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III. ELECTROMAGNETIC-COUPLING INSTABILITIES IN A COASTING BEAM

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ABSTRACT

The electromagnetic interaction of an intense relativistic coasting beam with itself, including the effect of a confining nonperfect vacuum tank, or a quiescent rf cavity, is investigated theoretically. It is shown that the resonances that may occur between harmonics of the particle circulation frequencies and the electromagnetic modes of the cavities can lead to a longitudinal instability of the beam. A criterion for stability of the beam against such longitudinal bunching is obtained as a restriction on the shunt impedance of the rf cavity, or the Q of the vacuum tank. This criterion contains the energy spread and intensity of the coasting beam, as well as the parameters of the accelerator. Numerical examples are given which indicate that in general the resonances with the vacuum tank will not cause instabilities, while those with an rf cavity can be prevented from causing instabilities by choosing the shunt impedance at a sufficiently low but still convenient value.

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I. INTRODUCTION

In the second article (Part II) of this series[1] it was shown that a resonance can occur between a beam of particles in an accelerator and the characteristic electromagnetic modes of the vacuum tank. It is possible that this resonance could lead to instabilitites in an intense relativistic coasting beam. This problem is distinguished from the longitudinal instabilities investigated previously by a number of authors[2, 3] because resonance can occur only with modes characterized by short wavelengths in the azimuthal direction. Thus we shall be dealing with perturbation frequencies that are very high harmonics of the particle circulation frequency.

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We shall again take a toroid with rectangular cross section as a model of the vacuum tank (Fig. 1), neglecting all windows, discontinuities, and straight sections. The conductivity of the walls is sufficiently high to allow the vanishing of the tangential electric field to be used as a boundary condition in the solution of Maxwell's equations. Therefore, we can use the results in Part II of this series.

The stability of the coasting beam also may be affected by the presence of an rf cavity through which the beam must pass. If the cavity has an eigen-frequency near a harmonic of the beam circulation frequency, a resonance condition exists between the beam and the cavity. Such a resonance generally occurs for a much lower harmonic than the resonance with the modes of the vacuum tank. For purposes of this calculation we assume that the cavity is not driven externally.

Transverse particle motion will be neglected throughout this work, except insofar as it contributes to the cross-sectional area of the beam. The density of particles in the unperturbed beam is taken as being uniform azimuthally. In Section II we assume an infinitesimal perturbation that preserves the cross-sectional dimensions of the beam. It is then possible to solve the linearized one-dimensional Vlasov equation to obtain a dispersion relation that gives the allowed values of the perturbation frequency. This dispersion relation contains the azimuthal electric field generated by the perturbation, and in Section III convenient expressions are cited for this component of the electric field, using results from Part II of this series.

Section IV is devoted to a discussion of the dispersion relation. A criterion for stability is derived that places an upper limit on the quality factor Q of the resonant mode of the vacuum tank. If the beam is near a resonance with the rf cavity, this criterion can be expressed as an upper limit on the input impedance of the cavity. These criteria contain the total number of particles in the machine as well as the energy spread of the coasting beam. Numerical estimates using the parameters of two quite different accelerators are given in Section V and indicate that instabilities arising from excitation of vacuum tank modes in general will not be a serious problem. Instabilities induced by an rf cavity, on the other hand, may place significant upper limits on the input impedance of rf cavities used in beam stacking schemes.

II. THE DISPERSION RELATION

It will be convenient in what follows to introduce the action variable W, which is defined by

$$W = \int_{E_0}^{E} \frac{dE}{f(E)} \quad . \tag{2.1}$$

Here E is the energy of the particle and f the instantaneous circulation frequency of the particle. The variable W is canonically conjugate to the angle variable ϕ describing the particle's position in azimuth. In the absence of an applied radio-frequency voltage, the equations of motion are given by

$$\dot{W} = 2\pi e R\, E_\phi$$

and

$$\dot{\phi} = 2\pi f \, . \tag{2.2}$$

The effective azimuthal electric field is designated by E_ϕ.

We may denote the distribution function for particles in synchrotron phase space by $\Psi(W, \phi, t)$, and it can then be shown[2] that Ψ satisfies the one-dimensional Vlasov equation,

$$\frac{\partial \Psi}{\partial t} + \dot{\phi}\, \frac{\partial \Psi}{\partial \phi} + 2\pi e \quad R\, E_\phi \quad \frac{\partial \Psi}{\partial W} = 0, \tag{2.3}$$

in the well-justified approximation of ignoring collisions between particles. In Eq. (2.3), $\langle R\, E_\phi \rangle$ involves the longitudinal electric field averaged over the beam cross section. For the investigation described here, we may safely replace this average by the orbit radius times the electric field at the center of the beam.

Since the umperturbed coasting beam is assumed to be uniform in azimuth, the unperturbed distribution of particles in $W - \phi$ space may be described by a function $\psi_0(W)$. We shall consider an infinitesimal perturbation such that the total distribution function $\Psi(\phi, W, t)$ may be written as

$$\Psi(\phi, W, t) = \psi_0(W) + \psi_1\, (n, W, \omega)\, e^{i(n\phi - \omega t)} \quad . \tag{2.4}$$

Note that the perturbation does not affect the transverse distribution of particles. Linearizing Eq. (2.3) leads to

$$\psi_1(n, W, \omega) = -2\pi i e \left\langle RE_\phi \right\rangle \frac{\partial \psi_0 / \partial W}{(\omega - n\dot{\phi})} . \tag{2.5}$$

The electric field in Eq. (2.5) arises from the charge and current densities of the perturbation only. The particle density associated with the perturbation is

$$\delta N = N_n e^{i(n\phi - \omega t)} , \tag{2.6}$$

where

$$N_n = \int \psi_1 (n, W, \omega) \, dW, \tag{2.7}$$

and the associated azimuthal electric field at the beam center may be written as

$$\left\langle RE_\phi \right\rangle = \epsilon N_n e^{i(n\phi - \omega t)} . \tag{2.8}$$

The quantity ϵ thus defined will be investigated in the next section. If we insert Eq. (2.8) into Eq. (2.5) and use Eq. (2.7), the condition for a solution to the Vlasov equation becomes

$$1 = -2\pi i e \, \epsilon \int \frac{d\psi_0}{dW} \frac{dW}{(\omega - n\dot{\phi})} \tag{2.9}$$

The particular dependence of ψ_0 upon W is not important as long as $d\psi_0/dW$ has no discontinuities. A completely realistic distribution function would necessarily vanish for values of W corresponding to particles moving faster than the velocity of light. For convenience, however, we shall take a Lorentz (resonance) line shape for ψ_0 and set

$$\psi_0 (W) = \frac{N(\Delta W)}{2\pi^2 \left[(W - W_0)^2 + (\Delta W)^2 \right]} . \tag{2.10}$$

This function falls off as W^{-2} for large values of W, and this behavior should not appreciably affect the results of the calculation. In Eq. (2.10), N is the total number of particles circulating in the machine. The distribution is centered about the value $W = W_0$ and has a characteristic width ΔW The integral in Eq. (2.9) may be evaluated by integration along the W axis, assuming ω has a small positive imaginary part,[2] and employing the relation

$$\dot{\phi}\ (W) = \dot{\phi}\ (W_0) + 2\pi f \left(\frac{df}{dE}\right) (W - W_0).$$

One obtains

$$\frac{d\psi_0}{dW} \frac{dW}{(\omega - n\dot{\phi})} = \frac{-nkN}{2\pi} \frac{1}{(\omega - n\omega_0 + i\tau)^2}, \qquad (2.11)$$

in which $k = 2\pi f\, df/dE$, ω_0 is the central frequency of the beam (equal to $\dot{\phi}\ (W_0)$), and $\tau = n|k|\Delta W$ is n/2 times the characteristic frequency spread of the distribution.

Having evaluated this integral, we have reduced Eq. (2.9) to the form

$$1 = i\, e\, k\, N\, n\, \epsilon(\omega - n\omega_0 + i\tau)^{-2}. \qquad (2.12)$$

The next section is devoted to a discussion of the quantity ϵ.

III. AZIMUTHAL ELECTRIC FIELD

For resonances with the accelerator tank we may use directly the result of II Eqs. (3.4), (3.5), (3.9) and (3.10) to obtain

$$\epsilon = B_t \frac{i\,\omega^2}{(\omega_i^2 - \omega^2) - i\,(\omega_i^2/Q_{TE})} , \tag{3.1}$$

with

$$B_t = \frac{2ec\,R^2 \mathcal{J}\left[\frac{dZ}{dr}\right]_B^2}{n\,q^2 h \int_a^b r\,Z^2\,dr} \tag{3.2}$$

for a resonance in the first possible mode. The notation is that of Part II of this series.

Although the contribution to the azimuthal electric field from the resonant mode is the major contribution, other contributions also arise from current and charge distributions that vary as exp ($in\phi$). These additional contributions may be attributed to the excitation of modes characterized by the same value of n, but having more than one wavelength in the r and z directions. A more general treatment of this problem, including the excitation of nonresonant modes, shows that the nonresonant contributions to the electric field have little effect on the results of the dispersion analysis. We shall therefore use the expression for ϵ of Eq. (3.1) in the dispersion relation.

For resonance with an rf cavity, we may proceed from Eq. (2.9) of I Part I of this series,[4] and write the effective azimuthal electric field as

$$E_\phi = -\frac{e\,\omega_r\,Z_c\,N_n\,e^{i(n\phi - \omega t)}}{2\pi R n} \tag{3.3}$$

which is a valid expression if the perturbation frequency is exactly equal to the resonant frequency of the cavity. If the cavity is being driven slightly off resonance, we may write

$$\left\langle r\, E_{\phi} \right\rangle = \frac{i\, e\, \omega_r\, \omega^2 \,(Z_c/Q_c)\; N_n\, e^{\,i(n\phi - \omega t)}}{2\pi n\,(\omega_r^{\,2} - \omega^2 - i\,\omega_r^{\,2}/Q_c)} \quad .$$

It follows immediately that

$$\epsilon = B_c \; \frac{i\,\omega^2}{\omega_r^{\,2} - \omega^2 - i\,\omega_r^{\,2}/Q_c} \tag{3.4}$$

with

$$B_c = \frac{e\, Z_c\, \omega_r}{2\pi n\, Q_c} \quad . \tag{3.5}$$

IV. CRITERION FOR STABILITY

Either for resonance with a tank mode or resonance with the rf cavity, we may write the dispersion relation, Eq. (2.12), in the form

$$-1 = e\,k\,N\,n\,B\,\omega^2 \left[\omega_r^2 - \omega^2 - \frac{i\,\omega_r^2}{Q}\right]^{-1} (\omega - n\,\omega_0 + i\,\tau)^{-2} \,, \qquad (4.1)$$

with B given by Eqs. (3.2) or (3.5). One can see that the fourth-order Eq. (4.1) has various roots corresponding to possible instabilities. One root is always stable ($\omega \sim -\omega_r$), two correspond to the longitudinal instability of a coasting beam treated previously, and the root in which $\omega \sim \omega_r$ corresponds to the possible instability associated with the electromagnetic mode with eigenfrequency ω_r. Setting $\omega = \omega_r + \nu$, we solve for ν by linearizing the dispersion relation in ν. The imaginary part of ν is then obtained as a function of $\omega_r - n\,\omega_0$. The criterion for stability is $\mathrm{Im}\,\nu < 0$, and since the $\mathrm{Im}\,\nu$ is largest for $\omega_r - n\,\omega_0 \approx \pm\,\tau$, we make this substitution to obtain

$$\nu = \frac{A\,\omega_r \pm 2\,\omega_r\,\tau^2/Q}{2\left(-\frac{\omega_r}{Q}\tau - A\right) \pm i\left(2\tau + \frac{\omega_r}{Q}\right)\tau} \,, \qquad (4.2)$$

where $A = e\,k\,N\,n\,B$. The plus or minus signs refer to the choice of $\omega_r - n\,\omega_0 = \pm\,\tau$.

Observing that $Q >> 1$, we have as a criterion for stability

$$\left[-A\,\omega_r \mp \frac{2\omega_r}{Q}\,\tau^2\right]\left[\left(\frac{\tau\,\omega_r}{Q} + A\right) \pm \left(2\,\tau + \frac{\omega_r}{Q}\right)\tau\,i\right] < 0 \qquad (4.3)$$

or

$$\pm\left(-A \pm \frac{2\tau^2}{Q}\right) < 0. \qquad (4.4)$$

By appropriate choice of the sign, depending upon whether k is positive or negative (corresponding to the beam's being below or above the transition energy), we obtain as the most stringent requirement for stability:[5]

$$Q < \frac{2\,\tau^2}{e\,|k|\,N\,n\,B} \quad . \tag{4.5}$$

We now may use the definitions of k and τ which, after substitution of B for the case of a tank resonance yields

$$Q_t < 1/2 \;\; \frac{n^4 \left|\frac{df}{dE}\right| \,(\Delta E)^2\, h}{N \quad f \;(m_0c^2)\; r_0 R^4} \left[\frac{\int_a^b r \quad Z^2 dr}{\left(dZ/dr\right)_B^2}\right] , \tag{4.6}$$

where r_0, the classical electron radius, is 2.8×10^{-13} cm.

For $w << R$ this may be written as

$$Q_t < \frac{1}{16} \;\; \frac{n^4 \left|\frac{df}{dE}\right| \,(\Delta E)^2\, h\, w^3}{N \quad f \quad (m_0C^2) r_0 \;\; R^3} \left[\frac{\int_{-1}^{1} \quad Z^2\, dx}{dZ/dx_B^2}\right] , \tag{4.7}$$

where a few illustrative values for the last bracket can be found in Part II, while tables of values are in Ref. 8 of Part II of this series.

For resonance with an rf cavity, we obtain as the condition of stability from Eq. (4.5) that the shunt impedance Z_c must satisfy

$$Z_c < \frac{n\,c\,\mathcal{Z}\left|\frac{d\,f}{d\,E}\right| \quad (\Delta\;E)^2}{(m_0c^2) r_0 N f^2} \quad . \tag{4.8}$$

The quantity $\mathcal{Z}$ is the impedance of free space, which is equal to 377 ohms in mks units, and $4\pi/c$ in cgs units.

V. NUMERICAL EXAMPLES

A. RF Cavity Resonances

As an example of a resonance with an rf cavity, we take the MURA 40-Mev electron model:

$$f \frac{df}{dE} = 1.1 \times 10^{12} \text{ Mev}^{-1} \text{ sec}^{-2} ,$$

$$f = 25 \times 10^{6} \text{ sec}^{-1},$$

$$n = 1$$

$$\Delta E = 3 \text{ Mev},$$

$$N = 1.5 \times 10^{13} .$$

This requires that the shunt impedance Z_c must be less than 3200 ohms, to prevent a longitudinal instability. It is sufficiently high to ensure no difficulty.

As a second example, we might consider a hypothetical proton storage ring for 15-Bev particles. As reasonable parameters, we take

$$\frac{df}{dE} = 0.70 \text{ Mev}^{-1} \text{ sec}^{-1} ,$$

$$f = 10^{6} \text{ sec}^{-1},$$

$$n = 10,$$

$$\Delta E = 300 \text{ Mev},$$

$$N = 10^{14}.$$

In this case the shunt impedance of an rf "maintaining cavity" must be less than 5.1×10^5 ohms, which would preclude the use of a very-high-Q cavity such as otherwise might have been used in such a device. For example, the cavities at the Cambridge electron accelerator have shunt impedances of 10^7 ohms.[6]

B. Tank Resonances

As a first example, we consider the MURA electron model in which the vacuum tank has a height of 5 cm, inner radius of 122 cm, and outer radius of 224 cm. The 38-Mev beam will be stacked at a radius of 203 cm. From Eq. (A-18) of Part II, the estimated n value for the first resonance is approximately 200, but the coupling factor $Z^2 dx \Big/ \left(\frac{dZ}{dx}\right)^2_B$ is so small that the restriction on Q_t is satisfied by a vacuum tank made of even the best conducting material imaginable.

As a second example, we consider a full-scale FFAG accelerator for which the following parameters might be typical:

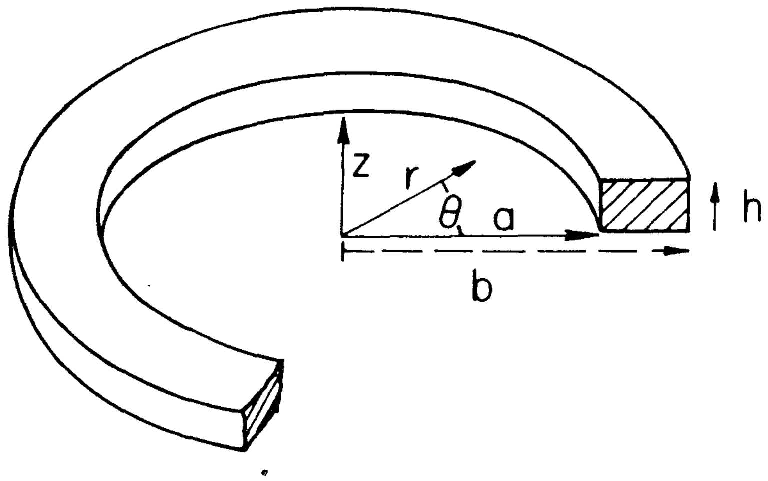
a	$= 7 \times 10^3$ cm,	f	$= 10^6$ sec^{-1}
b	= 7300 cm,	$\frac{df}{dE}$	= 0.70 Mev^{-1} sec^{-1}
R_B	= 7275 cm,	E	= 15 Mev,
h	= 15 cm,	ΔE	= 300 Mev,
		N	$= 10^{14}$.

The first resonance is at n = 30,000, and once again there need be no concern about a longitudinal instability for any physically realizable cavity.

REFERENCES

1. V.K. Neil, D.L. Judd, and L.J. Laslett, to be published.
2. A.A. Kolomensky and A.N. Lebedev, Proceedings of the CERN Symposium on High Energy Accelerators, Geneva (CERN Geneva 1959), p. 115.
3. C.E. Nielsen, A.M. Sessler, and K.R. Symon, Proceedings of the CERN Symposium on High Energy Accelerators, Geneva, 1959 (CERN, Geneva, 1959), p. 239.
4. V.K. Neil and A.M. Sessler, to be published.
5. If n is sufficiently large that resonances occur with higher-order electromagnetic modes, the coupling factors which enter in the coefficient B_t of Eq. (3.2), and which appear in expressions for evaluation of Q_{TE}, may be modified materially. From the WKB form of the function Z(r), however, estimates of the relevant factors can be obtained which suggest that the factor n^4 in Eqs. (4.6)and (4.7) will increase rapidly enough to ensure that no more stringent limitations on particle number or wall conductivity will result from the presence of such higher-order modes. With high-order resonances present, of course, more than one resonance can occur within a sufficiently small frequency interval that the coupling with the beam is enhanced, and a somewhat stronger limitation can result. But if n were high enough so that many resonances would fall within the range where interaction with the beam occurs, variation of phase amongst the several modes excited by the beam would appear to suppress the reactive feature of the coupling which permits instabilities to develop.
6. M.S. Livingston, Proceedings of the CERN Symposium on High Energy Accelerators, Geneva, 1959 (CERN, Geneva, 1959), p. 335.

Fig. 1. Cutaway view of toroidal cavity.



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