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Representation of Analysis Results Involving Aleatory and Epistemic Uncertainty

J. C. Helton, J. D. Johnson, W. L. Oberkampf, C.J. Sallaberry

Prepared by
Sandia National Laboratories
Albuquerque, New Mexico 87185 and Livermore, California 94550

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Representation of Analysis Results Involving Aleatory and Epistemic Uncertainty

Jon C. Helton
Department of Mathematics and Statistics, Arizona State University, Tempe, AZ 85287-1804 USA

Jay D. Johnson
ProStat, Mesa, AZ 85204-5326

William L. Oberkampf
Validation and Uncertainty Quantification Department
Sandia National Laboratories
PO Box 5800
Albuquerque, NM 87185-0828

Cedric J. Sallaberry
Total Systems Performance Assessment Department
Sandia National Laboratories
PO Box 5800
Albuquerque, NM 87185-0776

Abstract

Procedures are described for the representation of results in analyses that involve both aleatory uncertainty and epistemic uncertainty, with aleatory uncertainty deriving from an inherent randomness in the behavior of the system under study and epistemic uncertainty deriving from a lack of knowledge about the appropriate values to use for quantities that are assumed to have fixed but poorly known values in the context of a specific study. Aleatory uncertainty is usually represented with probability and leads to cumulative distribution functions (CDFs) or complementary cumulative distribution functions (CCDFs) for analysis results of interest. Several mathematical structures are available for the representation of epistemic uncertainty, including interval analysis, possibility theory, evidence theory and probability theory. In the presence of epistemic uncertainty, there is not a single CDF or CCDF for a given analysis result. Rather, there is a family of CDFs and a corresponding family of CCDFs that derive from epistemic uncertainty and have an uncertainty structure that derives from the particular uncertainty structure (i.e., interval analysis, possibility theory, evidence theory, probability theory) used to represent epistemic uncertainty. Graphical formats for the representation of epistemic uncertainty in families of CDFs and CCDFs are investigated and presented for the indicated characterizations of epistemic uncertainty.

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1. Introduction

The appropriate treatment of uncertainty in analyses of complex systems is a topic of great importance and hence widespread interest [1-14]. Such treatment is particularly important in computational analyses that are used to support important societal decisions on issues related to climate change [15-19], reactor safety [20-26], radioactive waste disposal [27-34], nuclear weapon safety [35-38], economic policy [39-43], environmental degradation [44-47], and many additional areas of concern and challenge. Indeed, it is difficult to envision how adequately informed decisions can be made on such issues without an appropriate assessment of the uncertainties present in the supporting analyses.

An immediate challenge in the development of an appropriate treatment of uncertainty in an analysis of a complex system is the selection of a mathematical structure to be used in the representation of uncertainty. Traditionally, probability theory has provided this structure [48-55]. However, in the last several decades, additional mathematical structures for the representation of uncertainty such as evidence theory [56-63], possibility theory [64-70], fuzzy set theory [71-75], and interval analysis [76-81] have been introduced. This introduction has been accompanied by a lively discussion of the strengths and weaknesses of the various mathematical structures for the representation of uncertainty [82-90]. For perspective, several comparative discussions of these different approaches to the representation of uncertainty are available [72; 91-98].

An additional and closely related challenge derives from the presence of two different types of uncertainty in most analyses for complex systems. The first type derives from an inherent randomness in the behavior of the system under study. For example, the weather conditions at the time of a major accident at a chemical plant could have a significant effect on the number of resultant off-site injuries but is essentially random in so far as our ability to predict the future is concerned. Uncertainty of this type is usually referred to as aleatory uncertainty; alternative designators include variability, stochastic, irreducible, and Type A [11; 53; 99-105]. The second type of uncertainty derives from a lack of knowledge about a quantity that is assumed to have a fixed, but poorly known, value in the context of a particular analysis. For example, the appropriate value to use for a spatially averaged permeability in an analysis involving groundwater flow has, by definition, a single value but this single "effective" value can never be known with certainty. Uncertainty of this type is usually referred to as epistemic uncertainty; alternative designators include state of knowledge, subjective, reducible, and Type B [11; 53; 99-105].

The challenges associated with the treatment of aleatory and epistemic uncertainty in the analysis of a complex system are twofold. First, it is necessary to select and then implement a mathematical structure to represent each of these uncertainties. The mathematical structures used to represent aleatory and epistemic uncertainty in a particular analysis are not necessarily the same. For example, probability theory could be, as is usually the case, used to represent aleatory uncertainty while, in the same analysis, evidence theory is used to represent epistemic uncertainty. Second, the mathematical structures used to represent aleatory and epistemic uncertainty must be propagated through the analysis in a manner that maintains an appropriate separation of these uncertainties in the final results of interest.

The purpose of this presentation is to discuss and illustrate the representation of analysis results involving aleatory and epistemic uncertainty. To this end, several mathematical structures for the representation of uncertainty are described (Sect. 2); the distinction between aleatory and epistemic uncertainty is discussed (Sect. 3); a simple example involving the reliability of a coastal dike is introduced for use in illustrating the representation of uncertainty (Sect. 4); the representation of unstructured epistemic uncertainty is discussed and illustrated (Sect. 5); and the representation of structured epistemic uncertainty is discussed and illustrated (Sect. 6). The presentation then ends with a concluding discussion (Sect. 7).

2. Representation of Uncertainty

This section provides a brief overview of the following mathematical structures that are used in the representation of uncertainty: interval analysis (Sect. 2.1), possibility theory (Sect. 2.2), evidence theory (Sect. 2.3), and probability theory (Sect. 2.4). For each structure, the following topics are considered: (i) the representation of uncertainty in a single variable x_i, (ii) the representation of uncertainty in a vector $\mathbf{x} = [x_1, x_2, \ldots, x_{nX}]$ of uncertain variables, and (iii) the representation of the uncertainty in a variable y defined by

$$y = F(\mathbf{x}), \mathbf{x} = [x_1, x_2, \ldots, x_{nX}], \tag{2.1}$$

where F is a function of the vector $\mathbf{x}$ of uncertain variables $x_1, x_2, \ldots, x_{nX}$. For this overview, no distinction is made between aleatory uncertainty and epistemic uncertainty. Then, the section concludes with a discussion of the use of sampling-based (i.e., Monte Carlo) procedures in the propagation of different mathematical structures for the representation of uncertainty (Sect. 2.5).

2.1 Interval Analysis

Interval analysis is based on the assumption that a set $\mathcal{X}_i$ of possible values for a variable x_i is known but with no specified uncertainty structure within the set $\mathcal{X}_i$ [76-81]. Thus, all that is assumed to be known about x_i is that its value is contained within the set $\mathcal{X}_i$. Usually, but not necessarily, $\mathcal{X}_i$ is defined by

$$\mathcal{X}_i = \{x_i : a_i \le x_i \le b_i\}, \tag{2.2}$$

where $[a_i, b_i]$ is an interval that contains the possible values for x_i.

For a vector $\mathbf{x} = [x_1, x_2, \ldots, x_{nX}]$ of variables known only to be contained in the sets $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_{nX}$, the set $\mathcal{X}$ of possible values is given by

$$\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_{nX}. \tag{2.3}$$

Given that there is no specified uncertainty structure for the sets $\mathcal{X}_1$, $\mathcal{X}_2$, …, $\mathcal{X}_{nX}$, there is also no uncertainty structure for the set $\mathcal{X}$ of possible values for $\mathbf{x}$. Further, the preceding representation for $\mathcal{X}$ is predicated on the assumption that no restrictions exist that preclude specific combinations of values for the individual variables contained in $\mathbf{x}$.

Propagation of the individual values of $\mathbf{x}$ contained in $\mathcal{X}$ through the function F results in the set

$$\mathcal{Y} = \{y : \mathbf{x} \in \mathcal{X} \text{ and } y = F(\mathbf{x})\} \tag{2.4}$$

of possible values for y. Given that there is no uncertainty structure for the set $\mathcal{X}$, there is also no uncertainty structure for the set $\mathcal{Y}$.

In most applications, the indicated propagation to produce the set $\mathcal{Y}$ is based on using algebraic procedures implemented with appropriate software. However, an interval analysis can also be thought of as an optimization process in which it is desired to find the minimum and maximum of the function F on the set $\mathcal{X}$. Alternatively, the uncertainty propagation associated with an interval analysis can be approximated with a sampling-based (i.e., Monte Carlo) procedure.

2.2 Possibility Theory

Possibility theory [64-70] provides a representation for uncertainty that permits the specification of more structure than interval analysis and is based on the specification of a pair $(\mathcal{X}_i, r_i)$ for a variable x_i, where (i) $\mathcal{X}_i$ is the set of possible values for x_i and (ii) r_i is a function defined on $\mathcal{X}_i$ such that $0 \le r_i(x_i) \le 1$ for $x_i \in \mathcal{X}_i$ and $\sup\{r(x_i): x_i \in \mathcal{X}_i\} = 1$. The function r_i provides a measure of the amount of "credence" or "confidence" that is assigned to each element of $\mathcal{X}_i$ and is referred to as the possibility distribution function for x_i. The pair $(\mathcal{X}_i, r_i)$ defines a possibility space for the variable x_i.

A value of $r(x_i) = 1$ indicates that there is no known information that refutes the "occurrence" or "appropriateness" of a specific value x_i contained in $\mathcal{X}_i$, and a value of $r(x_i) = 0$ indicates that known information completely refutes the "occurrence" or "appropriateness" of x_i. Further, increasing values for $r(x_i)$ between 0 and 1 indicate an increasing absence of information that refutes the "occurrence" or "appropriateness" of x_i. Intuitively, $r(x_i) = 1$ signifies that x_i is entirely possible in the sense that nothing is known that contradicts the possibility of x_i; $0 < r(x_i) < 1$ signifies that x_i is possible but with the amount of information indicating that x_i is not possible increasing as $r(x_i)$ approaches 0; and $r(x_i) = 0$ signifies that x_i is known to be impossible.

Possibility theory provides two measures of likelihood for subsets of $\mathcal{X}_i$: possibility and necessity. Specifically, possibility and necessity for a subset $\mathcal{U}$ of $\mathcal{X}_i$ are defined by

$$Pos_i(\mathcal{U}) = \sup\{r_i(x_i) : x_i \in \mathcal{U}\} \tag{2.5}$$

and

$$Nec_i(\mathcal{U}) = 1 - Pos_i(\mathcal{U}^c) = 1 - \sup\left\{r_i(x_i) : x_i \in \mathcal{U}^c\right\}, \tag{2.6}$$

respectively. In consistency with the properties of the possibility distribution function r_i, $Pos_i(\mathcal{U})$ provides a measure of the amount of information that does not refute the proposition that $\mathcal{U}$ contains the appropriate value for x_i, and $Nec_i(\mathcal{U})$ provides a measure of the amount of uncontradicted information that supports the proposition that $\mathcal{U}$ contains the appropriate value for x_i.

Relationships satisfied by possibility and necessity for the possibility space $(\mathcal{X}_i, r_i)$ include

$$1 = Nec_i(\mathcal{U}) + Pos_i(\mathcal{U}^c),\ Nec_i(\mathcal{U}) \le Pos_i(\mathcal{U}) \tag{2.7}$$

$$1 \le Pos_i(\mathcal{U}) + Pos_i(\mathcal{U}^c),\ Nec_i(\mathcal{U}) + Nec_i(\mathcal{U}^c) \le 1 \tag{2.8}$$

$$1 = \max\left\{Pos_i(\mathcal{U}), Pos_i(\mathcal{U}^c)\right\},\ 0 = \min\left\{Nec_i(\mathcal{U}), Nec_i(\mathcal{U}^c)\right\} \tag{2.9}$$

$$Pos_i(\mathcal{U}) < 1 \Rightarrow Nec_i(\mathcal{U}) = 0,\ Nec_i(\mathcal{U}) > 0 \Rightarrow Pos_i(\mathcal{U}) = 1 \tag{2.10}$$

for subsets $\mathcal{U}$ of $\mathcal{X}_i$ (see Ref. [106], p. 34).

Convenient graphical summaries of possibility spaces are provided by cumulative necessity functions (CNFs), complementary cumulative necessity functions (CCNFs), cumulative possibility functions (CPoFs), and complementary cumulative possibility functions (CCPoFs). Specifically, the CNF, CCNF, CPoF and CCPoF for the possibility space $(\mathcal{X}_i, r_i)$ are defined by the sets

$$\mathcal{CNF}_i = \left\{\left[x, Nec_i(\mathcal{U}_x)\right] : x \in \mathcal{X}_i\right\},\ \mathcal{CCNF}_i = \left\{\left[x, Nec_i\left(\mathcal{U}_x^c\right)\right] : x \in \mathcal{X}_i\right\} \tag{2.11}$$

$$\mathcal{CPoF}_i = \left\{\left[x, Pos_i(\mathcal{U}_x)\right] : x \in \mathcal{X}_i\right\},\ \mathcal{CCPoF}_i = \left\{\left[x, Pos_i\left(\mathcal{U}_x^c\right)\right] : x \in \mathcal{X}_i\right\}, \tag{2.12}$$

where

$$\mathcal{U}_x = \left\{\tilde{x} : \tilde{x} \in \mathcal{X}_i \text{ and } \tilde{x} \le x\right\}.$$

Plots of the curves defined by the points associated with $\mathcal{CNF}_i$, $\mathcal{CCNF}_i$, $\mathcal{CPoF}_i$ and $\mathcal{CCPoF}_i$ yield the CNF, CCNF, CPoF, and CCPoF for the possibility space $(\mathcal{X}_i, r_i)$ (Fig. 1).

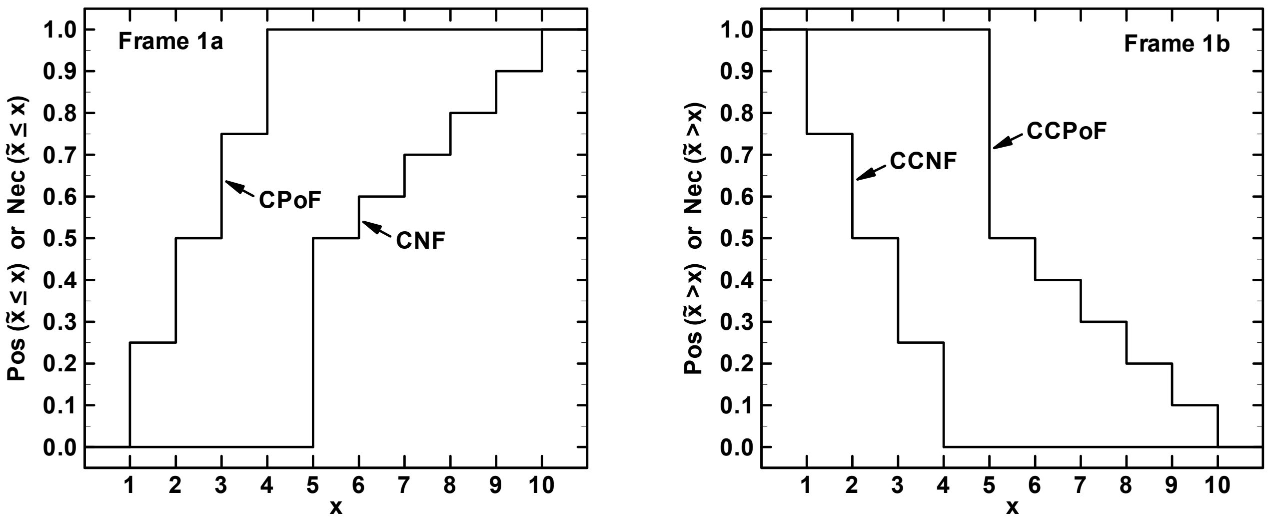

Fig. 1. Plots of CNF, CCNF, CPoF and CCPoF for possibility space $(\mathcal{X}, r)$ with (i) $\mathcal{X} = \{x: 1 \le x \le 10\}$, (ii) $r(x) = i/4$ for $i \le x \le i + 1$ and $i = 1, 2, 3, 4$, (iii) $r(x) = (10 - i)/10$ for $i \le x \le i + 1$ and $i = 5, 6, 7, 8, 9$, and (iv) $Pos(\tilde{x} \le x)$, $Nec(\tilde{x} \le x)$, $Pos(\tilde{x} > x)$ and $Nec(\tilde{x} > x)$ used as abbreviated notations for the expressions $Pos_i(\mathcal{U}_x)$, $Nec_i(\mathcal{U}_x)$, $Pos_i(\mathcal{U}_x^c)$ and $Nec_i(\mathcal{U}_x^c)$ in Eqs. (2.11) and (2.12): (a) CNF and CPoF, and (b) CCNF and CCPoF.

If the variables $x_1, x_2, \ldots, x_{nX}$ have associated possibility spaces $(\mathcal{X}_1, r_1), (\mathcal{X}_2, r_2), \ldots, (\mathcal{X}_{nX}, r_{nX})$, then the vector $\mathbf{x} = [x_1, x_2, \ldots, x_{nX}]$ also has an associated possibility space $(\mathcal{X}, r_X)$, where $\mathcal{X}$ is defined the same as in Eq. (2.3) and

$$r_X(\mathbf{x}) = \min\{r_1(x_1), r_2(x_2), \ldots, r_{nX}(x_{nX})\}. \tag{2.13}$$

The indicated definitions for $\mathcal{X}$ and r_X are predicated on the assumption that no restrictions involving possible combinations of values for the x_i's exist. If such restrictions exist, then the definition of r_X is more complex.

Once the possibility space $(\mathcal{X}, r_X)$ for $\mathbf{x}$ is defined, possibility $Pos_X(\mathcal{U})$ and necessity $Nec_X(\mathcal{U})$ for subsets $\mathcal{U}$ of $\mathcal{X}$ are defined as indicated in Eqs. (2.5) and (2.6). Further, the relationships indicated in Eqs. (2.7) – (2.10) also hold.

Propagation of the individual values of $\mathbf{x}$ contained in $\mathcal{X}$ through the function F indicated in Eq. (2.1) results in a set $\mathcal{Y}$ of possible values for y of the form shown in Eq. (2.4). Given that a possibility space $(\mathcal{X}, r_X)$ exists for $\mathbf{x}$, a resultant possibility space $(\mathcal{Y}, r_Y)$ also exists for the values of y. Specifically, the possibility distribution function r_Y is defined by

$$r_Y(y) = \sup\{r_X(\mathbf{x}) : \mathbf{x} \in \mathcal{X} \text{ and } y = F(\mathbf{x})\} = Pos_X\left\{F^{-1}(y)\right\} \tag{2.14}$$

for $y \in \mathcal{Y}$, where $F^{-1}(y)$ represents the set

$$F^{-1}(y) = \{\mathbf{x} : \mathbf{x} \in \mathcal{X} \text{ and } y = F(\mathbf{x})\}.$$

In turn, the possibility $Pos_Y(\mathcal{U})$ and necessity $Nec_Y(\mathcal{U})$ for subsets $\mathcal{U}$ of $\mathcal{Y}$ can be defined as indicated in Eqs. (2.5) and (2.6); further, the relationships indicated in Eqs. (2.7) – (2.10) also hold.

Provided y is real valued, the possibility space ($\mathcal{Y}$, r_Y) can be summarized by presentation of the corresponding CNF, CCNF, CPoF and CCPoF as discussed in conjunction with Eqs. (2.11) and (2.12). Specifically, the CNF, CCNF, CPoF and CCPoF for y are defined by the sets

$$\mathcal{CNF} = \left\{\left[y, Nec_Y\left(\mathcal{U}_y\right)\right] : y \in \mathcal{Y}\right\} = \left\{\left[y, Nec_X\left(F^{-1}\left[\mathcal{U}_y\right]\right)\right] : y \in \mathcal{Y}\right\}, \tag{2.15}$$

$$\mathcal{CCNF} = \left\{\left[y, Nec_Y\left(\mathcal{U}_y^c\right)\right] : y \in \mathcal{Y}\right\} = \left\{\left[y, Nec_X\left(F^{-1}\left[\mathcal{U}_y^c\right]\right)\right] : y \in \mathcal{Y}\right\}, \tag{2.16}$$

$$\mathcal{CPoF} = \left\{\left[y, Pos_Y\left(\mathcal{U}_y\right)\right] : y \in \mathcal{Y}\right\} = \left\{\left[y, Pos_X\left(F^{-1}\left[\mathcal{U}_y\right]\right)\right] : y \in \mathcal{Y}\right\}, \tag{2.17}$$

$$\mathcal{CCPoF} = \left\{\left[y, Pos_Y\left(\mathcal{U}_y^c\right)\right] : y \in \mathcal{Y}\right\} = \left\{\left[y, Pos_X\left(F^{-1}\left[\mathcal{U}_y^c\right]\right)\right] : y \in \mathcal{Y}\right\}, \tag{2.18}$$

where

$$\mathcal{U}_y = \{\tilde{y} : \tilde{y} \in \mathcal{Y} \text{ and } \tilde{y} \le y\}.$$

Plots of the curves defined by $\mathcal{CNF}$, $\mathcal{CCNF}$, $\mathcal{CPoF}$ and $\mathcal{CCPoF}$ produce a figure identical in concept to Fig. 1 and provide a visual representation of the uncertainty associated with y in terms of necessity and possibility.

2.3 Evidence Theory

Evidence theory, which is also known as Dempster-Shafer theory in recognition of the initial work done by these two individuals, provides a representation for uncertainty that permits the specification of more structure than possibility theory [56-63]. Evidence theory is based on the specification of a triple ($\mathcal{X}_i$, Ξ_i, m_i) for a variable x_i, where (i) $\mathcal{X}_i$ is the set of possible values for x_i, (ii) Ξ_i is a countable collection of subsets of $\mathcal{X}_i$, and (iii) m_i is a function defined for subsets $\mathcal{U}$ of $\mathcal{X}_i$ such that $m_i(\mathcal{U}) > 0$ if $\mathcal{U} \in \Xi_i$, $m_i(\mathcal{U}) = 0$ if $\mathcal{U} \notin \Xi_i$, and

$$\sum\nolimits_{\mathcal{U} \in \mathrm{X}_i} m_i(\mathcal{U}) = 1. \tag{2.19}$$

In the terminology of evidence theory, (i) $\mathcal{X}_i$ is the sample space or universal set, (ii) Ξ_i is the set of focal elements for $\mathcal{X}_i$ and m_i, and (iii) $m_i(\mathcal{U})$ is the basic probability assignment associated with a subset $\mathcal{U}$ of $\mathcal{X}_i$. In concept, the basic probability assignment $m_i(\mathcal{U})$ provides a measure of the amount of information (or credibility or probability) that can be associated with a subset $\mathcal{U}$ of $\mathcal{X}_i$ but which cannot be further decomposed over subsets of $\mathcal{U}$.

Evidence theory provides two measures of likelihood for subsets of $\mathcal{X}_i$: plausibility and belief. Specifically, the plausibility and belief for a subset $\mathcal{U}$ of $\mathcal{X}_i$ are defined by

$$Pl_i(\mathcal{U}) = \sum_{\mathcal{V} \cap \mathcal{U} \neq \varnothing} m_i(\mathcal{V}) \tag{2.20}$$

and

$$Bel_i(\mathcal{U}) = \sum_{\mathcal{V} \subset \mathcal{U}} m_i(\mathcal{V}), \tag{2.21}$$

respectively. As a result of the intersection requirement (i.e., $\mathcal{V} \cap \mathcal{U} \neq \varnothing$ in Eq. (2.20)), $Pl_i(\mathcal{U})$ provides a measure of the amount of information that could possibly be associated with $\mathcal{U}$. Similarly as a result of the subset requirement (i.e., $\mathcal{V} \subset \mathcal{U}$ in Eq. (2.21)), $Bel_i(\mathcal{U})$ provides a measure of the amount of information that is known to be associated with $\mathcal{U}$.

Relationships satisfied by plausibility and belief for the evidence space ($\mathcal{X}_i$, Ξ_i, m_i) include

$$Bel_i(\mathcal{U}) + Pl_i\left(\mathcal{U}^c\right) = 1, \tag{2.22}$$

$$Bel_i(\mathcal{U}) + Bel_i\left(\mathcal{U}^c\right) \leq 1 \tag{2.23}$$

and

$$Pl_i(\mathcal{U}) + Pl_i\left(\mathcal{U}^c\right) \geq 1 \tag{2.24}$$

for subsets $\mathcal{U}$ of $\mathcal{X}_i$.

Convenient graphical summaries of evidence spaces are provided by cumulative belief functions (CBFs), complementary cumulative belief functions (CCBFs), cumulative plausibility functions (CPFs), and complementary cumulative plausibility functions (CCPFs). Specifically, the CBF, CCBF, CPF and CCPF for the evidence space ($\mathcal{X}_i$, Ξ_i, m_i) are defined by the sets

$$\mathcal{CBF}_i = \left\{\left[x, Bel_i(\mathcal{U}_x)\right] : x \in \mathcal{X}_i\right\}, \mathcal{CCBF}_i = \left\{\left[x, Bel_i\left(\mathcal{U}_x^c\right)\right] : x \in \mathcal{X}_i\right\} \tag{2.25}$$

$$\mathcal{CPF}_i = \left\{\left[x, Pl_i(\mathcal{U}_x)\right] : x \in \mathcal{X}_i\right\}, \mathcal{CCPF}_i = \left\{\left[x, Pl_i\left(\mathcal{U}_x^c\right)\right] : x \in \mathcal{X}_i\right\}, \tag{2.26}$$

where $\mathcal{U}_x$ is defined the same as in conjunction with Eqs. (2.11) and (2.12). Plots of the curves defined by the points associated with $\mathcal{CBF}_i$, $\mathcal{CCBF}_i$, $\mathcal{CPF}_i$ and $\mathcal{CCPF}_i$ yield the CBF, CCBF, CPF and CCPF for the evidence space ($\mathcal{X}_i$, Ξ_i, m_i) (Fig. 2).

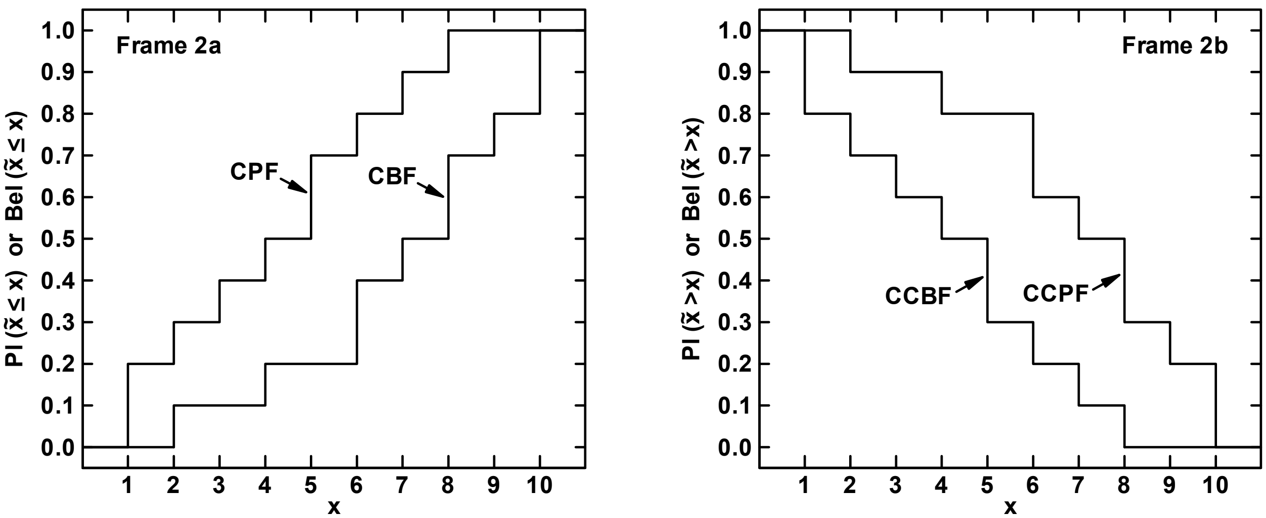

Fig. 2. Plots of CBF, CCBF, CPF and CCPF for evidence space $(\mathcal{X}, \Xi, m)$ with (i) $\mathcal{X} = \{x: 1 \le x \le 10\}$, (ii) $\Xi = \{\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_{10}\}$ with $\mathcal{U}_i = [i, 2i]$ for $i = 1, 2, 3, 4, 5$ and $\mathcal{U}_i = [i-1, i]$ for $i = 6, 7, 8, 9, 10$, (iii) $m(\mathcal{U}) = 1/10$ if $\mathcal{U} \in \Xi$ and $m(\mathcal{U}) = 0$ otherwise, and (iv) $Pl(\tilde{x} \le x)$, $Bel(\tilde{x} \le x)$, $Pl(\tilde{x} > x)$ and $Bel(\tilde{x} > x)$ used as abbreviated notations for $Pl_i(\mathcal{U}_x)$, $Bel_i(\mathcal{U}_x)$, $Pl_i(\mathcal{U}_x^c)$ and $Bel_i(\mathcal{U}_x^c)$ in Eqs. (2.25) and (2.26): (a) CBF and CPF, and (b) CCBF and CCPF.

If the variables $x_1, x_2, \ldots, x_{nX}$ have associated evidence spaces $(\mathcal{X}_1, \Xi_1, m_1)$, $(\mathcal{X}_2, \Xi_2, m_2)$, …, $(\mathcal{X}_{nX}, \Xi_{nX}, m_{nX})$, then the vector $\mathbf{x} = [x_1, x_2, \ldots, x_{nX}]$ also has an associated evidence space $(\mathcal{X}, \Xi, m_X)$, where (i) $\mathcal{X}$ is defined the same as in Eq. (2.3), (ii) $\mathcal{U} \in \Xi$ if, and only if,

$$\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_{nX} \tag{2.27}$$

with $\mathcal{U}_i \in \Xi_i$ for $i = 1, 2, \ldots, nX$, and (iii)

$$m_X(\mathcal{U}) = \prod_{i=1}^{nX} m_i(\mathcal{U}_i) \tag{2.28}$$

if $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_{nX} \in \Xi$ and $m_X(\mathcal{U}) = 0$ otherwise. The preceding definition for $(\mathcal{X}, \Xi, m_X)$ is predicated on the assumption that no restrictions involving possible combinations of values for the x_i exist. If such restrictions exist, then the definition of $(\mathcal{X}, \Xi, m_X)$ is more complex.

Once the evidence space $(\mathcal{X}, \Xi, m_X)$ for $\mathbf{x}$ is defined, the plausibility $Pl_X(\mathcal{U})$ and belief $Bel_X(\mathcal{U})$ for subsets $\mathcal{U}$ of $\mathcal{X}$ are defined as indicated in Eqs. (2.20) and (2.21). Further, the relationships indicated in Eqs. (2.22) – (2.24) also hold.

Propagation of the individual values of **x** contained in $\mathcal{X}$ through the function F indicated in Eq. (2.1) results in a set $\mathcal{Y}$ of possible values for y of the form shown in Eq. (2.4). Given that an evidence space ($\mathcal{X}$, Ξ, m_X) exists for **x**, a resultant evidence space ($\mathcal{Y}$, Ψ, m_Y) also exists for the value of y. Specifically, (i)

$$\mathrm{Y} = \left\{ F\left(\mathcal{V}_1\right), \quad F\left(\mathcal{V}_2\right), \ldots, F\left(\mathcal{V}_n\right) \right\} \tag{2.29}$$

where $\mathcal{V}_1$, $\mathcal{V}_2$, …, $\mathcal{V}_n$ correspond to the elements of Ξ, (ii)

$$m_Y\left(\mathcal{U}\right) = \sum_{k \in I(\mathcal{U})} m\left(\mathcal{V}_k\right) \tag{2.30}$$

if $\mathcal{U} \in \Psi$, where $k \in I(\mathcal{U})$ if, and only if, $\mathcal{U} = F(\mathcal{V}_k)$, and (iii) $m_Y(\mathcal{U}) = 0$ if $\mathcal{U} \notin \Psi$. The summation over k in the definition of $m_Y(\mathcal{U})$ in Eq. (2.30) is necessary to appropriately incorporate the possibility that $\mathcal{U} = F(\mathcal{V}_k)$ for more than one element $\mathcal{V}_k$ of Ξ. In turn, the plausibility $Pl_Y(\mathcal{U})$ and belief $Bel_Y(\mathcal{U})$ for subsets $\mathcal{U}$ of $\mathcal{Y}$ can be defined as indicated in Eqs. (2.20) and (2.21); further, the relationships indicated in Eqs. (2.22) – (2.24) also hold.

Provided y is real valued, the evidence space ($\mathcal{Y}$, Ψ, m_Y) can be summarized by presentation of the corresponding CBF, CCBF, CPF and CCPF as discussed in conjunction with Eqs. (2.25) and (2.26). Specifically, the CBF, CCBF, CPF and CCPF for y are defined by the sets

$$\mathcal{CBF} = \left\{ \left[y, Bel_Y\left(\mathcal{U}_y\right) \right] : y \in \mathcal{Y} \right\} = \left\{ \left[y, Bel_X\left(F^{-1}\left[\mathcal{U}_y\right] \right) \right] : y \in \mathcal{Y} \right\}, \tag{2.31}$$

$$\mathcal{CCBF} = \left\{ \left[y, Bel_Y\left(\mathcal{U}_y^c\right) \right] : y \in \mathcal{Y} \right\} = \left\{ \left[y, Bel_X\left(F^{-1}\left[\mathcal{U}_y^c\right] \right) \right] : y \in \mathcal{Y} \right\}, \tag{2.32}$$

$$\mathcal{CPF} = \left\{ \left[y, Pl_Y\left(\mathcal{U}_y\right) \right] : y \in \mathcal{Y} \right\} = \left\{ \left[y, Pl_X\left(F^{-1}\left[\mathcal{U}_y\right] \right) \right] : y \in \mathcal{Y} \right\}, \tag{2.33}$$

$$\mathcal{CCPF} = \left\{ \left[y, Pl_Y\left(\mathcal{U}_y^c\right) \right] : y \in \mathcal{Y} \right\} = \left\{ \left[y, Pl_X\left(F^{-1}\left[\mathcal{U}_y^c\right] \right) \right] : y \in \mathcal{Y} \right\}, \tag{2.34}$$

where $\mathcal{U}_\mathrm{y}$ is defined the same as in conjunction with Eqs. (2.15) – (2.18). Plots of the curves defined by the points associated with $\mathcal{CBF}$, $\mathcal{CCBF}$, $\mathcal{CPF}$ and $\mathcal{CCPF}$ produce a figure identical in concept to Fig. 2 and provide a visual representation of the uncertainty associated with y in terms of belief and plausibility.

2.4 Probability Theory

Probability theory provides a representation for uncertainty that involves the specification of more structure than evidence theory [48-55; 107-111]. Similarly to evidence theory, probability theory is based on the specification of a triple ($\mathcal{X}_i$, Ξ_i, p_i) for a variable x_i, where (i) $\mathcal{X}_i$ is the set of possible values for x_i, (ii) Ξ_i is a suitably restricted collection of subsets of $\mathcal{X}_i$ (i.e., if $\mathcal{U} \in \Xi_i$, then $\mathcal{U}^c \in \Xi_i$, and if $\mathcal{U}_1$, $\mathcal{U}_2$, … is a countable sequence of elements of Ξ_i, then $\cup_k \mathcal{U}_k \in \Xi_i$ and $\cap_k \mathcal{U}_k \in \Xi_i$),and (iii) p_i defines the probability for elements of $\mathcal{X}_i$ (i.e., $0 \le p_i(\mathcal{U}) \le 1$ if $\mathcal{U} \in \Xi_i$, $p_i(\mathcal{X}_i) = 1$, and $p_i(\cup_k \mathcal{U}_k) = \sum_k p_i(\mathcal{U}_k)$ if $\mathcal{U}_1$, $\mathcal{U}_2$, … is a countable sequence of nonintersecting elements of Ξ_i). However, in contrast to an evidence space ($\mathcal{X}_i$, Ξ_i, m_i), a probability space ($\mathcal{X}_i$, Ξ_i, p_i) involves the imposition of more structure on Ξ_i and p_i than is the case for Ξ_i and m_i for an evidence space. In the terminology of probability theory, (i) $\mathcal{X}_i$ is the sample space, (ii) the elements of Ξ_i are events and collectively constitute what is known as a σ-algebra, and (iii) p_i is a probability measure (Sects. IV.3 and IV.4, Ref. [111]). For notational and computational convenience, a probability space ($\mathcal{X}_i$, Ξ_i, p_i) is often summarized with a density function d_i, where

$$p_i(\mathcal{U}) = \int_{\mathcal{U}} d_i(x)\,\mathrm{d}x \tag{2.35}$$

for $\mathcal{U} \in \Xi_i$.

Unlike possibility theory and evidence theory, which provide two measures of likelihood (i.e., possibility and necessity in possibility theory and plausibility and belief in evidence theory), probability theory provides only one measure of likelihood: probability. The probabilities of a set and its complement are related by

$$p_i(\mathcal{U}) + p_i\left(\mathcal{U}^c\right) = 1 \tag{2.36}$$

for $\mathcal{U} \in \Xi_i$, which is a more restrictive requirement than shown in Eq. (2.8) for possibility and necessity and in Eqs. (2.23) an (2.24) for belief and plausibility.

Convenient graphical summaries of probability spaces are provided by cumulative distribution functions (CDFs) and complementary cumulative distribution functions (CCDFs). Specifically, the CDF and CCDF for the probability space ($\mathcal{X}_i$, Ξ_i, p_i) with the corresponding density function d_i are defined by the sets

$$\mathcal{CDF}_i = \left\{\left[x,\, p_i(\mathcal{U}_x)\right] : x \in \mathcal{X}_i\right\} = \left\{\left[x,\, \int_{\mathcal{U}_x} d_i(\tilde{x})\,\mathrm{d}\tilde{x}\right] : x \in \mathcal{X}_i\right\} \tag{2.37}$$

and

$$\mathcal{CCDF}_i = \left\{\left[x,\, p_i\left(\mathcal{U}_x^c\right)\right] : x \in \mathcal{X}_i\right\} = \left\{\left[x,\, \int_{\mathcal{U}_x^c} d_i(\tilde{x})\,\mathrm{d}\tilde{x}\right] : x \in \mathcal{X}_i\right\}, \tag{2.38}$$

where $\mathcal{U}_x$ is defined the same as in conjunction with Eqs. (2.11) and (2.12). Plots of the curves defined by the points associated with $\mathcal{CDF}_i$ and $\mathcal{CCDF}_i$ yield the CDF and CCDF for the probability space ($\mathcal{X}_i$, Ξ_i, p_i) (Fig. 3).

One interpretation of an evidence space ($\mathcal{X}$, Ξ, m) is that it is a characterization of a partially defined probability space. In general, there are many possible probability spaces ($\mathcal{X}$, Ξ, p) that are consistent with a given evidence space ($\mathcal{X}$, Ξ, m) in the sense that, if $\mathcal{U} \subset \mathcal{X}$ (i.e., technically, an element of the set Ξ associated with ($\mathcal{X}$, Ξ, p)), then

$$Bel(\mathcal{U}) \le p(\mathcal{U}) \le Pl(\mathcal{U}). \tag{2.39}$$

As a result of the preceding inequality, if a probability space ($\mathcal{X}$, Ξ, p) is consistent with an evidence space ($\mathcal{X}$, Ξ, m), then the CDF associated with ($\mathcal{X}$, Ξ, p) falls between the CBF and CPF associated with ($\mathcal{X}$, Ξ, m) and similarly the CCDF falls between the CCBF and CCPF.

For example, if $\mathcal{X}$ corresponds to a bounded interval $I = [a, b]$ and each focal element $\mathcal{U}_k$ associated with the evidence space ($\mathcal{X}$, Ξ, m) is a subinterval $I_k = [a_k, b_k]$ of I, then a probability space ($\mathcal{X}$, Ξ, p) consistent with the evidence space ($\mathcal{X}$, Ξ, m) is defined by the density function

$$d(x) = \sum_k \delta_k(x)\, m(\mathcal{U}_k)/(b_k - a_k), \tag{2.40}$$

where

$$\delta_k(x) = \begin{cases} 1 & \text{if } x \in \mathcal{U}_k \\ 0 & \text{otherwise.} \end{cases}$$

As a result, the CDF for ($\mathcal{X}$, Ξ, p) falls between the CBF and CPF for ($\mathcal{X}$, Ξ, m), and similarly, the CCDF for ($\mathcal{X}$, Ξ, p) falls between the CCBF and CCPF for ($\mathcal{X}$, Ξ, m) (Fig. 3).

If the variables $x_1, x_2, \ldots, x_{nX}$ have associated probability spaces ($\mathcal{X}_1$, Ξ_1, p_1), ($\mathcal{X}_2$, Ξ_2, p_2), …, ($\mathcal{X}_{nX}$, Ξ_{nX}, p_{nX}), then the vector $\mathbf{x} = [x_1, x_2, \ldots, x_{nX}]$ also has an associated probability space ($\mathcal{X}$, Ξ, p_X), where (i) $\mathcal{X}$ is defined the same as in Eq. (2.3), (ii) Ξ is developed from the sets contained in

$$\mathrm{C} = \left\{\mathcal{U} : \mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_{nX} \in \mathrm{X}_1 \times \mathrm{X}_2 \times \ldots \times \mathrm{X}_{nX}\right\} \tag{2.41}$$

(see Sect. IV.6, Ref. [111] and Sect. 2.6, Ref. [108]), and (iii) p_X is developed from the properties of $p_1, p_2, \ldots, p_{nX}$. Specifically, if the x_i are independent (i.e., if the occurrence of one x_i has no implications for the occurrence of the remaining x_j, $j \ne i$), then

$$p_X(\mathcal{U}) = \prod_{i=1}^{nX} p_i[\mathcal{U}_i] \tag{2.42}$$

for $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_{nX} \in \mathrm{X}$ and, more generally,

$$p_X(\mathcal{U}) = \int_{\mathcal{U}} d_X(\mathbf{x})\, \mathrm{d}X \tag{2.43}$$

for $\mathcal{U} \in \Xi$, where

$$d_X(\mathbf{x}) = \prod_{i=i}^{nX} d_i(x_i)$$

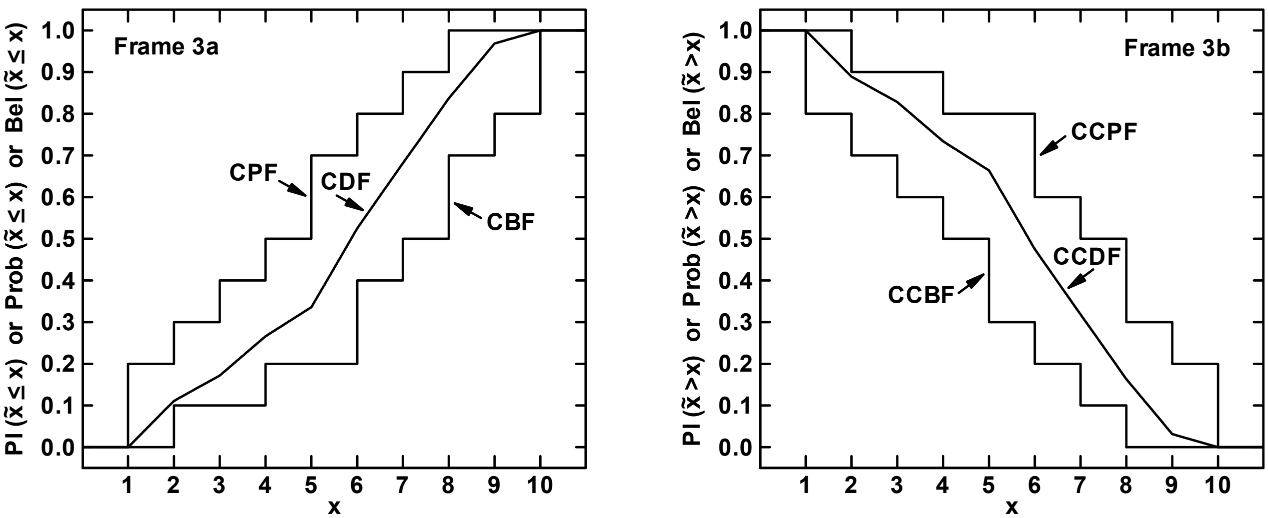

Fig. 3. Plots of (a) CBF, CCBF, CPF and CCPF for evidence space ($\mathcal{X}$, Ξ, m) with (i) $\mathcal{X} = \{x: 1 \le x \le 10\}$, (ii) $\Xi = \{\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_{10}\}$ with $\mathcal{U}_i = [i, 2i]$ for i = 1, 2, 3, 4, 5, $\mathcal{U}_i = [i-1, i]$ for i, 6, 7, 8, 9, and $\mathcal{U} = [1, 10]$, and (iii) $m(\mathcal{U}) = 1/10$ if $\mathcal{U} \in \Xi$ and $m(\mathcal{U}) = 0$ otherwise, and (iv) $Pl(\tilde{x} \le x)$, $Bel(\tilde{x} \le x)$, $Pl(\tilde{x} > x)$ and $Bel(\tilde{x} > x)$ used as abbreviated notations for $Pl_i(\mathcal{U}_x)$, $Bel_i(\mathcal{U}_x)$, $Pl_i(\mathcal{U}_x^c)$ and $Bel_i(\mathcal{U}_x^c)$ in Eqs. (2.25) and (2.26), and (b) CDF and CCDF for probability space (X, Ξ, p) with density function d defined as indicated in Eq. (2.40) with $Prob(\tilde{x} \le x)$ and $Prob(\tilde{x} > x)$ used as abbreviated notations for $p_i(\mathcal{U}_x)$ and $p_i(\mathcal{U}_x^c)$ in Eqs. (2.37) and (2.38).

is the density function associated with ($\mathcal{X}$, Ξ, p_X) and d_i is the density function associated with ($\mathcal{X}_i$, Ξ_i, p_i) for i = 1, 2, …, nX. The definition of p_X and d_X are more complex when the x_i are not independent and will not be considered here.

Propagation of the individual values of **x** contained in $\mathcal{X}$ through the function F indicated in Eq. (2.1) results in a set Y of possible values for y of the form shown in Eq. (2.4). Given that a probability space ($\mathcal{X}$, Ξ, p_X) exists for **x**, a resultant probability space ($\mathcal{Y}$, Ψ, p_Y) also exists for the values of y. In concept, the probability $p_Y(\mathcal{U})$ for a subset $\mathcal{U}$ of $\mathcal{Y}$ is given by

$$p_Y(\mathcal{U}) = p_X\left[F^{-1}(\mathcal{U})\right]. \tag{2.44}$$

A formal development of Ψ and p_Y would focus on the properties that F must possess to actually produce the probability space ($\mathcal{Y}$, Ψ, p_Y) (see Sect. IV. 4, Ref. [111], and Sects. 4.6 and 4.7, Ref. [108]); such details are outside the scope of this presentation.

Provided y is real valued, the probability space ($\mathcal{Y}$, Ψ, p_Y) can be summarized by the presentation of the corresponding CDF and CCDF. Specifically, the CDF and CCDF for y are defined by the sets

$$\mathcal{CDF} = \left\{ \left[y, \, p_Y \left(\mathcal{U}_y \right) \right] : y \in \mathcal{Y} \right\} = \left\{ \left[y, \, p_X \left(F^{-1} \left[\mathcal{U}_y \right] \right) \right] : y \in \mathcal{Y} \right\} \tag{2.45}$$

$$\mathcal{CCDF} = \left\{ \left[y, \, p_Y \left(\mathcal{U}_y^c \right) \right] : y \in \mathcal{Y} \right\} = \left\{ \left[y, \, p_X \left(F^{-1} \left[\mathcal{U}_y^c \right] \right) \right] : y \in \mathcal{Y} \right\}, \tag{2.46}$$

where $\mathcal{U}_y$ is defined the same as in conjunction with Eqs. (2.15) – (2.18). Plots of the curves defined by the points associated with $\mathcal{CDF}$ and $\mathcal{CCDF}$ produce a CDF and CCDF identical in concept to the CDF and CCDF in Fig. 3 and provide a visual representation of a probabilistic characterization of the uncertainty associated with y.

2.5 Sampling-Based Uncertainty Propagation

An analysis outcome $y = F(\mathbf{x})$ of the form indicated on Eq. (2.1) will have an uncertainty structure that derives from the uncertainty structure associated with $\mathbf{x}$. In particular, the uncertainty associated with y will have an interval representation, a possibility theory representation, an evidence theory representation or a probabilistic representation in consistency with an interval representation (Sect. 2.1), a possibility theory representation (Sect. 2.2), an evidence theory representation (Sect. 2.3) or a probabilistic representation (Sect. 2.4) for the uncertainty associated with $\mathbf{x}$. An exact determination of the uncertainty structure associated with y without any numerical or approximation error is usually not possible in a real analysis. However, the indicated uncertainty structures for y can be approximated with sampling-based procedures.

As indicated by the name, sampling-based (i.e., Monte Carlo) procedures involve the use of a sample

$$\mathbf{x}_i = \left[x_{i1}, x_{i2}, \ldots, x_{i,nX} \right], i = 1, 2, \ldots, nS, \tag{2.47}$$

from the set $\mathcal{X}$ of possible values of $\mathbf{x}$ in the estimation of the uncertainty structure associated with $y = F(\mathbf{x})$ that derives from the uncertainty structure associated with $\mathbf{x}$ [112-119]. For uncertainty propagations involving interval analysis, possibility theory and evidence theory, it is important that the sample provide an "adequate" coverage of $\mathcal{X}$ but there are no requirements for a specific structure for this sample. Of course, what constitutes adequate coverage of $\mathcal{X}$ depends on properties of $\mathcal{X}$ and the function $F(\mathbf{x})$. In the case of an evidence theory representation of the uncertainty associated with $\mathbf{x}$, adequate coverage of $\mathbf{x}$ corresponds to a sample that provides a reasonable estimate of the minimum and maximum value of $F(\mathbf{x})$ for each focal element in the evidence space defined for $\mathcal{X}$. However, for a probabilistic representation of the uncertainty associated with $\mathbf{x}$, the sample in Eq. (2.47) must be generated in consistency with the probability distribution defined for $\mathbf{x}$. An exception to this is when importance sampling is used in the propagation of a probabilistic representation of uncertainty; in this situation, a specially selected distribution is used for sampling and the effects of this distribution must then be compensated for to obtain the desired uncertainty propagation [120-126].

Once an appropriate sample of the form indicated in Eq. (2.47) is generated, an interval representation for the uncertainty associated with y is given by

$$[y_{mn}, y_{mx}] = [\inf(\mathcal{Y}), \sup(\mathcal{Y})]$$

$$\cong [\min\{y_i : i = 1, 2, \ldots, nS\}, \max\{y_i : i = 1, 2, \ldots, nS\}], \quad (2.48)$$

where $\mathcal{Y}$ is the set of possible values for y defined in Eq. (2.4) and $y_i = y(\mathbf{x}_i)$ for $i = 1, 2, \ldots, nS$. It is emphatically emphasized that the preceding procedure will not be the most computationally efficient method for estimating $[y_{mn}, y_{mx}]$ in many analyses. However, it is presented here for consistency with the sampling-based procedures described below for use in conjunction with possibility theory, evidence theory and probability theory representations of the epistemic uncertainty in $\mathbf{x}$ and hence in y.

If the epistemic uncertainty associated with $\mathbf{x}$ is characterized by a possibility space $(\mathcal{X}, r_X)$, then the corresponding possibility space $(\mathcal{Y}, r_Y)$ for y can be summarized by its associated CNF, CCNF, CPoF and CCPoF (see Eqs. (2.15) – (2.18)). Specifically, the CNF, CCNF, CPoF and CCPoF associated with $(\mathcal{Y}, r_Y)$ can be approximated with use of the sample in Eq. (2.47) through the relationships

$$\mathcal{CNF} = \left\{\left[y, Nec_Y\left(\mathcal{U}_y\right)\right] : y \in \mathcal{Y}\right\}$$

$$\cong \left\{\left[y, 1 - \hat{P}os_Y\left(\mathcal{U}_y^c\right)\right] : y \in \hat{\mathcal{Y}}\right\}$$

$$= \left\{\left[y, 1 - Pos_X\left(\{\mathbf{x}_i : 1 \le i \le nS \text{ and } y_i > y\}\right)\right] : y \in \hat{\mathcal{Y}}\right\}, \quad (2.49)$$

$$\mathcal{CCNF} = \left\{\left[y, Nec_Y\left(\mathcal{U}_y^c\right)\right] : y \in \mathcal{Y}\right\}$$

$$\cong \left\{\left[y, 1 - \hat{P}os_Y\left(\mathcal{U}_y\right)\right] : y \in \hat{\mathcal{Y}}\right\}$$

$$= \left\{\left[y, 1 - Pos_X\left(\{\mathbf{x}_i : 1 \le i \le nS \text{ and } y_i \le y\}\right)\right] : y \in \hat{\mathcal{Y}}\right\}, \quad (2.50)$$

$$\mathcal{CPoF} = \left\{\left[y, Pos_Y\left(\mathcal{U}_y\right)\right] : y \in \mathcal{Y}\right\}$$

$$\cong \left\{\left[y, \hat{P}os_Y\left(\mathcal{U}_y\right)\right] : y \in \hat{\mathcal{Y}}\right\}$$

$$= \left\{\left[y, Pos_X\left(\{\mathbf{x}_i : 1 \le i \le nS \text{ and } y_i \le y\}\right)\right] : y \in \hat{\mathcal{Y}}\right\}, \quad (2.51)$$

$$CCPoF = \left\{\left[y, Pos\left(\mathcal{U}_y^c\right)\right] : y \in \mathcal{Y}\right\}$$

$$\cong \left\{\left[y, \hat{P}os_Y\left(\mathcal{U}_y^c\right)\right] : y \in \hat{\mathcal{Y}}\right\}$$

$$= \left\{\left[y, Pos_X\left(\left\{\mathbf{x}_i : 1 \le i \le nS \text{ and } y_i > y\right\}\right)\right] : y \in \hat{\mathcal{Y}}\right\} \tag{2.52}$$

as indicated in conjunction with Table 2 of Ref. [93], where (i) $\hat{\mathcal{Y}}$ is a set that at least contains the approximation to the interval $[y_{mn}, y_{mx}]$ defined in Eq. (2.48) but may correspond to a larger interval for plotting purposes, (ii) $\mathcal{U}_y$ denotes a subset of $\mathcal{Y}$ or $\hat{\mathcal{Y}}$ as appropriate of the form defined in conjunction with Eqs. (2.15) – (2.18), and (iii) $\hat{P}os_Y(\mathcal{U})$ denotes an approximation to $Pos_Y(\mathcal{U})$ for subsets $\mathcal{U}$ of $\mathcal{Y}$ and $\hat{\mathcal{Y}}$. As the sample values for $\mathbf{x}$ become increasingly dense in $\mathcal{X}$, the approximations in Eqs. (2.49) – (2.52) will approach the CNF, CCNF, CPoF and CCPoF for y.

If the epistemic uncertainty associated with $\mathbf{x}$ is characterized by an evidence space $(\mathcal{X}, \Xi, m_X)$, then the corresponding evidence space $(\mathcal{Y}, \Psi, m_Y)$ for y can be summarized by its associated CBF, CCBF, CPF and CCPF (see Eqs. (2.31) – (2.34)). Specifically, the CBF, CCBF, CPF and CCPF associated with $(\mathcal{Y}, \Psi, m_Y)$ can be approximated with use of the sample in Eq. (2.47) through the relationships

$$CBF = \left\{\left[y, Bel_Y\left(\mathcal{U}_y\right)\right] : y \in \mathcal{Y}\right\}$$

$$\cong \left\{\left[y, 1-\hat{P}l_Y\left(\mathcal{U}_y^c\right)\right] : y \in \hat{\mathcal{Y}}\right\}$$

$$= \left\{\left[y, 1-Pl_X\left(\left\{\mathbf{x}_i : 1 \le i \le nS \text{ and } y_i > y\right\}\right)\right] : y \in \hat{\mathcal{Y}}\right\}, \tag{2.53}$$

$$CCBF = \left\{\left[y, Bel_Y\left(\mathcal{U}_y^c\right)\right] : y \in \mathcal{Y}\right\}$$

$$\cong \left\{\left[y, 1-\hat{P}l_Y\left(\mathcal{U}_y\right)\right] : y \in \hat{\mathcal{Y}}\right\}$$

$$= \left\{\left[y, 1-Pl_X\left(\left\{\mathbf{x}_i : 1 \le i \le nS \text{ and } y_i < y\right\}\right)\right] : y \in \hat{\mathcal{Y}}\right\}, \tag{2.54}$$

$$CPF = \left\{\left[y, Pl_Y\left(\mathcal{U}_y\right)\right] : y \in \mathcal{Y}\right\}$$

$$\cong \left\{\left[y, \hat{P}l_Y\left(\mathcal{U}_y\right)\right] : y \in \hat{\mathcal{Y}}\right\}$$

$$= \left\{\left[y, Pl_X\left(\left\{\mathbf{x}_i : 1 \le i \le nS \text{ and } y_i \le y\right\}\right)\right] : y \in \hat{\mathcal{Y}}\right\}, \tag{2.55}$$

$$CCPF = \left\{ \left[y,\, Pl_Y\left(\mathcal{U}_y^c\right) \right] : y \in \mathcal{Y} \right\}$$

$$\cong \left\{ \left[y,\, \hat{P}l_Y\left(\mathcal{U}_y^c\right) \right] : y \in \hat{\mathcal{Y}} \right\}$$

$$= \left\{ \left[y,\, Pl_X\left(\left\{ \mathbf{x}_i : 1 \le i \le nS \text{ and } y_i > y \right\}\right) \right] : y \in \hat{\mathcal{Y}} \right\} \qquad (2.56)$$

as indicated in conjunction with Table 1 of Ref. [93], where (i) $\hat{\mathcal{Y}}$ and $\mathcal{U}_y$ are defined the same as in Eqs. (2.49) – (2.52) and (ii) $\hat{P}l_Y(\mathcal{U})$ denotes an approximation to $Pl_Y(\mathcal{U})$ for subsets $\mathcal{U}$ of $\mathcal{Y}$ and $\hat{\mathcal{Y}}$. As the sample values for $\mathbf{x}$ become increasingly dense in $\mathcal{X}$ and, in particular, approach the values at which F has its minimum and maximum values for the individual focal elements in Ξ, the approximations in Eq. (2.53) – (2.56) will approach the CBF, CCBF, CPF and CCPF for y.

If the epistemic uncertainty associated with $\mathbf{x}$ is characterized by a probability space $(\mathcal{X}, \Xi, p_X)$, then the corresponding probability space $(\mathcal{Y}, \Psi, p_Y)$ for y can be summarized by its associated CCDF and CDF (see Eqs. (2.45) – (2.46)). If the sample in Eq. (2.47) is generated in consistency with the distribution for $\mathbf{x}$ defined by the probability space $(\mathcal{X}, \Xi, p_X)$, then the CCDF and CDF associated with $(\mathcal{Y}, \Psi, p_Y)$ can be approximated through the standard sampling-based relationships

$$CCDF = \left\{ \left[y,\, p_Y\left(\mathcal{U}_y^c\right) \right] : y \in \mathcal{Y} \right\}$$

$$\cong \left\{ \left[y,\, \hat{p}_Y\left(\mathcal{U}_y^c\right) \right] : y \in \hat{\mathcal{Y}} \right\}$$

$$= \left\{ \left[y,\, \sum_{i=1}^{nS} \overline{\delta}_y\left(y_i\right) / nS \right] : y \in \hat{\mathcal{Y}} \right\}, \qquad (2.57)$$

$$CDF = \left\{ \left[y,\, p_Y\left(\mathcal{U}_y\right) \right] : y \in \mathcal{Y} \right\}$$

$$\cong \left\{ \left[y,\, \hat{p}_Y\left(\mathcal{U}_y\right) \right] : y \in \hat{\mathcal{Y}} \right\}$$

$$= \left\{ \left[y,\, \sum_{i=1}^{nS} \underline{\delta}_y\left(y_i\right) / nS \right] : y \in \hat{\mathcal{Y}} \right\}, \qquad (2.58)$$

where (i) $\hat{\mathcal{Y}}$ and $\mathcal{U}_y$ are defined the same as in Eqs. (2.49) – (2.52), (ii) $\hat{p}_Y(\mathcal{U})$ denotes an approximation to $p_Y(\mathcal{U})$ for subsets $\mathcal{U}$ of $\mathcal{Y}$ and $\hat{\mathcal{Y}}$, and (iii) the indicator functions $\overline{\delta}_y$ and $\underline{\delta}_y$ are defined by

$$\overline{\delta}_y\left(\tilde{y}\right) = \begin{cases} 1 & \text{if } y < \tilde{y} \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \underline{\delta}_y\left(\tilde{y}\right) = \begin{cases} 1 & \text{if } \tilde{y} \le y \\ 0 & \text{otherwise,} \end{cases}$$

respectively. As the sample size increases, the approximations in Eqs. (2.57) and (2.58) will approach the CCDF and CDF for y.

When appropriately designed, sampling-based uncertainty propagations also provide a mapping between analysis inputs and analysis results that can be explored with a variety of sensitivity analysis procedures [127-129].

3. Aleatory and Epistemic Uncertainty

The primary focus of this presentation is on the representation of uncertainty in analyses that involve both aleatory and epistemic uncertainty. Conceptually, such analyses involve three distinct mathematical entities: (i) a characterization of aleatory uncertainty, (ii) a function that predicts results of interest, and (iii) a characterization of epistemic uncertainty [130; 131]. This presentation assumes that probability theory provides the mathematical structure used to represent aleatory uncertainty (Sect. 2.4). However, four different mathematical structures are considered as alternatives for the representation of epistemic uncertainty: interval analysis (Sect. 2.1), possibility theory (Sect. 2.2), evidence theory (Sect. 2.3), and probability theory (Sect. 2.4).

The function that predicts results of interest can be represented by

$$z = f\left(\mathbf{a}\middle|\mathbf{e}_M\right) = f\left(a_1, a_2, \ldots, a_{nA}\middle|e_{M1}, e_{M2}, \ldots, e_{M,nM}\right), \tag{3.1}$$

where z is the result of interest, $\mathbf{a} = [a_1, a_2, \ldots, a_{nA}]$ is the vector of variables included in the analysis that are assumed to be uncertain in an aleatory sense, and $\mathbf{e}_M = [e_{M1}, e_{M2}, \ldots, e_{M,nM}]$ is the vector of variables included in the analysis that are involved in the evaluation of the function f and are assumed to be uncertain in an epistemic sense. In addition, there is often epistemic uncertainty with respect to the appropriate values to use for the parameters that define the distributions that characterize the aleatory uncertainty in the elements of $\mathbf{a}$. As a result, there is also a vector $\mathbf{e}_D = [e_{D1}, e_{D2}, \ldots, e_{D,nD}]$ of epistemically uncertain variables used in the definition of the distributions that characterize the aleatory uncertainty associated with the elements of $\mathbf{a}$. Notationally, the distribution for $\mathbf{a}$ conditional on a specific realization for $\mathbf{e}_D$ can be represented by a density function $d_A(\mathbf{a}|\mathbf{e}_D)$. In turn, the vector

$$\mathbf{e} = \left[\mathbf{e}_M, \mathbf{e}_D\right] = \left[e_1, e_2, \ldots, e_{nE}\right] \tag{3.2}$$

contains all the epistemically uncertain variables under consideration with $nE = nM + nD$.

The uncertainty characterization associated with each element e_i of $\mathbf{e}$ starts with a set $\mathcal{E}_i$ of possible values for e_i. In turn, the set of all possible values for $\mathbf{e}$ is given by

$$\mathcal{E} = \mathcal{E}_1 \times \mathcal{E}_2 \times \ldots \times \mathcal{E}_{nE}, \tag{3.3}$$

although in general there can potentially be additional restrictions that limit the possible combinations of values for specific elements of $\mathbf{e}$. What distinguishes the various alternatives for the representation of epistemic uncertainty (e.g., interval analysis, possibility theory, evidence theory, probability theory) is the type of internal uncertainty structure imposed on the sets $\mathcal{E}_i$ and hence on the set $\mathcal{E}$. In practice, this internal uncertainty structure is typically developed through some type of expert review process [132-144].

A specific element $\mathbf{e} = [\mathbf{e}_M, \mathbf{e}_D]$ of $\mathcal{E}$ results in (i) a specific definition for the function $f(\mathbf{a}|\mathbf{e}_M)$ in which $\mathbf{e}_M$ is fixed and (ii) a specific definition of the density function $d_A(\mathbf{a}|\mathbf{e}_D)$, which corresponds to the aleatory distribution for $\mathbf{a}$, in which $\mathbf{e}_D$ is fixed. Further, associated with the density function $d_A(\mathbf{a}|\mathbf{e}_D)$ is a set $\mathcal{A}$ of possible values for $\mathbf{a}$. In general, $\mathcal{A}$ could be a different set for each possible value of $\mathbf{e}_D$; however, this potential dependency will be suppressed for notational simplicity. Or, equivalently, it can be assumed that $d_A(\mathbf{a}|\mathbf{e}_D) = 0$ for vectors $\mathbf{a}$ that are not possible for a given value of $\mathbf{e}_D$.

With $\mathbf{e} = [\mathbf{e}_M, \mathbf{e}_D]$ fixed as indicated, a single distribution for z results. This distribution is often presented as a CDF defined by the points $[z, Prob_A(\tilde{z} \le z \,|\, \mathbf{e})]$ with

$$Prob_A\left(\tilde{z} \le z \,|\, \mathbf{e}\right) = \int_{\mathcal{A}} \underline{\delta}_z\left[f\left(\mathbf{a}|\mathbf{e}_M\right)\right] d_A\left(\mathbf{a}|\mathbf{e}_D\right) \mathrm{d}A \tag{3.4}$$

and

$$\underline{\delta}_z\left[f\left(\mathbf{a}|\mathbf{e}_M\right)\right] = \begin{cases} 1 & \text{if } f\left(\mathbf{a}|\mathbf{e}_M\right) \le z \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

or as a CCDF defined by the points $[z, Prob_A(\tilde{z} > z \,|\, \mathbf{e})]$ with

$$Prob_A\left(\tilde{z} > z \,|\, \mathbf{e}\right) = \int_{\mathcal{A}} \bar{\delta}_z\left[f\left(\mathbf{a}|\mathbf{e}_M\right)\right] d_A\left(\mathbf{a}|\mathbf{e}_D\right) \mathrm{d}A \tag{3.6}$$

and

$$\bar{\delta}_z\left[f\left(\mathbf{a}|\mathbf{e}_M\right)\right] = \begin{cases} 1 & \text{if } f\left(\mathbf{a}|\mathbf{e}_M\right) > z \\ 0 & \text{otherwise} \end{cases} \tag{3.7}$$

(Fig. 4). The function $\bar{\delta}_z$ is the same as the Heaviside function except that $\bar{\delta}_z(z) = 0$ rather than 1/2. In the preceding, the subscript A is used to indicate that the probabilities $Prob_A(\tilde{z} \le z|\mathbf{e})$ and $Prob_A(\tilde{z} > z|\mathbf{e})$ are characterizing aleatory uncertainty. Similarly, the distribution for z conditional on a specific realization for $\mathbf{e}$ can be represented by a density function $d_A(z|\mathbf{e})$.

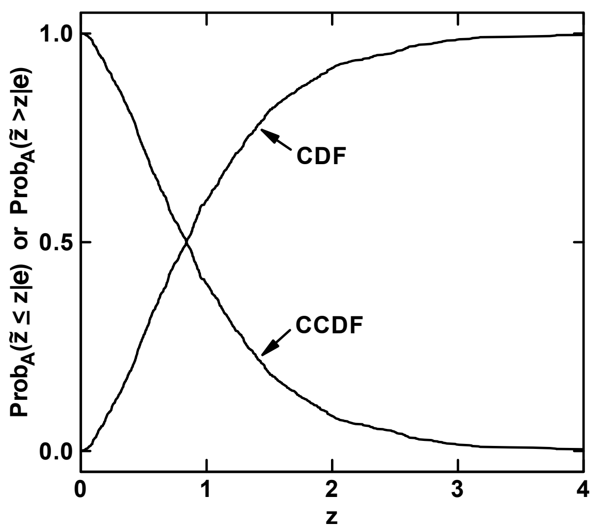

Fig. 4. Example CDF and CCDF defined by points $\left[z, Prob_A\left(\tilde{z} \le z \,|\, \mathbf{e}\right)\right]$ and $\left[z, Prob_A\left(\tilde{z} > z \,|\, \mathbf{e}\right)\right]$, respectively.

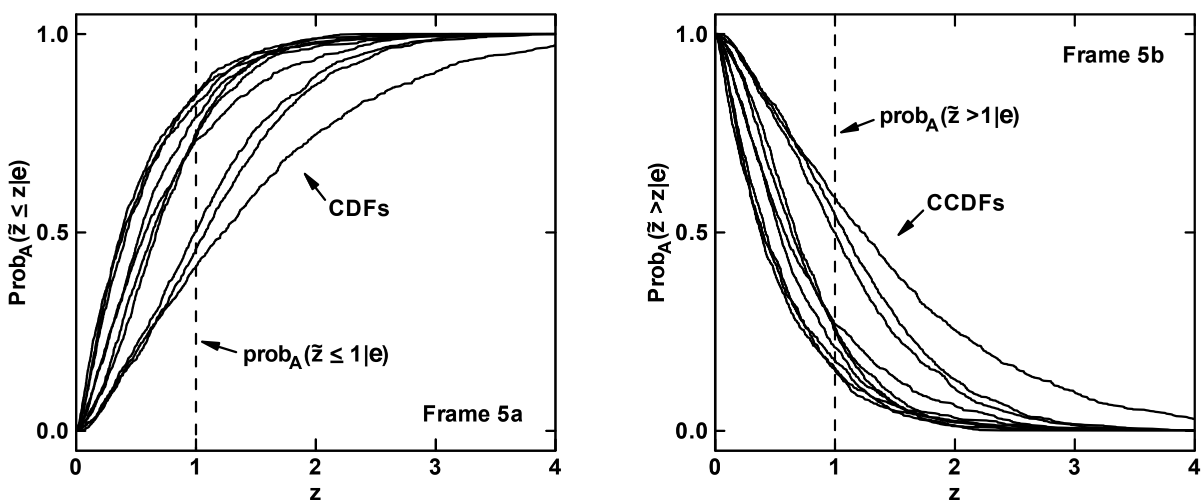

Fig. 5. Example CDFs and CCDFs that result for different values of the vector $\mathbf{e} = [\mathbf{e}_M, \mathbf{e}_D]$ of epistemically uncertain analysis inputs: (a) CDFs, and (b) CCDFs.

Different values for $\mathbf{e} = [\mathbf{e}_M, \mathbf{e}_D]$ result in different distributions for z. Thus, as $\mathbf{e}$ takes on different values from the set $\mathcal{E}$, a set of epistemically uncertain distributions for z will result (Fig. 5). In general, the cardinality (i.e., number of elements) of the resultant set of distributions could, but may not, equal the cardinality of $\mathcal{E}$.

The discussions that follow will focus primarily on the uncertainty structure associated with sets of the form

$$\underline{\mathcal{P}}(z) = \left\{ p : p = Prob_A\left(\tilde{z} \le z \middle| \mathbf{e}\right) \text{ for } \mathbf{e} \in \mathcal{E} \right\} \tag{3.8}$$

and

$$\bar{\mathcal{P}}(z) = \left\{ p : p = Prob_A\left(\tilde{z} > z \middle| \mathbf{e}\right) \text{ for } \mathbf{e} \in \mathcal{E} \right\}. \tag{3.9}$$

Thus, $\bar{\mathcal{P}}(z)$ is the set of all probabilities for a value $\tilde{z}$ less than or equal to z, and $\bar{\mathcal{P}}(z)$ is the set of all probabilities for a value $\tilde{z}$ greater than z. The individual probabilities in $\underline{\mathcal{P}}(z)$ and $\bar{\mathcal{P}}(z)$ derive from aleatory uncertainty as indicated in Eqs. (3.4) and (3.6); however, the sets $\underline{\mathcal{P}}(z)$ and $\bar{\mathcal{P}}(z)$ derive from epistemic uncertainty that results from the multiple values for $\mathbf{e}$ contained in the set $\mathcal{E}$. Specifically, the sets $\underline{\mathcal{P}}(z)$ and $\bar{\mathcal{P}}(z)$ contain the probabilities for different values of $\mathbf{e}$ that are associated with vertical lines drawn through the CDFs and CCDFs in Figs. 5a and 5b, respectively.

Before continuing, it is important to recognize that the study of the uncertainty associated with the sets $\underline{\mathcal{P}}(z)$ and $\bar{\mathcal{P}}(z)$ defined in Eqs. (3.8) and (3.9) is actually just a special case of the study of the uncertainty associated with the set

$$\mathcal{Y} = \left\{ y : \mathbf{x} \in \mathcal{X} \text{ and } y = F(\mathbf{x}) \right\} \tag{3.10}$$

previously defined in Eq. (2.4) and discussed extensively in Sect. 2. For this presentation, y corresponds to a probability as defined in Eqs. (3.4) and (3.6), and the function $F(\mathbf{x})$ is defined by the integrals in Eqs. (3.4) and (3.6).

4. Example Problem

This presentation employs as an example a coastal dike reliability problem originally introduced by Hussaarts et al. [145] and subsequently used with modifications by Hall and Lowry [146] and Ferson and Tucker [147]. In this problem, the reliability of a dike (Fig. 6) is based on the force balance equation

$$z = \Delta D - H \tan(\alpha) \Big/ \left[\cos(\alpha) M \sqrt{s} \right], \tag{4.1}$$

where

Δ = relative density of the revetment blocks on the front face of the dike (dimensionless),

D = thickness of revetment blocks (m),

H = significant wave height (m), which is the average height of the highest one third of the waves in a storm event,

α = slope of revetment (radians),

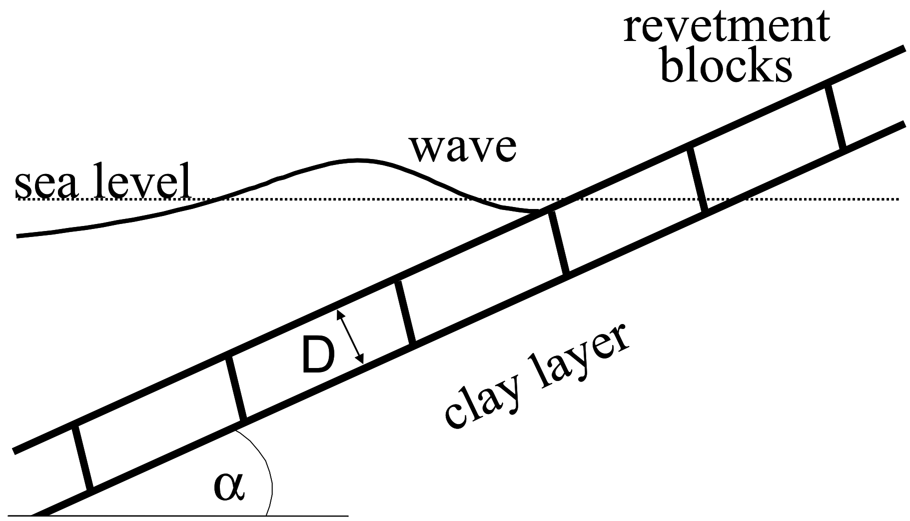

Fig. 6. Dike revetment (Fig. 12 in Ferson and Tucker [147] as redrawn from Hussaarts et al. [145]).

M = introduced parameter used to represent model uncertainty (dimensionless),

s = offshore peak wave steepness (dimensionless).

Specifically, the dike is assumed to fail if z is negative, which corresponds to a situation in which the force pushing the revetment blocks away from the face of the dike exceeds the force pushing the revetment blocks against the face of the dike (see Sect. 3, Ref. [145]).

With respect to the definition of z in Eq. (4.1), the quantities Δ, D, α and M are epistemically uncertain quantities related to properties of the dike, and H and s are aleatory quantities with distributions that derive in large part from weather variability. The quantities Δ, D, α and M are assigned the following sets of values in Ferson and Tucker with no specified uncertainty structure within these sets:

$$\mathcal{E}_1 = \{\Delta : 1.60 \le \Delta \le 1.65\}, \mathcal{E}_2 = \{D : 0.68 \le D \le 0.72 \text{ m}\} \tag{4.2}$$

$$\mathcal{E}_3 = \{\alpha : 0.309 \le \alpha \le 0.328 \text{ radians}\}, \mathcal{E}_4 = \{M : 3.0 \le M \le 5.2\}. \tag{4.3}$$

Further, H and s are assigned probability distributions with epistemically uncertain defining parameters. Specifically, the aleatory uncertainty in H is assumed to be characterized by a Weibull distribution with epistemically uncertain values for the scale factor sc and the shape factor sh, and the aleatory uncertainty in s is assumed to be characterized by a normal distribution with epistemically uncertain values for the mean μ and the standard deviation σ. With respect to the defining parameters for the density function

$$p_X(x) = c\alpha^{-1}\{(x-\xi_0)/\alpha\}^{c-1}\exp\left[-\{(x-\xi_0)/\alpha\}^c\right], \xi_0 < x, \tag{4.4}$$

of a Weibull distribution in Sect. 20.1 of Ref. [148], $sc = \alpha$, $sh = c$, and $\delta_0 = 0$. The quantities sc, sh, μ and σ are assigned the following sets of possible values in Ferson and Tucker [147]:

$$\mathcal{E}_5 = \{sc : 1.2 \le sc \le 1.5\}, \mathcal{E}_6 = \{sh : 10.0 \le sh \le 12.0\} \tag{4.5}$$

$$\mathcal{E}_7 = \{\mu : 0.039 \le \mu \le 0.041\}, \mathcal{E}_8 = \{\sigma : 0.005 \le \sigma \le 0.006\}. \tag{4.6}$$

As for Δ, D, α and M, no uncertainty structure was specified within these sets.

It is important to recognize exactly what the distribution assigned to H is characterizing. Specifically, this distribution is characterizing the year-to-year variability in the maximum annual value for H. Or, put another way, the distribution for H when converted to a CCDF gives the probabilities of H exceeding different values in a single given year. In turn, the distribution for s is for conditions associated with a large value for H (i.e., the maximum value for H in a specific year) but is assumed to be independent of the specific value for this maximum (p. 326, Ref. [145]).

In the context of the notation introduced in Sect. 3, the function f in Eq. (3.1) is given by

$$z = f(\mathbf{a}|\mathbf{e}_M) = \Delta D - H\tan(\alpha)\Big/\left[\cos(\alpha)M\sqrt{s}\right] \tag{4.7}$$

with $\mathbf{a} = [H, s]$ and $\mathbf{e}_M = [\Delta, D, \alpha, M]$. Further, the density function $d_A(\mathbf{a}|\mathbf{e}_D)$ for $\mathbf{a}$ is defined by the assumptions that H and s follow Weibull and normal distributions, respectively, with parameters defined by the elements of the vector $\mathbf{e}_D = [sc, sh, \mu, \sigma]$. In particular,

$$\mathbf{a} = [H, s] \tag{4.8}$$

is a vector of $nA = 2$ aleatory variables, and

$$\mathbf{e} = [\mathbf{e}_M, \mathbf{e}_D] = [\Delta, D, \alpha, M, sc, sh, \mu, \sigma] \tag{4.9}$$

is a vector of $nE = 8$ epistemically uncertain variables.

As already indicated, the aleatory variables H and s have specified probability distributions with the epistemically uncertain parameters that constitute the elements of $\mathbf{e}_D$. The $nE = 8$ epistemically uncertain variables that constitute the elements of $\mathbf{e} = [\mathbf{e}_M, \mathbf{e}_D]$ in Eq. (2.8) have ranges (i.e., sets of possible values $\mathcal{E}_1$, $\mathcal{E}_2$, …, $\mathcal{E}_8$) as indicated in Eqs. (4.2), (4.3), (4.5) and (4.6). However, no uncertainty structure was specified for these ranges in Ferson and Tucker.

The fundamental quantity of interest in this example is the (annual) probability that the dike will fail, which corresponds to the probability that the quantity z in Eq. (4.1) is negative. In turn, this probability is given by the integral defining $Prob_A(\tilde{z} \le 0 \,|\, \mathbf{e})$ in Eq. (3.4) for each possible value of $\mathbf{e}$, and the set of all possible values for this probability is represented by the set $\underline{\mathcal{P}}(0)$ in Eq. (3.8). Probabilities $Prob_A(\tilde{z} \le z \,|\, \mathbf{e})$ and sets $\underline{\mathcal{P}}(z)$ for other values of z are defined similarly. If desired, probabilities and sets of the form $Prob_A(\tilde{z} > z \,|\, \mathbf{e})$ and $\overline{\mathcal{P}}(z)$ in Eqs. (3.6) and (3.9) can also be defined.

5. Unstructured Epistemic Uncertainty

In the presentation by Ferson and Tucker [147] no uncertainty structure is specified for the sets $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_8$ containing the possible values for the $nE = 8$ epistemically uncertain variables under consideration, which corresponds to an uncertainty specification of the form on which interval analysis (Sect. 2.1) is predicated. This uncertainty information can also be converted into uncertainty representations of the form used in possibility theory (Sect. 2.2), evidence theory (Sect. 2.3), and probability theory (Sect. 2.4), respectively.

For possibility theory, the resultant distribution function r_i for variable e_i is given by

$$r_i(e_i) = \begin{cases} 1 & \text{if } e_i \in \mathcal{E}_i \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

For evidence theory, the resultant BPA m_i for subsets $\mathcal{U}$ of $\mathcal{E}_i$ is given by

$$m_i(\mathcal{U}) = \begin{cases} 1 & \text{if } \mathcal{U} = \mathcal{E}_i \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

For probability theory, the resultant probability distribution for e_i is obtained by recourse to the Laplacian concept of insufficient reason, which asserts that a uniform distribution should be used to characterize epistemic uncertainty when only a set of possible values is specified (pp. 52 – 55, Ref. [49]). This recourse results in the assignment of the density function

$$d_i(e_i) = \begin{cases} 1 \big/ \left[\sup(\mathcal{E}_i) - \inf(\mathcal{E}_i) \right] & \text{if } e_i \in \mathcal{E}_i \\ 0 & \text{otherwise} \end{cases} \tag{5.3}$$

to represent the uncertainty in e_i.

Collectively, the sets $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_8$ give rise to the set

$$\mathcal{E} = \mathcal{E}_1 \times \mathcal{E}_2 \times \ldots \times \mathcal{E}_8 \tag{5.4}$$

of vectors of the form $\mathbf{e} = [e_1, e_2, \ldots, e_8]$ shown in Eq. (4.9). In turn, a possibility space ($\mathcal{E}$, r_E) results from the definition of $r_i(e_i)$ in Eq. (5.1) and the assignment

$$r_E(\mathbf{e}) = \min\{r_1(e_1), r_2(e_2), \ldots, r_8(e_8)\} = 1; \tag{5.5}$$

an evidence space ($\mathcal{E}$, $\mathbb{E}$, m_E) results from the definition of $m_i(\mathcal{U})$ in Eq. (5.2) and the assignments $\mathbb{E} = \{\mathcal{E}\}$ and

$$m_E(\mathcal{U}) = \begin{cases} m_1(\mathcal{X}_1)\, m_2(\mathcal{X}_2) \ldots m_8(\mathcal{X}_8) = 1 & \text{if } \mathcal{U} = \mathcal{E} \in \mathbb{E} \\ 0 & \text{otherwise;} \end{cases} \tag{5.6}$$

and a probability space ($\mathcal{E}$, $\mathbb{E}$, p_E) results from the definition of $d_i(e_i)$ in Eq. (5.3) and the assignment of

$$d_E(\mathbf{e}) = d_1(e_1)\, d_2(e_2) \ldots d_8(e_8) \tag{5.7}$$

as the defining density function for $\mathbb{E}$ and p_E.

Each element $\mathbf{e}$ of the set $\mathcal{E}$ defined in Eq. (5.4) gives rise to a CDF for z, notationally represented by $\mathcal{CDF}(\mathbf{e})$, defined by probabilities of the form indicated in Eq. (3.4) (see Eq. (2.45) for a discussion of CDFs for model predictions). As a reminder,

$$\begin{aligned} \mathcal{CDF}(\mathbf{e}) &= \left\{\left[z,\, prob_A\left(\tilde{z} \le z \middle| \mathbf{e}\right)\right] : \mathbf{e} = [\mathbf{e}_M, \mathbf{e}_D] \in \mathcal{E} \text{ and } -\infty < z < \infty\right\} \\ &= \left\{\left[z,\, \int_{\mathcal{A}} \underline{\delta}_z\left[f\left(\mathbf{a}\middle|\mathbf{e}_M\right)\right] d_A\left(\mathbf{a}\middle|\mathbf{e}_D\right) \mathrm{d}A\right] : \mathbf{e} = [\mathbf{e}_M, \mathbf{e}_D] \in \mathcal{E} \text{ and } -\infty < z < \infty\right\}, \end{aligned} \tag{5.8}$$

where $\mathcal{A}$ is the set of possible values for $\mathbf{a}$ associated with the density function $d_A(\mathbf{a}|\mathbf{e}_D)$ and $\underline{\delta}_z[f(\mathbf{a}\,|\,\mathbf{e}_M)]$ is defined in Eq. (3.5). In turn, there exists a set

$$\mathcal{C} = \left\{\mathcal{CDF} : \mathbf{e} = [\mathbf{e}_M, \mathbf{e}_D] \in \mathcal{E} \text{ and } \mathcal{CDF} = \mathcal{CDF}(\mathbf{e})\right\} \tag{5.9}$$

of possible CDFs for z. The set $\mathcal{C}$ can be viewed in the context of interval analysis, possibility theory, evidence theory, or probability theory.

In the context of interval analysis, $\mathcal{C}$ is the set of possible CDFs associated with the set $\mathcal{E}$ of epistemically uncertain variables and nothing more can be said. For possibility theory, there is a possibility space ($\mathcal{C}$, r_C), where

$$r_C(\mathcal{CDF}) = \sup\left\{\mathbf{e} : \mathbf{e} \in \mathcal{E} \text{ and } \mathcal{CDF} = \mathcal{CDF}(\mathbf{e})\right\} = 1 \tag{5.10}$$

for $\mathcal{CDF} \in \mathcal{C}$. Similarly for evidence theory, there is an evidence space ($\mathcal{C}$, $\mathbb{X}$, m_C), where $\mathbb{X} = \{\mathcal{C}\}$ and

$$m_C(\mathcal{U}) = \begin{cases} 1 & \text{if } \mathcal{U} = \mathcal{C} \\ 0 & \text{otherwise} \end{cases} \tag{5.11}$$

for subsets $\mathcal{U}$ of $\mathcal{C}$. Because the spaces ($\mathcal{E}$, r_E) and ($\mathcal{E}$, E, m_E) are degenerate (i.e., $r_E(\mathbf{e}) = 1$ for $\mathbf{e} \in \mathcal{E}$ and $m_E(\mathcal{E}) = 1$), the corresponding spaces ($\mathcal{C}$, r_C) and ($\mathcal{C}$, X, m_C) are also degenerate (i.e., $r_C(\mathcal{CDF}) = 1$ for $\mathcal{CDF} \in \mathcal{C}$ and $m_C(\mathcal{C}) = 1$) and are effectively equivalent to the outcome of an interval analysis. In contrast, the probability space ($\mathcal{C}$, X, p_C) is not degenerate (i.e., there does not exist an element $\mathcal{CDF}$ of $\mathcal{C}$ such that $p_C(\{\mathcal{CDF}\}) = 1$) because of the structure of the probability space ($\mathcal{E}$, E, p_E). Specifically,

$$p_C(\mathcal{U}) = p_E\left(\left\{\mathbf{e} : \mathbf{e} \in \mathcal{E} \text{ and } \mathcal{CDF}(\mathbf{e}) \in \mathcal{U}\right\}\right) \tag{5.12}$$

for $\mathcal{U} \in$ X, where the formal properties of X would follow from the properties of E and p_E.

An analogous development is also possible for CCDFs and indeed for any property such as an expected value or a quantile that can be extracted from a CDF or a CCDF. However, it is worth noting that the consideration of a particular property extracted from a CDF are CCDF (e.g., an expected value or a quantile) is equivalent to studying the set of all CDFs or CCDFs with the extracted quantity serving as an index to identify individual CDFs or CCDFs. In this example, the primary quantity of interest is the probability for values less than $z = 0$. As discussed at the end of Sect. 4, this set of probabilities is represented by $\underline{\mathcal{P}}(0)$ for $z = 0$ and by $\underline{\mathcal{P}}(z)$ for an arbitrary value of z.

For this example, sampling-based (i.e., Monte Carlo) methods are used to both propagate epistemic uncertainty and integrate over aleatory uncertainty to estimate the CDFs in the set $\mathcal{C}$ defined in Eq. (5.9) and thus obtain the probabilities in the sets $\underline{\mathcal{P}}(z)$ and $\overline{\mathcal{P}}(z)$ defined in Eqs. (3.8) and (3.9). Specifically, a random sample

$$\mathbf{e}_i = \left[e_{1i}, e_{2i}, \ldots, e_{8i}\right], i = 1, 2, \ldots, nSE_1, \tag{5.13}$$

of size $nSE_1 = 10^4$ was generated from $\mathcal{E}$ with a uniform distribution assigned to each element of $\mathbf{e}$ (i.e., distributions of the form defined in Eq. (5.3)). In addition, a sample

$$e_i = \left[e_{1i}, e_{2i}, \ldots, e_{8i}\right],\ i = nSE_1 + 1, nSE_1 + 2, \ldots, nSE_1 + nSE_2 \tag{5.14}$$

of size $nSE_2 = 2^8 = 256$ was generated from $\mathcal{E}$ by taking all possible combinations of the endpoints of the intervals that correspond to the sets $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_8$. The purpose of this second sample is to include extreme combinations of parameter values that would not be obtained with random sampling. The result is a sample of size $nSE = nSE_1 + nSE_2 = 10{,}256$ from $\mathcal{E}$.

The sample

$$\mathbf{e}_i = \left[e_{1i}, e_{2i}, \ldots, e_{8i}\right], i = 1, 2, \ldots, nSE = nSE_1 + nSE_2, \tag{5.15}$$

provides the basis for the propagation of epistemic uncertainty. Then, aleatory uncertainty is propagated conditional on each element $\mathbf{e}_i$ of the preceding sample. Specifically, this involves the evaluation of integrals of the form appearing in Eqs. (3.4) and (3.6) to obtain CDFs and CCDFs for z and, correspondingly, elements of the sets $\underline{\mathcal{P}}(z)$ and $\overline{\mathcal{P}}(z)$. As a reminder, the indicated CDFs and CCDFs derive from the vector $\mathbf{a} = [H, s]$ of aleatory variables; further, the density function $d_A(\mathbf{a}|\mathbf{e}_D)$ for $\mathbf{a}$ is conditional on the vector

$$\mathbf{e}_D = [e_5, e_6, e_7, e_8] = [sc, sh, \mu, \sigma] \tag{5.16}$$

of epistemically uncertain variables, and the evaluation of the function $f(\mathbf{a}|\mathbf{e}_M)$ in Eq. (4.7) is conditional on the vector

$$\mathbf{e}_M = [e_1, e_2, e_3, e_4] = [\Delta, D, \alpha, M] \tag{5.17}$$

of epistemically uncertain variables.

A sampling-based procedure is also used to evaluate the CDF and CCDF for z conditional on each sample element $\mathbf{e}_i$. Specifically, a random sample

$$\mathbf{a}_{ij} = [H_{ij}, s_{ij}],\ j = 1, 2, \ldots, nSA, \tag{5.18}$$

of size $nSA = 10^7$ is generated from the set $\mathcal{A}$ of possible values for $\mathbf{a}$ in consistency with the density function $d_A(\mathbf{a}|\mathbf{e}_{Di})$. Then, the probabilities that define the CDF and CCDF for z conditional on a specific element $\mathbf{e}_i$ of the sample indicated in Eq. (5.15) are given by

$$Prob_A(\tilde{z} \le z|\mathbf{e}_i) \cong \sum_{j=1}^{nSA} \bar{\delta}_z\left[f(\mathbf{a}_{ij}|\mathbf{e}_{Mi})\right] \Big/ nSA \tag{5.19}$$

and

$$Prob_A(\tilde{z} > z|\mathbf{e}_i) \cong \sum_{j=1}^{nSA} \bar{\delta}_z\left[f(\mathbf{a}_{ij}|\mathbf{e}_{Mi})\right] \Big/ nSA, \tag{5.20}$$

respectively. The result is nSE = 10,256 CDFs for z and a corresponding number of CCDFs (Fig. 7).

For this example, CDFs are more meaningful entities to consider than CCDFs because dike failure is associated with negative values of z and the primary result of interest is how likely z is to be close to or below zero. Therefore, the following discussion will focus on CDFs and the corresponding sets $\underline{\mathcal{P}}(z)$. However, the associated ideas and representations are equally applicable to CCDFs and the corresponding sets $\overline{\mathcal{P}}(z)$. Indeed, in most risk assessments, CCDFs are the primary summary outcomes of interest because they answer the question "How likely is it to be this large or larger?"

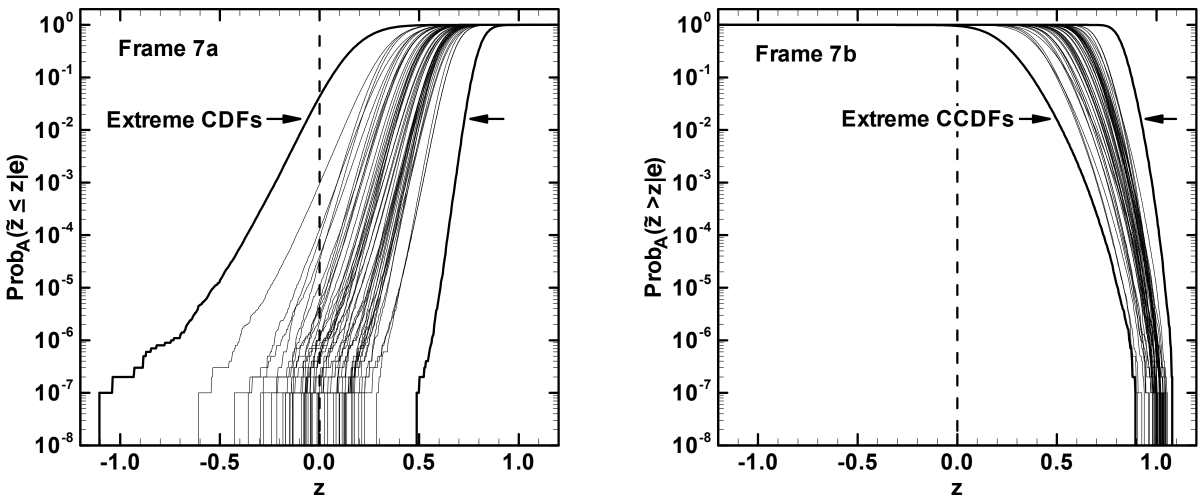

Fig. 7. Illustration of 50 of the nSE = 10,256 CDFs and CCDFs generated for the sample in Eq. (5.15): (a) CDFs, and (b) CCDFs.

For illustration, this discussion will focus on the set $\mathcal{P} = \underline{\mathcal{P}}(0)$. However, the ideas and associated result structure are the same for $\underline{\mathcal{P}}(z)$ with other values of z. The elements (i.e., probabilities) contained in $\mathcal{P}$ correspond to the probabilities associated with the vertical line through $z = 0$ in Fig. 7a. The outcome for interval analysis is simply the range of probabilities associated with the indicated vertical line, which corresponds to the interval

$$\left[\inf\left\{p : p \in \mathcal{P}\right\}, \sup\left\{p : p \in \mathcal{P}\right\}\right] \cong \left[0.0, 0.043\right]. \qquad (5.21)$$

For possibility theory, evidence theory and probability theory, the same set $\mathcal{P}$ of possible probabilities is under consideration. In concept, possibility theory, evidence theory and probability theory result in more internal uncertainty structure within $\mathcal{P}$ than is the case for interval analysis. However, in the example of this section, additional uncertainty structure within $\mathcal{P}$ only exists for probability theory.

A possibility space ($\mathcal{P}$, r_P), an evidence space ($\mathcal{P}$, Π, m_P) and a probability space ($\mathcal{P}$, Π, p_P) are associated with the set $\mathcal{P}$. In concept, the sampling-based procedures described in Sect. 2.5 can be used to estimate the CNF, CCNF, CPoF and CCPoF for the possibility space ($\mathcal{P}$, r_P), the CBF, CCBF, CPF and CCPF for the evidence space ($\mathcal{P}$, Π, m_P), and the CDF and CCDF for the probability space ($\mathcal{P}$, Π, p_P). However, the spaces ($\mathcal{P}$, r_P) and ($\mathcal{P}$, Π, m_P) are so simple this is hardly necessary. Specifically, since the possibility space ($\mathcal{E}$, r_E) is degenerate in the sense that $r_E(\mathbf{e}) = 1$ for $\mathbf{e} \in \mathcal{E}$ and the evidence space ($\mathcal{E}$, E, m_E) is degenerate in the sense that $m_E(\mathcal{E}) = 1$, it follows immediately that ($\mathcal{P}$, r_P) is degenerate in the sense that $r_P(p) = 1$ for $p \in \mathcal{P}$ and similarly that ($\mathcal{P}$, Π, m_P) is degenerate in the sense that $m_P(\mathcal{P}) = 1$. As a result, the CNF, CCNF, CPoF and CCPoF associated with ($\mathcal{P}$, r_P) and the CBF, CCBF, CPF and CCPF associated with ($\mathcal{P}$, Π, m_P) have simple forms that indicate no uncertainty structure within the set $\mathcal{P}$ (Fig. 8). Indeed, in this simple example, interval analysis, possibility theory and evidence theory provide the same infor-

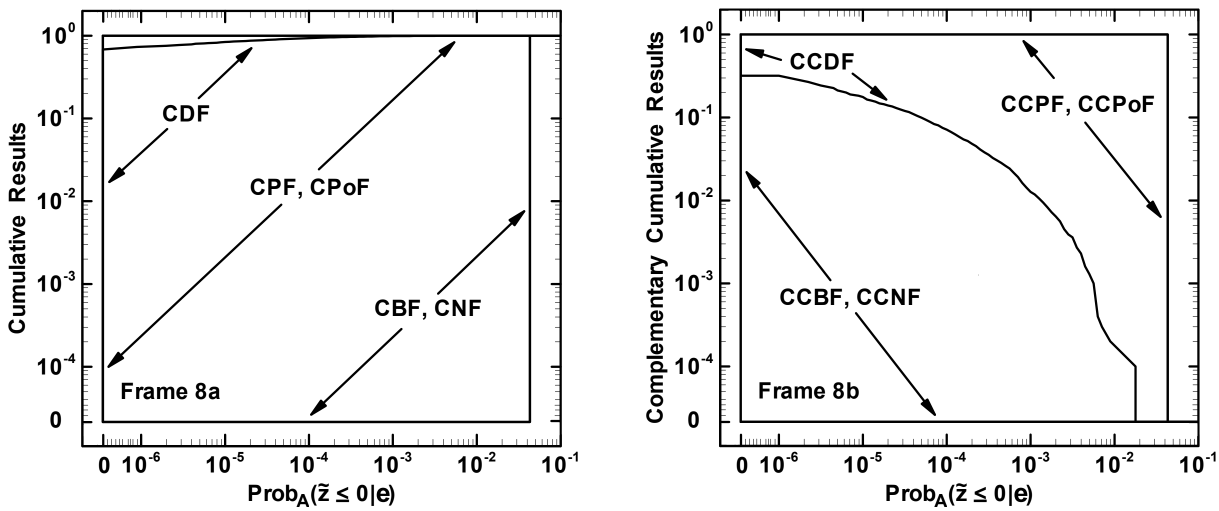

Fig. 8. Illustration of (i) CNF, CCNF, CPoF and CCPoF for unstructured possibility space ($\mathcal{P}$, r_P) defined in Sect. 5 (see Eqs. (5.1) and (5.5)), (ii) CBF, CCBF, CPF and CCPF for unstructured evidence space ($\mathcal{P}$, Π, m_P) defined in Sect. 5 (see Eqs. (5.2) and (5.6)), and (iii) CDF and CCDF for uniform probability space ($\mathcal{P}$, Π, p_P) defined in Sect. 5 (see Eqs. (5.3) and (5.7)): (a) CDF, CPF, CPoF, CBF and CNF, and (b) CCDF, CCPF, CCPoF, CCBF and CCNF.

mation: namely, within the limits of sampling error, the values for p contained in $\mathcal{P}$ fall in the interval [0.0, 0.045] and no uncertainty structure exists within this interval.

In elaboration, the probabilities contained in the set $\mathcal{P} = \underline{\mathcal{P}}(0)$ are represented on the abscissas of Figs. 8a and 8b and fall within the interval [0.0, 0.043]. As a reminder, the epistemically uncertain probabilities contained in the set $\mathcal{P}$ correspond to the probabilities associated with the vertical line through $z = 0$ in Fig 7a. For any value p on the abscissas of Figs. 8a and 8b, the possibility and plausibility for the set

$$\mathcal{P}_p = \left\{ \tilde{p} : \tilde{p} \le p \right\} \tag{5.21}$$

are given by

$$Pos_P\left(\mathcal{P}_p\right) = Pl_P\left(\mathcal{P}_p\right) = \begin{cases} 1 & \text{if } p \ge 0 \\ 0 & \text{if } p < 0, \end{cases} \tag{5.22}$$

where Pos_P denotes the possibility measure associated with the possibility space ($\mathcal{P}$, r_P) and Pl_P denotes the plausibility measure associated with the evidence space ($\mathcal{P}$, Π, m_P). As a result, plots of $Pos_P(\mathcal{P}_p)$ and $Pl_P(\mathcal{P}_p)$ overlay and form the top and left side of the box in Fig. 8a. Similarly,

$$Nec_P\left(\mathcal{P}_p\right) = Bel_P\left(\mathcal{P}_p\right) = \begin{cases} 1 & \text{if } p \geq 0.043 \\ 0 & \text{if } p < 0.043, \end{cases} \tag{5.23}$$

where Nec_P denotes the necessity measure associated with the possibility space ($\mathcal{P}$, r_P) and Bel_P denotes the belief measure associated with the evidence space ($\mathcal{P}$, Π, m_P). As a result, plots of $Nec_P(\mathcal{P}_p)$ and $Bel_P(\mathcal{P}_p)$ overlay and form the right side and bottom of the box in Fig. 8a. The structure of Fig. 8b is similar, with

$$Pos_P\left(\mathcal{P}_p^c\right) = Pl_P\left(\mathcal{P}_p^c\right) = \begin{cases} 1 & \text{if } p \leq 0.043 \\ 0 & \text{if } p > 0.043 \end{cases} \tag{5.24}$$

overlaying and forming the top and right side of the box in Fig. 8b and

$$Nec_P\left(\mathcal{P}_p^c\right) = Bel_P\left(\mathcal{P}_p^c\right) = \begin{cases} 1 & \text{if } p \leq 0 \\ 0 & \text{if } p > 0 \end{cases} \tag{5.25}$$

overlaying and forming the left side and bottom of the box in Fig. 8b. As stated, the inequalities involving p in Eqs. (5.21) – (5.25) can be viewed as tacitly extending the definition of $\mathcal{P}$ to the interval (–∞, ∞); however, it is really the interval [0.0, 0.043] that is of primary interest.

Unlike the spaces ($\mathcal{P}$, r_P) and ($\mathcal{P}$, Π, m_P), the probability space ($\mathcal{P}$, Π, p_P) does involve an uncertainty structure on the set $\mathcal{P}$ that derives from the probability space ($\mathcal{E}$, E, p_E) and the associated uniform distributions assigned to the elements of $\mathbf{e}$ in Eq. (5.3). As indicated in Sect. 2.5, the sample elements in Eq. (5.13) and associated estimates for $Prob_A(\tilde{z} \leq 0 \,|\, \mathbf{e}_i)$ indicated in Eq. (5.19) can be used to estimate the CDF and CCDF for p that derives from the probability space ($\mathcal{E}$, E, p_E) (Fig. 8). In this example, the probability space ($\mathcal{P}$, Π, p_P) is never fully determined in the sense of giving complete definitions for Π and p_P; rather, a sampling-based procedure is used to estimate the associated CDF and CCDF. This approximation is evident as the maximum probability obtained with the random sample of size $nSE_1 = 10^4$ used in the probabilistic calculation to obtain CDFs and CCDFs is approximately 0.018, while the maximum value obtained when the $nSE_2 = 256$ extreme value combinations of the elements of $\mathbf{e}$ are included is approximately 0.043. To obtain probabilities closer to 0.043 in the probabilistic calculation requires either (i) use of a value for nSE_1 that is considerably larger than 10^4 or (ii) use of importance sampling.

In elaboration, the CDF in Fig. 8a is a plot of the probabilities $p_P(\mathcal{P}_p)$, and the CCDF in Fig. 8b is a plot of the probabilities $p_P(\mathcal{P}_p^c)$. Because most values for $p_P(\mathcal{P}_p)$are very close to 1, the resultant CDF is barely discernable in the upper left corner of the box in Fig. 8a. In contrast, the small values for $p_P(\mathcal{P}_p^c)$ result in a CCDF that is clearly displayed in Fig. 8b with the log transformation used on the ordinate. In this example, the CCDF rather than the CDF for the probability p that z is less than zero is the quantity of greater relevance because increasing values for p correspond to increasing likelihoods that the system will fail. In concept, and for the same reason, the CCPoF, CCNF, CCPF and CCBF in Fig. 8b are also of greater relevance than the CPoF, CNF, CPF and CBF in Fig. 8a, although, in the current example, these quantities are not very interesting because of their degenerate structure; an example in which the CPoF, CPF, CCPoF and CCPF for *p* have more structure is presented in the next section (Sect. 6).

6. Structured Epistemic Uncertainty

As illustrated in Sect. 5, the absence of an internal uncertainty structure for the sets $\mathcal{E}_1$, $\mathcal{E}_2$, …, $\mathcal{E}_8$ results in analyses based on possibility theory and evidence theory that are effectively identical to results based on interval analysis. Thus, to help differentiate between results obtained with interval analysis, possibility theory, evidence theory and probability theory, additional uncertainty structure is now assumed and illustrated for several elements of **e**. Specifically, additional uncertainty structure is assumed for the sets $\mathcal{E}_2$, $\mathcal{E}_4$ and $\mathcal{E}_5$ that contain possible values for *D*, *M* and *sc*, respectively.

For convenience, the same uncertainty structure is imposed on $\mathcal{E}_2$, $\mathcal{E}_4$ and $\mathcal{E}_5$ (i.e., on *D*, *M* and *sc*). For use in describing this structure, I_1, I_2, I_3, I_4 and I_5 denote subintervals (i.e., subsets) of an interval [*a*, *b*] defined by

$$\begin{aligned} &I_1 = \left[a, a+6(b-a)/10\right], I_2 = \left[a+(b-a)/10, a+7(b-a)/10\right], I_3 = \left[a+2(b-a)/10, a+8(b-a)/10\right], \\ &I_4 = \left[a+3(b-a)/10, a+9(b-a)/10\right], I_5 = \left[a+4(b-a)/10, b\right] \end{aligned} \tag{6.1}$$

and illustrated in Fig. 9. For this example, it is assumed that each of the indicated subintervals of [*a*, *b*] is equally likely to contain the correct value for the quantity under consideration. Notionally, such a situation could arise from five equally credible experts expressing different intervals of possible values for the quantity under consideration but with no specified internal uncertainty structure for the individual intervals.

In turn, the indicated intervals and the assumptions of equal likelihood for the individual intervals can be converted into uncertainty representations in the context of possibility theory, evidence theory and probability theory, respectively. For possibility theory, a distribution function *r* can be defined by

$$r(x) = \sum_{i=1}^{5} \delta_i(x), \tag{6.2}$$

where

$$\delta_i(x) = \begin{cases} 1/5 & \text{if } x \in I_i \\ 0 & \text{otherwise.} \end{cases}$$

For evidence theory, a BPA *m* can be defined by

$$m(\mathcal{U}) = \begin{cases} 1/5 & \text{if } \mathcal{U} = I_1, I_2, I_3, I_4 \text{ or } I_5 \\ 0 & \text{otherwise} \end{cases} \tag{6.3}$$

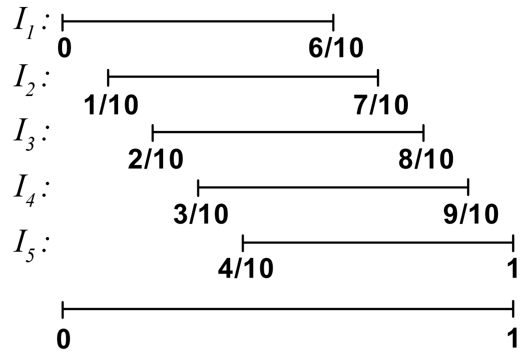

Fig. 9. Graphical illustration of sets I_1, I_2, I_3, I_4, I_5 defined in Eq. (6.1) with the interval $[a, b]$ normalized to [0, 1] for notational convenience.

for subsets $\mathcal{U}$ of $[a, b]$. For probability theory with recourse to the Laplacian concept of insufficient reason, a density function d can be defined by

$$d(x) = \sum_{i=1}^{5} \delta_i(x) / L(I_i) \tag{6.4}$$

where $\delta_i(x)$ is defined in conjunction with Eq. (6.2) and $L(I_i)$ is the length of the interval I_i.

As previously indicated, the same uncertainty structure is being imposed on $\mathcal{E}_2$, $\mathcal{E}_4$ and $\mathcal{E}_5$. Specifically, the intervals that correspond to $\mathcal{E}_2$, $\mathcal{E}_4$ and $\mathcal{E}_5$ (i.e., [0.68, 0.72], [3.0, 5.2] and [1.20, 1.5]) are subdivided as indicated in Eq. (6.1) and the resultant uncertainty characterizations for possibility theory, evidence theory and probability theory are defined as shown in Eq. (6.2), (6.3) and (6.4), respectively. This results in new definitions for (i) r_2, m_2 and d_2 for D, (ii) r_4, m_4 and d_4 for M, and (iii) r_5, m_5 and d_5 for sf. In turn, this results in new definitions for the spaces ($\mathcal{E}$, r_E), ($\mathcal{E}$, E, m_E) and ($\mathcal{E}$, E, p_E) introduced in Sect. 5 and, as a result, also for the corresponding spaces ($\mathcal{P}$, r_P), ($\mathcal{P}$, Π, m_P) and ($\mathcal{P}$, Π, p_P) with additional internal uncertainty structure imposed on the set $\mathcal{P} = \underline{\mathcal{P}}(0)$. However, the set $\mathcal{P}$ itself remains unchanged. Similar expansions also result for the corresponding spaces associated with the sets $\underline{\mathcal{P}}(z)$ and $\bar{\mathcal{P}}(z)$.

As indicated in Sect. 5 and discussed in greater detail in Sect. 2.5, sampling-based procedures can be used to propagate the uncertainty representations provided by ($\mathcal{E}$, r_E), ($\mathcal{E}$, E, m_E) and ($\mathcal{E}$, E, p_E). For this propagation, a random sample

$$\mathbf{e}_i = \left[e_{1i}, e_{2i}, \ldots, e_{8i}\right], i = 1, 2, \ldots, nSE_1, \tag{6.5}$$

of size $nSE_1 = 10^4$ is again generated from $\mathcal{E}$ but now with the redefined distributions for the elements of $\mathbf{e}$ (see Eq. (6.4)). In addition, a second sample

$$\mathbf{e}_i = [e_{1i}, e_{2i}, \ldots, e_{8i}], i = nSE_1 + 1, nSE_1 + 2, \ldots, nSE_1 + nSE_2, \tag{6.6}$$

of size $nSE_2 = 2^5\ 10^3 = 32{,}000$ is again generated from $\mathcal{E}$ by taking all possible combinations of the endpoints of the focal elements contained in E_1, E_2, …, E_8. The purpose of the second sample is to assure coverage of the end points of the focal elements contained in E_1, E_2, …, E_8. The result is a sample of size $nSE = nSE_1 + nSE_2 = 42{,}000$ from $\mathcal{E}$. In turn, this results in nSE CDFs and nSE corresponding CCDFs of the form illustrated in Fig. 7 and corresponding approximations to the sets $\underline{\mathcal{P}}(z)$ and $\overline{\mathcal{P}}(z)$.

The sampling-based procedures described in Sect. 2.5 can be used (i) with the combined sample $\mathbf{e}_i$, $i = 1, 2, \ldots, nSE = nSE_1 + nSE_2$, to estimate the CNF, CCNF, CPoF and CCPoF for the possibility space ($\mathcal{P}$, r_P) and the CBF, CCBF, CPF and CCPF for the evidence space ($\mathcal{P}$, Π, m_P) and (ii) with the random sample $\mathbf{e}_i$, $i = 1, 2, \ldots, nSE_1$, to estimate the CDF and CCDF for the probability space ($\mathcal{P}$, Π, p_P) (Fig. 10). Because interval analysis assumes no uncertainty structure internal to $\mathcal{E}_1$, $\mathcal{E}_2$, …, $\mathcal{E}_8$ and the set $\mathcal{P}$ is unchanged from Sect. 5, the interval analysis result is still approximated by the interval [0.0, 0.043]. As a result of their increasing levels of uncertainty structure, the uncertainty representations from the possibility space ($\mathcal{P}$, r_P) contain the uncertainty representations from the evidence space ($\mathcal{P}$, Π, m_P) (i.e., the CBF and CPF for ($\mathcal{P}$, Π, m_P) fall between the CNF and CPoF for ($\mathcal{P}$, r_P) and similarly the CCBF and CCPF fall between the CCNF and CCPoF), and the uncertainty representations from ($\mathcal{P}$, Π, m_P) contain the uncertainty representations from the probability space ($\mathcal{P}$, Π, p_P) (i.e., the CDF for ($\mathcal{P}$, Π, p_P) falls between the CBF and CPF for ($\mathcal{P}$, Π, m_P) and similarly the CCDF falls between the CCBF and CCPF).

In elaboration, the results in Fig. 10a show the changes to the CPoF, CNF, CPF, CBF and CDF in Fig. 8a that result when the added uncertainty structure associated with D, M and sc is incorporated into the definitions of the possibility space ($\mathcal{P}$, r_P), the evidence space ($\mathcal{P}$, Π, m_P) and the probability space ($\mathcal{P}$, Π, p_P). Specifically, the CPoF, CPF and CDF are not substantially changed (actually, the CDF has changed but this change is not apparent at the resolution of Figs. 8a and 10a,c). However, the CNF and CBF now display a structure that was completely lacking in Fig. 8a. For example, the necessity and belief in Fig. 8a that p is less than 0.02 are 0; in contrast, the corresponding values in Fig. 10a are 0.80 and 0.976, respectively. Similarly, the results in Figs. 10b,c,d show the changes to the CCPoF, CCNF, CCPF, CCBF and CCDF in Fig. 8b that result from the changed definitions for ($\mathcal{P}$, r_P), ($\mathcal{P}$, Π, m_P) and ($\mathcal{P}$, Π, p_P). The results in Figs. 10b,c,d are the same, but different scalings on the abscissa and ordinate are being used to better display small numerical values. Specifically, linear scales are used on the abscissa and ordinate in Fig. 10b; linear and log scales are used on the abscissa and ordinate, respectively, in Fig. 10c; and log scales are used on the abscissa and the ordinate in Fig. 10d. The CCNF and CCBF in Figs. 10b,c,d are the same as the CCNF and CCBF in Fig. 8b. However, the CCPoF, CCPF and CCDF are substantially changed. For example, the possibility and plausibility that p is greater than 0.02 are 1.0; in contrast, the corresponding values in Figs. 10b,c,d are 0.2 and 0.024, respectively. In both Fig. 8b and Figs. 10b,c,d, the probability that p exceeds 0.02 is beneath the numerical resolution of the sample size from $\mathcal{E}$ in use (i.e., $nSE_1 = 10^4$ as indicated in conjunction with Eq. (6.5)); estimation of the probability that p exceeds 0.02 in the analyses presented in Figs. 8b and 10b,c,d would require either a much larger random sample from $\mathcal{E}$ or the use of some type of importance sampling procedure. However, as another comparison, the probability that p exceeds 0 is approximately 0.09 in Fig. 8b and approximately 0.043 in Figs. 10b,c,d, with this difference resulting from the changed definitions for the probability space ($\mathcal{P}$, Π, p_P).

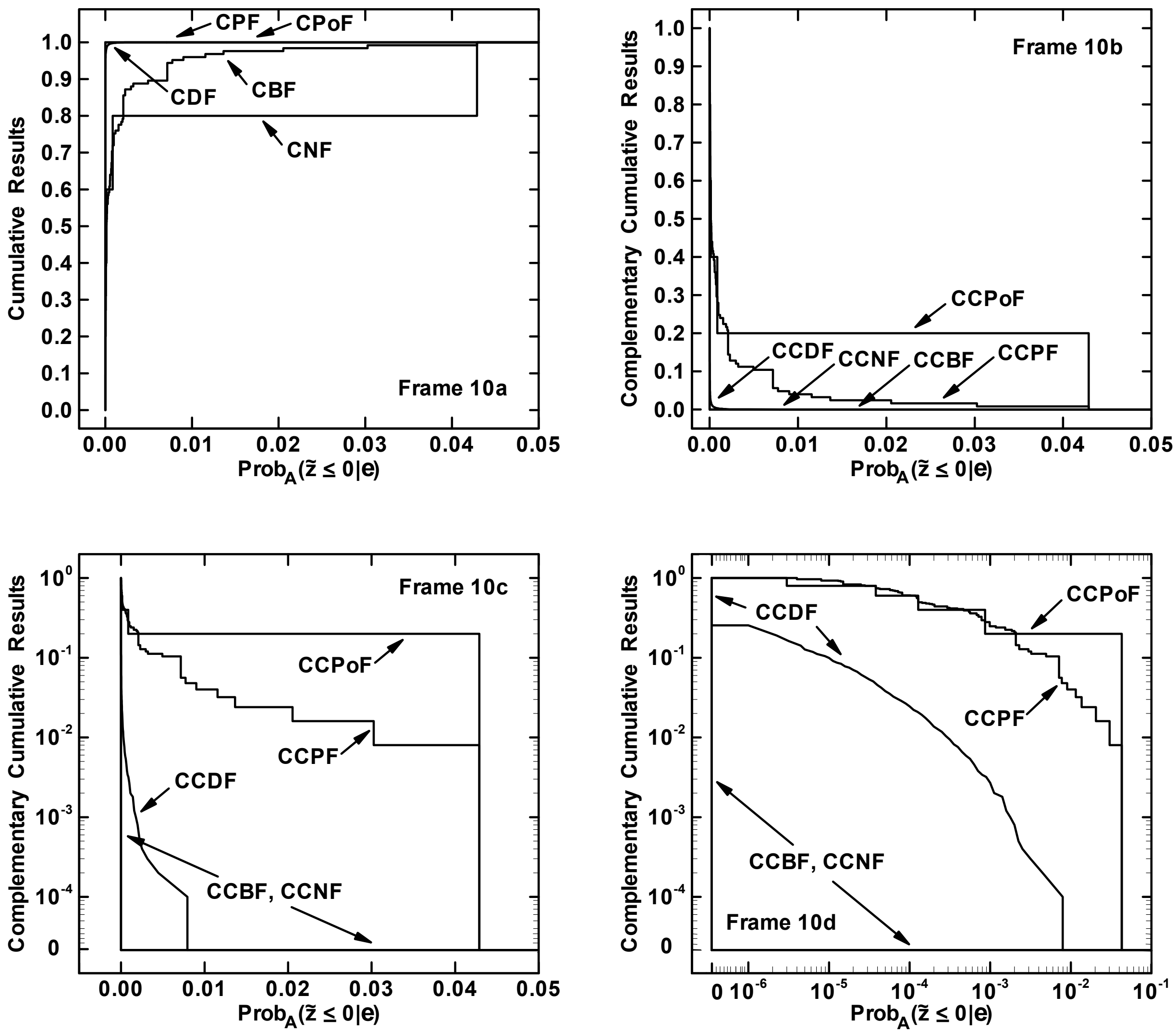

Fig. 10. Illustration of (i) CNF, CCNF, CPoF and CCPoF for structured possibility space ($\mathcal{P}$, r_P) defined in Sect. 6 (see Eq. (6.2) and resultant definition for r_E), (ii) CBF, CCBF, CPF and CCPF for structured evidence space ($\mathcal{P}$, Π, m_P) defined in Sect. 6 (see Eq. (6.3) and resultant definition for m_E), and (iii) CDF and CCDF for nonuniform probability space ($\mathcal{P}$, Π, p_P) defined in Sect. 6 (see Eq. (6.4) and resultant definition for density function d_E corresponding to p_E): (a) CPoF, CNF, CPF, CBF and CDF with linear scales on abscissa and ordinate, (b) CCPoF, CCNF, CCPF, CCBF and CCDF with linear scales on abscissa and ordinate, (c) same as (b) but with log scale on ordinate, and (d) same as (b) but with log scales on abscissa and ordinate.

The uncertainty representations in Fig. 10 are for the probabilities contained in the set $\mathcal{P} = \underline{\mathcal{P}}(0)$. Analogous representations are also possible for the probabilities contained in the sets $\underline{\mathcal{P}}(z)$ and $\overline{\mathcal{P}}(z)$ for other values of z and provide a representation of the uncertainty associated with the CDFs and CCDFs in Fig. 7. For illustration, the sets $\underline{\mathcal{P}}(z)$ are considered. An analogous development is possible, but not shown, for the sets $\overline{\mathcal{P}}(z)$. Let $\underline{\mathcal{P}}_p(z)$ denote the set defined by

$$\underline{\mathcal{P}}_p(z) = \left\{ \tilde{p} : \mathbf{e} \in \mathcal{E} \text{ and } \tilde{p} = Prob_A\left(\tilde{z} \le z \middle| \mathbf{e}\right) > p \right\} \tag{6.7}$$

with $Prob_A(\tilde{z} \le z \mid \mathbf{e})$ defined in Eq. (3.4) and again in Eq. (5.8). In words, $\underline{\mathcal{P}}_p(z)$ is the set of possible values for $Prob_A(\tilde{z} \le z \mid \mathbf{e})$ that are larger than p. In turn, the possibility space ($\mathcal{E}$, r_E), the evidence space ($\mathcal{E}$, E, m_E) and the probability space ($\mathcal{E}$, E, p_E) give rise to a possibility $Pos_E[\underline{\mathcal{P}}_p(z)]$, a necessity $Nec_E[\underline{\mathcal{P}}_p(z)]$, a plausibility $Pl_E[\underline{\mathcal{P}}_p(z)]$, a belief $Bel_E[\underline{\mathcal{P}}_p(z)]$ and a probability $Prob_E[\underline{\mathcal{P}}_p(z)]$ for each set $\underline{\mathcal{P}}_p(z)$, with the subscript E added to emphasize that epistemic uncertainty is being represented. For perspective, the CCPoF, CCNF, CCPF, CCBF and CCDF in Figs. 10b–d correspond to plots of the points $\{p, Pos_E[\underline{\mathcal{P}}_p(z)]\}$, $\{p, Nec_E[\underline{\mathcal{P}}_p(z)]\}$, $\{p, Pl_E[\underline{\mathcal{P}}_p(z)]\}$, $\{p, Bel_E[\underline{\mathcal{P}}_p(z)]\}$ and $\{p, Prob_E[\underline{\mathcal{P}}_p(z)]\}$, respectively, for $z = 0$ and $0 \le p \le 0.043$. As discussed previously, the indicated uncertainty measures are obtained from the spaces ($\mathcal{E}$, r_E), ($\mathcal{E}$, E, m_E) and ($\mathcal{E}$, E, p_E) by mapping $\underline{\mathcal{P}}_p(z)$ back to a subset of $\mathcal{E}$ and then determining the uncertainty measure of this set.

However, the presentation of the preceding representations for multiple values of z is inefficient and unwieldy. A more effective presentation is to display plots of quantiles that derive from the probability spaces associated with the sets $\underline{\mathcal{P}}(z)$ and $\overline{\mathcal{P}}(z)$ and analogous quantities that derive from the possibility and evidence spaces associated with $\underline{\mathcal{P}}(z)$ and $\overline{\mathcal{P}}(z)$.

For the probability space ($\mathcal{E}$, E, p_E), the resultant probability $prob_E[\underline{\mathcal{P}}_p(z)]$ is a nonincreasing function of p because $\underline{\mathcal{P}}_v(z) \subseteq \underline{\mathcal{P}}_u(z)$ for $0 \le u \le v \le 1$. As a result, the value $Prb_q(z)$ for the q quantile (e.g., q = 0.1, 0.2, …, 0.9) of the set, $\underline{\mathcal{P}}(z)$ can be informally defined as the element of p of $\underline{\mathcal{P}}(z)$ for which the approximate equality

$$Prob_E\left[\underline{\mathcal{P}}_p(z)\right] \cong q \tag{6.8}$$

most closely holds and can be formally defined by

$$Prb_q(z) = \inf\left\{ p : p \in \underline{\mathcal{P}}(z) \text{ and } Prob_E\left[\underline{\mathcal{P}}_p(z)\right] \ge q \right\}. \tag{6.9}$$

With $\underline{\mathcal{P}}(0)$ used as an example, the preceding corresponds graphically to (i) starting at the value q on the ordinate of Fig. 10d (or, equivalently, Fig. 10b or 10c), (ii) drawing a horizontal line to the CCDF, and then (iii) drawing a vertical line down to the ordinate to determine the value of $p = Prob_A(\tilde{z} \le 0|\mathbf{e})$ that is the q quantile value $Prb_q(0)$. In this context, quantiles are being associated with the probabilities of exceeding specified values rather than with the probabilities of being less than specified values.

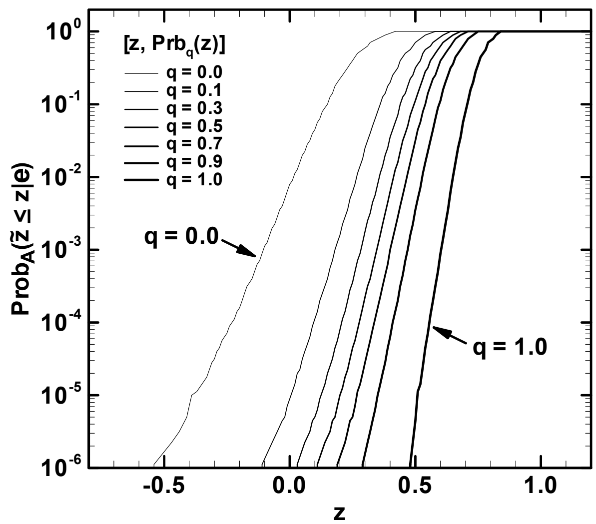

Fig. 11. Illustration of quantile (probability) curves defined by $[z, Prb_q(z)]$ for $z_{mn} \le z \le z_{mx}$ and q = 0.0, 0.1, 0.3, 0.5, 0.7, 0.9 and 1.0.

In words, there is an epistemic (i.e., degree of belief) probability q (e.g., q = 0.1, 0.2, …, 0.9) that the value for an element p of $\underline{\mathcal{P}}(z)$ is larger than $Prb_q(z)$. More specifically, this implies a probability of q that the correct value for $Prob_A(\tilde{z} \le z)$ is greater than or equal to $Prb_q(z)$. In turn, the epistemic uncertainty associated with the set $\mathcal{C}$ of CDFs defined in Eq. (5.9) and illustrated in Fig. 7a can be summarized with plots of the quantile (probability) curves defined by $[z, Prb_q(z)]$ for $z_{mn} \le z \le z_{mx}$ and selected values of q (e.g., for q = 0.0, 0.1, 0.3, 0.5, 0.7, 0.9 and 1.0 as illustrated in Fig. 11).

For the possibility space $(\mathcal{E}, r_E)$, the quantities $Pos_q(z)$ and $Nec_q(z)$ for the set $\underline{\mathcal{P}}(z)$ are defined by

$$Pos_q(z) = \inf\left\{p : p \in \underline{\mathcal{P}}(z) \text{ and } Pos_E\left[\underline{\mathcal{P}}_p(z)\right] \ge q\right\} \tag{6.10}$$

and

$$Nec_q(z) = \inf\left\{p : p \in \underline{\mathcal{P}}(z) \text{ and } Nec_E\left[\underline{\mathcal{P}}_p(z)\right] \ge q\right\}, \tag{6.11}$$

respectively. The quantities $Pos_q(z)$ and $Nec_q(z)$ are analogous to $Prb_q(z)$ and are amenable to similar intuitive descriptions except that they correspond to values of p with an exceedance possibility and an exceedance necessity of q rather than a value of p with an exceedance probability of value q. In turn, the epistemic uncertainty associated with the set $\mathcal{C}$ of CDFs defined in Eq. (5.9) can be summarized with plots of the possibility curves and necessity curves defined by $[z, Pos_q(z)]$ and $[z, Nec_q(z)]$, respectively, for $z_{mn} \le z \le z_{mx}$ and selected values of q (Fig. 12). Values for q are given a step size of 0.2 in Fig. 12 because of the discretized nature of possibility and necessity in this example.

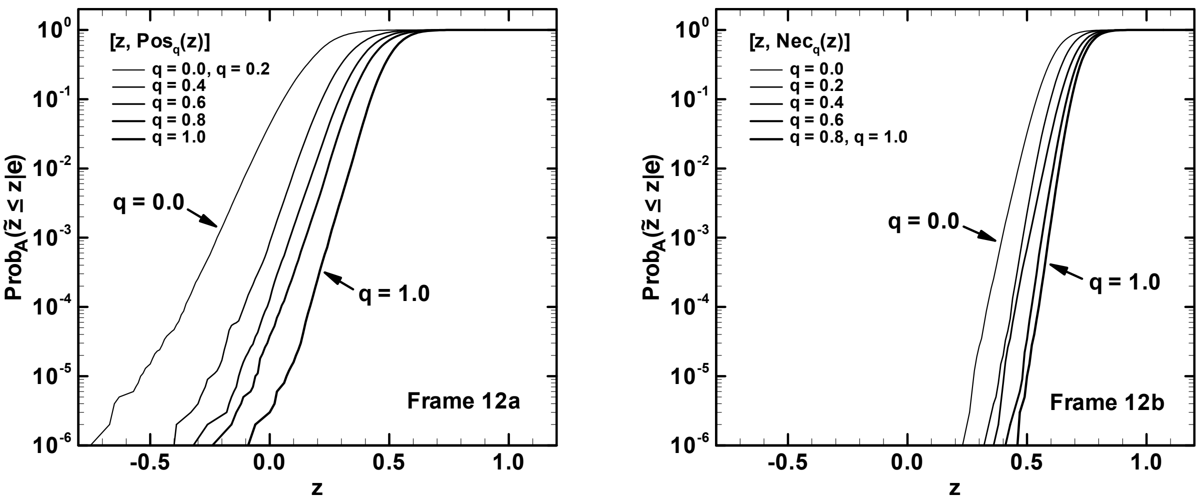

Fig. 12. Illustration of possibility curves and necessity curves defined by [z, $Pos_q(z)$] and [z, $Nec_q(z)$], respectively, for $z_{mn} \le z \le z_{mx}$ and q = 0.0, 0.2, 0.4, 0.6, 0.8 and 1.0: (a) Possibility curves, and (b) Necessity curves.

Similarly for the evidence space ($\mathcal{E}$, E, m_E), the quantities $Pl_q(z)$ and $Bel_q(z)$ for the set $\underline{\mathcal{P}}(z)$ are defined by

$$Pl_q(z) = \inf\left\{p : p \in \underline{\mathcal{P}}(z) \text{ and } Pl_E\left[\underline{\mathcal{P}}_p(z)\right] \ge q\right\} \tag{6.12}$$

and

$$Bel_q(z) = \inf\left\{p : p \in \underline{\mathcal{P}}(z) \text{ and } Bel_E\left[\underline{\mathcal{P}}_p(z)\right] \ge q\right\}, \tag{6.13}$$

respectively. The quantities $Pl_q(z)$ and $Bel_q(z)$ are analogous to $Prb_q(z)$, $Pos_q(z)$ and $Nec_q(z)$ and are amenable to similar intuitive descriptions except that they correspond to values of p with an exceedance plausibility and an exceedance belief of q rather than values of p with an exceedance probability, an exceedance possibility and an exceedance necessity of q. As in Figs. 11 and 12 for $Prb_q(z)$, $Pos_q(z)$ and $Nec_q(z)$, the epistemic uncertainty associated with the set $\mathcal{C}$ of CDFs defined in Eq. (5.9) can be summarized with plots of the plausibility curves and belief curves defined by [z, $Pl_q(z)$] and [z, $Bel_q(z)$], respectively, for $z_{mn} \le z \le z_{mx}$ and selected values of q (Fig. 13).

For interval analysis, there is no internal uncertainty structure associated with the set $\mathcal{C}$ of CDFs. Thus, all that can be said is that the elements of $\mathcal{C}$ fall between the bounding (i.e., extreme) CDFs indicated in Fig. 7.

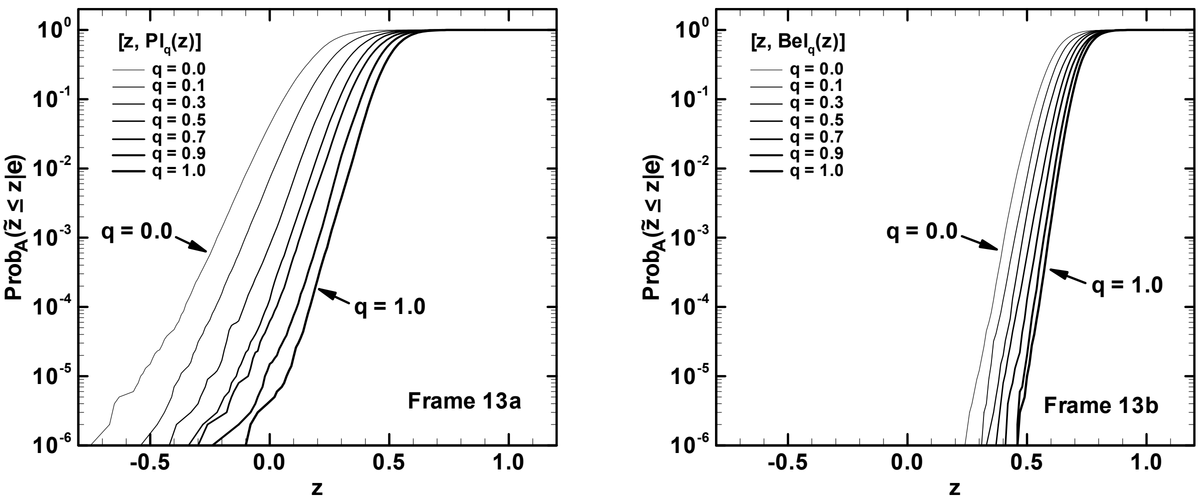

Fig. 13. Illustration of plausibility curves and belief curves defined by $[z, Pl_q(z)]$ and $[z, Bel_q(z)]$, respectively, for $z_{mn} \le z \le z_{mx}$ and q = 0.0, 0.1, 0.3, 0.5, 0.7, 0.9 and 1.0: (a) Plausibility curves, and (b) Belief curves.

7. Summary Discussion

The appropriate incorporation and representation of the effects and implications of aleatory and epistemic uncertainty are fundamental parts of modern performance and risk studies.

Traditionally, probability theory has provided the mathematical structure used to characterize both aleatory and epistemic uncertainty. For example, probability is used to characterize aleatory uncertainty and epistemic uncertainty in the U.S. Nuclear Regulatory Commission's reassessment of the risks posed by commercial nuclear power stations [25; 26; 149] and in the U.S. Department of Energy's successful compliance certification application for the Waste Isolation Pilot Plant [131; 150]. With this approach to the representation of uncertainty, aleatory uncertainty in analysis outcomes of interest is typically represented with CDFs or CCDFs and, in turn, epistemic uncertainty leads to distributions of these curves. Specifically, the outcome is a probabilistic characterization of the epistemic uncertainty associated with families of CDFs and CCDFs, which in turn are probabilistic characterizations of aleatory uncertainty [101; 151; 152].

In the last several decades, a number of alternatives to probability theory for the representation of epistemic uncertainty have been proposed, including interval analysis, possibility theory and evidence theory. These alternatives permit a less detailed representation of epistemic uncertainty than is possible with probability theory. As a result, these alternatives may more appropriately characterize epistemic uncertainty in the presence of limited information than probability theory. In particular, the use of probability to characterize epistemic uncertainty in the presence of limited information can imply the presence of more knowledge than is actually present.

This presentation illustrates the use of interval analysis, possibility theory, evidence theory and probability theory in the representation of the epistemic uncertainty associated with CDFs and CCDFs that summarize the effects of aleatory uncertainty. As the presented examples show, the resultant representation of epistemic uncertainty and the associated implications of this uncertainty can be very different depending on the mathematical structure used to characterize epistemic uncertainty in analysis inputs.

Although possibility theory, evidence theory and probability theory provide different mathematical structures for the representation of epistemic uncertainty, the uncertainty results that derive from these different structures can be summarized in conceptually similar formats. Specifically, cumulative and complementary cumulative uncertainty representations are possible for each of these theories. With this format, the outcomes of an uncertainty analysis based on possibility theory can be represented with CNFs, CCNFs, CPoFs and CCPoFs; the outcomes of an uncertainty analysis based on evidence theory can be represented with CBFs, CCBFs, CPFs and CCPFs; and, as is usually done, the outcomes of an uncertainty analysis based on probability theory can be represented with CDFs and CCDFs. Cumulative and complementary cumulative uncertainty representations provide compact and informative summaries of uncertainty information. Further, as illustrated in this presentation, cumulative and complementary cumulative uncertainty representations provided a common format that can be used to compare uncertainty results obtained when different mathematical structures are used to characterize epistemic uncertainty.

Possibility theory and evidence theory provide uncertainty representations with less internal structure than probability theory However, the propagation of these representations through a model to obtain the resultant uncertainty representations for model results can require more computation (i.e., model evaluations) than is the case when probability is used to represent uncertainty. This computational requirement results when a large number of discontinuities are present in a possibility or evidence theory representation for epistemic uncertainty. For example, an evidence theory representation for uncertainty can rapidly expand to involve a huge number of focal elements as the number of uncertain variables increases (e.g., an evidence space constructed from 10 uncertain variables with 10 focal elements for each variable has 10^{10} focal elements). This presentation has used a computationally simple model for illustration. As a result, large numbers of model evaluations were possible.

In most real analyses, this level of naïve computation is unlikely to be possible. Rather, some type of efficient computational strategy will have to be developed to support the large number of model evaluations required to propagate uncertainty representations based on possibility theory or evidence theory. For example, sensitivity analysis procedures can be used to identify the variables that dominate the uncertainty in analysis results of interest [127-129; 153-159]. Then, only these important variables can be included in the uncertainty propagation. This reduces the dimensionality of the input space, and as a result, can significantly reduce the number of model evaluations required in an uncertainty propagation. A related approach is to perform a stepwise uncertainty propagation in which the full uncertainty representation is used for the most important input variable and all other variables are assigned degenerate representations; the analysis is then repeated with the full uncertainty representation used for the two most important variables and all other variables assigned degenerate uncertainty representations; this process then continues until the inclusion of full uncertainty representations for additional variables results in no significant changes in the uncertainty representations for analysis results of interest, with the analysis stopping at this point [160]. Again, this approach reduces the dimensionality of the input space, and as a result, can significantly reduce the number of model evaluations required in an uncertainty propagation. Computational savings can also be achieved by reducing the complexity of the uncertainty representations in use (e.g., by replacing an evidence space with many focal elements with a related evidence space with fewer focal elements) [160]. Again, this results in computational savings by reducing the complexity of the input space. Finally, significant computational savings can be achieved by using nonparametric regression techniques and other related procedures to develop computationally efficient approximations to numerically demanding models [161-170].

The results of performance and risk analyses for complex systems are usually presented as CDFs and CCDFs that summarize the effects of aleatory uncertainty. In turn, the presence of epistemic uncertainty results in many possible values for these CDFs and CCDFs. If possibility theory and evidence theory are to have a role in characterizing epistemic uncertainty in the results of such analyses, these theories must be able to provide uncertainty characterizations for sets of epistemically uncertain CDFs and CCDFs. As illustrated in this presentation, such characterizations can be obtained with possibility and evidence theory.

However, three challenges remain to the use of possibility theory and evidence theory in performance and risk analyses for complex systems. First, it is necessary to convince the supporters (i.e., funders) of these analyses of the appropriateness and value of the use of an alternative to probability for the representation of epistemic uncertainty. This is likely to involve a large educational effort as few funders or users of such analyses will be familiar with these alternatives to probability for the representation of epistemic uncertainty. Second, most analysts who participate in analyses of this type will not be familiar with these alternative uncertainty representations. Again, a significant educational effort is likely to be necessary before the desired uncertainty representations for analysis inputs can be obtained. Third, computationally practicable methods must be developed and implemented for the propagation of the uncertainty representations through the analysis. This development and implementation is likely to be analysis-specific.

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