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Proof that stable monotonic equilibrium distributions in a continuous focusing channel are necessarily axisymmetric

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The transverse Vlasov equilibrium distribution function of an unbunched ion beam propagating in a continuous focusing channel is specified by a function $f_\perp(H_\perp)$, where $H_\perp$ is the single-particle Hamiltonian. In standard treatments of continuous focusing equilibria in Vlasov-Poisson electrostatic models, it is assumed that a stable beam equilibrium specified by monotonic $f_\perp(H_\perp)$ with $\partial f_\perp(H_\perp)/\partial H_\perp \leq 0$ is axisymmetric (no variation in azimuthal angle, i.e., with $\partial/\partial\theta = 0$). In this paper a simple, but rigorous, proof is presented that *only* axisymmetric equilibrium solutions are possible in Vlasov-Poisson models for any physical choice of $f_\perp(H_\perp)$ with $\partial f_\perp(H_\perp)/\partial H_\perp \leq 0$ if the confining boundary of the system (the beam pipe) is axisymmetric or if the geometry is radially unbounded.

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The continuous-focusing model has been extensively studied by Davidson [1] and Reiser[2], and extensive reviews can be found in US Particle Accelerator School courses[3]. Although the model can only be regarded as a highly idealized representation of more realistic periodic focusing lattices, it is nevertheless useful to illustrate basic physics and scaling properties. The proof we present that a stable, continuous focusing equilibrium distribution formed from a monotonic decreasing function of the Hamiltonian can only produce axisymmetric beams improves the rigorous understanding of equilibrium properties in the continuous focusing model. The proof parallels an analysis of a similar equilibrium equation used to model a strongly magnetized, pure electron plasma described by $\mathbf{E}\times\mathbf{B}$ flow which is confined in a Penning trap[4].

We consider an infinitely long, unbunched ($\partial/\partial z = 0$) beam of ions of charge q and rest mass m. All particles propagate with axial velocity $\beta_b c = \text{const}$. Here, c is the speed of light *in vacuo*. The beam phase-space is described as a function of the axial coordinate s in terms of the transverse spatial coordinates $\mathbf{x}_\perp$ of the particles and the angles $\mathbf{x}'_\perp$ that the particles make with the axis of the system. We adapt a Vlasov description, where the beam is modeled by a continuous, single-particle distribution function $f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$. Within the paraxial approximation, $f_\perp$ evolves as an incompressible fluid in 4D transverse phase-space according to the nonlinear Vlasov equation[1–3]

$$\left\{\frac{\partial}{\partial s} + \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp}\cdot\frac{\partial}{\partial \mathbf{x}_\perp} - \frac{\partial H_\perp}{\partial \mathbf{x}_\perp}\cdot\frac{\partial}{\partial \mathbf{x}'_\perp}\right\} f_\perp = 0. \tag{1}$$

Here,

$$H_\perp = \frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp + \frac{q}{m\gamma_b^3\beta_b^2 c^2}\phi \tag{2}$$

is the single-particle Hamiltonian, $k^2_{\beta 0} = \text{const} > 0$ is the continuous, applied focusing force of the channel, $\gamma_b = 1/\sqrt{1-\beta_b^2} = \text{const}$ is the relativistic gamma factor, and $\phi(\mathbf{x}_\perp, s)$ is the self-field potential generated by the beam space-charge. The potential ϕ satisfies the transverse Poisson equation

$$\nabla^2_\perp \phi = -\frac{q}{\epsilon_0}\int d^2x'_\perp\, f_\perp, \tag{3}$$

with ϕ subject to the appropriate boundary conditions on the transverse machine aperture. Here, ϵ_0 is the permittivity of free-space.

The Vlasov-Poisson system given by Eqs. (1)–(3) model the transverse beam evolution in the continuum approximation. The system is solved as an initial value problem where $f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$ is specified at some initial value of $s = s_i$. The transverse particle Hamiltonian $H_\perp$ given by Eq. (2) is a single-particle constant of the motion with $H_\perp = \text{const}$. Therefore, any function $f_\perp = f_\perp(H_\perp)$ satisfying $f_\perp \geq 0$ at $s = s_i$ will form a valid stationary continuous-focusing equilibrium solution to the Vlasov-Poisson system. Functional bounds can be applied to show that the monotonicity

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condition $\partial f_\perp(H_\perp)/\partial H_\perp \leq 0$ is a *sufficient* condition for stability of the continuous-focusing equilibrium to both small- and large-amplitude perturbations[1, 5–7].

It is convenient to define an effective potential[1–3]

$$\psi \equiv \frac{1}{2}k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}. \tag{4}$$

Then,

$$H_\perp = \frac{1}{2}\mathbf{x}_\perp'^2 + \psi \tag{5}$$

and, without loss in generality, the beam density can be calculated as

$$n = \int d^2x'_\perp \; f_\perp(H_\perp) \; = 2\pi \int_\psi^\infty dH_\perp \; f_\perp(H_\perp), \tag{6}$$

to recast the Poisson equation (3) of the equilibrium as

$$\nabla_\perp^2 \psi = 2k_{\beta 0}^2 - \frac{2\pi q^2}{m\epsilon_0\gamma_b^3\beta_b^2c^2}\int_\psi^\infty dH_\perp \; f_\perp(H_\perp). \tag{7}$$

If the system is confined in a cylindrical, conducting pipe of radius $r = \sqrt{x^2+y^2} = r_p$ held at potential $\phi = V = \text{const}$, the boundary condition on on ψ is

$$\psi(r = r_p) = \frac{1}{2}k_{\beta 0}^2 r_p^2 + \frac{qV}{m\gamma_b^3\beta_b^2c^2}. \tag{8}$$

For the special case of an equilibrium with a finite radial extent and line-charge $\lambda = q\int d^2x_\perp \; n$ in a radially unbounded system (i.e., free space), the boundary condition (8) is replaced by the requirement that

$$\left.\frac{\partial\psi}{\partial r}\right|_{r\gg r_b} = k_{\beta 0}^2 r - \frac{q\lambda}{2\pi\epsilon_0\gamma_b^3\beta_b^2c^2}\frac{1}{r},$$

where $r = r_b$ is the characteristic transverse radius of the equilibrium beam.

Equation (7) is highly nonlinear and explicit solution for ψ must generally be numerically constructed for a specific choice of equilibrium function $f_\perp(H_\perp)$. Specific examples of solutions are analyzed in detail in Ref. [8]. In spite of these general difficulties, it is possible to show that Eqs. (7) and (8) admit only axisymmetric $[\partial/\partial\theta = 0$ where $\theta = \tan^{-1}(y,x)]$ solutions for any physical choice of equilibrium function $f_\perp(H_\perp)$. Paralleling Smith et. al[4], we assume that a nonaxisymmetric $(\partial/\partial\theta \neq 0)$ solution $\psi = \psi_1$ exists to Eqs. (7) and (8). Because the boundary condition (8) is invariant under the rotation, another solution $\psi = \psi_2$ can be generated by actively rotating ψ_1 through any angle where the solution does not map back onto itself by symmetry. We define a positive definite functional

$$F \equiv \int_{\text{pipe}} d^2x_\perp \; \left|\frac{\partial}{\partial\mathbf{x}_\perp}(\psi_1 - \psi_2)\right|^2 > 0. \tag{9}$$

Integrating by parts, and applying the Divergence theorem with the boundary condition (8), and then the Poisson equation (7) obtains

$$\begin{aligned} F &= -\int_{\text{pipe}} d^2x_\perp \; (\psi_1 - \psi_2)\nabla_\perp^2(\psi_1 - \psi_2) \\ &= \frac{2\pi q^2}{m\epsilon_0\gamma_b^3\beta_b^2c^2}\int_{\text{pipe}} d^2x_\perp \; (\psi_1 - \psi_2)[G(\psi_1) - G(\psi_2)], \end{aligned} \tag{10}$$

where

$$G(\psi) \equiv \int_\psi^\infty dH_\perp \; f_\perp(H_\perp).$$

If $f_\perp$ is a monotonic decreasing function of $H_\perp$ with $\partial f_\perp(H_\perp)/\partial H_\perp \leq 0$, then G must also be a monotonic decreasing function of ψ. Thus, wherever $\psi_1 \geq \psi_2$, $G(\psi_1) \leq G(\psi_2)$, and wherever $\psi_1 \leq \psi_2$, $G(\psi_1) \geq G(\psi_2)$. Consequently, the integrand in Eq. (10) satisfies $(\psi_1 - \psi_2)[G(\psi_1) - G(\psi_2)] \leq 0$ for all $\mathbf{x}_\perp$ in the pipe, which contradicts the requirement that $F > 0$. Therefore, the assumption that a nonaxisymmetric solution exists is invalid and any solution to Eqs. (7) and (8) when $\partial f_\perp(H_\perp)/\partial H_\perp \leq 0$ is necessarily axisymmetric.

This proof is easily modified to cover the case of a beam with finite radial extent in radially unbounded space (i.e., free space). The fact that system axisymmetry and stability are connected in simple, continuous focusing systems is not surprising because the $H_\perp$ depends only on $|\mathbf{x}'_\perp|$ and the spatial $\mathbf{x}_\perp$ and angle $\mathbf{x}'_\perp$ degrees of freedom are strongly connected in an equilibrium. If either: the beam pipe is replaced by a nonaxisymmetric conducting pipe, the radial focusing force represented by $k^2_{\beta 0}$ is replaced by focusing forces differing in two orthogonal directions, or if $f_\perp(H_\perp)$ is a non-monotonic function; then no symmetry restrictions can be immediately obtained from the method presented.

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