -\frac{256\pi^2}{27}\, \frac{\alpha_s\langle\bar{u}u\rangle^2}{Q^2 M^6} \int\!\!\int_0^1 d\beta dy\, \frac{\beta(\bar{\beta}-\beta)y}{(1-y\beta)^3}\, e^{y\beta(q^2\bar{\beta}-Q^2)/((1-y\beta)M^2)}. \qquad (7.15)$$

[5]Note, that in the chiral limit $m_u = m_d = 0$, the $\rho$- and $\omega$-contributions cannot be distinguished and are equal to each other (cf. [18]).

The non-contact terms give

$$\Phi^{5ef(\rho)} = -\frac{256\pi^2}{27}\frac{\alpha_s\langle\bar{u}u\rangle}{Q^2M^6}\frac{(f_\rho^V m_\rho)f_\rho^T}{m_\rho^2+q^2}\frac{3}{4}q^2 \times \tag{7.16}$$

$$\times \int\int\int_0^1 d\alpha d\beta dy\, \frac{y\alpha\beta(\overline{y\beta}-y\bar{\beta})}{(1-y\alpha\beta)^3}\, \varphi_\rho^T(y)\; e^{y\alpha\beta(q^2\overline{y\beta}-Q^2)/(1-y\alpha\beta)M^2}$$

where $\overline{y\beta} \equiv 1-y\beta,\quad \bar{\beta}\equiv 1-\beta$.

Analyzing the remaining bilocal contributions capable of producing a coefficient function of the three-propagator type (see diagrams of Fig.5 $b, c, i, l, o, p$), we found that the contact type terms are either absent or do not terms with the tensor structure $p_\alpha\, \epsilon_{\mu\nu q_1 q_2}$. Contact terms are also absent for the bilocals corresponding to the one- and two-propagator coefficient functions.

For $q^2=0$ the contribution of the contact terms reads:

$$\Phi^C(Q^2,M^2) = -\frac{256\pi^2\alpha_s\langle\bar{q}q\rangle^2}{27Q^6M^6}\int_0^\infty e^{-s/M^2}\left[\ln\frac{s+Q^2}{s} - 2\frac{Q^2}{s+Q^2}\right] s ds. \tag{7.17}$$

Representing again $\ln s = \lim_{\lambda^2\to 0}\ln(s+\lambda^2)$ and using eq.(6.19), we obtain

$$\Phi^C(Q^2,M^2) = -\frac{256\pi^2\alpha_s\langle\bar{q}q\rangle^2}{27Q^6M^2}\lim_{\lambda^2\to 0}\left[\ln\frac{Q^2}{\lambda^2} - 3 + \int_0^\infty e^{-s/M^2}\left(\frac{s^2+3sQ^2+6Q^4}{(s+Q^2)^3} - \frac{1}{s+\lambda^2}\right) ds\right]. \tag{7.18}$$

# 8 QCD sum rule in the small-$q^2$ kinematics

Collecting now all the contributions, we obtain the theoretical part (the modified OPE) of the QCD sum rule for the form factor $F_{\gamma^*\gamma^*\to\pi^0}(q^2,Q^2)$ (see Fig.3):

$$\begin{aligned}\Phi(q^2,Q^2,M^2) = {} & \Phi^{PT}(q^2,Q^2,M^2) + \Phi_{5b}^{\langle GG\rangle} + \Phi_{5c}^{\langle GG\rangle} + \Phi_{5f,g}^{\langle GG\rangle} + \\ & + \Phi_{6g,h,i}^{\langle\bar{q}q\rangle} + \Phi_{6d,e,f}^{\langle\bar{q}q\rangle} + \Phi_{5a,b,c}^{\langle\bar{q}q\rangle} + \Phi_{5d}^{\langle\bar{q}q\rangle} + \\ & + \Phi^{B(\rho)} + \left(\Phi^{B(cont)} - \Phi^{B(PT)}\right) - \Phi_{5b}^B - \Phi_{5c}^B - \Phi_{6d,f}^B + \\ & + \Phi^{\rho(2)} - \Phi_{5f,g}^B - \Phi_{6e}^B - \Phi_{5a}^B + \\ & + \Phi^{\rho(3)} - \Phi_{5b,c}^B + \left[\Phi^{5ef(C)} + \Phi^{5ef(\rho)}\right],\end{aligned} \tag{8.1}$$

where the first two rows correspond to the original OPE valid for symmetric kinematics. Each of the next rows represents the additional terms corresponding to different types of the coefficient functions. As explained in Section 4, all the terms of the standard OPE, which are non-analytic in the $q^2 \to 0$ limit, are cancelled by the corresponding B-contributions. As a result, the coefficient functions of the SD-regime are analytic functions of $q^2$ (compare with [20]). Substituting explicit expressions for all the terms which appear in eq.(8.1) gives the following expression for the SVZ-Borel transformed OPE for the three-point correlator valid in the region of small $q^2$:

$$\Phi(q^2,Q^2,M^2) = \frac{1}{\pi M^2}\left\{\int_0^1 dy\; e^{-Q^2y/M^2\bar{y}} \left\{(1+\frac{q^2y}{M^2}e^{q^2y/M^2}) + \right.\right.$$

$$+e^{q^2y/M^2}\left[\frac{2y}{M^2}\left(q^2\ln\frac{(\sigma_o+q^2)y}{M^2}-\sigma_o\right)+\frac{y^2}{M^4}\left(q^4\ln\frac{(\sigma_o+q^2)y}{M^2}-q^2\sigma_o+\frac{\sigma_o^2}{2}\right)\right]-$$

$$\left.-\sum_{n=1}^{\infty}\left(\frac{q^2y}{M^2}\right)^n\frac{\psi(n)(n+1)}{(n-1)!}\right\}+$$

$$+\frac{\pi^2}{9}\langle\frac{\alpha_s}{\pi}GG\rangle\left[\frac{1}{2M^2Q^2}+\frac{1}{M^4}\int_0^1 dy\,\frac{y}{\bar{y}^2}\,e^{-Q^2y/M^2\bar{y}}\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{q^2y}{M^2}\right)^{n-1}\right]+$$

$$+\frac{64\pi^3}{243}\,\alpha_s\langle\bar{q}q\rangle^2\,\frac{q^2}{Q^4M^2}\;+\;\frac{64\pi^3}{27}\,\alpha_s\langle\bar{q}q\rangle^2\,\frac{1}{2Q^2M^4}\;+$$

$$+\frac{4\pi^2}{3}\frac{f_\rho^V m_\rho}{m_\rho^2+q^2}\int_0^1 dy\,\frac{1}{\bar{y}^2M^2}\,e^{-Q^2y/M^2\bar{y}}\,e^{q^2y/M^2}$$

$$\times\left[-a_{V1}\,f_\rho^V m_\rho\left(\varphi_{\rho\perp}^{V1}(y)-\frac{4C_{V51}}{\bar{y}M^2}\varphi_{\rho\perp}^{V51}(y)\right)-f_\rho^A(1+2\bar{y})\left(\varphi_{\rho\perp}^{A}(y)-\frac{4C_{A5}}{\bar{y}M^2}\varphi_{\rho\perp}^{A5}(y)\right)\right]$$

$$+\frac{8\pi^2}{3}\frac{f_\rho^V m_\rho}{m_\rho^2+q^2}\int_0^1 d\alpha\,\alpha\int_0^1 d\beta\int_0^1[dy]_3\;e^{\;b/aM^2}$$

$$\times\left\{f_{3\rho}^A\,\varphi_{3\rho}^A(y_1,y_2;y_3)\left[\frac{c_1}{a^2M^2}-\frac{d_1}{2a^3M^4}\right]-f_{3\rho}^V\,\varphi_{3\rho}^V(y_1,y_2;y_3)\left[\frac{c_2}{a^2M^2}-\frac{d_2}{2a^3M^4}\right]\right\}$$

$$-\frac{64\pi^3}{27}\frac{\alpha_s\langle\bar{u}u\rangle}{M^4}\frac{m_\rho f_\rho^V f_\rho^T}{m_\rho^2+q^2}\iint_0^1 dy d\beta\,\frac{\beta\,\varphi_\rho^T(y)}{(1-y\beta)^3}\;e^{\beta(q^2y\bar{y}-Q^2y)/((1-y\beta)M^2)}$$

$$-\frac{256\pi^3}{27}\frac{\alpha_s\langle\bar{u}u\rangle^2}{Q^2M^4}\iint_0^1 d\beta dy\,\frac{\beta(\beta-\beta)y}{(1-y\beta)^3}\;e^{y\beta(q^2\bar{\beta}-Q^2)/((1-y\beta)M^2)}$$

$$\left.-\frac{64\pi^3}{9}\frac{\alpha_s\langle\bar{u}u\rangle}{Q^2M^4}\frac{m_\rho f_\rho^V f_\rho^T}{m_\rho^2+q^2}q^2\iiint_0^1 d\alpha d\beta dy\,\frac{y\alpha\beta(\bar{y}\bar{\beta}-y\beta)}{(1-y\alpha\beta)^3}\,\varphi_\rho^T(y)\;e^{y\alpha\beta(q^2\bar{y}\bar{\beta}-Q^2)/(1-y\alpha\beta)M^2}\right\} \quad (8.2)$$

This expression can be used as a starting point for constructing sum rules for the form factor $F_{\gamma^*\gamma^*\pi^0}(q^2,Q^2)$ at $q^2=0$, its derivative $\frac{\partial}{\partial q^2}F_{\gamma^*\gamma^*\pi^0}(q^2,Q^2)|_{q^2=0}$ or even for studying the $q^2$-dependence of $F_{\gamma^*\gamma^*\pi^0}(q^2,Q^2)$ in the region of small $q^2 \lesssim 1\,GeV^2$ for fixed values of the large virtuality $Q^2 \gtrsim 2\,GeV^2$.

For the constants present in eq.(8.2) we use the following numerical values: $f_\rho^V = 0.2$ GeV, $m_\rho = 0.77$ GeV ; the constants $f_\rho^A = -f_\rho^V m_\rho/4$, $a_{V1} = 1/40$ are obtained from the equations of motion (see Appendix C), the values $f_{3\rho}^A = 0.6 \cdot 10^{-2}\,\text{GeV}^2, f_{3\rho}^V = 0.25 \cdot 10^{-2}\,\text{GeV}^2$ are taken from the QCD sum rule estimates given in ref. [41]. The quark and gluon condensate values are standard: $\langle(\alpha_s/\pi)GG\rangle = 0.012\,GeV^4$, $\alpha_s\langle\bar{q}q\rangle^2 = 1.8\cdot 10^{-4}\,GeV^6$. For the continuum threshold in the $\rho$-channel we take the standard value $\sigma_o \simeq 1.5\,\text{GeV}^2$ obtained from the QCD sum rule for the $\rho$-decay constant $f_\rho^V$ [18]. This value was also extracted from the QCD sum rule analysis of the first few moments of the $\rho$-meson wave functions [41]. To estimate relative importance of various contributions, we take the asymptotic forms for the following $\rho$-meson wave functions (see Appendix C):

$$
\begin{aligned}
\varphi_{V1} &= \varphi_{V1}^{as} = 60y\bar{y}\,(2y-1),\\
\varphi_A &= \varphi_A^{as} = 6y\bar{y},\\
\varphi_{3A} &= \varphi_{3A}^{as} = 360y_1y_2y_3^2,\\
\varphi_{3V} &= \varphi_{3V}^{as} = 7!(y_1-y_2)y_1y_2y_3^2.
\end{aligned}
\tag{8.3}
$$

Numerical analysis shows that the most important contributions in (8.2) come from: *a*) SD-regime (first five rows of (8.2)) and *b*) $\rho^o$-meson contribution with the leading twist wave functions (B-regime) in diagonal and nondiagonal correlators. The terms associated with three-particle twist-5 wave functions are small: their contribution into the sum rule is of the order of a few percent. We expect that terms corresponding to the next-to-leading two-particle wave functions (also twist-5) are similarly suppressed. Contact-type power corrections were also found to be small.

Below, we consider only the simplest sum rule for $F_{\gamma^*\gamma^*\pi^0}(q^2=0,Q^2)$, keeping in it only the most important terms listed above and contact terms. All the necessary expressions substituting the terms from eq.(8.2) by their $q^2=0$ limit were given in preceding sections. Combining them together, we obtain the QCD sum rule for the $\gamma\gamma^* \to \pi^0$ form factor:

$$
\begin{aligned}
\pi f_\pi F_{\gamma\gamma^*\pi^0}(Q^2) = &\int_0^{s_0}\left\{1 - 2\frac{Q^2-2s}{(s+Q^2)^2}\left(s_\rho - \frac{s_\rho^2}{2m_\rho^2}\right)\right.\\
&\left. + 2\frac{Q^4-6sQ^2+3s^2}{(s+Q^2)^4}\left(\frac{s_\rho^2}{2} - \frac{s_\rho^3}{3m_\rho^2}\right)\right\} e^{-s/M^2}\frac{Q^2 ds}{(s+Q^2)^2}\\
+ &\frac{\pi^2}{9}\langle\frac{\alpha_s}{\pi}GG\rangle\left\{\frac{1}{2Q^2M^2} + \frac{1}{Q^4} - 2\int_0^{s_0} e^{-s/M^2}\frac{ds}{(s+Q^2)^3}\right\}\\
&+\frac{64}{27}\pi^3\alpha_s\langle\bar{q}q\rangle^2 \lim_{\lambda^2\to 0}\left\{\frac{1}{2Q^2M^4} + \frac{12}{Q^4m_\rho^2}\left[\log\frac{Q^2}{\lambda^2} - 2\right.\right.\\
&\left.\left. + \int_0^{s_0} e^{-s/M^2}\left(\frac{s^2+3sQ^2+4Q^4}{(s+Q^2)^3} - \frac{1}{s+\lambda^2}\right)ds\right]\right.
\end{aligned}
$$

$$-\frac{4}{Q^6}\left[\log\frac{Q^2}{\lambda^2}-3+\int_0^{s_0}e^{-s/M^2}\left(\frac{s^2+3sQ^2+6Q^4}{(s+Q^2)^3}-\frac{1}{s+\lambda^2}\right)ds\right]\Bigg\}. \qquad (8.4)$$

The sum rule must be taken in the limit $\lambda^2 \to 0$ of the parameter $\lambda^2$ specifying the regularization which we used to calculate the integrals with the $(1/s)_+$ distribution. Furthermore, in this sum rule we model the continuum by an effective spectral density $\rho^{eff}(s,Q^2)$ rather than by $\rho^{PT}(s,Q^2)$, with $\rho^{eff}(s,Q^2)$ including all the spectral densities which are nonzero for $s>0$, *i.e.*, $\rho_3^B(s,Q^2)$ (6.30), $\rho_3^\varrho(s,Q^2)$ (6.33), $\rho_5^B(s,Q^2)$ (6.31), $\rho_5^\varrho(s,Q^2)$ (6.35), $\rho^T(s,Q^2)$ (6.58), $\rho^C(s,Q^2)$ (7.18) and also an analogous contribution from the gluon condensate term.

We studied the stability of our sum rule with respect to variations of the SVZ-Borel parameter $M^2$ in the region $M^2 > 0.6\,GeV^2$. Good stability was observed not only for the "canonical" value $s_0^\pi \approx 0.7\,GeV^2$ but also for smaller values of $s_0$, even as small as $0.4\,GeV^2$. Since our results are sensitive to the $s_0$-value, we incorporated a more detailed model for the spectral density, treating the $A_1$-meson as a separate resonance at $s = 1.6\,GeV^2$, with the continuum starting at some larger value $s_A$. The results obtained in this way have a good $M^2$-stability and, for $M^2 < 1.2\,GeV^2$, show no significant dependence on $s_A$. Numerically, they practically coincide with the results obtained from the sum rule (8.4) for $s_0 = 0.7\,GeV^2$.

In Fig.11, we present a curve for $Q^2F_{\gamma\gamma^*\pi^0}(Q^2)/4\pi f_\pi$ calculated from eq.(8.4) for $s_0 = 0.7\,GeV^2$ and $M^2 = 0.8\,GeV^2$. It is rather close to the curve corresponding to the Brodsky-Lepage interpolation formula $\pi f_\pi F_{\gamma\gamma^*\pi^0}(Q^2) = 1/(1+Q^2/4\pi^2f_\pi^2)$ and to that based on the $\rho$-pole approximation $\pi f_\pi F(Q^2) = 1/(1+Q^2/m_\rho^2)$. It should be noted, however, that the closeness of our results to the $\rho$-pole behaviour in the $Q^2$-channel has nothing to do with the explicit use of the $\rho$-contributions in our models for the correlators in the $q^2$-channel: the $Q^2$-dependence of the $\rho$-pole type emerges due to the fact that the pion duality interval $s_0 \approx 0.7\,GeV^2$ is numerically close to $m_\rho^2 \approx 0.6\,GeV^2$.

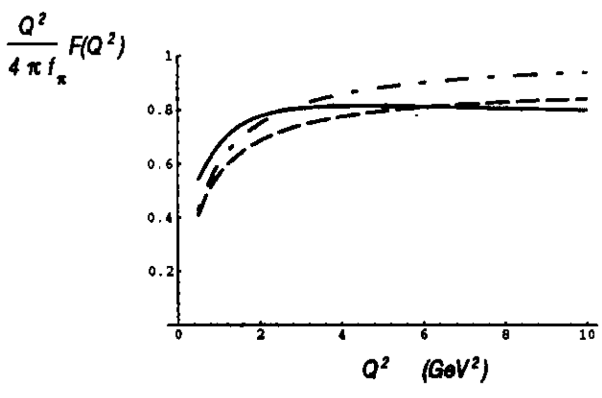


Figure 11: Combination $Q^2F_{\gamma\gamma^*\pi^0}(Q^2)/4\pi f_\pi$ as calculated from the QCD sum rule (solid line), $\rho$-pole model (dashed line) and Brodsky-Lepage interpolation (dash-dotted line).

For $Q^2 < 3\,GeV^2$, our curve goes slightly above those based on the $\rho$-pole dominance and BL-interpolation (which are close to the data [8]). This overshooting is a consequence of our assumption that $Q^2$ can be treated as a large variable: in some terms, $1/Q^2$ serves as an expansion parameter. Such an approximation for these terms is invalid for small $Q^2$ and appreciably overestimates them for $Q^2 \sim 1\, GeV^2$ producing enlarged values for $F_{\gamma\gamma^*\pi^0}(Q^2)$.

In the region $Q^2 > 3\, GeV^2$, our curve for $Q^2 F_{\gamma\gamma^*\pi^0}(Q^2)$ is practically constant, supporting the pQCD expectation (2.19). The absolute magnitude of our prediction gives $I \approx 2.4$ for the $I$-integral. Of course, this value has some uncertainty: it will drift if we change our models for the photon distribution amplitudes (bilocals). The strongest sensitivity is to the choice of $\varphi_\rho^T(y)$ in the tensor contribution (6.55). However, even rather drastic changes in the form of $\varphi_\rho^T(y)$ do not increase our result for $I$ by more than 20%. The basic reason for this stability is that the potentially large $1/q^2$ factor from the relevant contribution in the original sum rule (3.11) is substituted in (2.7) by a rather small (and non-adjustible) factor $1/m_\rho^2$.

Comparing the value $I = 2.4$ with $I^{as} = 3$ and $I^{CZ} = 5$, we conclude that our result favours a pion distribution amplitude which is narrower than the asymptotic form. Parametrizing the width of $\varphi_\pi(x)$ by a simple model $\varphi_\pi(x) \sim [x(1-x)]^n$, we get that $I = 2.4$ corresponds to $n = 2.5$. The second moment $\langle \xi^2 \rangle$ ($\xi$ is the relative fraction $\xi = x - \bar{x}$) for such a function is 0.125. This low value (recall that $\langle \xi^2 \rangle^{as} = 0.2$ while $\langle \xi^2 \rangle^{CZ} = 0.43$) agrees, however, with the lattice calculation [44] and also with the recent result [45] obtained from the analysis of a non-diagonal correlator.

# 9 Conclusions

Thus, the QCD sum rules support the expectation that the $Q^2$-dependence of the transition form factor $F_{\gamma\gamma^*\pi^0}(Q^2)$ is rather close to a simple interpolation between the $Q^2 = 0$ value (fixed by the ABJ anomaly) and the large-$Q^2$ pQCD behaviour $F(Q^2) \sim Q^{-2}$. Moreover, the QCD sum rule approach enables us to calculate the absolute normalization of the $Q^{-2}$ term. The value produced by the QCD sum rule is close to that corresponding to the asymptotic form $\varphi_\pi^{as}(x) = 6 f_\pi x(1-x)$ of the pion distribution amplitude. Our curve for $F_{\gamma\gamma^*\pi^0}(Q^2)$ is also in satisfactory agreement with the CELLO data [8] and in good agreement with preliminary high-$Q^2$ results from CLEO [9]. Hence, there is a very solid evidence, both theoretical and experimental, that $\varphi_\pi(x)$ is a rather narrow function.

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# Appendix

## A Alpha-representation and asymptotic behaviour of the three-point function

To study the asymptotic behaviour of the perturbative amplitudes in the limit when some momentum invariants are large, one can use different types of integral representations for the relevant diagrams. For the purposes of a general analysis, one of the most effective approaches is that using the the "alpha-representation" for the relevant Feynman integral. To get the alpha-representation, one should write the denominator of each propagator of the Feynman diagram as

$$\frac{1}{m_\sigma^2 - k_\sigma^2 - i\epsilon} = i\int_0^\infty \exp\{\alpha_\sigma(k_\sigma^2 - m_\sigma^2 + i\epsilon)\}d\alpha_\sigma \tag{A.1}$$

where $\sigma$ numerates the lines of the diagram, and then take the resulting Gaussian integration over all the virtual momenta $k_\sigma$. As a result, for each diagram contributing to $T(q_1, q_2)$ (see Fig.7), one gets the expression having the following structure:

$$T(q_1,q_2) = \frac{P(\text{c.c.})}{(4\pi)^{zd/4}}\int_0^\infty \prod_\sigma d\alpha_\sigma D^{-d/2}(\alpha)$$
$$G(\alpha, q_1, q_2; m_\sigma)\exp\left\{ip^2\frac{A_0(\alpha)}{D(\alpha)} + iq_1^2\frac{A_1(\alpha)}{D(\alpha)} + iq_2^2\frac{A_2(\alpha)}{D(\alpha)} - i\sum_\sigma \alpha_\sigma(m_\sigma^2 - i\epsilon)\right\} \tag{A.2}$$

where $d$ is the space-time dimension, $P(\text{c.c.})$ is the relevant product of the coupling constants, $z$ is the number of loops of the diagram; $D, Q, G$ are functions of the $\alpha$-parameters uniquely determined by the structure of the diagram. In particular, $D(\alpha)$ is a sum of products of the $\alpha$-parameters, each term in the sum containing $z$ $\alpha$-factors. In our case, all the functions $A_i(\alpha)$ are also the sums of products of the $\alpha$-parameters, with $z+1$ $\alpha$-parameters in each product. Hence, $D(\alpha)$ and all $A_i(\alpha)$ are positive for positive $\alpha$'s. The preexponential factor $G(\alpha, q_1, q_2; m_\sigma)$ is a polynomial in $\alpha$'s, $p^2$, $q_1^2$ and $q_2^2$.

In the region where one of the momentum variables $p_i^2$ is large, all the contributions having a power-type behaviour on that variable can only come from the integration region where the relevant $A_i/D$ factor vanishes: if $A_i/D$ is larger than some constant $\rho$ in the region of integration, the resulting contribution is $\sim \exp(p_i^2\rho)$, *i.e.*, it is exponentially suppressed.

When all $A_i$'s are non-negative, there are two basic possibilities to arrange $A_i/D = 0$. In the first case, called the "short-distance regime", $A_i$ vanishes faster than $D$ when some of the $\alpha$-parameters tend to zero (small $\alpha$ correspond to large virtualities $k^2$, *i.e.*, to short distances). The second possibility, called the "infrared regime", occurs if $D$ goes to infinity faster than $A_i$ when some of the $\alpha$-parameters tend to infinity (large $\alpha$ correspond to small momenta $k$, *i.e.* to the infrared limit). One can also imagine a combined regime, when $A_i/D = 0$ because some $\alpha$-parameters vanish and some are infinite.

In fact, there exists a simple rule using which one can easily find the lines $\sigma$ whose $\alpha$-parameters must be set to zero and whose must be taken infinite to assure that $A_i/D = 0$. First, one should realize that $A_i/D = 0$ means that the corresponding diagram of a scalar theory (in which $G = 1$) has no dependence on the momentum invariant $p_i^2$. As the second step, one should incorporate the well-known analogy between the Feynman diagrams and electrical circuits [46]: the $\alpha_\sigma$-parameters may be interpreted as the resistances of the corresponding lines $\sigma$. In other words, $\alpha_\sigma = 0$ corresponds to the short-circuiting the line $\sigma$ while $\alpha_\sigma = \infty$ corresponds to its removal from the diagram. Hence, the problem is to find the sets of lines $\{\sigma\}_{SD}$, $\{\sigma\}_{IR}$ whose contraction into point (for $\{\sigma\}_{SD}$) or removal from the diagram (for $\{\sigma\}_{IR}$) produces the diagram which, in a scalar theory, does not depend on $p_i^2$.

Thus, the rule determining possible topological types of the short-distance factorizable contributions is the following: if the part of the diagram corresponding to a short-distance subprocess is contracted into point, the resulting effective diagram should have no dependence on the large momentum invariants.

The simplest situation is when the short-distance part coincides with the whole diagram. This configurion is allowed for any relation between $q_1^2, q_2^2$ and $p^2$. In this case, all the currents $J_\mu(X)$, $J_\nu(0)$ and $j_\alpha^5(Y)$ (see Fig.7) are close to each other, *i.e.*, all the intervals $X^2, Y^2, (X-Y)^2$ are small. However, if, say, the variable $q_1^2$ is small, the dependence on large variables $q_2^2$ and $p^2$ can be eliminated by contracting into point a subgraph containing the vertices corresponding to momenta $q_2$ and $p$. In this situation, the interval $Y^2$ is small while $X^2$ and $(X-Y)^2$ are large. Such a configuration is sensitive to the long-distance effects in the $q_1$-channel. They can be described by introducing distribution amplitudes for the $q_1$-photon. Finally, another interesting situation is when both the photon virtualities are much larger than $p^2$. Then there exists a short-distance subprocess which includes only the photon vertices: the interval $X^2$ is small, while $Y^2$ and $(X-Y)^2$ may be large. In this case, there are long-distance effects in the axial current channel, which are usually described/parameterized by the pion distribution amplitude.

# B Calculation of some useful momentum integrals

We shall calculate our integrals using dimensional regularization. The basic, well-known, integral reads:

$$I(L,r) = \int d^D\hat{p} \frac{(p^2)^r}{[p^2+S]^L} = \frac{i(-1)^{r-L}}{(4\pi)^{D/2}} \frac{\mu^{4-D}}{\Gamma(L)} \frac{\Gamma(r+D/2)}{\Gamma(D/2)} \frac{\Gamma(L-r-D/2)}{(-S)^{L-r-D/2}} \tag{B.1}$$

where $D = 4-2\epsilon$ and $d^D\hat{p} \equiv d^Dp/(2\pi)^D$. It is more convenient to define the integral:

$$R(L,r) \equiv I(L,r) 2^r \frac{\Gamma(D/2)}{\Gamma(r+D/2)} \tag{B.2}$$

The integrals of interest are of the form:

$$\int d^D\hat{p} \frac{(py)^n\{1, p_\rho, p_\rho p_\sigma, \ldots\}}{(p^2)^\alpha (p-q)^{2\beta}} = \int_0^1 dx x^{\alpha-1} \bar{x}^{\beta-1} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int d^D\hat{p} \frac{(py)^n\{1, p_\rho, p_\rho p_\sigma, \ldots\}}{[(p-\tilde{q})^2+S]^L} \tag{B.3}$$

where $L=\alpha+\beta, \tilde{q}=q\bar{x}, S=q^2x\bar{x}, \bar{x}\equiv 1-x$. Omitting for a moment the integration over $x$ we are left with:

$$\{J(L,n), J_\rho(L,n), J_{\rho\sigma}(L,n),\ldots\} = \int d^D\hat{p}\frac{(py)^n\{1,p_\rho,p_\rho p_\sigma,\ldots\}}{[(p-\tilde{q})^2+S]^L} \tag{B.4}$$

After a shift of the integration variable we expand in a standard way:

$$(p+\tilde{q}.y)^n = (\tilde{q}.y)^n + C_n^1(\tilde{q}.y)^{n-1}(py) + C_n^2(\tilde{q}.y)^{n-2}(py)^2+\cdots \tag{B.5}$$

where $C_n^m \equiv n!/m!(n-m)!$ are the binomial coefficients. Now the integration over $d^D\hat{p}$ is straightforward and we obtain up to terms $\sim y^2$:

$$J(L,n) = (\tilde{q}.y)^n R(L,0) + C_n^2(\tilde{q}.y)^{n-2}y^2R(L,1) + \bar{O}(y^4), \tag{B.6}$$

$$\begin{aligned} J_\rho(L,n) &= \tilde{q}_\rho\left\{(\tilde{q}.y)^n R(L,0) + C_n^2(\tilde{q}.y)^{n-2}y^2R(L,1)+\bar{O}(y^4)\right\} + \\ &+ y_\rho\left\{C_n^1(\tilde{q}.y)^{n-1}R(L,1) + C_n^3(\tilde{q}.y)^{n-3}3y^2R(L,2)+\bar{O}(y^4)\right\}, \end{aligned} \tag{B.7}$$

$$\begin{aligned} J_{\rho\sigma}(L,n) &= \tilde{q}_\rho\tilde{q}_\sigma\left\{(\tilde{q}.y)^n R(L,0) + C_n^2(\tilde{q}.y)^{n-2}y^2R(L,1)+\bar{O}(y^4)\right\} + \\ &+ (\tilde{q}_\rho y_\sigma + \tilde{q}_\sigma y_\rho)\left\{C_n^1(\tilde{q}.y)^{n-1}R(L,1) + C_n^3(\tilde{q}.y)^{n-3}3y^2R(L,2)+\bar{O}(y^4)\right\} + \\ &+ g_{\rho\sigma}(\tilde{q}.y)^n R(L,1) + C_n^2(\tilde{q}.y)^{n-2}R(L,2)\left\{2y_\rho y_\sigma + g_{\rho\sigma}y^2\right\} + \\ &+ C_n^4(\tilde{q}.y)^{n-4}R(L,3)\left\{12y_\rho y_\sigma y^2 + \bar{O}(y^4)\right\} \end{aligned} \tag{B.8}$$

# C Equations of motion and $\rho$-meson wave functions

Here we demonstrate how one can use equations of motion to obtain relations between the moments of the $\rho$-meson wave functions of different twist. A similar analysis was done in refs. [39] and [47].

Consider the identity:

$$\langle 0|\bar{\psi}_\alpha(0)\,(i\hat{\nabla}-m)_{\beta\rho}\,\psi_\rho(z)|\rho\rangle = 0, \tag{C.1}$$

where $i\hat{\nabla}_{\beta\alpha} = (i\hat{\partial}_z + g\hat{A}(z))_{\beta\alpha}$. Applying the Fiertz transformation we rewrite (C.1) as

$$\begin{aligned} &(i\hat{\nabla}-m)_{\beta\alpha}S(z) + \left[(i\hat{\nabla}-m)\,\gamma_5\right]_{\beta\alpha}P(z) + \left[(i\hat{\nabla}-m)\,\gamma_\mu\right]_{\beta\alpha}V_\mu(z) - \\ &- \left[(i\hat{\nabla}-m)\,\gamma_\mu\gamma_5\right]_{\beta\alpha}A_\mu(z) + \frac{1}{2}\left[(i\hat{\nabla}-m)\,\sigma_{\mu\delta}\right]_{\beta\alpha}T_{\mu\delta}(z) = 0. \end{aligned} \tag{C.2}$$

To obtain a relation between wave functions, one should substitute in (C.2) the expressions for the bilocal matrix elements like (6.8),(6.9),(6.38)(6.39),(6.53), differentiate with respect to $z$ and put $z^2=0$. By contraction with $[\sigma_{\nu\rho}]_{\alpha\beta}$, we extract a combination of the V-, A- and T-projections. There are three independent tensor structures

$$z_\nu\varepsilon_\rho - z_\rho\varepsilon_\nu,\ (\varepsilon z)\,(z_\nu p_\rho - z_\rho p_\nu),\ p_\nu\varepsilon_\rho - p_\rho\varepsilon_\nu$$

and, as a result, we get three systems of equations:

$$f_\rho^A m_\rho^2 \langle x^{n+1}\rangle_A \;=\; f_{3\rho}^A m_\rho^2 \int_0^1 d\beta\,\beta\, n\,\langle\, [x_3\beta + x_2]^{n-1}\rangle_{3A} \;+$$

$$+\; 2 f_\rho^A C_{A5}\, n\,\langle x^{n-1}\rangle_{A5} \;-\; - 2C_{V5}\, f_\rho^V m_\rho \langle x^n\rangle_{V5} + \frac{3}{2}\, f_\rho^{V25}\,\langle x^n\rangle_{V25}, \quad \text{(C.3)}$$

$$f_\rho^A\, C_{A5}\,\langle x^n\rangle_{A5} \;=\; a_{V1}\, f_\rho^V m_\rho\, C_{V15}\,\langle x^n\rangle_{V15} \;+\; \frac{1}{2}\, f_\rho^{V25}\,\langle x^{n+1}\rangle_{V25}, \quad \text{(C.4)}$$

$$(n+2)\, f_\rho^A\,\langle x^n\rangle_A \;=\; -\, f_\rho^V m_\rho \langle x^{n+1}\rangle_V \;-\; a_{V1}\, f_\rho^V m_\rho \langle x^n\rangle_{V1} \;-$$

$$-\, f_{3\rho}^A \int_0^1 d\beta\,\beta\, n\,\langle\, [x_3\beta + x_2]^{n-1}\rangle_{3A}$$

$$+\, f_{3\rho}^V \int_0^1 d\beta\,\beta\, n\,\langle\, [x_3\beta + x_2]^{n-1}\rangle_{3V} \;+\; 4m_q\, f_\rho^T\,\langle x^n\rangle_T. \quad \text{(C.5)}$$

Eq.(C.5) was derived also in [39], but the constant $a_{V1}$ was missed there. In the chiral limit, we can neglect the last term in (C.5). Taking the infinite limit for the renormalization parameter, $\mu^2 \to \infty$, we obtain the equation relating the moments of the asymptotic twist-3 wave functions:

$$(n+2)\, f_\rho^A\,\langle x^n\rangle_A \;=\; -\, f_\rho^V m_\rho \langle x^{n+1}\rangle_V \;-\; a_{V1}\, f_\rho^V m_\rho \langle x^n\rangle_{V1}. \quad \text{(C.6)}$$

Taking into account the normalization conditions (6.10), we conclude that there exists the only solution:

$$\varphi_V^{as} = \frac{3}{2}(1 - 2x\bar{x}),\;\; \varphi_{V1}^{as} = 60x\bar{x}\,(2x-1),\;\; \varphi_A^{as} = 6x\bar{x} \quad \text{(C.7)}$$

with

$$f_\rho^A = -\frac{f_\rho^V\, m_\rho}{4}, \quad a_{V1} = \frac{1}{40}. \quad \text{(C.8)}$$

Note, that $\varphi_V^{as}, \varphi_{V1}^{as}$, given by eqs. (C.7) and (C.8), obey the condition that for a longitudinally polarized $\rho^o$-meson (*i.e.*, when $\varepsilon_\sigma^{\lambda=0} \simeq i\,p_\sigma/m_\rho + \mathcal{O}\,(m_\rho/p_z)\,$, as $p_z \to \infty$), the leading-twist part in eq.(6.8) provides the well known asymptotic twist-2 vector wave function (cf. eq.(6.4)). The value of $f_\rho^A$ was also calculated in the SR method [41], and the result is in a good agreement with that dictated by equations of motion.

Substituting eq.(A.8) into eq.(C.6), we get

$$\frac{(n+2)}{4}\,\langle x^n\rangle_A \;=\; \langle x^{n+1}\rangle_V \;+\; \frac{1}{40}\,\langle x^n\rangle_{V1}. \quad \text{(C.9)}$$

# D Some properties of traceless combinations

To construct the orthogonal projection operators $P_{(n)}$ onto the subspace of traceless symmetric Lorentz tensors of rank $n$, we use the techniques similar to those in [42]. Here we list some useful formulas concerning these projectors as well as some contractions that appear in the paper.

By definition, for an arbitrary Lorentz tensor $T$ we have [42]:

$$\left[P_{(n)}T\right]^{\mu_1\dots\mu_n} = P_{(n)}{}^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_n}\, T^{\nu_1\dots\nu_n}. \tag{D.1}$$

It is straightforward to derive the formula

$$\left(P_{(n)}T\right)^{\mu_1\dots\mu_n} = \frac{1}{n}\sum_{i=1}^{n} T^{\mu_1\dots[\mu_i]\dots\mu_n\mu_i} - \frac{1}{n^2}\sum_{i<j}^{n} g^{\mu_i\mu_j}\, T^{\mu_1\dots[\mu_i]\dots[\mu_j]\dots\mu_n\alpha\,\alpha}, \tag{D.2}$$

where $T^{\nu_1\dots\nu_{n-1}\,\alpha}$ is now traceless and symmetric in its first $n-1$ indices and $[\mu_i]$ means that the corresponding index is absent. Choosing $T^{\{\nu_1\dots\nu_{n-1}\}\alpha} \equiv s^\alpha\,\{q_1^{\nu_1}\dots q_1^{\nu_{n-1}}\}$ we have:

$$\begin{aligned}\{s^{\mu_1}q_1^{\mu_2}\dots q_1^{\mu_n}\} &= \frac{1}{n}\sum_{i=1}^{n} s^{\mu_i}\,\{q_1^{\mu_1}\dots[q_1^{\mu_i}]\dots q_1^{\mu_n}\} - \\ &- \frac{1}{n^2}\sum_{i<j}^{n} g^{\mu_i\mu_j}\, s^\alpha\,\{q_1^\alpha q_1^{\mu_1}\dots[q_1^{\mu_i}]\dots[q_1^{\mu_j}]\dots q_1^{\mu_n}\}.\end{aligned} \tag{D.3}$$

Making use of the Nachtmann's [48] contraction

$$z^{\mu_1}\dots z^{\mu_n}\,\{q_1^{\mu_1}\dots q_1^{\mu_n}\}_n = \left(\frac{q_1^2z^2}{4}\right)^{n/2} C_n^{(1)}(\eta) \tag{D.4}$$

and some recursion relations for the Gegenbauer polynomials $C_n^{(\lambda)}(\eta)$ [49], one can derive the formula:

$$\begin{aligned}z^{\mu_1}\dots z^{\mu_{n-1}}\,\{q_1^\alpha q_1^{\mu_1}\dots q_1^{\mu_{n-1}}\} &= \frac{1}{n}\frac{\partial}{\partial z^\alpha}\, z^{\mu_1}\dots z^{\mu_n}\,\{q_1^{\mu_1}\dots q_1^{\mu_n}\} = \\ &= \frac{1}{n}\left[\frac{z^\alpha}{Z^2}\,\tau^n\,(-2\,C_{n-2}^{(2)}(\eta)\,) + q_1^\alpha\,\tau^{n-1}\,C_{n-1}^{(2)}(\eta)\right],\end{aligned} \tag{D.5}$$

where $\eta = i\,(q_1 z)/\sqrt{-z^2q_1^2}\tau = -i\sqrt{-z^2q_1^2}/2$.

Using (D.3) – (D.5), one gets for an arbitrary 4-vector $s$:

$$z^{\mu_1}\dots z^{\mu_n}\,\{s^{\mu_1}q_1^{\mu_2}\dots q_1^{\mu_n}\} = \frac{(zs)}{n}\,\tau^{n-1}\,C_{n-1}^{(2)}(\eta) - \frac{(q_1 s)}{n}\,\frac{z^2}{2}\,\tau^{n-2}\,C_{n-2}^{(2)}(\eta). \tag{D.6}$$

# E Contact terms

Here we derive eq.(7.11). Before considering the relevant contraction, let us note that, incorporating the relation

$$\{z\partial\}^n = (z\partial)\{z\partial\}^{n-1} - \frac{\partial^2 z^2}{4}\frac{(n-2)}{n}\,\{z\partial\}^{n-2}, \tag{E.1}$$

and neglecting higher twist contributions, one can substitute the original correlator (7.6) by

$$\Pi_{\mu\{\mu_1\dots\mu_n\}}(q_1) = \Pi_{\mu\mu_1\{\mu_2\dots\mu_n\}}(q_1) + \cdots\ . \tag{E.2}$$

As a result,

$$
\begin{aligned}
&\Pi_{\mu\{\mu_1\dots\mu_n\}}(q_1)\, g_{\mu\mu_1} \simeq \Pi_{\mu\mu\{\mu_2\dots\mu_n\}}(q_1) \\
&= \int dx e^{-iq_1 x} \langle 0|T\{J_\mu(x)\,\bar{u}(0)\overleftarrow{\partial}_\mu\{\overleftarrow{\partial}_{\mu_2}\dots\overleftarrow{\partial}_{\mu_n}\}u(0)\}|0\rangle \\
&= \frac{1}{2}\int dx e^{-iq_1 x} \langle 0|T\{J_\mu(x)\,\bar{u}(0)\overleftarrow{\hat{\partial}}\gamma_\mu\{\overleftarrow{\partial}_{\mu_2}\dots\overleftarrow{\partial}_{\mu_n}\}u(0)\}|0\rangle \qquad \text{(E.3)} \\
&-\frac{1}{2}\int dx e^{-iq_1 x} \langle 0|T\{J_\mu(x)\,\bar{u}(0)\gamma_\mu\{\overleftarrow{\partial}_{\mu_2}\dots\overleftarrow{\partial}_{\mu_n}\}\overrightarrow{\hat{\partial}}u(0)\}|0\rangle \qquad \text{(E.4)} \\
&-\frac{i}{2}q_{1\epsilon}\int dx e^{-iq_1 x} \langle 0|T\{J_\mu(x)\,\bar{u}(0)\gamma_\mu\gamma_\epsilon\{\overleftarrow{\partial}_{\mu_2}\dots\overleftarrow{\partial}_{\mu_n}\}u(0)\}|0\rangle, \qquad \text{(E.5)}
\end{aligned}
$$

where we have made use of the identity:

$$
\begin{aligned}
&\int dx e^{-iq_1 x}\left[\langle 0|T\{J_\mu(x)\,\bar{u}(0)\overleftarrow{\partial}_\epsilon\hat{\Gamma}u(0)\}|0\rangle + \langle 0|T\{J_\mu(x)\,\bar{u}(0)\hat{\Gamma}\overrightarrow{\partial}_\epsilon u(0)\}|0\rangle\right] = \\
&\qquad = -i\,q_{1\epsilon}\int dx e^{-iq_1 x}\langle 0|T\{J_\mu(x)\,\bar{u}(0)\hat{\Gamma}u(0)\}|0\rangle. \qquad \text{(E.6)}
\end{aligned}
$$

Applying (7.3) to (E.3) and (E.4), and integrating by parts in (E.3) we get

$$
\text{(C.3)} = -2\,(-i)^{n-1}\,\{q_{1\mu_2}\dots q_{1\mu_n}\}\,\langle\bar{u}u\rangle. \qquad \text{(E.7)}
$$

Now, taking into account that $\langle 0|\bar{u}(0)\{\overleftarrow{\partial}_{\mu_2}\dots\overleftarrow{\partial}_{\mu_n}\}u(0)\}|0\rangle = 0$ for all $n$, we obtain:

$$
\text{(E.4)} = 0\ . \qquad \text{(E.8)}
$$

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