SPACES OF OPERATORS CONTAINING c_o AND/OR ℓ_∞ WITH AN APPLICATION OF VECTOR MEASURES

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The Banach spaces L(X, Y), K(X, Y), $L_{w*}(X^*, Y)$, and $K_{w*}(X^*, Y)$ are studied to determine when they contain the classical Banach spaces c_o or ℓ_∞. The complementation of the Banach space K(X, Y) in L(X, Y) is discussed as well as what impact this complementation has on the embedding of c_o or ℓ_∞ in K(X, Y) or L(X, Y). Results concerning the complementation of the Banach space $K_{w*}(X^*, Y)$ in $L_{w*}(X^*, Y)$ are also explored and how that complementation affects the embedding of c_o or ℓ_∞ in $K_{w*}(X^*, Y)$ or $L_{w*}(X^*, Y)$. The ℓ_p spaces for $1 \leq p < \infty$ are studied to determine when the space of compact operators from one ℓ_p space to another contains c_o. The paper contains a new result which classifies these spaces of operators. Results of Kalton, Feder, and Emmanuele concerning the complementation of K(X, Y) in L(X, Y) are generalized. A new result using vector measures is given to provide more efficient proofs of theorems by Kalton, Feder, Emmanuele, Emmanuele and John, and Bator and Lewis as well as a new proof of the fact that ℓ_∞ is prime.

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CHAPTER 1

INTRODUCTION

If each of X and Y is a Banach space, let $L(X, Y)$ (resp. $K(X, Y)$) be the space of all continuous (resp. compact) linear transformations from X to Y. The norm of $L(X, Y)$ is defined by $\|T\| = sup\{\|T(x)\| : x \in X, \|x\| \leq 1\}$. An extensive list of definitions can be found in Chapter 2.

The isomorphic structure of $K(X, Y)$, and especially $L(X, Y)$, can be quite complex. That is, there may be many classical Banach spaces Z and linear homeomorphisms $W : Z \to L(X, Y)$. Although the structure of $K(X, Y)$ is less complicated in general, this space may contain copies of the sequence space c_o.

To illustrate these points and to introduce topics of central importance in this paper, consider the spaces $K(\ell_2, \ell_2)$ and $L(\ell_2, \ell_2)$, where ℓ_2 is the separable, infinite dimensional Hilbert space. Specifically, ℓ_2 is the vector space of all real sequences $x = (x_n)$ satisfying

$$\sum_{n=1}^{\infty} |x_n|^2 < \infty.$$

If ℓ_2 is equipped with pointwise operations and the norm

$$\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2} < \infty,$$

then ℓ_2 is a separable, infinite dimensional Banach space.

If $x = (x_n) \in \ell_2$ and $e_n = (e_{n_i})_{i=1}^{\infty}$ is defined by $e_{n_i} = 1$ if $i = n$ and $e_{n_i} = 0$ otherwise, i.e. $(e_n) = (0, 0, \ldots, 0, \underbrace{1}_{n^{th}}, 0, \ldots)$, then $x = \sum_{n=1}^{\infty} x_n e_n$. In fact, this infinte series converges unconditionally to x in the metric topology ($\|f - g\|_2 = \left(\sum_{n=1}^{\infty} |f(n) - g(n)|^2 \right)^{1/2}$); i.e., any rearrangement converges. Moreover, if A is a non-empty subset of $\mathbf{N}$ and $P_A : \ell_2 \to \ell_2$ is defined by $P_A(x) = \sum_{n \in A} x_n e_n$, then P_A is linear, $\|P_A\| \leq 1$, and $P_A^2 = P_A$; i.e. P_A is a projection. If $A = \{1, \ldots, n\}$, then P_A is usually denoted by P_n. If A is finite, then $P_A \in K(\ell_2, \ell_2)$. Define a projection $Q_n : \ell_2 \to \ell_2$ by $Q_n(x) = x_n e_n$. Let $\mathcal{F}$ be the finite -

co-finite algebra of subsets of $\mathbf{N}$, and define $\mu : \mathcal{F} \to K(\ell_2, \ell_2)$ by

$$\mu(A)(x) = \begin{cases} \sum_{n\in A} Q_n(x) \text{ if } A \text{ is finite} \\ -\sum_{n\notin A} Q_n(x) \text{ if } \mathbf{N} \setminus A \text{ is finite} \end{cases}$$

(Of course, $\mu(\emptyset)$ is defined to be 0.) It is not difficult to see that μ is bounded and finitely additive. It is also clear that $\|\mu(\{n\})\| \not\to 0$; i.e., μ is not strongly additive. The algebra version of the Diestel-Faires Theorem [5] then guarantees that $K(\ell_2, \ell_2)$ contains an isomorphic copy of c_o. Observe that $c_o \not\hookrightarrow \ell_2^* = X^*$ and $c_o \not\hookrightarrow \ell_2 = Y$.

A stronger conclusion can be drawn if one relaxes the requirement that the operators be compact. Let $\mathcal{P}$ be the power class of $\mathbf{N}$ and define $\mu : \mathcal{P} \to L(\ell_2, \ell_2)$ by

$$\mu(A)(x) = \begin{cases} 0 & \text{if } A = \emptyset \\ \sum_{n\in A} Q_n(x) & \text{if } A \neq \emptyset \end{cases}$$

for $x \in \ell_2$. If $x = (x_n) \in \ell_2$, then the unconditional convergence of the series $\sum_{n=1}^{\infty} x_n e_n$ guarantees that μ is well defined. As above, μ is bounded, finitely additive, and not strongly additive. Since μ is defined on a σ-algebra of sets, the Diestel-Faires Theorem ensures that $L(\ell_2, \ell_2)$ contains an isomorphic copy of ℓ_∞. Since ℓ_∞ contains an isometric copy of all separable Banach spaces, it follows that $L(\ell_2, \ell_2)$ contains an isomorphic copy of each separable Banach space. Thus, the isomorphic structure of $L(\ell_2, \ell_2)$ is very complex. However, we observe that $\ell_\infty \not\hookrightarrow K(\ell_2, \ell_2)$ (see Theorem 1.1 below).

Let X^* be the continuous linear dual of X. To see how $X^* \hookrightarrow K(X, Y)$, consider the following. Choose $y \in Y$ with $\|y\| = 1$. For $x^* \in X^*$, define $(x^* \otimes y)(x) = x^*(x)y$. Note that these are rank one. Thus, each x^* maps to a compact operator in $K(X, Y)$. And we have $\|x^* \otimes y\| = \|x^*\|$.

Numerous authors have discussed the structure of $K(X, Y)$ and whether $K(X, Y)$ is complemented in $L(X, Y)$ when the two spaces of operators are different. A subspace H of X is said to be *complemented* if there is a projection P on X with $P(X) = H$. Specifically, we note the papers by Feder [11], Kalton [15], Drewnowski [6], Emmanuele

[7] and [8], Emmanuele and John [9], and John [14]. The presense of isomorphic copies of c_o in $K(X,Y)$ has been central in these discussions. See Theorem 1.3 below.

Earlier remarks indicated that the structure of $K(X,Y)$ is, in general, simpler than that of $L(X,Y)$. Kalton established the following fundamental result in [15].

THEOREM 1.1. *$K(X,Y)$ contains a copy of ℓ_∞ if and only if $\ell_\infty \hookrightarrow X^*$ or $\ell_\infty \hookrightarrow Y$.*

Note that by a theorem of Bessaga and Pelczynski [2], $\ell_\infty \hookrightarrow X^*$ if and only if X contains a complemented isomorphic copy of ℓ_1.

Kalton's results about complementation include the following:

THEOREM 1.2. *(Kalton [15])*
Suppose X contains a complemented subspace isomorphic to ℓ_1 and that Y is infinite-dimensional. Then $K(X,Y)$ is uncomplemented in $L(X,Y)$.

An elementary proof of Theorem 1.3 using vector measures will be given later.

THEOREM 1.3. *(Kalton [15])*
Let X be a Banach space with an unconditional finite-dimensional expansion of the identity $\{A_n\}$. If Y is any infinite-dimensional Banach space, then the following are equivalent:

(i) $K(X,Y) = L(X,Y)$
(ii) *$K(X,Y)$ contains no copy of c_o.*
(iii) *$L(X,Y)$ contains no copy of ℓ_∞.*
(iv) *$K(X,Y)$ is complemented in $L(X,Y)$.*
(v) *For $T \in L(X,Y)$ the series $\sum_{n=1}^{\infty} TA_n$ converges in norm.*

Results of Moshe Feder ([10], [11]) have played a prominent role in the study of complementation of $K(X,Y)$ in $L(X,Y)$. Further, Feder's results have naturally led to an investigation of whether c_o embeds isomorphically into $K(X,Y)$.

THEOREM 1.4. *(Feder [10]) Let X be weakly compactly generated and suppose there exists a sequence (T_n) in $K(X,Y)$ such that $\sum_{n=1}^{\infty} T_n(x)$ converges unconditionally to $T(x)$ for every $x \in X$ where $T \in L(X,Y)$ is non-compact. Then $K(X,Y)$ is uncomplemented in $L(X,Y)$.*

A separable Banach space X has the *bounded approximation property* (BAP) if there is a sequence (T_n) of finite-rank operators so that

$$\lim_{n\to\infty} \|x - T_n(x)\| = 0$$

for all $x \in X$. If $\sum_{n=1}^{\infty} T_n(x)$ converges unconditionally for all $x \in X$, then (T_n) is called an *unconditional finite dimensional expansion of the identity* (u.f.d.e.i.) of X. A sequence (T_n) in $K(X,Y)$ is said to be an *unconditional compact expansion* of $T \in L(X,Y)$ if $\sum_{n=1}^{\infty} T_n(x)$ converges unconditionally to $T(x)$ for every $x \in X$.

A basis $(x_n)_{n=1}^{\infty}$ of a Banach space X is called *unconditional* if for each $x \in X$ the series $\sum_{n=1}^{\infty} x_n^*(x)x_n$ converges unconditionally to x.

Feder showed that the non-complementation of $K(X,Y)$ in $L(X,Y)$ is equivalent to $K(X,Y) \neq L(X,Y)$ in the cases where X is reflexive (i.e., $X = X^{**}$) and when X or Y has the bounded approximation property and Y is a subspace of a space with an unconditional basis. Note that for a reflexive Banach space X with the approximation property, X has an unconditional finite dimensional expansion of the identity if and only if X is a subspace of a space with an unconditional basis.

Theorem 1.5. *(Feder* [11]*)*
For Banach spaces X and Y, if there exists a non-compact operator $T \in L(X,Y)$ admitting an unconditional compact expansion, then $K(X,Y)$ is uncomplemented in $L(X,Y)$.

Emmanuele stated that Feder's hypothesis in the previous result is equivalent to the existence of copies of c_o inside $K(X,Y)$. In 1992, Emmanuele [7] and John [14] independently proved

Theorem 1.6. *If $K(X,Y) \neq L(X,Y)$ and $c_o \hookrightarrow K(X,Y)$, then $K(X,Y)$ is not complemented in $L(X,Y)$.*

An elementary and self-contained proof of Theorem 1.6 will be presented in Chapter 4.

Emmanuele and John subsequently collaborated on an expanded study of complemented spaces of operators and established the following theorem. Their paper contains refinements of previous results as well as an extensive bibliography.

THEOREM 1.7. *(Emmanuele and John* [9]*)*
If $c_o \hookrightarrow K(X,Y)$, then $K(X,Y)$ is uncomplemented in $L(X,Y)$, and if there is a non-compact operator from X to Y that factors through a space with an unconditional finite dimensional expansion of the identity, then $c_o \hookrightarrow K(X,Y)$.

The space $L_{w^*}(X^*,Y)$ consists of those bounded linear operators $W : X^* \to Y$ which are $w^* - w$ continuous; $K_{w^*}(X^*,Y)$ is defined analogously.

THEOREM 1.8. *(Emmanuele* [8]*)*
If $c_o \hookrightarrow K_{w^*}(X^*,Y)$, then either $K_{w^*}(X^*,Y) = L_{w^*}(X^*,Y)$ or $K_{w^*}(X^*,Y)$ is uncomplemented in $L_{w^*}(X^*,Y)$.

Emmanuele also proved the following result similar to Theorem 1.3.

THEOREM 1.9. *(Emmanuele* [8]*)*
Let Y be a Banach space with an unconditional finite-dimensional expansion of the identity $\{A_n\}$. If Y is any infinite-dimensional Banach space, then the following are equivalent:

(i) $K_{w^*}(X^*,Y) = L_{w^*}(X^*,Y)$
(ii) $K_{w^*}(X^*,Y)$ *contains no copy of* c_o.
(iii) $L_{w^*}(X^*,Y)$ *contains no copy of* ℓ_∞.
(iv) $K_{w^*}(X^*,Y)$ *is complemented in* $L_{w^*}(X^*,Y)$.

Theorem 1.1, due to Kalton, was generalized by Drewnowski who demonstrated

THEOREM 1.10. *(Drewnowski* [6]*)*
$K_{w^*}(X^*,Y)$ contains a copy of ℓ_∞ if and only if either $\ell_\infty \hookrightarrow X$ or $\ell_\infty \hookrightarrow Y$.

Note that since $K_{w^*}(X^*,Y)$ contains isometric copies of both X and Y, the *if* part is trivial.

Papers previously cited had noted that if $c_o \hookrightarrow K(X,Y) \neq L(X,Y)$, then $\ell_\infty \hookrightarrow L(X,Y)$. In 2005 Lewis [16] showed

THEOREM 1.11. *If X is infinite dimensional and $c_o \hookrightarrow L(X,Y)$, then $\ell_\infty \hookrightarrow L(X,Y)$.*

Observe that $c_o \hookrightarrow L_{w^*}(X^*, Y)$ does not imply $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$. To see this consider the following example. Let $c_o = X$ and $\ell_1 = Y$ (Recall ℓ_1 is a Schur space). Then $c_o \hookrightarrow L_{w^*}(c_o^*, \ell_1) = K_{w^*}(c_o^*, \ell_1)$ but $\ell_\infty \not\hookrightarrow L_{w^*}(c_o^*, \ell_1)$ since $\ell_\infty \not\hookrightarrow c_o^*$ and $\ell_\infty \not\hookrightarrow \ell_1$.

Particularly in view of Theorem 1.11 and the preceding example, one is motivated to study what additional conditions on X and Y will ensure that $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$ when $c_o \hookrightarrow K_{w^*}(X^*, Y)$. Emmaunele proved:

Theorem 1.12. *(Emmanuele* [8]*)*
Let $c_o \hookrightarrow K_{w^*}(X^*, Y)$. Then $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$, provided X and Y do not have the Schur property.

Also, see Theorem 3.5 (below) in this context.

The principal new results of this paper are Theorems 3.3, 3.4, 3.7, and 4.3. In particular, note that Theorem 4.3 leads to new, accessible, and streamlined proofs of Theorem 1.6, results of Feder [10], Kalton [15], Bator and Lewis [3], Lewis [17], Drewnowski [6], and the fact that ℓ_∞ is prime.

CHAPTER 2

HISTORICAL PERSPECTIVE

Throughout this paper X and Y will represent real Banach spaces, and X^* will represent the continuous linear dual of X. A linear functional $f : X \to \mathbf{R}$ is continuous if and only if $\{f(x) : x \in X, \|x\| \leq 1\}$ is bounded in $\mathbf{R}$. Analogously, a linear function $W : X \to Y$ is continuous if and only if $\{W(x) : x \in X, \|x\| \leq 1\}$ is bounded in Y. It is customary to refer to continuous linear functionals as bounded linear functionals and to refer to continuous linear functions as bounded linear operators or more simply operators. For convenience, let $B_X = \{x \in X : \|x\| \leq 1\}$, and as in the introduction, let $L(X, Y)$ denote the linear space (pointwise operations) of all operators $T : X \to Y$. If $T \in L(X, Y)$, set $\|T\| = sup\{\|T(x)\| : x \in B_X\}$. An operator $T : X \to Y$ is said to be compact, or $T \in K(X, Y)$, if $T(B_X)$ has compact closure in Y. In particular, a non-zero compact operator does not have compact range. Observe that in a Banach space, X, B_X is compact if and only if X is finite dimensional.

An operator T is an *embedding* of X into Y if T is an isomorphism (= linear homeomorphism) onto its image $T(X)$. In this case we say that X embeds in Y or that Y contains an isomorphic copy of X. If $T : X \to Y$ is an embedding such that $\|T(x)\|_Y = \|x\|_X$ for all $x \in X$, then T is said to be an *isometric embedding* and is referred to as an *isometric isomorphism.*

For a normed linear space X, the *weak topology* of X is the weakest topology on X such that each functional $x^* \in X^*$ is continuous. It is common to write $(x_\alpha) \stackrel{w}{\to} x$ to denote that a net (x_α) from X converges to x in the weak topology. Let $\eta : X \to X^{**}$ be the natural embedding of a Banach space X into its second dual, given by $\eta(x)(x^*) = x^*(x)$. The space X is identified with $\eta(X) \subset X^{**}$. The *weak** *topology* on a dual space X^* is the topology induced on X^* by X, i.e. it is the weakest topology on X^* that makes all elements of $\eta(X)$ continuous. We say that a net (x^*_α) from X^* converges *weak** to $x^* \in X^*$, and we write $(x^*_\alpha) \stackrel{w^*}{\to} x^*$ if for each $x \in X$, $x^*_\alpha(x) \to x^*(x)$.

An operator $T : X^* \to Y$ is said to be *weak* −to−weak continuous* if $(T(x_\alpha^*)) \xrightarrow{w} T(x^*)$ in the weak topology whenever $(x_\alpha^*) \xrightarrow{w^*} x^*$ in the *weak**-topology on X^*. The space of all compact $w^* - w$ continuous linear operators from X^* into Y will be represented by $K_{w^*}(X^*, Y)$, and the space of all bounded $w^* - w$ continuous linear operators from X^* into Y will be represented by $L_{w^*}(X^*, Y)$. The norm of an operator $T \in L_{w^*}(X^*, Y)$ is defined by $\|T\| = sup\{\|T(x^*)\| : x^* \in X^*, \|x^*\| \leq 1\}$.

A Banach space X has the *Schur property* (or X is a *Schur space*) if a sequence $(x_n)_{n=1}^\infty$ in X converges to 0 weakly if and only if $(x_n)_{n=1}^\infty$ converges to 0 in norm, i.e., weak and norm sequential convergence coincide. It is well known that ℓ_1 has the Schur property and that every infinite dimensional Schur space contains an isomorphic copy of ℓ_1. That is, if X is an infinite dimensional Schur space, then there is a linear homeomorphism T from ℓ_1 into X.

The space c_o is the linear space of all sequences $x = (\alpha_n)_{n=1}^\infty$ converging to 0 with the sup norm. The space ℓ_1 is defined as the linear space of all sequences $x = (\alpha_n)_{n=1}^\infty$ of scalars for which the norm $\|x\| = \sum_{n=1}^\infty |\alpha_n| < \infty$. Note that c_o^* is isometrically isomorphic to ℓ_1, i.e. $c_o^* \cong \ell_1$. Furthermore, $\ell_1^* \cong \ell_\infty$, the space of all bounded sequences $x = (\alpha_n)_{n=1}^\infty$ of scalars with the sup norm.

In this paper the following two fundamental results from vector measure theory will prove useful in identifying isomorphic copies of c_o and ℓ_∞: *Rosenthal's lemma* and the *Diestel-Faires Theorem*. Rosenthal's lemma will be stated and then used to give a detailed proof of the Diestel-Faires Theorem. To paraphrase the words of Diestel [4], Rosenthal's lemma is the sharpest general disjointification result in existence.

Let $\mathcal{F}$ be a field of subsets of the set Ω and let $\mu : \mathcal{F} \longrightarrow X$ be a vector measure. A vector measure μ is said to be *strongly additive* provided the series $\sum_{n=1}^\infty \mu(E_n)$ converges in norm whenever (E_n) is a sequence of pairwise disjoint members of $\mathcal{F}$. In fact, any rearrangement of the infinite series converges and has the same sum, i.e. $\sum_{n=1}^\infty \mu(E_n)$ is unconditionally convergent. An equivalent formulation of strong additivity is that whenever (E_n) is a pairwise disjoint sequence from $\mathcal{F}$, $\|\mu(E_n)\| \to 0$. The ℓ_∞ norm of a bounded vector measure μ is defined by $\|\mu\|_\infty = sup\,\{\|\mu(A)\| : A \in \mathcal{F}\} < \infty$.

Lemma 2.1. *(Rosenthal's lemma* [4]*)*

Let $\mathcal{F}$ be a field of subsets of the set Ω. Let (μ_n) be a uniformly bounded sequence of finitely additive scalar-valued measures defined on $\mathcal{F}$. Then if (E_n) is a disjoint sequence of memebers of $\mathcal{F}$ and $\epsilon > 0$ there is a subsequence (E_{n_j}) of (E_n) such that

$$|\mu_{n_j}|\left(\bigcup_{\substack{k\neq j\\ k\in\Delta}} E_{n_k}\right) < \epsilon$$

for all finite subsets Δ of $\boldsymbol{N}$ and for all $j = 1, 2, 3, \ldots.$

If, in addition, $\mathcal{F}$ is a σ-field, then the subsequence $\left(E_{n_j}\right)$ may be chosen such that

$$|\mu_{n_j}|\left(\bigcup_{k\neq j} E_{n_k}\right) < \epsilon$$

for all $j = 1, 2, 3, \ldots.$

Theorem 2.2. *(Diestel-Faires* [5]*)*

Let $\mathcal{F}$ be a field of subsets of the set Ω, and let X be a Banach space. If $\mu : \mathcal{F} \longrightarrow X$ is a bounded (finitely additive) vector measure which is not strongly additive, then there is a linear homeomorphism $T : c_o \longrightarrow X$ and a pairwise disjoint sequence (E_n) in $\mathcal{F}$ so that $T(e_n) = \mu(E_n)$ for every $n \in \boldsymbol{N}$. If, in addition, $\mathcal{F}$ is a σ-field, then the above statement remains true if c_o is replaced by ℓ_∞.

Proof. Assume first that $\mathcal{F}$ is a field. If μ is not strongly additive, then there exists a sequence of disjoint memebers (E_n) of $\mathcal{F}$ and an $\epsilon > 0$ so that $\|\mu(E_n)\| > \epsilon$ for every $n \in \mathbf{N}$.

Choose (x_n^*) in X^* with $\|x_n^*\| = 1$ such that $x_n^*\mu(E_n) = \|\mu(E_n)\| > \epsilon$ for every $n \in \mathbf{N}$. Since μ is bounded, $(x_n^*\mu)$ is a uniformly bounded sequence.

By Rosenthal's lemma, there exists a subsequence $(x_{n_j}^*\mu)$ such that

$$\left|x_{n_j}^*\mu\right|\left(\bigcup_{\substack{i\neq j\\ i\in\Delta}} E_{n_i}\right) < \frac{\epsilon}{2}$$

for all $j \in \mathbf{N}$ and all finite subsets Δ of $\mathbf{N}$.

Relabel $x_{n_j}^*$ by x_j^* and E_{n_j} by E_j. Therefore,

$$\left|x_{n_j}^*\mu\right|\left(\bigcup_{\substack{i\neq j\\ i\in\Delta}} E_{n_i}\right) < \frac{\epsilon}{2}, \qquad (1)$$

for all $j \in \mathbf{N}$ and all finite subsets Δ of $\mathbf{N}$, and

$$\left|x_j^*\mu(E_j)\right| > \epsilon, \qquad (2)$$

for all $j \in \mathbf{N}$.

For a finitely supported sequence $(\alpha_i) \in c_o$ define $T : c_o \to X$ by

$$T((\alpha_i)) = \sum_{i=1}^{\infty} \alpha_i \mu(E_i).$$

The unconditional covergence of $\sum_{i=1}^{\infty} \mu(E_i)$ guarantees that $\sum_{i=1}^{\infty} \alpha_i\mu(E_i)$ converges. This also shows that T is well-defined and continuous, and T is linear on the dense linear subspace of c_o consisting of the finitely supported sequences.

We want to show T is bounded. Suppose $\alpha = (\alpha_i) \in c_o$ is finitely supported and $x^* \in B_{X^*}$. Then

$$\begin{aligned}
\|x^*T(\alpha)\| &= \left\|\left\langle x^*, \sum_{i=1}^{\infty} \alpha_i\mu(E_i)\right\rangle\right\| \\
&= \left\|\sum_{i=1}^{\infty} \alpha_i x^*\mu(E_i)\right\| \\
&\leq \sum_{i=1}^{\infty} |\alpha_i| \cdot |x^*\mu(E_i)| \\
&\leq \sum_{i=1}^{\infty} \|\alpha\|_\infty \cdot |x^*\mu(E_i)| \\
&= \|\alpha\|_\infty \sum_{i=1}^{\infty} |x^*\mu(E_i)| \\
&\leq \|\alpha\|_\infty \|\mu\|(\Omega)
\end{aligned}$$

Therefore, $\|T(\alpha)\| \leq \|\alpha\|_\infty \|\mu\|(\Omega)$, $\|T\| \leq \|\mu\|(\Omega)$, and hence, T is bounded on a dense subspace of c_o. Thus, T has a bounded extension to all of c_o, still denoted by T, with norm $\leq \|\mu\|(\Omega)$. Therefore, if $\alpha = (\alpha_i) \in c_o$, then $T(\alpha) = \sum_{i=1}^{\infty} \alpha_i\mu(E_i)$. Note that $T(e_n) = \mu(E_n)$.

We want to show T is an isomorphism (topologically); i.e., for all $\alpha \in c_o$, $A\|\alpha\|_\infty \leq \|T(\alpha)\| \leq B\|\alpha\|_\infty$ for some $A, B > 0$. Let $B = \|\mu\|(\Omega)$

Suppose $\alpha = (\alpha_i) \in c_o$. Then

$$\begin{aligned}
\left|x_j^* T(\alpha)\right| &= \left|\sum_{i=1}^{\infty} \alpha_i x_j^* \mu(E_i)\right| \\
&= \left|\lim_{m\to\infty}\left(\sum_{i=1}^{m} \alpha_i x_j^* \mu(E_i)\right)\right| \\
&\geq \left|\alpha_j x_j^* \mu(E_j)\right| - \left|\lim_{m\to\infty}\left(\sum_{\substack{i=1\\ i\neq j}}^{m} \alpha_i x_j^* \mu(E_i)\right)\right| \\
&\geq \left|\alpha_j\right| \cdot \left|x_j^* \mu(E_j)\right| - \left\|\alpha\right\|_\infty \cdot \lim_{m\to\infty}\left(\sum_{\substack{i=1\\ i\neq j}}^{m} \left|x_j^* \mu\right|(E_i)\right) \\
&\geq \left|\alpha_j\right| \epsilon - \left\|\alpha\right\|_\infty \cdot \tfrac{\epsilon}{2}
\end{aligned}$$

by (1) and (2). Whence, $\left|x_j^* T(\alpha)\right| \geq \left|\alpha_j\right| \epsilon - \left\|\alpha\right\|_{c_o} \cdot \frac{\epsilon}{2}$, for all $j \in \mathbf{N}$ and

$$\begin{aligned}
\left\|T(\alpha)\right\| &= sup\left\{\left|x^* T(\alpha)\right| \,|\, x^* \in B_{X^*}\right\} \\
&\geq sup_j \left|x_j^* T(\alpha)\right| \\
&\geq sup_j \left\{\left|\alpha_j\right| \epsilon - \left\|\alpha\right\|_{c_o} \cdot \tfrac{\epsilon}{2}\right\} \\
&= \left\|\alpha\right\|_\infty \epsilon - \left\|\alpha\right\|_\infty \tfrac{\epsilon}{2} \\
&= \epsilon \tfrac{1}{2} \left\|\alpha\right\|_\infty.
\end{aligned}$$

Consequently, $\left\|T(\alpha)\right\| \geq \frac{\epsilon}{2} \left\|\alpha\right\|_\infty$ and, T is an isomorphism. Let $A = \frac{\epsilon}{2}$.

Now, suppose $\mathcal{F}$ is a σ-field. Proceed as above to produce (with the help of the σ-field version of Rosenthal's lemma) an $\epsilon > 0$, (x_n^*) in B_{X^*} and a sequence (E_j) of pairwise disjoint members of $\mathcal{F}$ such that

$$\left|x_j^* \mu(E_j)\right| > \epsilon$$

and

$$\left|x_j^* \mu\right| \left(\bigcup_{i\neq j} E_i\right) < \frac{\epsilon}{2}$$

for all $j \in \mathbf{N}$.

Let E be the set of finitely-valued sequences $\alpha = (\alpha_i) \in \ell_\infty$. If $\alpha = (\alpha_i) \in \ell_\infty$ is a finitely-valued sequence, write $\alpha = (\alpha_i) = \sum_{j=1}^{n} \beta_j \chi_{A_j}$ where $A_1, \ldots A_n$ are pairwise disjoint sets of positive integers.

Define $T : E \to X$ by $T(\alpha) = \sum_{j=1}^{n} \beta_j \mu \left(\bigcup_{k \in A_j} E_k \right)$, if $\alpha \in E$ with $\alpha = \sum_{j=1}^{n} \beta_j \chi_{A_j}$. Then T is linear on the dense linear subspace E of ℓ_∞. We want to show T is bounded on E. If $x^* \in B_{X^*}$ and $\alpha \in E$, then

$$\begin{aligned} \|x^* T(\alpha)\| &= \left\| \left\langle x^*, \sum_{i=1}^{\infty} \alpha_i \mu(E_i) \right\rangle \right\| \\ &= \left| \sum_{j=1}^{n} \beta_j x^* \mu \left(\bigcup_{k \in A_j} E_k \right) \right| \\ &\leq \sum_{j=1}^{n} |\beta_j| \cdot \left| x^* \mu \left(\bigcup_{k \in A_j} E_k \right) \right| \\ &\leq \sum_{j=1}^{n} \|\alpha\|_\infty \cdot \left| x^* \mu \left(\bigcup_{k \in A_j} E_k \right) \right| \\ &\leq \|\alpha\|_\infty \|\mu\| (\Omega). \end{aligned}$$

Therefore, $\|T(\alpha)\| \leq \|\alpha\|_\infty \|\mu\| (\Omega)$, and $\|T\| \leq \|\mu\| (\Omega)$.

We want to show T is an isomorphism. To this end, we will show $\|T(\alpha)\| \geq A \|\alpha\|_\infty$ for some $A > 0$.

$$
\begin{aligned}
\left|x_j^* T(\alpha)\right| &= \left|\sum_{i=1}^{n} \beta_i x_j^* \mu\left(\bigcup_{k\in A_j} E_k\right)\right| \\
&= \left|\beta_j x_j^* \mu(E_j) + \beta_j x_j^* \mu\left(\bigcup_{\substack{k\in A_j\\ k\neq j}} E_k\right) + \left(\sum_{\substack{i=1\\ i\neq j}}^{n} \beta_i x_j^* \mu\left(\bigcup_{\substack{k\in A_i\\ k\neq j}} E_k\right)\right)\right| \\
&\geq \left|\beta_j x_j^* \mu(E_j)\right| - \left|\beta_j x_j^* \mu\left(\bigcup_{\substack{k\in A_j\\ k\neq j}} E_k\right) + \left(\sum_{\substack{i=1\\ i\neq j}}^{n} \beta_i x_j^* \mu\left(\bigcup_{\substack{k\in A_i\\ k\neq j}} E_k\right)\right)\right| \\
&\geq \left|\beta_j\right| \epsilon - \|\alpha\|_\infty \left[\left|x_j^* \mu\left(\bigcup_{\substack{k\in A_j\\ k\neq j}} E_k\right)\right| + \left|\left(\sum_{\substack{i=1\\ i\neq j}}^{n} \beta_i x_j^* \mu\left(\bigcup_{\substack{k\in A_i\\ k\neq j}} E_k\right)\right)\right|\right] \\
&\geq \left|\beta_j\right| \epsilon - \|\alpha\|_\infty \left|x_j^* \mu\right| \left(\bigcup_{i\neq j} E_i\right) \\
&\geq \left|\beta_j\right| \epsilon - \|\alpha\|_\infty \tfrac{\epsilon}{2}, \text{ for all } 1 \leq j \leq n
\end{aligned}
$$

Therefore, $\|T(\alpha)\| \geq \|\alpha\|_\infty \epsilon - \|\alpha\|_\infty \frac{\epsilon}{2} = \frac{\epsilon}{2} \|\alpha\|_\infty$, and T is an isomorphism. □

Another well-known result which will be useful in proving several theorems in Chapter 4 is the Josefson-Nissenzweig Theorem which was independently discovered by B. Josefson and A. Nissenzweig in 1975.

THEOREM 2.3. *(Josefson-Nissenzweig Theorem* [4]*) If X is an infinite dimensional Banach space, then there exists a weak*-null sequence of norm-one vectors in X^*.*

In Chapter 4 we will also make use of Phillips's Theorem [4]. Recall that a Banach space Y is called *injective* if whenever X is a Banach space, E is a closed subspace of X, and $T : E \to Y$ is a bounded operator, then there is a bounded linear operator $\widetilde{T} : X \to Y$ which is an extension of T. Y is called *isometrically injective* if $\widetilde{T}$ can be additionally chosen to have $\left\|\widetilde{T}\right\| = \|T\|$.

THEOREM 2.4. *(Phillips's Theorem) Let R be a linear subspace of the Banach space X and suppose $T : R \to \ell_\infty$ is a bounded linear operator. Then T may be extended to a bounded linear operator $S : X \to \ell_\infty$.*

PROOF. Observe that the operator T must be of the form $T(r) = (r_n^*(r))$ for some bounded sequence (r_n^*) in R^*. If, for each $n \in \mathbf{N}$, we let x_n^* be a Hahn-Banach extension of r_n^* to all of X, then the operator $S(x) = (x_n^*(x))$ is a bounded linear extension of T. □

Note that if ℓ_∞ is a closed linear subspace of a Banach space X, we can extend the identity operator $id : \ell_\infty \to \ell_\infty$ to an operator $S : X \to \ell_\infty$ with $\|S\| = 1$. The operator S is naturally a norm-one projection of X onto ℓ_∞.

A formal series $\sum_{n=1}^{\infty} x_n$ in a Banach space X is *weakly unconditionally convergent* (wuc) (or *weakly unconditionally Cauchy*) if for every x^* in X^*

$$\sum_{n=1}^{\infty} |x^*(x_n)| < \infty.$$

Note that $\sum_{n=1}^{\infty} e_n$ is wuc in c_o. In fact, Bessaga and Pelczynski [2] showed that $c_o \hookrightarrow X$ when X contains any wuc series which is not unconditionally convergent.

The *strong operator topology* in $L(X, Y)$ is the topology defined by the basic set of neighborhoods

$$N(T; A, \epsilon) = \{R : R \in L(X, Y), \|(T - R)x\| < \epsilon, x \in A\},$$

where A is an arbitrary finite subset of X and $\epsilon > 0$ is arbitrary. Thus, in the strong operator topology, a generalized sequence (T_α) converges to T if and only if $(T_\alpha(x))$ converges to $T(x)$ for every x in X.

The space $K_{w^*}(X^*, Y)$ was originally introduced as the $\epsilon-$product $X \otimes_\epsilon Y$ of X and Y, where $X \otimes_\epsilon Y$ is the least crossnorm tensor product completion of X and Y. It has the advantage over $K(X, Y)$ in that, as far as methods of proof are concerned, it is conceptually easier to deal with than $K(X, Y)$ itself, and it comprises not only spaces of the type $K(X, Y)$ but also concrete spaces of analysis such as spaces of vector-valued continuous functions and of vector-valued measures. More specifically, we have the following well-known fundamental isometric isomorphism and isometric embedding, respectively:

$$K(X,Y) \cong K_{w^*}(X^{**},Y)$$

by

$$k \longmapsto k^{**},$$

and

$$X \otimes_\epsilon Y \hookrightarrow K_{w^*}(X^*,Y)$$

by

$$(x \otimes y)(x^*) \longmapsto x^*(x)y$$

See Ruess [20] for a detailed discussion of $K_{w^*}(X^*,Y)$.

Let $(x_n)_{n=1}^{\infty}$ be a sequence in a Banach space X. Suppose there is a sequence $(x_n^*)_{n=1}^{\infty}$ in X^* such that

(i) $x_k^*(x_j) = 1$ if $j = k$ and $x_k^*(x_j) = 0$ otherwise, for any k, $j \in \mathbf{N}$,

(ii) $x = \sum_{n=1}^{\infty} x_n^*(x)x_n$ for each $x \in X$.

Then $(x_n)_{n=1}^{\infty}$ is called a *Schauder basis* (or simply basis) for X and the functionals $(x_n^*)_{n=1}^{\infty}$ in X^* are called the *biorthogonal functionals* associated with $(x_n)_{n=1}^{\infty}$. Let $P_n : X \to X$ be defined by $P_n(x) = \sum_{i=1}^{n} x_i^*(x)x_i$. Note that $P_n^2 = P_n$. Thus, P_n is a projection, and the closed linear span of $\{x_i : 1 \le i \le n\}$ is complemented in X for each $n \in \mathbf{N}$. Denote the closed linear span of $\{x_i : 1 \le i \le n\}$ by $[x_i : 1 \le i \le n]$. If (x_n) is a basis for X, we have $sup_n \|P_n\| < \infty$. If (x_n) is a basis for a Banach space X, then the number $K = sup_n \|P_n\|$ is called the *basis constant* for (x_n). Further, a basis (x_n) is said to be *seminormalized* if there exists positive constants A and B so that $A \le x_n \le B$ for every $n \in \mathbf{N}$.

Let (x_n) be a basis for a Banach space X. Suppose that (p_n) is a strictly increasing sequence of integers with $p_o = 0$ and that (a_n) are scalars. Then a sequence of nonzero vectors (u_n) in X of the form

$$u_n = \sum_{j=p_{n-1}+1}^{p_n} a_j x_j$$

is called a *block basic sequence* of (x_n).

The following theorem known as the *Bessaga-Pelczynksi Selection Principle* guarantees the existence of a plethora of basic sequences in a non-Schur space.

THEOREM 2.5. *(Bessaga-Pelczynksi Selection Principle) Let $(e_n)_{n=1}^{\infty}$ be a basis for a Banach space X with basis constant K and dual functionals $(e_n^*)_{n=1}^{\infty}$. Suppose $(x_n)_{n=1}^{\infty}$ is a sequence in X such that*

(i) $\inf_n \|x_n\| > 0$, *but*
(ii) $\lim_{n\to\infty} e_k^*(x_n) = 0$ *for all* $k \in \boldsymbol{N}$.

Then $(x_n)_{n=1}^{\infty}$ contains a subsequence $(x_{n_k})_{n=1}^{\infty}$ which is equivalent to some block basic sequence $(y_k)_{k=1}^{\infty}$ of $(e_n)_{n=1}^{\infty}$. Furthermore, for every $\epsilon > 0$ it is possible to choose $(n_k)_{k=1}^{\infty}$ so that $(x_{n_k})_{n=1}^{\infty}$ has a basis constant at most $K + \epsilon$. In particular, the same result holds if $(x_n)_{n=1}^{\infty}$ converges to 0 weakly but not in the norm topology.

Recall that a basis $(x_n)_{n=1}^{\infty}$ of a Banach space X is called *unconditional* if for each $x \in X$ the series $\sum_{n=1}^{\infty} x_n^*(x)x_n$ converges unconditionally to x. Let $a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N$ be scalars satisfying $|a_n| \le |b_n|$ for $n = 1, \dots, N$, then the least constant K such that the inequality

$$\left\| \sum_{n=1}^{N} a_n x_n \right\| \le K \left\| \sum_{n=1}^{N} b_n x_n \right\|$$

holds is called the unconditional basis constant. Let X have an unconditional seminormalized Schauder basis, (x_n), and suppose $\sum_{i=1}^{\infty} x_i^*(x)x_i = x$. If (x_n) is an unconditional basis for X, then for $Q_i = P_i - P_{i-1}$ we have $sup_F \left\| \sum_{i\in F} Q_i \right\| < \infty$ for F finite. In fact, there exists a constant K such that if $F \subseteq N$ and F is finite, then $\left\| \sum_{i\in F} Q_i \right\| < K$. ($K$ is an unconditional basis constant.)

Two bases (or basic sequences) $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in the respective Banach spaces X and Y are *equivalent* if whenever we take a sequence of scalars $(a_n)_{n=1}^{\infty}$, then $\sum_{n=1}^{\infty} a_n x_n$ coverges if and only if $\sum_{n=1}^{\infty} a_n y_n$ coverges.

Recall the *Uniform Boundedness Principle* states that given a family (T_α) of bounded linear operators from a Banach space X into a normed linear space Y, if $sup\{\|T_\alpha(x)\| : \alpha \in \Gamma\}$ is finite for each x in X, then $sup\{\|T_\alpha\| : \alpha \in \Gamma\}$ is finite.

We conclude Chapter 2 by using the Diestel-Faires Theorem to give quick proofs of two major results which are closely related to themes of this paper.

THEOREM 2.6. *(Feder [10])*
Suppose $(T_n)_{n=1}^{\infty}$ is a sequence of operators in $K(X,Y)$, $\sum_{n=1}^{\infty} T_n(x)$ converges unconditionally for every x in X, and the pointwise limit operator, $T(x)$, is not compact. Then $c_o \hookrightarrow K(X,Y)$.

PROOF. Note that the unconditional convergence of $\sum_{n=1}^{\infty} T_n(x)$ and the Uniform Boundedness Principle implies that

$$\left\{ \sum_{i \in F} T_i : F \subseteq \mathbf{N}, |F| < \infty \right\}$$

is norm bounded. Using the finite - co-finite algebra of subsets of $\mathbf{N}$, the non-strongly additive vector measure $\mu(F) = \sum_{i \in F} T_i$ and the fact that $\sum T_i$ is not Cauchy with respect to the norm, we can conclude that $c_o \hookrightarrow K(X,Y)$. □

The Orlicz-Pettis Theorem is one of the more celebrated results in classical Banach space theory and vector measure theory. The theorem was known by the Polish functional analysis school by 1930. However, the result was not widely known until it was independently discovered by Pettis in 1938. Although three different proofs of this theorem appear in Diestel's book [4] and a fourth proof is in Diestel and Uhl [5], the following argument is particularly efficient and emphasizes important ideas in this paper.

COROLLARY 2.7. *(Orlicz-Pettis Theorem) Let $\sum_{n=1}^{\infty} x_n$ be a formal series in X such that every subseries of $\sum_{n=1}^{\infty} x_n$ is weakly convergent. Then $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent in norm.*

Consequently, a weakly countably additive vector measure on a σ-field is norm countably additive.

PROOF. Let $X_o = [x_n : n \geq 1]$, set $\mu(A) = w - \lim \sum_{i \in A} x_i$, and observe that μ is bounded and finitely additive. Since X_o is weakly closed, μ maps $\mathcal{P}$ into X_o. Since X_o is separable, the Diestel-Faires Theorem tells us that μ is strongly additive. Therefore, $\sum_{n=1}^{\infty} \mu(\{n\}) = \sum_{n=1}^{\infty} x_n$ is unconditionally covergent. □

CHAPTER 3

CONTAINMENT OF c_o AND ℓ_∞ IN SPACES OF OPERATORS

We first note that the same argument in the introduction for the containment of c_o in $K(\ell_2, \ell_2)$ also works to show that $K(\ell_p, \ell_p)$ contains c_o for $1 \leq p < \infty$. In fact, the same argument works for any infinite dimensional Banach space X with a seminormalized unconditional basis: *If X is an infinite dimensional Banach space with a seminormalized unconditional basis, then $c_o \hookrightarrow K(X, X)$ and $\ell_\infty \hookrightarrow L(X, X)$.*

The following theorems are used to prove a new result. Recall that a basis (x_n) for X is perfectly homogeneous ([22]) if it is seminormalized and every seminormalized block basic sequence with respect to (x_n) is equivalent to (x_n). A perfectly homogeneous basis is unconditional. The unit vector bases of c_o and ℓ_p for $1 \leq p < \infty$ are, up to equivalence, the only perfectly homogeneous bases (Zippin [22]). We begin with a well-known result by Pitt. The proof is from [1].

THEOREM 3.1. *(Pitt) For $1 \leq q < p$, $L(\ell_p, \ell_q) = K(\ell_p, \ell_q)$. In other words, for $1 \leq q < p < \infty$, if X is a closed subspace of ℓ_p and $T : X \to \ell_q$ is a bounded operator, then T is compact.*

PROOF. Observe that ℓ_p is reflexive, hence X is reflexive and so B_X is weakly compact. Therefore in order to prove that T is compact it suffices to show that $T_{|B_X}$ is weak-to-norm continuous. Since the weak topology of X restricted to B_X is metrizable, it suffices to see that whenever $(x_n)_{n=1}^\infty \subset B_X$ is weakly convergent to some x in B_X then $(T(x_n))_{n=1}^\infty$ converges in norm to $T(x)$.

We need only show that if $(x_n)_{n=1}^\infty$ is a weakly null sequence in X then $\lim_{n\to\infty} \|T(x_n)\| = 0$. If this fails, we may suppose the existence of a weakly null sequence $(x_n)_{n=1}^\infty$ with $\|x_n\| = 1$ such that $\|T(x_n)\| \geq \delta > 0$ for all n. By passing to a subsequence we may suppose that $(x_n)_{n=1}^\infty$ is a basic sequence equivalent to the canonical ℓ_p basis.

But then, since $(T(x_n))_{n=1}^{\infty}$ is also weakly null, by passing to a further subsequence we may suppose that $\left(\frac{T(x_n)}{\|T(x_n)\|}\right)_{n=1}^{\infty}$, and hence $(T(x_n))_{n=1}^{\infty}$, is basic and equivalent to the canonical ℓ_q-basis. Since T is bounded we have effectively shown that the identity map $i : \ell_p \to \ell_q$ is bounded, which is absurd. Thus the theorem is proved. □

Quite a few theorems in this paper involve c_0 and ℓ_∞ embedding in various spaces of operators. Kalton showed in Corollary 2 of [15] that if X and Y are reflexive and $K(X, Y) = L(X, Y)$, then $L(X, Y)$ is reflexive. Observe that if $1 < p < q < \infty$, then $L(\ell_q, \ell_p) = K(\ell_q, \ell_p)$ is a reflexive space, and c_0 does not embed in $L(\ell_q, \ell_p)$ as c_o is not reflexive.

The next result from [13] will be used in several subsequent theorems. First, we need a few definitions. A bounded subset A of X is called a *limited subset* of X if each *weak**-null sequence in X^* tends to zero uniformly on A. If every limited subset of X is relatively compact, then we say that X has the *Gelfand-Phillips property.* Separable Banach spaces have the Gelfand-Phillips property.

Theorem 3.2. *(Ghenciu and Lewis* [13]*)*
Let X and Y be Banach spaces satisfying the following assumption: there exists a Banach space G with an unconditional basis (g_n) and biorthogonal coefficients (g_n^*) and two operators $R : G \to Y$ and $S : G^* \to X$ such that $(R(g_i))$ and $(S(g_i^*))$ are seminormalized sequences and either $(R(g_i))$ or $(S(g_i^*))$ is a basic sequence. Then $c_o \hookrightarrow K_{w^*}(X^*, Y)$ (indeed, in any subspace H of $L_{w^*}(X^*, Y)$ which contains $X \otimes_\lambda Y$).

Moreover, if $(R(g_i))$ and $(S(g_i^*))$ are basic and Y (or X) has the Gelfand-Phillips property, then $K_{w^*}(X^*, Y)$ contains a complemented copy of c_o.

Emmanuele observed in [7] that if X is a Banach space containing ℓ_1 and Y is a Banach space containing ℓ_p, for some $p \geq 2$, then $c_o \hookrightarrow K(X, Y)$ and $K(X, Y)$ cannot be complemented in $L(X, Y)$. He also noted that the closed subspace $\ell_p \otimes_\epsilon \ell_q$ embeds in $K(\ell_{p'}, \ell_q)$, for $\frac{1}{p} + \frac{1}{p'} = 1$, and $\ell_p \otimes_\epsilon \ell_q$ contains a complemented copy of c_o provided $1 < p' \leq q < \infty$. We refer to p' as the conjugate of p, i.e. $(\ell_p)^* = \ell_{p'}$.

THEOREM 3.3. *Suppose* $1 < p < \infty$, p' *is conjugate to* p, *and* $S : \ell_p \longrightarrow X$ *is a non-compact operator. For* $p' \leq p \leq q$ *or* $p \leq p' \leq q$, *if* $R : \ell_q \longrightarrow Y$ *is a non-compact operator, then* $c_o \hookrightarrow K_{w^*}(X^*, Y)$. *Furthermore, if* X *or* Y *is Gelfand-Phillips (separability is sufficient) then* $c_o \overset{c}{\hookrightarrow} K_{w^*}(X^*, Y)$. *However, if* $p < q < p'$, *then there exists spaces* X *and* Y *and appropriate operators* S *and* R *so that* $c_o \not\hookrightarrow K_{w^*}(X^*, Y)$.

PROOF. Case 1: $p' \leq p \leq q$.

Since $S : \ell_p \longrightarrow X$ is a non-compact operator, we can find a $\delta > 0$ and a sequence (x_n) in B_{ℓ_p} such that $\|S(x_n) - S(x_m)\| > \delta$ if $n \neq m$. Since ℓ_p is reflexive, B_{ℓ_p} is weakly compact. Thus, without loss of generality we may assume $(a_n) = (x_n - x_{n+1})$ is weakly null.

Observe that $\|S(a_n)\| > \delta$ for all $n \in \mathbf{N}$. Thus, $(a_n) \not\to 0$. Hence (a_n) is weakly null and seminormalized. By the Bessaga-Pelycznski Selection Principle (a_n) contains a subsequence (a_{n_k}) which is equivalent to a block basic sequence (h_n) of (e_n^p).

Note that ℓ_p is perfectly homogeneous for all $1 \leq p < \infty$, so we may assume (a_n) is equivalent to (e_n^p). Thus, (a_n) is basic. Since $p' \leq p$, there is a natural injection, J, from $\ell_{p'}$ into ℓ_p which sends $(e_n^{p'})$ to (a_n). Note that the Bessaga-Pelczynski Selection Principle also applies to the sequence $(S(a_n))$. Hence, we have (a_n) equivalent to $(J((e_n^{p'})))$, and without loss of generality $(S(J((e_n^{p'})))) = (S(a_n))$ is a seminormalized basic sequence in X.

Similarly, one can find a weakly null, seminormalized sequence (b_n) equivalent to (e_n^q) in ℓ_q so that $(R(b_n))$ is a seminormalized basic sequence in Y. Since $p \leq q$, there is a natural injection, U, from ℓ_p into ℓ_q which sends (e_n^p) to (b_n). Hence, we have (b_n) equivalent to $(U((e_n^p)))$, and without loss of generality $(R(U((e_n^p)))) = (R(b_n))$ is a weakly null, seminormalized basic sequence in Y. (The Bessaga-Pelczynski Selection Principle applies to the sequence $(R(b_n))$.) Therefore, by Theorem 3.2, $c_o \hookrightarrow K_{w^*}(X^*, Y)$.

Case 2: $p \leq p' \leq q$.

Since $S : \ell_p \longrightarrow X$ is a non-compact operator, we can find a $\delta > 0$ and a sequence (x_n) in B_{ℓ_p} such that $\|S(x_n) - S(x_m)\| > \delta$ if $n \neq m$. Since ℓ_p is reflexive, B_{ℓ_p} is weakly compact. Thus, without loss of generality we may assume $(a_n) = (x_n - x_{n+1})$ is weakly null.

Observe that $\|S(a_n)\| > \delta$ for every $n \in \mathbf{N}$. Thus, $(a_n) \not\to 0$. Hence (a_n) is weakly null and seminormalized. By the Bessaga-Pelycznski Selection Principle (a_n) contains a subsequence (a_{n_k}) which is equivalent to a block basic sequence (h_n) of (e_n^p).

Note that ℓ_p is perfectly homogeneous for all $1 \leq p < \infty$, so we may assume (a_n) is equivalent to (e_n^p). Thus, (a_n) is basic. Without loss of generality $(S(a_n))$ is a weakly null, seminormalized basic sequence in X.

Similarly, one can find a weakly null, seminormalized sequence (b_n) equivalent to (e_n^q) in ℓ_q. Since $p' \leq q$, there is a natural injection, U, from $\ell_{p'}$ into ℓ_q which sends $(e_n^{p'})$ to (b_n). Hence, we have (b_n) equivalent to $U((e_n^{p'}))$, and without loss of generality $R(U((e_n^{p'}))) = (R(b_n))$ is a seminormalized basic sequence in Y. Therefore, by Theorem 3.2, $c_o \hookrightarrow K_{w^*}(X^*, Y)$.

Case 3: $p < q < p'$.

Since $p < q < p'$, every operator from ℓ_q to ℓ_p is compact and every operator from $\ell_{p'}$ to ℓ_q is compact; i.e., $K_{w^*}((\ell_p)^*, \ell_q) = K(\ell_{p'}, \ell_q) = L(\ell_{p'}, \ell_q)$. In fact, this space of compact operators is reflexive. Thus c_o cannot embed in $K_{w^*}((\ell_p)^*, \ell_q)$. In this case, let $X = \ell_p$, $Y = \ell_q$, and let $S : \ell_p \to \ell_p$ and $R : \ell_q \to \ell_q$ be identity operators. $\square$

In Lemma 3 of [15], Kalton showed that $K(X,Y) \stackrel{c}{\not\hookrightarrow} L(X,Y)$ if Y is infinite dimensional and $\ell_1 \stackrel{c}{\hookrightarrow} X$. If Y is infinite dimensional, then B_Y is not compact ([4], Chapter 1). Thus, there is a $\delta > 0$ and a sequence (y_n) in B_Y with $\|y_n - y_m\| > \delta$ if $n \neq m$. The operator $T : \ell_1 \to Y$ defined by $T(\lambda) = \sum_{n=1}^{\infty} \lambda_i y_i$ is clearly non-compact. Kalton made crucial use of these strong properties on ℓ_1 and Y in the result just cited. The next theorem extends Kalton's result.

Theorem 3.4. *Suppose $\ell_p \stackrel{c}{\hookrightarrow} X$ where $1 < p < \infty$ and there exists a noncompact operator $T : \ell_p \to Y$. Then $K(X,Y) \stackrel{c}{\not\hookrightarrow} L(X,Y)$.*

Proof. The hypotheses guarantee that we can find a non-compact operator T, and there exists a sequence (x_n) in B_{ℓ_p} and a $\delta > 0$ such that $\|T(x_n) - T(x_m)\| > \delta$ if $n \neq m$. Since ℓ_p is reflexive, B_{ℓ_p} is weakly compact. Thus, without loss of generality we may assume that $(x_n - x_{n+1}) = (a_n)$ is weakly null. Observe that $\|T(a_n)\| > \delta$ for every

n. Thus, $(a_n) \not\to 0$. Hence, (a_n) is weakly null and seminormalized. By the Bessaga-Pelcznski Selection Principle (a_n) contains a subsequence (a_{n_k}) which is equivalent to a block basic sequence (h_n) of (e_n^p). Now, using the fact that ℓ_p is perfectly homogeneous, without loss of generality (a_n) is equivalent to (e_n^p). Thus, (a_n) is basic. Without loss of generality, the Bessaga-Pelcznski Selection Principle allows one to assume that $(T(a_n))$ is a seminormalized weakly null basic sequence in Y. Since (a_n) is an unconditional basic sequence, $c_o \hookrightarrow K(X,Y)$ by Theorem 20 of [13]. Hence, $K(X,Y) \stackrel{c}{\not\hookrightarrow} L(X,Y)$ by Theorem 3 of [17]. □

As noted in the remark immediately following Theorem 1.11, there are examples where c_o embeds in $K_{w^*}(X^*,Y)$ and ℓ_∞ does not embed in $L_{w^*}(X^*,Y)$. Emmanuele [8] showed that if X and Y are not Schur and c_o embeds in $K_{w^*}(X^*,Y)$, then ℓ_∞ embeds in $L_{w^*}(X^*,Y)$. The next theorem shows that this implication is also valid if neither X nor Y contains c_o.

THEOREM 3.5. *If $c_o \hookrightarrow L_{w^*}(X^*,Y)$, $c_o \not\hookrightarrow X$, and $c_o \not\hookrightarrow Y$, then $\ell_\infty \hookrightarrow L_{w^*}(X^*,Y)$.*

PROOF. Let $\phi : c_o \longrightarrow L_{w^*}(X^*,Y)$ be an isomorphic embedding.

Since $\sum_{n=1}^{\infty} e_n$ is wuc, we have

$$\sum_{n=1}^{\infty} |\langle \phi(e_n)(x^*), y^* \rangle| < \infty,$$

for every $x^* \in X^*, y^* \in Y^*$. Thus, $\sum_{n=1}^{\infty} \phi(e_n)(x^*)$ is wuc in Y, and since $c_o \not\hookrightarrow Y$, $\sum_{n=1}^{\infty} \phi(e_n)(x^*)$ is unconditionally converging in Y.

Therefore if M is a nonempty subset of $\mathbf{N}$, then $\sum_{n\in M} \phi(e_n)$ converges unconditionally in the strong operator topology (but certainly not in norm).

By the Uniform Boundedness Principle,

$$\left\{ \sum_{n\in M} \phi(e_n) : M \subseteq \mathbf{N}, M \neq \emptyset \right\}$$

is bounded in the norm topology.

Let $\mathcal{P}$ denote the power class of $\mathbf{N}$, and define $\mu : \mathcal{P} \to L_{w^*}(X^*, Y)$ by

$$\mu(M) = \begin{cases} 0 & \text{if } M = \emptyset \\ \sum_{n=1}^{\infty} \phi(e_n) \text{ (sot)} & \text{if } M \neq \emptyset \end{cases}$$

Observe that $\mu(M)$ is $w^* - w$ continuous, bounded, and finitely additive. However, μ is not strongly additive as $(\mu(\{n\})) = (\phi(e_n)) \not\to 0$.

Hence, by the σ-algebra version of the Diestel-Faires Theorem, $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$. $\square$

Recall that a Banach space X is *uniformly convex* if given $\epsilon > 0$ there is a $\delta > 0$ such that whenever $x, y \in S_X$ and $\|x - y\| = \epsilon$, then $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$. Note that the idea of uniform convexity involves keeping uniform control of convex combinations of points on the sphere.

Theorem 3.6. *(N. and V. Gurarii [4]) If the normalized Schauder basis $(x_n)_{n=1}^\infty$ spans a uniformly convex space X, then there is a $p > 1$ and an $A > 0$ such that $\sum_{n=1}^{\infty} a_n x_n \in X$ whenever $(a_n)_{n=1}^\infty \in \ell_p$ and*

$$\left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq A \left\|(a_n)_{n=1}^\infty\right\|_p.$$

Theorem 3.7. *If X is an infinite dimensional uniformly convex Banach space, then there is a $p > 1$ so that if $1 \leq q \leq p$ or $1 \leq q' \leq p$, then $K(\ell_q, X) \stackrel{c}{\not\hookrightarrow} L(\ell_q, X)$ and $K(\ell_{q'}, X) \stackrel{c}{\not\hookrightarrow} L(\ell_{q'}, X)$.*

Proof. By [4] p.39 we may choose a normalized basic sequence $(x_n)_{n=1}^\infty$ in X. Note that the sequence (x_n) cannot converge to anything as $\|x_n\| = 1$ for every $n \in \mathbf{N}$ and $(x_n) \xrightarrow{w} 0$. Use Theorem 3.6 and find $p > 1$ and an $A > 0$ satisfying

$$\left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq A \left\|(a_n)_{n=1}^\infty\right\|_p.$$

If $1 \leq q \leq p$, then the linear map $L : \ell_q \to X$ which sends $e_n^q \longmapsto e_n^p \longmapsto x_n$ for all $n \geq 1$ is continuous, non-compact and satisfies the requirements of Theorem 3 of [17], i.e. (e_n^p) is an unconditional basic sequence in ℓ_q such that $[e_n^p] \stackrel{c}{\hookrightarrow} \ell_q$, and $\{L(e_n^p) : n \in \mathbf{N}\}$ is not relatively compact. Therefore, $K(\ell_q, X) \stackrel{c}{\not\hookrightarrow} L(\ell_q, X)$.

On the other hand, suppose q' is conjugate to q and $1 \leq q' \leq p$. Then define maps $J : \ell_{q'} \to X$ which sends $e_n^{q'} \longmapsto x_n$ for $n \geq 1$ and the identity, id, on ℓ_q. Observe that $(J(e_n^{q'}))$ is a weakly null, seminormalized basic sequence in X equivalent to (x_n), and $id(e_n^q) = (e_n^q)$ is obviously a basic sequence. Hence, these maps satisfy the hypotheses of Theorem 3.2.

Note that since $q' \leq p$ every operator from ℓ_p to $\ell_{q'}$ is compact; i.e., $K_{w^*}((\ell_q)^*, \ell_p) = K(\ell_{q'}, \ell_p)$.

Hence, $c_o \overset{c}{\hookrightarrow} K(\ell_{q'}, X)$. Therefore, $K(\ell_{q'}, X) \overset{c}{\not\hookrightarrow} L(\ell_{q'}, X)$ by Theorem 3.4. $\square$

CHAPTER 4

A VECTOR-VALUED MEASURE INTERPRETATION AND APPLICATIONS TO SPACES OF OPERATORS

As noted in the introduction, results of Feder have been instrumental in the study of the complementation of $K(X,Y)$ in $L(X,Y)$. Specifically, Feder established the following result in [10].

THEOREM 4.1. *For Banach spaces X and Y, if there exists a non-compact operator $T \in L(X,Y)$ admitting an unconditional compact expansion, then $K(X,Y)$ is uncomplemented in $L(X,Y)$.*

Suppose that $T_n \in K(X,Y)$ for each $n \in \mathbf{N}$, $T \notin K(X,Y)$, and $\sum_{n=1}^{\infty} T(x)$ converges unconditionally to $T(x)$ for all $x \in X$. Feder considered two cases: (a) There is a $y^* \in Y^*$ so that $\sum_{n=1}^{\infty} T_n^*(y^*)$ is not weakly subseries convergent, i.e. $\sum_{n=1}^{\infty} T_n^*(y^*)$ is not unconditionally convergent, and (b) $\sum_{n=1}^{\infty} T_n^*(y^*)$ is weakly subseries convergent.

In both cases, Feder appealed to results of Kalton [15] to conclude that $K(X,Y)$ is not complemented in $L(X,Y)$. Moreover, Feder noticed that if X is infinite dimensional and $c_o \hookrightarrow Y$, then the hypotheses of Theorem 4.1 are satisfied. Emmanuele [7] subsequently showed that Theorem 4.1 implies that $c_o \hookrightarrow K(X,Y)$. See Theorem 2.6 in this paper for an efficient proof using the Diestel-Faires Theorem. Of course, as remarked previously, Emmanuele and John showed that if $c_o \hookrightarrow K(X,Y) \neq L(X,Y)$ (regardless of how c_o finds its way isomorphically into the space of compact operators), then $K(X,Y) \overset{c}{\not\hookrightarrow} L(X,Y)$. None of these papers contained the following.

Theorem 4.2. *Suppose that $K(X,Y) \neq L(X,Y)$ and $K(X,Y) \overset{c}{\hookrightarrow} L(X,Y)$. If (T_n) is a sequence in $K(X,Y)$, then $\sum_{n=1}^{\infty} T_n$ is unconditionally convergent in the norm of $K(X,Y)$ if and only if $\sum_{n=1}^{\infty} T_n(x)$ is unconditionally converging in Y for every $x \in X$.*

For the sake of continuity, we give a proof of Theorem 4.2 using Theorem 1.6. However, we note that subsequently in this chapter we give a proof of Theorem 1.6 which does not involve results from any of these papers.

Proof. (of Theorem 4.2) Observe that evaluation at a point is a continuous and linear operator. Therefore, if $\sum_{n=1}^{\infty} T_n$ is unconditionally converging, then $\sum_{n=1}^{\infty} T_n(x)$ is unconditionally converging for every $x \in X$.

Conversely, if $\sum_{n=1}^{\infty} T_n(x)$ is unconditionally converging for every $x \in X$ and $\sum_{n=1}^{\infty} T_n$ is not unconditionally converging, let $\pi : \mathbf{N} \to \mathbf{N}$ be a permutation of the natural numbers, let $\epsilon > 0$, and let (p_i), (q_i) be intertwining sequences of positive integers such that $\left\| \sum_{n=p_i}^{q_i} T_{\pi(n)} \right\| > \epsilon$ for every $i \in \mathbf{N}$. Let $L_i = \left\| \sum_{n=p_i}^{q_i} T_{\pi(n)} \right\| > \epsilon$. Then $\sum L_i(x)$ is unconditionally converging for every $x \in X$. Let $\mathcal{F}$ be the finite - co-finite algebra of subsets of $\mathbf{N}$, and note that the uniform boundedness principle and the pointwise unconditional convergence of the series above guarantees that if

$$\mu(A) = \begin{cases} \sum_{i \in A} L_i \text{ if } A \text{ is finite} \\ -\sum_{i \notin A} L_i \text{ if } \mathbf{N} \setminus A \text{ is finite,} \end{cases}$$

then μ is bounded and finitely additive. Since $\|\mu(i)\| \not\to 0$, μ is not strongly additive, and thus $c_o \hookrightarrow K(X,Y)$. This is a clear contradiction of Theorem 1.6. □

Kalton showed in [15] that for a bounded linear operator $T : \ell_\infty \to \ell_\infty$ with $T(e_n) = 0$ for every $n \in \mathbf{N}$, there exists an infinite subset M of $\mathbf{N}$ such that $T(x) = 0$ for every $x \in \ell_\infty(M)$. Kalton then established an operator theoretic version of this theorem: *If X is separable and $T : \ell_\infty \to L(X, \ell_\infty)$ is an operator so that $T(e_n) = 0$ for all $n \in \boldsymbol{N}$, then there is an infinite set $M \subseteq \boldsymbol{N}$ so that $T(x) = 0$ for all $x \in \ell_\infty(M)$.*

If X is separable, then there are countably many functionals which separate the points of $L(X, \ell_\infty)$, i.e. if D is a countable dense subset of X and $0 \neq T \in L(X, \ell_\infty)$, then there exists an $x \in D$ and $n \in \mathbf{N}$ so that $\langle T(x), e_n \rangle \neq 0$.

The first theorem in this chapter is a measure theoretic generalization of Kalton's results. Several corollaries which indicate applications to topics considered in this paper will follow. The reader should note that any operator $T : \ell_\infty \to X$ generates an X-valued measure [5] via $T(\chi_A) = m(A)$ for $A \subseteq \mathbf{N}$ and $T(x) = \int_{\mathbf{N}} x \, dm$.

THEOREM 4.3. *If $\mu : \mathcal{P} \to X$ is a bounded, finitely additive vector measure with $\mu(\{n\}) = 0$ for every $n \in \boldsymbol{N}$ and there are countably many elements of X^* which separate the range of a vector measure μ, then there exists an infinite set $M \subseteq \boldsymbol{N}$ such that $\mu(A) = 0$ for every $A \subseteq M$.*

PROOF. Partition $\mathbf{N}$ into uncountably many infinite sets $(U_\alpha)_{\alpha \in \Delta}$ such that $U_\alpha \bigcap U_\beta$ is finite if $\alpha \neq \beta$. Note that $\mu\left(\bigcup_{i \in F} U_i\right) = \sum_{i \in F} \mu(U_i)$ for all finite subsets $F \subseteq \mathbf{N}$.

We want to show that there exists an $\alpha \in \Delta$ such that $\mu(B) = 0$ for every $B \subseteq U_\alpha$.

Suppose not. Then for every $\alpha \in \Delta$ we can find $B_\alpha \subseteq U_\alpha$ with $\mu(B_\alpha) \neq 0$.

Thus, we can find $i \in \mathbf{N}$ such that $\{\alpha : x_i^*(\mu(B_\alpha)) \neq 0\}$ is uncountable. Without loss of generality suppose $x_i^*(\mu(B_\alpha)) \neq 0$ for every $\alpha \in \Delta$. Hence, without loss of generality suppose $x_i^*(\mu(B_\alpha)) > 0$ for every $\alpha \in \Delta$.

Choose $p > 0$ with $W = \{\alpha : x_i^*(\mu(B_\alpha)) > p\}$ is uncountable. This produces a contradiction since if $F \subseteq W$ and F is finite, then

$$\begin{aligned}
|F| \cdot p &\leq \left| x_i^* \left(\sum_{\alpha \in F} \mu(B_\alpha) \right) \right| \\
&\leq \left| x_i^* \left(\mu \left(\bigcup_{\alpha \in F} (B_\alpha) \right) \right) \right| \\
&\leq \|\mu\| \\
&= sup \{ \|\mu(A)\| : A \subseteq \mathbf{N} \}
\end{aligned}$$

For $|F|$ large enough we have a contradiction to the inequality, and thus, the theorem is proved. ∎

The same conclusion can be obtained if the range space is replaced by $L(X,Y)$ with X a separable space and Y the dual of a separable space.

If (y_n) is a basic sequence in Y, (y_n) is equivalent to (e_n), and X is infinite dimensional, then Theorem 2.3 produces a normailzed w^*-null sequence in X^* which immediately leads to a family T_M of non-compact operators: $T_M(x) = \sum_{i \in M} x_i^*(x) y_i$, where M is any infinite subset of $\mathbf{N}$. Feder used this construction to show that $K(X,Y) \overset{c}{\not\hookrightarrow} L(X,Y)$ if $c_o \hookrightarrow Y$. The following corollary generalizes Feder's result: If X is infinite dimensional and $c_o \hookrightarrow Y$, then $K(X,Y) \overset{c}{\not\hookrightarrow} L(X,Y)$.

Corollary 4.4. *Suppose Y contains a seminormalized unconditional basic sequence (y_i), and $(P_M)_{M \subseteq \boldsymbol{N}}$ is the family of projections associated with (y_i). If $U : X \to [y_i : i \geq 1]$ is an operator and X_o is a separable subspace of X so that $P_M U_{|X_o}$ is a non-compact operator for every infinite subset $M \subseteq \boldsymbol{N}$, then $K(X,Y) \overset{c}{\not\hookrightarrow} L(X,Y)$.*

Proof. By Phillips's Theorem (Theorem 2.4), let $J : Y \to \ell_\infty$ be an operator so that J is an isometry on $[y_i : i \geq 1]$. Suppose by way of contradiction that $Q : L(X,Y) \to K(X,Y)$ is a projection and define $\mu : \mathcal{P} \to L(X,Y)$ by $\mu(A) = P_A U$. Note that μ is bounded and finitely additive. Also, $JQ\mu(\{n\}) = J\mu(\{n\})$ for each $n \in \mathbf{N}$. Apply Theorem 4.3 and let M be an infinite subset of $\mathbf{N}$ so that $J\mu(M)_{|X_o}$ is compact. But neither $\mu(M)_{|X_o}$ nor $J\mu(M)_{|X_o}$ is compact, and we have a contradiction. $\square$

The next result is a new proof of a theorem of Kalton. It follows as a corollary of Theorem 4.3.

Corollary 4.5. *If $\ell_1 \overset{c}{\hookrightarrow} X$ and Y is infinite dimensional, then $K(X,Y) \overset{c}{\not\hookrightarrow} L(X,Y)$.*

Proof. Let $Q : X \to \ell_1$ be a projection. By way of contradiction, suppose $K(X,Y) \overset{c}{\hookrightarrow} L(X,Y)$. Since Y is infinite dimensional we can find a sequence (y_n) in B_Y and a $\delta > 0$ such that $\|y_n - y_m\| > \delta$ for $n \neq m$. Define $\mu : \mathcal{P} \to L(X,Y)$ by

$$\mu(M)(x) = \sum_{n \in M} (Q(x))_n y_n$$

for $M \neq \emptyset$. Suppose, by way of contradiction, there is a projection $P : L(X,Y) \to K(X,Y)$. By Phillips's Theorem (Theorem 2.4) let $J : Y \to \ell_\infty$ be an operator such that $J_{|[y_i]}$ is an isometry. Observe that $J\mu(A) : X \to \ell_\infty$, $JP\mu(A) : X \to \ell_\infty$ and $J\mu(\{n\}) = JP\mu(\{n\})$ for every $n \in \mathbf{N}$. The equations remain valid if X is replaced by ℓ_1, and we note that ℓ_1 is separable. Thus, Theorem 4.3 applies, and we obtain an infinite subset M so that $J\mu(M)$ is compact. Let $n \in M$ and consider $\mu(M)(e_n^*) = y_n$. Observe that $\{y_n : n \in M\}$ is not relatively compact. Thus, we have a contradiction. Therefore, no such projection, P, exists, and $K(X,Y) \stackrel{c}{\not\hookrightarrow} L(X,Y)$. □

The following corollary generalizes three results simultaneously: Corollary 4.5, Theorem 3.4 of this paper, and Theorem 3(ii) in Lewis [17]. See the remarks preceeding Corollary 4.4.

Corollary 4.6. *Suppose (x_n) is an unconditional basic sequence, $[x_n] \stackrel{c}{\hookrightarrow} X$, $(P_M)_{M \subseteq \mathbf{N}}$ the family of projections associated with (x_n), and $T : [x_n : n \geq 1] \to Y$ is an operator so that no subsequence of $(T(x_n))$ converges. Then $K(X,Y) \stackrel{c}{\not\hookrightarrow} L(X,Y)$.*

Proof. Let $Q : X \to [x_n]$ be a projection, and let $J : Y \to \ell_\infty$ be an operator such that J is an isometry (by Phillips's Theorem (Theorem 2.4)) on $[T(x_n) : n \geq 1]$. Define $\mu : \mathcal{P} \to L(X,Y)$ by $\mu(A) = TP_AQ$. If A is finite, $\mu(A)$ is compact. Suppose by way of contradiction that $P : L(X,Y) \to K(X,Y)$ is a projection. Now μ and $P\mu$ are bounded and finitely additive, and $\mu(\{n\}) - P\mu(\{n\}) = 0$ for every $n \in \mathbf{N}$. Let M be an infinite set such that $J\mu(M)_{|[x_n]} = JP\mu(M)_{|[x_n]}$. But TP_MQ and JTP_MQ are not compact. Thus, we have a contradiction. □

Corollary 4.7. *(Emmanuele [7], John [15])*
If $c_o \hookrightarrow K(X,Y)$, then $K(X,Y)$ is uncomplemented in $L(X,Y)$

Proof. By way of contradiction suppose $P : L(X,Y) \to K(X,Y)$ is a projection. By Corollary 4.4, $c_o \not\hookrightarrow Y$. Let (T_n) be a sequence in $K(X,Y)$ equivalent to (e_n). Observe that $\sum T_n(x)$ is weakly unconditionally converging in Y. Hence, $\sum T_n(x)$ is unconditionally converging in Y since $c_o \not\hookrightarrow Y$. Define $\mu : \mathcal{P} \to L(X,Y)$ by $\mu(A)(x) = \sum_{n \in A} T_n(x)$ for $A \subseteq \mathbf{N}$. Let $\nu = P \circ \mu : \mathcal{P} \to K(X,Y)$. This measure is bounded and finitely additive but not strongly additive as $\|\nu(\{n\})\| = \|P(\mu(\{n\}))\| = \|P(\{e_n\})\| \not\to 0$.

Therefore, by the Diestel-Faires Theorem $\ell_\infty \hookrightarrow K(X,Y)$. By a result of Kalton [15] this implies either $\ell_\infty \hookrightarrow X^*$ or $\ell_\infty \hookrightarrow Y$.

If $\ell_\infty \hookrightarrow Y$, then we must have $c_o \hookrightarrow Y$, but this is a contradiction. Thus, $\ell_\infty \hookrightarrow X^*$.

Hence, $\ell_\infty \hookrightarrow X^*$, and $\ell_1 \overset{c}{\hookrightarrow} X$ ([2], [15]).

Therefore, by Corollary 4.6, we have $K(X,Y) \overset{c}{\not\hookrightarrow} L(X,Y)$. $\square$

COROLLARY 4.8. *(Bator and Lewis [3])*
Suppose $\ell_1 \overset{c}{\hookrightarrow} X$ and Y is a non-reflexive Banach space. Then $W(X,Y) \overset{c}{\not\hookrightarrow} L(X,Y)$.

PROOF. Let $Q : X \to \ell_1$ be a projection. Since Y is infinite dimensional we can find a sequence (y_n) in B_Y with no weakly convergent subsequence. Define $\mu : \mathcal{P} \to L(X,Y)$ by

$$\mu(M)(x) = \sum_{n\in M} (Q(x))_n y_n$$

for $M \neq \emptyset$. By way of contradiction, suppose that there is a projection $P : L(X,Y) \to W(X,Y)$. Let $J : Y \to \ell_\infty$ be an operator such that $J_{|[y_i]}$ is an isometry (by Theorem 2.4). Observe that $J\mu(A) : X \to \ell_\infty$, $JP\mu(A) : X \to \ell_\infty$ and $J\mu(\{n\}) = JP\mu(\{n\})$ for every $n \in \mathbf{N}$. The equations remain valid if X is replaced by ℓ_1, and we note that ℓ_1 is separable. Thus, Theorem 4.3 applies, and we obtain an infinite subset M so that $J\mu(M)$ is weakly compact. Let $n \in M$ and consider $\mu(M)(e_n^*) = y_n$. Observe that $\{y_n : n \in M\}$ is not relatively weakly compact. Thus, we have a contradiction. Therefore, no such projection, P, exists, and $W(X,Y) \overset{c}{\not\hookrightarrow} L(X,Y)$. $\square$

A Banach space X is said to be *prime* if every infinite-dimensional complemented subspace of X is isomorphic to X. For A any subset of $\mathbf{N}$ we denote by $\ell_\infty(A)$ the subspace of ℓ_∞ given by

$$\ell_\infty(A) = \{\xi = (\xi(k))_{k=1}^\infty \in \ell_\infty : \xi(k) = 0 \text{ if } k \notin A\}.$$

In 1960, Pelczynski [19] showed that c_o and ℓ_p, for $1 \leq p < \infty$, are prime. In particular, he proved: *If E is a subspace of c_o or ℓ_p, for $1 \leq p < \infty$, X is a subspace complmented in E and X contains a subspace Y complemented in X and isomorphic to E, then X is isomorphic to E.* In his main theorem he proved the following: *Let E be one of the spaces s, ℓ_p, for $1 \leq p < \infty$, or c_o. Then each subspace complemented* *in E is isomorphic to E or is of finite dimension.* In 1967, Lindenstrauss [18] proved that ℓ_∞ is also prime. One might ask if c_o is complemented in ℓ_∞ as it is certainly a subspace of ℓ_∞ (its bidual). As stated in the introduction, ℓ_∞ contains an isometric copy of all separable Banach spaces, and a separable Banach space can only be injective if it is isomorphic to a complemented subspace of ℓ_∞. Certainly, c_o will be injective if and only if it is complemented in ℓ_∞. However, in 1940, R. S. Phillips proved that c (the space of convergent sequences) is *not* complemented in ℓ_∞, and the following year A. Sobczyk proved the result for c_o. Classifying the subspaces of ℓ_∞ has been an important topic of study. The following is a vector measure theoretic version of the proof that ℓ_∞ is prime and is significantly shorter than the original proof by Lindenstrauss.

Let $S(\Sigma, X)$ be the linear space of all X-valued simple functions defined on a σ-algebra, Σ, i.e. $S(\Sigma, X) = \left\{ \sum_{i=1}^{n} \alpha_i \chi_{A_i} : A_i \in \Sigma, \alpha_i \in \mathbf{R} \right\}$. Then for any $f \in S(\Sigma, X)$, $f(\chi) = \sum_{i=1}^{n} \chi_{A_i} x_i$ for $A_i \in \Sigma, x_i \in X$. Let $T : S(\Sigma, X) \to Y$ be a continuous linear transformation defined by $T(\chi_A(x)) = \mu(A)(x) = \int \chi_A d\mu$. Thus, the measure is literally defined directly by the action of the operator on the simple functions. Let $U(\Sigma, X)$ be the uniform closure of $S(\Sigma, X)$. The simple functions are dense in $U(\Sigma, X)$, and any continuous linear map from the simple functions can be extended to the uniform closure. Observe that $\ell_\infty = U(\mathcal{P}, \mathbf{R})$ where $\mathcal{P}$ is the power class of $\mathbf{N}$, and $c_o = U(\mathcal{R}, \mathbf{R})$ where $\mathcal{R}$ is the ring of all finite subsets of $\mathbf{N}$. Note that $\mathcal{R}$ is closed under finite unions, relative complements and intersections.

Corollary 4.9. *If X is a complemented subspace of ℓ_∞ and $e_n \in X$ for every $n \in \boldsymbol{N}$, then X contains an isomorphic (and complemented) subspace isomorphic to ℓ_∞.*

Proof. Suppose $P : \ell_\infty \to X$ is a projection, and let $\nu : \mathcal{P} \to \ell_\infty$ be defined by $\nu(A) = P(\chi_A)$, $A \in \mathcal{P}$. Let $\mu : \mathcal{P} \to \ell_\infty$ be defined by $\mu(A) = \chi_A$, $A \in \mathcal{P}$.

If $m(A) = \mu(A) - \nu(A)$, $A \in \mathcal{P}$, then $m(\{n\}) = 0$ for every $n \in \mathbf{N}$. Observe m is bounded, finitely additive, and m takes its values in ℓ_∞, a space with a countable separating family of functionals. Therefore, by Theorem 4.3, there exists an infinite subset $M \subseteq \mathbf{N}$ such that $m(A) = 0$ for every $A \subseteq M$. Whence, $\nu(A) = \mu(A)$ for every $A \subseteq M$. Now, $\ell_\infty(M)$ is obviously isomorphic to ℓ_∞ and $\int \psi d\nu = \psi$ for all simple functions $\psi \in \ell_\infty(M)$ (i.e., ν generates the identity). It follows that $\int \xi d\nu = \xi$ for all $\xi \in \ell_\infty(M)$. Therefore, $\ell_\infty(M) \subseteq X$. □

Let $E = \ell_p$, $1 \leq p < \infty$, c_o, or ℓ_∞. As noted previously, Pelczynski showed that if X is complemented in E and X contains a subspace Y complemented in X and Y is isomorphic to E, then X is isomorphic to E [19].

The previous theorem showed that if X is complemented in ℓ_∞ and c_o embeds in X, then X contains $\ell_\infty(M)$ for some infinite $M \subseteq \mathbf{N}$. Clearly, $\ell_\infty(M)$ is complemented in ℓ_∞ and is therefore complemented in X, and $\ell_\infty(M)$ is isomorphic (even isometric) to ℓ_∞.

Now suppose that X is complemented in ℓ_∞ and c_o does not embed in X. Let P be a projection from ℓ_∞ onto X, and let m be the vector measure generated by P, i.e. $P \leftrightarrow m : \mathcal{P} \to X$. Note that m must be strongly additive by the Diestel-Faires Theorem since c_o does not embed in X. Thus, by Theorem 1, p. 148 of [5], P is weakly compact. Now, ℓ_∞ is a $C(K)$-space, and all weakly compact operators on any $C(K)$-space map weakly convergent sequences to norm convergent sequences (p. 113 of [4]). If (x_n) is a sequence in B_X, then there exists a subsequence (x_{n_i}) converging weakly to some element in B_X. This implies $P((x_{n_i})) = (x_{n_i})$ is norm convergent. Therefore, P is completely continuous. Thus, P^2 must be compact. Consequently, the unit ball of X must be relatively compact. (P is a projection onto X, and P must map the unit ball of ℓ_∞ onto the unit ball of X.) Thus, X must be finite dimensional.

Consequently, if X is a complemented subspace of ℓ_∞, then X is infinite dimensional if and only if X contains c_o and X is isomorphic to ℓ_∞.

In Chapter 1 strongly additive measures were discussed in detail; in particular, the Diestel-Faires Theorem was used to demonstrate applications of strongly additive measures. Theorem 4.3 provides an easy way to see that a strongly additive vector measure is almost countably additive.

Corollary 4.10. *(Drewnowski [5], p. 38) If X is separable, Σ is a σ-algebra of sets, $\mu : \Sigma \to X$ is strongly additive, and (A_i) is a pairwise disjoint sequence from Σ, then there is a subsequence (A_{n_i}) so that μ is countably additive on the σ-ring generated by this subsequence.*

Proof. Let $S = \bigcup_{i\in S} A_i$ where the $\{A_i\}$ are a pairwise disjoint sequence from Σ. Identify i with A_i and set $\nu(S) = \sum_{i\in S} \mu(A_i)$. Then $\nu(\{i\}) = \mu(\{i\})$ for all $i \in S$. Apply Theorem 4.3. □

The Diestel-Faires Theorem provided us with an elegant, almost existential, proof of Theorem 4.2. We conclude this paper by using Theorem 4.3 to give a constructive proof of Theorem 4.2.

Suppose X and Y are separable, $\{x_n : n \geq 1\}$ is dense in X, $\{y_n : n \geq 1\}$ is dense in Y, and $\{y_n^* : n \geq 1\}$ is a norming set of functionals in B_{Y^*}. That is, $\|y\| = sup\,\{y_n^*(y) : n \geq 1\}$.

Let $U : X \to Y$ be an operator, and suppose $U^* \neq 0$. Therefore, $U \neq 0$, and there exists a $y_n^* \in Y^*$ such that $U^*(y_n^*) \neq 0$. Therefore, there exists $k \in \mathbf{N}$ such that $\langle U^*(y_n^*), x_k\rangle \neq 0$. Note that $U^{**}(\eta(x_k)) = \eta(U(x_k)) \neq 0$.

Now suppose $\sum_{i=1}^{\infty} T_i$ is unconditionally converging with respect to the strong operator topology, and $\sum_{i=1}^{\infty} T_i^*$ is unconditionally converging with respect to the strong operator topology. We assert that $\left(\sum_{i\in A} T_i(sot)\right)^* = \sum_{i\in A} T_i^*(sot)$. (The operators are defined pointwise.) In fact,

$$\begin{aligned}
\left\langle \left(\sum_{i\in A} T_i(sot)\right)^* (y^*), x \right\rangle &= \left\langle \left(\sum_{i\in A} T_i(sot)\right) x, y^* \right\rangle \\
&= \left\langle \sum_{i\in A} T_i(x), y^* \right\rangle \\
&= \sum_{i\in A} \langle T_i(x), y^*\rangle \\
&= \sum_{i\in A} \langle T_i^*(y^*), x\rangle \\
&= \left\langle \sum_{i\in A} T_i^*(y^*), x \right\rangle.
\end{aligned}$$

Therefore, $\left(\sum_{i\in A} T_i(sot)\right)^* = \sum_{i\in A} T_i^*(sot)$.

Theorem 4.11. *Suppose $T_n : X \to Y$ is compact for every $n \in \boldsymbol{N}$, X and Y are separable, $\sum_{n=1}^{\infty} T_n(x)$ is unconditionally converging for every $x \in X$ and $\sum_{n=1}^{\infty} T_n^*(y^*)$ is* *unconditionally converging for every $y^* \in Y^*$. Further suppose $T_n \in K(X,Y)$ for every $n \in \boldsymbol{N}$ and $P : L(X,Y) \to K(X,Y)$ is a projection.*

Proof. Define $\mu : \mathcal{P} \to L(Y^*, X^*)$ by

$$\begin{aligned}
\mu(A) &= \left(\sum_{i\in A} T_i(sot) - P\left(\sum_{i\in A} T_i(sot)\right)\right)^* \\
&= \left(\sum_{i\in A} T_i(sot)\right)^* - \left(P\left(\sum_{i\in A} T_i(sot)\right)\right)^* \\
&= \sum_{i\in A} T_i^*(sot) - \left(P\left(\sum_{i\in A} T_i(sot)\right)\right)^*,
\end{aligned}$$

for all $A \subseteq \mathbf{N}$. Certainly, $\mu(\{n\}) = 0$ for every $n \in \mathbf{N}$. Now, if $\mu(A) \neq 0$, then

$$\sum_{i\in A} T_i(sot) - P\left(\sum_{i\in A} T_i(sot)\right) \neq 0.$$

Therefore, there exists x_n and y_k^* such that

$$\left\langle \sum_{i\in A} T_i(sot) - P\left(\sum_{i\in A} T_i(sot)\right) x_n, y_k^* \right\rangle \neq 0.$$

Thus, $\mu(\mathcal{P})$ is countably separated. Hence, there exists an infinite subset M of $\mathbf{N}$ such that $\mu(A) = 0$ for every $A \subseteq M$.

Whence, $\sum_{i\in A} T_i^*(y^*) = P\left(\sum_{i\in A} T_i(sot)\right)^* (y^*)$ for every $y^* \in Y^*$. Therefore, $\sum_{i\in A} T_i^*(y^*)$ defines a compact operator for every $A \subseteq M$.

Now consider Corollary 3 (page 269) of Kalton [15] with repect to $\sum_{i\in M} T_i^*(sot)$ and $\sum_{i\in A} T_i^*(sot)$ for $A \subseteq M$. We know by the Orlicz-Pettis Theorem (Theorem 2.7) that every subseries converging weakly to a point in the space implies the series is unconditionally converging. Therefore, the series $\sum_{i\in M} T_i$ is unconditionally converging. Thus, the full series $\sum_{n=1}^{\infty} T_n$ is unconditionally converging. □

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