SLAC-PUB-13130
2/2008

A simple low-frequency BPM

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Abstract

Detection of the field induced by a beam outside of the beam pipe can be used for the beam diagnostics. A wire in a longitudinal slot in the beam pipe wall can be used as a beam position monitor with a very small coupling impedance avoiding complications of the feed-through. The signal can be reasonably high at low frequencies. We calculate the beam-coupling impedance due to a long longitudinal slot and the signal induced in a wire placed in such a slot and shielded by a thin screen from the beam. Results can be relevant for impedance calculations of the slot to ante-chamber and slots of the DIPs.

PACS numbers: 29.27.Bd, 29-20-Dh, 41-60, 52-59-f

1 Introduction

The electro-magnetic (EM) field induced by a beam outside of a thin beam pipe may be quite noticeable. The analytical solution for electromagnetic fields in a round beam pipe can be find elsewhere [1]. Fig. (1) shows the time profile of the field induced by a bunch on the inner side (pancake thin red line) and on the outer side (blue line) of a stainless steel tube. The bunch length is 10 mm, the tube radius is 5 mm, the wall thickness is 0.1 mm. It can be seen that the field amplitude outside of the pipe decreases only by 100 times.

†Work supported by Department of Energy contract DE-AC05-96OR22464

*Work supported by US Department of Energy contract DE-AC02-76SF00515

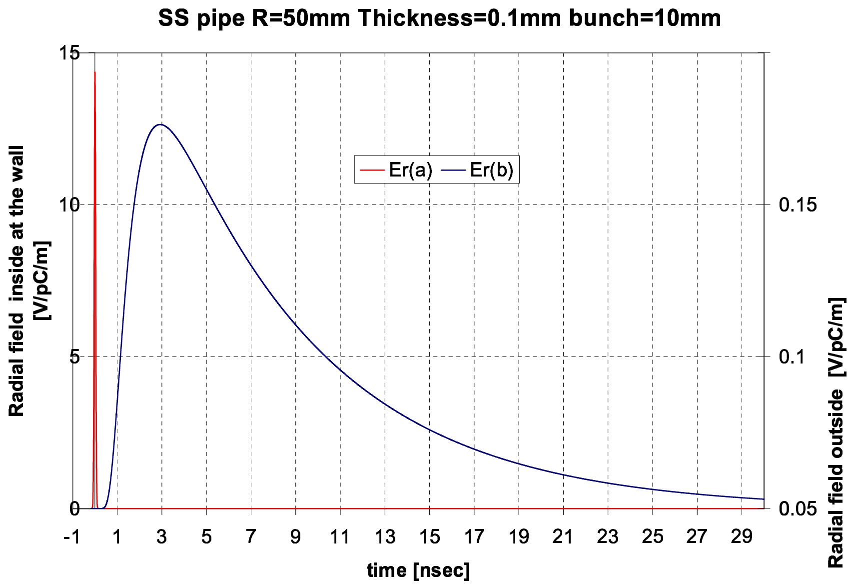

Figure 1: Field on the inner (ed line) and outer (blue line) sides of a beam pipe.

Another example is given in Fig. (2) for an aluminum chamber of radius 2.5 mm with a tube thickness of 0.5 mm (parameters of the LCLS [2], round chamber). The signal outside the pipe in this case may reach amplitude of 35 V/m for 1 nC bunch.

In both examples, the main contribution to the signal is given by the low frequencies modes which can penetrate through the wall. Such frequencies for short bunches are much lower than the width of the bunch spectrum. Therefore, the signal is practically independent on the bunch length what simplifies design of the BPM electronics. Another common feature of both results is the time delay between the signals on the inner and outer sides defined by the diffusion time of the magnetic field through the wall (about 3 ns in Fig. (1) and 200 ns in in Fig. (2)).

The field of a bunch outside of the beam pipe can be detected and used to build a a beam position monitor (BPM) without any feed-through preserving the smooth beam pipe wall seen by the beam. An idea of a BPM based on detection of the EM field behind a thin foil was suggested long ago [3]. Based on this approach, a low impedance BPM was proposed and tested by one of the authors (A.A.) for the VEPP-5 collider, a B-factory project considered to be built in Novosibirsk [4]. To prove feasibility of the approach an experimental model was built as shown in Fig. (3). The experimental signal measured outside of the beam pipe with the 15 mm inner radius [4] is shown in Figure (4). At that time, the full solution for EM fields was not obtained but simple estimates were used to derive the signal amplitude and duration. Dipole mode of the beam EM field was simulated by a short pulse propagating in a two wire transmission line. The transmission line was inserted into an aluminum pipe with central part of the pipe replaced with a

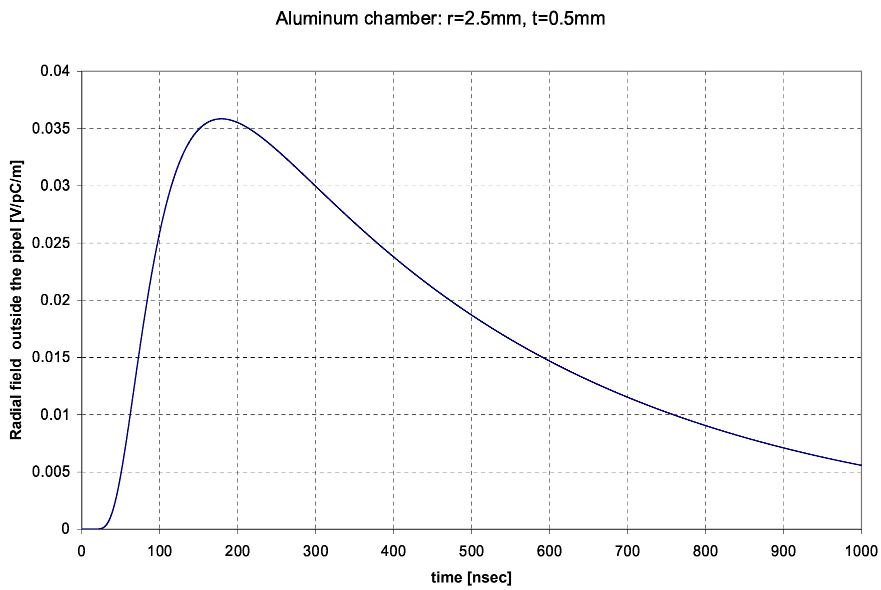

Figure 2: Field outside of the Al 0.5 mm round beam pipe.

50 μm thick stainless steel foil. Magnetic field penetrating through the foil was measured using 12 turns coil with 2×2 cm cross-section. Oscilloscope snapshot of the current pulse in the transmission line and the signal measured by the coil are shown in Fig.(4) (a) and (b), respectively [4]. Measured signal amplitude and duration were in good agreement with expected values.

Such kind of a BPM may be used in free-electron lasers like LCLS where the wall thickness can be as small as 0.5 mm [2]. In general, a BPM can be made as a loop of wire set into a thin longitudinal groove (or several grooves) in the outer side of the beam-pipe wall, see Fig. (5). For simplicity, we consider a round beam pipe denoting the inner radius a, the thickness of a screen Δ, and the wall conductivity σ_w.

2 EM fields in a beam pipe with a slot

Let us start calculations of the EM fields in a pipe with a slot from the Maxwell equations for a particle moving in a round beam pipe along the z axes with the offset r_0 and velocity v. Assuming dependence on time in the form $e^{-i\omega t}$, equations for the ω-frequency components of EM fields generated by the particle are

$$\nabla \times \mathrm{E} = \frac{i\omega}{c} B, \quad \nabla \times B = \frac{4\pi}{c}(j^b + \sigma \mathrm{E}) - \frac{i\omega}{c}\mathrm{E}, \tag{1}$$

where σ is the wall conductivity considered as a constant over ω, ρ_b is the particle density, and j_b is the current. The second equation can be written introducing $D = \epsilon E$

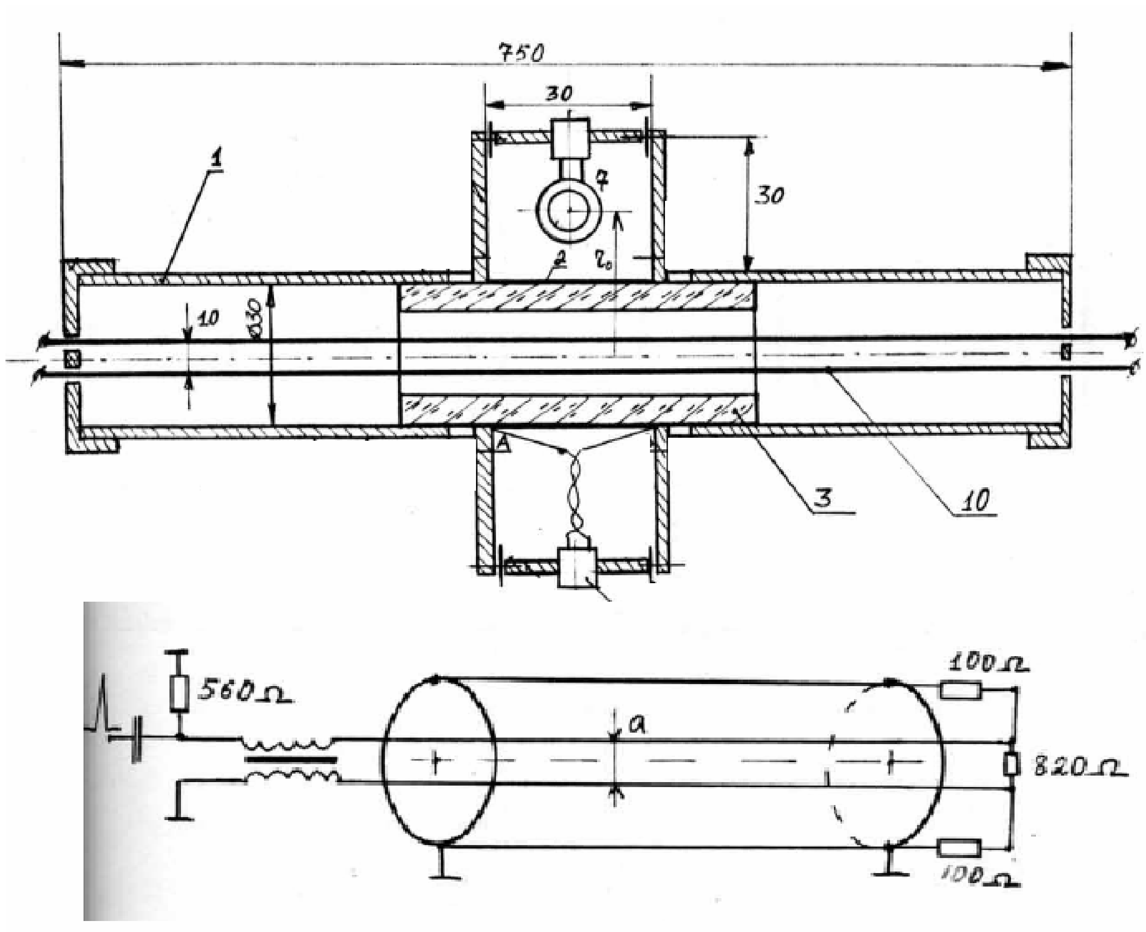

Figure 3: Schematic view of the experimental set-up and electrical diagram of the transmission line. 1- aluminum pipe, 2 - 50um stainless steel foil, 3 - Plexiglas foil support, 7 - measuring coil, 10 - two wire transmission line.

in the form

$$\nabla \times B = \frac{4\pi}{c} j^b - \frac{i\omega}{c} D,$$
$$\epsilon = 1 + i\frac{4\pi\sigma}{\omega}. \tag{2}$$

The first of Eqs. (1) gives $divB = 0$, and from Eq. (2) and the continuity equation

$$-i\omega\rho^b + divj_b = 0, \tag{3}$$

it follows that $divD = 4\pi\rho^b$.

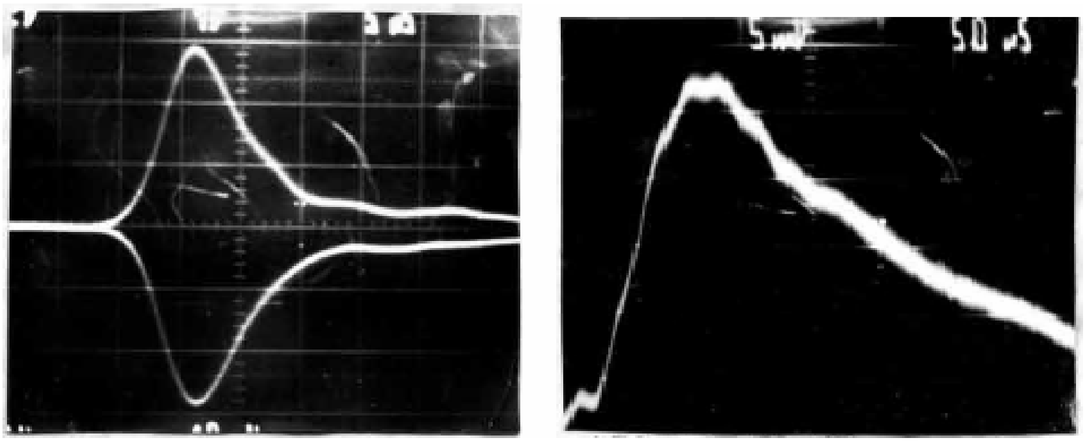

Figure 4: (a) Left pane: the signal from the transmission line terminating resistors, 5ns/div horizontal scale, and 20V/div vertical scale. (b) Right pane: the signal from the measuring coil, 50 ns/div horizontal scale, and 20 mV/div vertical scale.

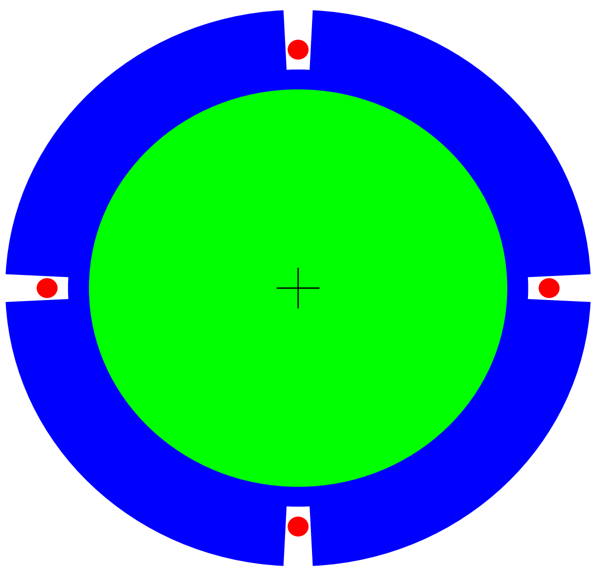

Figure 5: Sketch of the BPM design.

In cylindrical coordinates with the polar axes along the beam pipe axes, the current $j_b = \hat{z} v \rho^b$, and

$$\rho^b = \frac{e}{vr}\,\delta(r - r_0)\,\delta(\phi)\,e^{i\omega z/v}. \tag{4}$$

The wave equation follows from $\nabla \times \nabla \times B = \nabla(\nabla . B) - \triangle B$ and Maxwell equations.

In the regions of a constant ϵ,

$$\triangle B + (\frac{\omega}{c})^2 \, \epsilon \, B = -\frac{4\pi}{c} \, \nabla \times j^b,$$
$$(\nabla \times j^b)_r = \frac{v}{r}(\frac{\partial \rho}{\partial \phi}), \quad (\nabla \times j^b)_\phi = -v(\frac{\partial \rho}{\partial r}), \quad (\nabla \times j^b)_z = 0. \tag{5}$$

At the boundaries, where ϵ changes its value, tangential components of E and B have to be continuous. Then the normal components of D and B are continuous automatically.

Let us expand $B(r, \phi, z)$ over the azimuthal harmonics and assume dependence on z in the form $e^{i\omega z/v}$,

$$B(r, \phi, z) = e^{i\omega z/v} \sum_{m=-\infty}^{\infty} [\hat{r} B^r_m(r) + \hat{\phi} B^\phi_m(r) + \hat{z} B^z_m(r)] \, e^{im\phi}, \tag{6}$$

where $\hat{r}(\phi), \hat{\phi}(\phi)$ and $\hat{z}$ are unit vectors. Eq. (5) written for the components $B^\pm_m(r) = B^r_m \pm i B^\phi_m$ and B^z_m takes the form

$$[\frac{1}{r}\frac{\partial}{\partial r} r(\frac{\partial}{\partial r}) - (\frac{m \pm 1}{r})^2 - (\frac{\omega}{v})^2 (1 - \frac{v^2}{c^2} \, \epsilon)] B^\pm_m(r) = -\frac{4\pi i v}{c} \, (\frac{m \rho_m(r)}{r} \mp \frac{\partial \rho_m(r)}{\partial r}),$$
$$[\frac{1}{r}\frac{\partial}{\partial r} r(\frac{\partial}{\partial r}) - (\frac{m}{r})^2 - (\frac{\omega}{v})^2 (1 - \frac{v^2}{c^2} \, \epsilon)] B^z_m(r) = 0. \tag{7}$$

where

$$\rho_m(r) = \frac{e}{2\pi v r} \, \delta(r - r_0). \tag{8}$$

Inside of the beam pipe, $\epsilon = 1$. In the ultra-relativistic case, Eq. (7) are simplified to

$$[\frac{1}{r}\frac{\partial}{\partial r} r(\frac{\partial}{\partial r}) - (\frac{m \pm 1}{r})^2] B^\pm_m(r) = -4\pi i \, (\frac{m \rho_m(r)}{r} \mp \frac{\partial \rho_m(r)}{\partial r}),$$
$$[\frac{1}{r}\frac{\partial}{\partial r} r(\frac{\partial}{\partial r}) - (\frac{m}{r})^2] B^z_m(r) = 0. \tag{9}$$

where $k = \omega/c$.

In the region $r < r_0$ solution does not contain the singular term ar $r \to 0$ and has to be matched with the solution in the region $r_0 < r < a$. Conditions for matching at $r = r_0$ are defined by the right-hand-side (RHS) of Eq. (9),

$$B^{\pm,>}_m(r_0) - B^{\pm,<}_m(r_0) = \pm \frac{ieZ_0}{2\pi r_0}, \quad [\frac{\partial B^{\pm,>}_m(r)}{\partial r} - \frac{\partial B^{\pm,<}_m(r}{\partial r}]_{r=r_0} = -\frac{ieZ_0(m \pm 1)}{2\pi r_0^2}. \tag{10}$$

Here $Z_0 = 4\pi/c = 120\pi$ Ohm.
The fields at $r < r_0$ are:

$$\begin{aligned} B_m^{+,<}(r) &= c_m^+ \, (\frac{r}{r_0})^{m+1}, \\ B_m^{-,<}(r) &= (\frac{ieZ_0}{2\pi r_0} + c_m^-) \, (\frac{r}{r_0})^{m-1}, \\ B_m^{z,<}(r) &= i\frac{m+1}{kr_0} \, c_m^+ \, (\frac{r}{r_0})^m. \end{aligned} \tag{11}$$

$$\begin{aligned} E_m^{r,<}(r) &= -\frac{i}{2k^2 r_0^2} \, (\frac{r}{r_0})^{m-1} \, \{c_m^+ \, (2m + 2m^2 + k^2 r^2) - c_m^- \, k^2 r_0^2 - ieZ_0 k^2 r_0\}, \\ E_m^{\phi,<}(r) &= \frac{1}{2k^2 r_0^2} \, (\frac{r}{r_0})^{m-1} \, \{c_m^+ \, (2m + 2m^2 - k^2 r^2) - c_m^- \, k^2 r_0^2 - ieZ_0 k^2 r_0\}, \\ E_m^{z,<}(r) &= \frac{m+1}{kr_0} (\frac{r}{r_0})^m c_m^+. \end{aligned} \tag{12}$$

The fields at $r_0 < r < a$ are:

$$\begin{aligned} B_m^{+,>}(r) &= c_m^+ \, (\frac{r}{r_0})^{m+1} + \frac{iZ_0}{2\pi r_0} \, (\frac{r_0}{r})^{m+1}, \\ B_m^{-,>}(r) &= c_m^- \, (\frac{r}{r_0})^{m-1}, \\ B_m^{z,>}(r) &= i\frac{m+1}{kr_0} \, c_m^+ \, (\frac{r}{r_0})^m. \end{aligned} \tag{13}$$

$$\begin{aligned} E_m^{r,>}(r) &= \frac{i}{2k^2 r r_0} \, \{-c_m^+ \, (2m + 2m^2 + k^2 r^2) \, (\frac{r}{r_0})^m + k^2 r_0^2 \, (c_m^- \, (\frac{r}{r_0})^m - \frac{ieZ_0}{2\pi r_0} (\frac{r_0}{r})^m)\}, \\ E_m^{\phi,>}(r) &= \frac{1}{2k^2 r r_0} \, \{c_m^+ \, (2m + 2m^2 - k^2 r^2) \, (\frac{r}{r_0})^m - k^2 r_0^2 \, (c_m^- \, (\frac{r}{r_0})^m + \frac{ieZ_0}{2\pi r_0} (\frac{r_0}{r})^m)\}, \\ E_m^{z,>}(r) &= \frac{m+1}{kr_0} \, (\frac{r}{r_0})^m \, c_m^+. \end{aligned} \tag{14}$$

Inside of the beam-pipe wall, $r > a$, Eq. (7) in the ultra-relativistic case takes the form

$$[\frac{1}{r}\frac{\partial}{\partial r} r (\frac{\partial}{\partial r}) - (\frac{m \pm 1}{r})^2 + k_w^2] B_m^{\pm}(r) = R_m^{\pm}(r). \tag{15}$$

Here $k_w^2 = (\frac{\omega}{c})^2 \, (\epsilon - 1)$,

$$k_w^2 = i\frac{4\pi\sigma}{\omega}(\frac{\omega}{c})^2, \quad k_w = \frac{1+i}{\delta_\omega}, \tag{16}$$

where $\delta_\omega = c/\sqrt{2\pi\sigma\omega}$ is the skin depth.

The RHS $R_m^{\pm}(r)$ is equal to zero in the metal. In the slots, $r > a + \Delta$, $|\phi| < \alpha/2$, the RHS is

$$\begin{aligned} R_m^{\pm,z}(r) &= k_w^2 \sum_n \int_{-\alpha/2}^{\alpha/2} \frac{d\phi}{2\pi} B_n^{\pm,z}(r) e^{i(n-m)\phi} \\ &= \frac{\alpha k_w^2}{2\pi} \sum_{n=-\infty}^{\infty} s(n-m) B_n^{\pm,z}(r) \end{aligned} \tag{17}$$

where α is the angular slot width, and

$$s(n-m) = \frac{\sin[(n-m)\alpha/2]}{(n-m)\alpha/2}. \tag{18}$$

Solution of the homogeneous Eq. (15) at $r > a$ is given in terms of the Bessel functions

$$\begin{aligned} B_m^{0,\pm}(r) &= \alpha_m^{\pm} H_{m\pm1}^{(1)}(k_w r) + \beta_m^{\pm} H_{m\pm1}^{(2)}(k_w r), \\ B_m^{0,z}(r) &= -\frac{ik_w}{2k}(\alpha_m^+ - \alpha_m^-) H_m^{(1)}(k_w r) - \frac{ik_w}{2k}(\beta_m^+ - \beta_m^-) H_m^{(2)}(k_w r). \end{aligned} \tag{19}$$

This solution is valid in the metal for $a < r < a+\Delta$, and the tangential components of the fields B and $E = (i/k\epsilon)\nabla \times B$ has to be matched with the solution inside of the beam pipe at the beam-pipe radius $r = a$. For frequencies for which the skin depth $\delta_\omega << a$, we can use the asymptotic of the Bessel functions,

$$\begin{aligned} B_m^{0,\pm}(r) &= \sqrt{\frac{2}{\pi k_w r}}\{\alpha_m^{\pm}\, e^{i[k_w r - \frac{\pi}{2}(m\pm1) - \frac{\pi}{4}]} + \beta_m^{\pm}\, e^{-i[k_w r - \frac{\pi}{2}(m\pm1) - \frac{\pi}{4}]}\}, \\ B_m^{0,z}(r) &= \frac{ik_w}{2k}\sqrt{\frac{2}{\pi k_w r}}\{(\alpha_m^+ - \alpha_m^-)\, e^{i[k_w r - \frac{\pi}{2}m - \frac{\pi}{4}]} + (\beta_m^+ - \beta_m^-)\, e^{-i[k_w r - \frac{\pi}{2}m - \frac{\pi}{4}]}\}. \end{aligned} \tag{20}$$

Solution of the inhomogeneous Eq. (16) in the wall for $r > a + \Delta$ can be obtained using the Green's function $G_m^{\pm}(r, r')$,

$$\begin{aligned} &B_m^{\pm}(r) = B_m^{0,\pm}(r) + \int_{a+\Delta}^{\infty} r' dr' G_m^{\pm}(r, r') R_m^{\pm}(r'), \\ &[\frac{1}{r}\frac{\partial}{\partial r} r(\frac{\partial}{\partial r}) - (\frac{m\pm1}{r})^2 + k_w^2] G_m^{\pm}(r, r') = \frac{\delta(r - r')}{r}. \end{aligned} \tag{21}$$

Explicitly,

$$G_m^{\pm}(r,r') = -i\frac{\pi}{4}\theta(r-r')\,[H_{m\pm1}^{(1)}(k_w r)H_{m\pm1}^{(2)}(k_w r') - H_{m\pm1}^{(1)}(k_w r')H_{m\pm1}^{(2)}(k_w r)]. \qquad (22)$$

Here $\theta(r-r')$ is the step function, $\theta(r-r') = 1$ for $r > r'$ and zero otherwise. Eq.(21) takes form of the integral equation,

$$B_m^{\pm}(r) = B_m^{0,\pm}(r) - i\frac{\alpha k_w^2}{8}\sum_{n=-\infty}^{\infty} s(n-m)\int_{a+\Delta}^{r} r'dr'[H_{m\pm1}^{(1)}(k_w r)H_{m\pm1}^{(2)}(k_w r') - H_{m\pm1}^{(1)}(k_w r')H_{m\pm1}^{(2)}(k_w r)]B_n^{\pm}(r'). \qquad (23)$$

Using asymptotic values for the Bessel functions and defining $b^{\pm}(r) = \sqrt{r}B_m^{\pm}(r)$, Eq.(23) for $r > a+\Delta$ takes the form

$$b_m^{\pm}(r) = \sqrt{r}B_m^{0,\pm}(r) + \frac{\alpha k_w}{2\pi}\sum_{n=-\infty}^{\infty} s(n-m)\int_{a+\Delta}^{r} dr' \sin[k_w(r-r')]b_n^{\pm}(r'). \qquad (24)$$

For $r < a+\Delta$, $b_m^{\pm}(r) = \sqrt{r}B_m^{0,\pm}(r)$.

In the case of a thick wall, $b_m^{\pm}(r)$ has to decay at large r. If there is no slots, then $\beta_m^{\pm} = 0$, $B_m^{\pm}(r) = B_m^{0,\pm}(r)$,

$$B_m^{\pm}(r) = \alpha_m^{\pm}\, H_m^{(1)}(k_w r), \quad B_m^{z}(r) = \frac{ik_w}{2k}\,(\alpha_m^{+} - \alpha_m^{-})\, H_m^{(1)}(k_w r). \qquad (25)$$

For a beam pipe wall with a slot the condition $\beta_m^{\pm} = 0$ is not valid because the integral term in Eq. (24) gives an exponentially growing contribution. Therefore, $\beta_m^{\pm}$ can be defined only after Eq. (24) is solved.

To proceed further, we can expect that azimuthal harmonics $b_n^{\pm}$ get smaller for larger n. That is, certainly, the case when there are no slots. In this case, if the beam has zero offset $r_0 = 0$ there is only $n = 0$ harmonics, and with small r_0 harmonics $b_n^{\pm} \propto (r_0/a)^n$. For narrow slots such a hierarchy still exists although non-zero harmonics may be present even for the zero offset.

Let us consider the mode $m = 0$ neglecting coupling to the nonzero modes assuming that the latter are small. Eq. (24) takes the form of the Volterra integral equation of the second kind

$$b_0^{\pm}(r) = f_0^{\pm}(r) + \frac{\alpha k_w}{2\pi}\,\theta(r-a-\Delta)\int_{a+\Delta}^{r} dr' \sin[k_w(r-r')]b_0^{\pm}(r'), \qquad (26)$$

where $f_0^{\pm}(r)$ is the field in the beam-pipe wall with no slots

$$f_0^{\pm}(r) = \sqrt{\frac{2}{\pi k_w}} \{\alpha_0^{\pm}\, e^{i[k_w r \mp \frac{\pi}{2} - \frac{\pi}{4}]} + \beta_0^{\pm}\, e^{-i[k_w r \mp \frac{\pi}{2} - \frac{\pi}{4}]}\}. \qquad (27)$$

Solution of Eq. (26) at $r > a + \Delta$ can be obtained using Laplace transform, defining

$$\tilde{b}(p) = \int_{a+\Delta}^{\infty} dr e^{-pr} b_0^{\pm}(r), \quad \tilde{f}_0(p) = \int_{a+\Delta}^{\infty} dr e^{-pr} f_0^{\pm}(r). \qquad (28)$$

Integrating by parts, we get

$$\tilde{b}(p) = \frac{\tilde{f}_0(p)}{1 - K(p)} \qquad (29)$$

where $K(p)$ is the Laplace transform of the kernel in Eq. (26),

$$\begin{aligned} K(p) &= \frac{\alpha k_w}{2\pi} \int_0^{\infty} dr\, e^{-pr} \sin(k_w r) \\ &= \frac{\alpha}{2\pi} \frac{k_w^2}{p^2 + k_w^2}. \end{aligned} \qquad (30)$$

Inverse Laplace transform gives at $r > a + \Delta$

$$b_0^{\pm}(r) = f_0^{\pm}(r) + \int_{a+\Delta}^{r} dr' R(r - r')\, f_0^{\pm}(r'), \qquad (31)$$

where

$$R(r) = \int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} \frac{dp}{2\pi i}\, e^{pr}\, \frac{K(p)}{1 - K(p)}. \qquad (32)$$

Here $\varepsilon > 0$ and the contour is to the right of the integrand singularities.

Simple calculations give

$$R(r) = \frac{\alpha k_w}{2\pi\kappa} \sin[\kappa k_w r], \qquad (33)$$

where $\kappa = \sqrt{1 - \alpha/2\pi}$.

Eq. (31) gives

$$\begin{aligned} &b_0^{\pm}(r) = b_g(r) + b_d(r), \\ &b_g(r) = \frac{1}{\kappa\sqrt{2\pi k_w}}\, e^{-ik_w(r-a-\Delta)\kappa - ik_w(a+\Delta) \mp i\frac{\pi}{2} - i\frac{\pi}{4}} * \\ &(\alpha_0^{\pm}\,(-1+\kappa)\, e^{2ik_w(a+\Delta)} + \beta_0^{\pm}\,(1+\kappa)\, e^{2i(\pm\frac{\pi}{2}+\frac{\pi}{4})}), \\ &b_d(r) = \frac{1}{\kappa\sqrt{2\pi k_w}}\, e^{ik_w(r-a-\Delta)\kappa - ik_w(a+\Delta) \mp i\frac{\pi}{2} - i\frac{\pi}{4}} \\ &(\alpha_0^{\pm}\,(1+\kappa)\, e^{2ik_w(a+\Delta)} + \beta_0^{\pm}\,(-1+\kappa)\, e^{2i(\pm\frac{\pi}{2}+\frac{\pi}{4})}). \end{aligned} \qquad (34)$$

The term $b_g(r)$ grows exponentially with r and has to be cancelled out. That defines

$$\beta_0^{\pm} = i\alpha_0^{\pm}\kappa_0\, e^{2ik_w(a+\Delta)}, \tag{35}$$

where

$$\kappa_0 = \frac{1-\kappa}{1+\kappa}, \quad \kappa = \sqrt{1-\frac{\alpha}{2\pi}}. \tag{36}$$

Hence,

$$B_0^{\pm}(r) = \mp\alpha_0^{\pm}(1+i)\sqrt{\frac{1}{\pi k_w r}}\,\{e^{ik_w r} + \frac{1-\kappa}{1+\kappa}\,e^{-ik_w r+2ik_w(a+\Delta)}\},\ (a<r<a+\Delta)$$
$$B_0^{\pm}(r) = \mp\alpha_0^{\pm}\,\frac{(1+i)}{\sqrt{\pi k_w r}}\,\frac{2}{1+\kappa}\,e^{ik_w(r-a-\Delta)\kappa+ik_w(a+\Delta)},\ (r>a+\Delta).$$

Note, that $B_0^{\pm}(r)$ and its derivative are continues at $r=a+\Delta$.

Calculations of $B_0^z(r)$ give for $r<a+\Delta$

$$B_0^z(r) = \frac{(1+i)}{2k}\sqrt{\frac{k_w}{\pi r}}\,(\alpha_0^+ - \alpha_0^-)\,(e^{ik_w r} - \kappa_0 e^{-ik_w r+2ik_w(a+\Delta)}), \tag{37}$$

For the harmonics $m>0$, calculations can be carried out in the similar way. For $m>0$ and $a<r<a+\Delta$, the solution is $b_m^{\pm}(r) = f_m^{\pm}(r)$,

$$f_m^{\pm}(r) = \sqrt{\frac{2}{\pi k_w}}\{\alpha_m^{\pm}\,e^{i(k_w r-\frac{3\pi}{4}-\frac{\pi m}{2})} + \beta_m^{\pm}\,e^{-i(k_w r-\frac{3\pi}{4}-\frac{\pi m}{2})}\}, \tag{38}$$

with constants $\alpha_m^{\pm}$ and $\beta_m^{\pm}$.

For $m>0$ and $r>a+\Delta$, Eq. (24) gives taking into account coupling to $m=0$ mode

$$b_m^{\pm}(r) = h_m^{\pm}(r) + \frac{\alpha k_w}{2\pi}\,\theta(r-a-\Delta)\int_{a+\Delta}^{r} dr'\sin[k_w(r-r')]b_m^{\pm}(r'), \tag{39}$$

where

$$h_m^{\pm}(r) = f_m^{\pm}(r) + \frac{\alpha k_w}{2\pi}\,s(m)\theta(r-a-\Delta)\int_{a+\Delta}^{r} dr'\sin[k_w(r-r')]b_0^{\pm}(r'). \tag{40}$$

Note that solution of Eq. (39) is automatically matched with Eq. (38) ar $r=a+\Delta$ with its derivative. The explicit form of the solution can be obtained again with Laplace transform,

$$b_m^{\pm}(r) = h_m^{\pm}(r) + \frac{\alpha k_w}{2\pi\kappa}\,\theta(r-a-\Delta)\int_{a+\Delta}^{r} dr'\sin[\kappa k_w(r-r')]h_m^{\pm}(r'). \tag{41}$$

Eq. (31) can be obtained from here replacing $h_m^{\pm}(r)$ with $f_0^{\pm}(r)$. Eq. (41) can be simplified using Eq. (31),

$$b_m^{\pm}(r) = f_m^{\pm}(r) + \frac{\alpha k_w}{2\pi\kappa}\,\theta(r-a-\Delta)\int_{a+\Delta}^{r} dr'\sin[\kappa k_w(r-r')]f_m^{\pm}(r') +$$
$$(1-\delta_{m,0})s(m)\theta(r-a-\Delta)\frac{\alpha k_w}{2\pi\kappa}\int_{a+\Delta}^{r} dr'\sin[\kappa k_w(r-r')]b_0^{\pm}(r'). \quad (42)$$

To cancel the exponentially growing terms at $r\to\infty$, we put

$$\beta_m^{\pm} = i\kappa_0\{\alpha_m^{\pm} + \frac{s(m)}{\kappa}\alpha_0^{\pm}\,e^{im\pi/2}\}\,e^{2ik_w(a+\Delta)-i\pi m}. \quad (43)$$

That defines the fields in the wall in terms of the coefficients $\alpha_m^{\pm}$ and $c_m^{\pm,>}$, $m=0,1,...$ These coefficients are determined by the matching the tangential components of the E and B fields at $r=a$ for each m. Calculations are straightforward but cumbersome. Here we give explicit expressions in the limit $k_w >> k$, $k^2a/k_w << 1$. In this case,

$$c_m^{+} = -\frac{eZ_0}{2\pi a}\,(\frac{r_0}{a})^{m+1}\,(\frac{k}{k_w})^2\,(\frac{k_w a}{(1+m)(1+\kappa_0\,e^{2ik_w\Delta})})\,\{(\frac{r_0}{a})^m(1-\kappa_0\,e^{2ik_w\Delta}) -$$
$$e^{2ik_w\Delta}\,\frac{s(m)\kappa_0}{(m+1)\kappa g_0}[2i(m+1) - \frac{k^2a}{k_w}\,(1-\kappa_0\,e^{2ik_w\Delta})]\}, \quad (44)$$

$$c_m^{-} = \frac{eZ_0}{2\pi a}\,(\frac{r_0}{a})^{m-1}\,\{-i(\frac{r_0}{a})^m - \frac{s(m)}{k_w a g_0}\,\frac{\kappa_0}{\kappa}$$
$$(\frac{e^{2ik_w\Delta}}{(1+m)(1-\kappa_0 e^{2ik_w\Delta})})\,[\frac{k^2a}{k_w} + i(k^2a^2 - 2m(m+1))\,(1-\kappa_0 e^{2ik_w\Delta})]\}, \quad (45)$$

$$\alpha_m^{+} = (1+i)\frac{eZ_0}{4}\sqrt{\frac{k_w}{\pi a}}\,\frac{e^{-ik_w a+im\pi/2}}{1+\kappa_0 e^{2ik_w\Delta}}\,\{-(\frac{r_0}{a})^m +$$
$$e^{2ik_w\Delta}\,\frac{\kappa_0 s(m)}{\kappa g_0}\,[i + \frac{m}{k_w a} - \frac{k^2 a}{(1+m)k_w}]\}, \quad (46)$$

$$\alpha_m^{-} = \alpha_m^{+}(1-\frac{2k^2}{k_w^2}) + \frac{(1+i)eZ_0k^2s(m)}{2g_0k_w^3a^2(1-\kappa_0\,e^{2ik_w\Delta})^2}\,(\frac{\kappa_0}{\kappa})\sqrt{\frac{k_w a}{\pi}}\,e^{-ik_w a+2ik_w\Delta+im\pi/2}. \quad (47)$$

where

$$g_0 = i(1+\kappa_0\,e^{2ik_w\Delta}) + \frac{k^2a}{k_w}(1-\kappa_0\,e^{2ik_w\Delta}). \quad (48)$$

Coefficients $\alpha_0^{\pm}$ for the harmonics $m=0$ can be obtained from here putting $s(m)\to 0$ and then $m\to 0$.

3 Results

Eqs. (38)-(45) define the fields in the wall and the beam-pipe. For example, the azimuthal component $B^{\phi}_m(r)$ in the range $a < r < a + \Delta$ is

$$B^{\phi}_m(r) = \frac{e^{3i\pi/4+ik_w r}}{\kappa(1+\kappa)}\sqrt{\frac{1}{2\pi k_w r}}\{-e^{-2ik_w(r-a-\Delta)}(-1+\kappa)s(m)(\alpha^+_0+\alpha^-_0)+$$
$$\kappa e^{-im\pi/2}\,[-e^{-2ik_w(r-a-\Delta)}(-1+\kappa)+(1+\kappa)](\alpha^+_m+\alpha^-_m)\}. \tag{49}$$

The field $B^{\phi}_m(r) = (-i/2)(B^+_m - B^-_m)$ in the range $r > a + \Delta$ is defined by

$$B^{\pm}_m(r) = \frac{(1+i)}{\kappa(1+\kappa)}\,\frac{e^{ik_w[a+\Delta+\kappa(r-a-\Delta)]}}{\sqrt{\pi k_w r}}\,\{\pm is(m)(\kappa-1)(-i-k_w(r-a-\Delta)(1+\kappa))\alpha^{\pm}_0$$
$$e^{-im\pi/2} \mp 2\kappa\alpha^{\pm}_m\}. \tag{50}$$

The radial dependence of the $m = 1$ harmonics of $B^{\phi}(r)$ within the wall is illustrated in Fig. (6) for three values of the screen thickness (a) $\Delta = 0.01$ cm, (b) $\Delta = 0.2$ cm, and (c) $\Delta = 0.5$ cm. Other parameters were: $a = 5$ cm, stainless-steel wall conductivity $\sigma = 1.4\,10^4\ Ohm^{-1}cm^{-1}$, $\alpha = 0.04$, and the offset $r_0 = 0.001$ cm. At frequency 1 MHz, $\delta(\omega) = 0.042$ cm. The radial behavior shows the resonance character caused by reflection from the slot.

The magnetic flux $\Phi(\omega)$ through the wire contour with the length L shown in Fig. (5) is obtained integrating

$$\Phi_m(\omega) = L\int_{a+\Delta}^{\infty} dr B^{\phi}_m(r) \tag{51}$$
$$= \frac{L(1+i)}{2k_w\kappa^2(1+\kappa)}\,\frac{e^{ik_w(a+\Delta)-im\pi/2}}{\sqrt{\pi k_w(a+\Delta)}}\,\{s(m)(\frac{(1+2\kappa)(\kappa-1)}{\kappa}(\alpha^+_0+\alpha^-_0)\,e^{im\pi/2}-2\kappa(\alpha^+_m+\alpha^-_m)\}.$$

$\Phi(\omega)$ defines the harmonics of the voltage $V_m(\omega)$ induced in the contour,

$$\frac{dV_m(\omega)}{d\omega} = \oint E_\omega.dl = (\frac{ik}{2\pi})\,\Phi_m(\omega). \tag{52}$$

The spectral density is shown in Fig. (7).

Induced voltage $U_m(t)$ in the contour by a single beam particle lagging at z_i from the bunch center is obtained integrating over frequencies

$$U_m(t) = 2Re[\int_0^{\infty}\frac{dV_m(\omega)}{d\omega}\,e^{-i\omega z_i-i\omega t}\,d\omega]. \tag{53}$$

The signal from a bunch is obtained by summing up contributions of all particles and, for a Gaussian bunch with the rms length σ_b, replacing the sum by the spectrum density of the bunch,

$$\sum_i e^{-i\omega z_i} \rightarrow e^{-\frac{1}{2}(\frac{\omega\sigma_b}{c})^2}. \tag{54}$$

Then,

$$U_m^{bunch}(t) = 2Re[\int_0^\infty \frac{dV_m(\omega)}{d\omega}\, e^{-\frac{1}{2}(\frac{\omega\sigma_b}{c})^2}\, e^{-iks}\, d\omega]. \tag{55}$$

Result of integration per single electron for $m = 1$ mode is shown in Fig. (8) for $L = 10$ acm and $a = 5$ cm.

The longitudinal beam impedance $Z_m^l(\omega)$ per unit length is given by the coefficient c_m^+,

$$Z_m^l(\omega) = -\frac{1}{e}\,\frac{(m+1)c_m^+}{kr_0}\,(\frac{r}{r_0})^m. \tag{56}$$

$$Z_m^l(\omega) = \frac{Z_0}{2\pi a}\,(\frac{r}{a})^m\,(\frac{k}{k_w})\,(\frac{1}{1+\kappa_0\, e^{2ik_w\Delta}})\,\{(\frac{r_0}{a})^m(1-\kappa_0\, e^{2ik_w\Delta}) - e^{2ik_w\Delta}\,\frac{s(m)\kappa_0}{(m+1)\kappa g_0}[2i(m+1) - \frac{k^2 a}{k_w}\,(1-\kappa_0\, e^{2ik_w\Delta})]\}, \tag{57}$$

The longitudinal impedance is dominated by the contribution of $m = 0$ mode.

$$Z^l(\omega) = \frac{Z_0}{2\pi a}\,(\frac{k}{k_w})\,(\frac{1-\kappa_0\, e^{2ik_w\Delta}}{1+\kappa_0\, e^{2ik_w\Delta}}). \tag{58}$$

In the limit $\Delta \rightarrow \infty$ Eq. (58) gives the usual result for the longitudinal impedance per unit length

$$Z^l(\omega) = (1-i)\frac{Z_0}{2\pi a}\,(\frac{k\delta(\omega)}{2}). \tag{59}$$

For an open slot $\Delta \rightarrow 0$, result Eq. (59) is modified by additional factor. For small $\alpha << 1$,

$$Z^l(\omega) = (1-i)\frac{Z_0}{2\pi a}\,(\frac{k\delta(\omega)}{2})\,(\frac{\alpha}{4\pi}). \tag{60}$$

Similarly, the dominant contribution to the transverse impedance per unit length is given by the dipole mode $m = 1$,

$$Z^{tr}(\omega) = \frac{Z^l(\omega)}{kr_0^2}. \tag{61}$$

For small $\alpha << 1$,

$$Z^{tr}(\omega) = (1-i)\frac{Z_0\delta(\omega)}{4\pi a^3}\{\frac{1-\kappa_0\, e^{2ik_w\Delta}}{1+\kappa_0\, e^{2ik_w\Delta}} -$$
$$\frac{\kappa_0}{2\kappa g_0}(\frac{a}{r_0})\frac{e^{2ik_w\Delta}}{1+\kappa_0\, e^{2ik_w\Delta}}\,[4i - \frac{k^2 a}{k_w}\,(1-\kappa_0\, e^{2ik_w\Delta})]\}. \qquad (62)$$

For large $\Delta >> \delta(\omega)$, expression in the curly brackets is equal to one giving the usual transverse resistive wall impedance per unit length. Eq. (62) shows that, contrary to the usual resistive wall impedance, $Z^{tr}(\omega)$ for a beam pipe with a slot depends on the offset r_0. This dependence is weak provided

$$\frac{r_0}{a} >> \frac{\alpha}{4\pi}\, e^{-2\Delta/\delta}, \qquad (63)$$

but may be give

$$Z^{tr}(\omega) = (1-i)\frac{Z_0\delta(\omega)}{4\pi a^3}\,[1-\frac{\alpha}{4\pi}(\frac{a}{r_0})], \qquad (64)$$

for an open slot $\Delta \rightarrow 0$. The second term in the impedance Eq. (64) corresponds to a constant force acting on the beam due to a longitudinal slot in the beam pipe. Calculating the wake field due to the last term in Eq. (64),

$$W_t r(z) = i\int \frac{d\omega}{2\pi} Z^{tr}(\omega)\, e^{-i\omega z/c},$$
$$W_{tr}(z) = -(\frac{\alpha}{2\pi a^2 r_0}\,\frac{1}{\sqrt{\pi Z_0 \sigma z}}, \qquad (65)$$

and using equation of motion for a particle located at the distance z from the head of a bunch with the bunch population N_b

$$\frac{d^2x}{ds^2} + g(s)x(s) = \frac{N_b r_e}{\gamma} W_{tr}(z) r_0, \qquad (66)$$

we get for the average perturbation of the closed orbit

$$\langle x(s)\rangle = -(\frac{N_b r_e}{\gamma}\,(\frac{\alpha}{2\pi})\,\langle(\frac{\beta_x}{a})^2\rangle\,\frac{1}{\sqrt{\pi Z_0 \sigma z}}. \qquad (67)$$

Here r_e is classical electron radius, and β_x is the horizontal beta function. Effect is very small, $\langle x(s)\rangle \simeq 0.5\ \mu m$ for a particle at the distance $z = 1$ cm for parameters $N_b = 6.\,1010$, $\gamma = 6.0\,10^3$, $\beta_x = 15$ m, $a = 5$ cm, and stainless steel conductivity.

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4 Appendix 1: Derivation of the BPM impedance

The geometric impedance derived above does not take into account the current induced in the wire inserted in the slot. To take it into account, we follow G. Lambertson arguments.

The reciprocity theorem states that beam current $j_b(\omega, \overrightarrow{r})$ and the current induced in the wire $j_w(\omega, \overrightarrow{r})$ generate the fields $E_b(\omega, \overrightarrow{r})$ and $E_w(\omega, \overrightarrow{r})$, respectively, are related,

$$\int dV j_b(\omega, \overrightarrow{r}) E_w(\omega, \overrightarrow{r}) = \int dV j_w(\omega, \overrightarrow{r}) E_b(\omega, \overrightarrow{r}). \tag{68}$$

The Fourier component of the beam current $j_b(\omega, \overrightarrow{r}) = ec\rho(\overrightarrow{r})\, e^{-i\omega s/c}$ of a single ultra-relativistic particle gives for a point-like particle moving along the s-axes with the offset r_0

$$\int dV j_b(\omega, \overrightarrow{r}) E_w(\omega, \overrightarrow{r}) = e \int ds\, E_w(\omega, r_0, s)\, e^{-i\omega s/c}. \tag{69}$$

Comparing this expression with the definition of the beam impedance $Z(\omega)$ generated by the field $E(\omega, \overrightarrow{r})$ of a source ultra-relativistic particle moving along the s-axes with the offset r_0

$$Z(\omega) = \frac{1}{e} \int ds\, E(\omega, r_0, s)\, e^{-i\omega s/c}, \tag{70}$$

we obtain from Eqs. (69)-(68) result for the beam impedance induced by the wire,

$$Z(\omega) = \frac{1}{e} \int dV j_w(\omega, \overrightarrow{r}) E_b(\omega, \overrightarrow{r}). \tag{71}$$

For a thin wire, the integral in the right-hand-side

$$\int dV j_w(\omega, \overrightarrow{r}) E_b(\omega, \overrightarrow{r}) = I_w(\omega) V_w(\omega). \tag{72}$$

Here $V_w(\omega)$ is the voltage induced in the wire given by the contour integral along the wire

$$V_w(\omega) = \oint dl E_b(\omega, r(l)), \tag{73}$$

where E_b is taken at the location of the wire. The wire current $I_w(\omega)$ in Eq. (72) is depends on the impedance of the wire loop $Z_w(\omega)$,

$$I(\omega) = \frac{V_w(\omega)}{Z_w(\omega)}. \tag{74}$$

From the Maxwell equation, the contour integral can be converted to the surface integral over the plane enclosed by the wire

$$\oint dl E_b(\omega, r(l)) = i\frac{\omega}{c} \int \overrightarrow{dS}.\overrightarrow{B_b}(\omega, S). \tag{75}$$

For a flat wire loop in the (r, s) plane with the length L_w in the s-direction, we obtain neglecting variation of the azimuthal component B_b^ϕ along the beam pipe

$$\oint dl E_b(\omega, r(l)) = i\frac{\omega L_s}{c} \int dr B_b^\phi(\omega, r), \tag{76}$$

where the integral is taken over the span of r inclosed by the loop.

Combining Eqs. (71)-(76), we get

$$\begin{aligned} V_w(\omega) &= i\frac{\omega L_s}{c}\, e\, Z_0\, F(\omega), \\ Z(\omega) &= \frac{(V_w(\omega))^2}{e^2 Z_w}, \end{aligned} \tag{77}$$

where we introduced dimensionless

$$F(\omega) = \frac{1}{eZ_0} \int dr B_b^\phi(\omega, r). \tag{78}$$

Note that $F(\omega)$ and $Z(\omega)$ are independent of the particle charge e.

The field $B_b^\phi(\omega, r)$ can be approximated by the field of the beam. For a wire in a slot in the wall $a + w_s < r < a + w$, $w_s < w$, $B_b^\phi(\omega, r)$ is approximately given by the field outside of the beam pipe with the inner radius a and the wall thickness w_s.

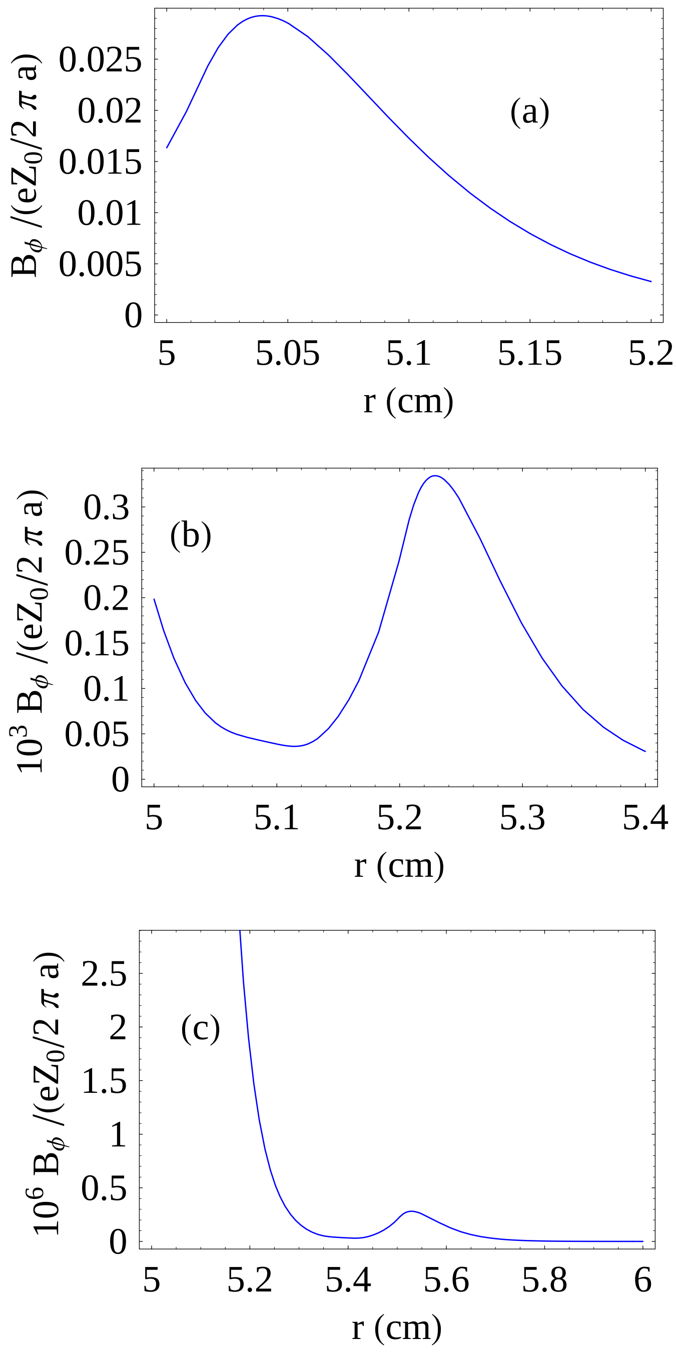

Figure 6: Radial dependence of the $m = 1$ harmonics of $B^{\phi}(r)$ within the wall for (a) $\Delta = 0.01$ cm, (b) $\Delta = 0.2$ cm, and (c) $\Delta = 0.5$ cm. Note the difference in scale. Other parameters are given in the text.

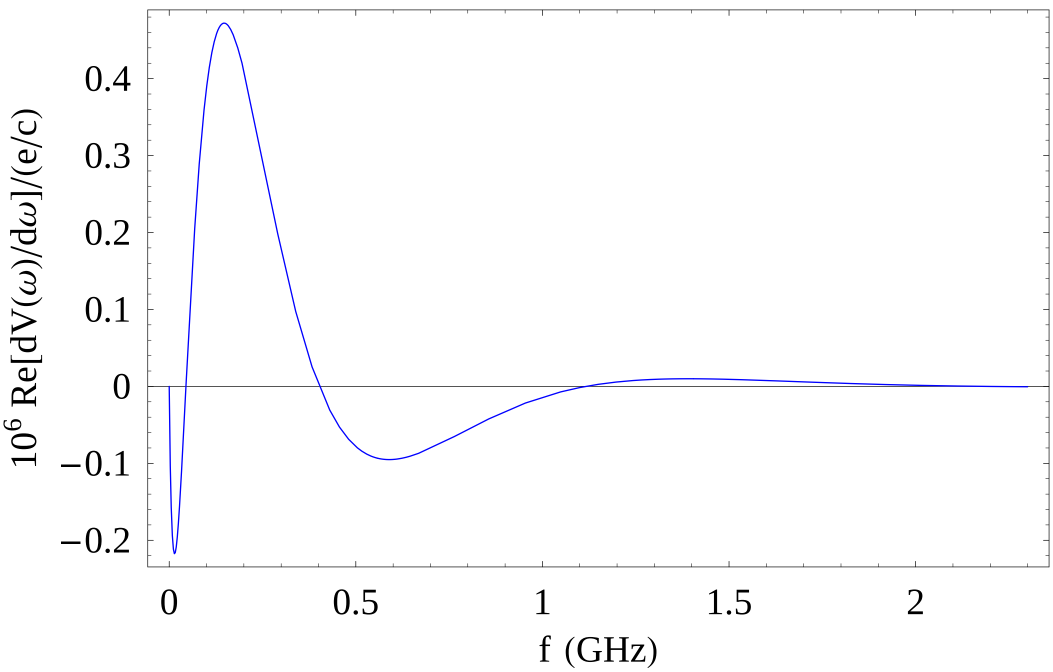

Figure 7: Spectral density $dV_m/d\omega$ of the dipole harmonics $m = 1$ of the azimuthal B^ϕ field per single beam particle.

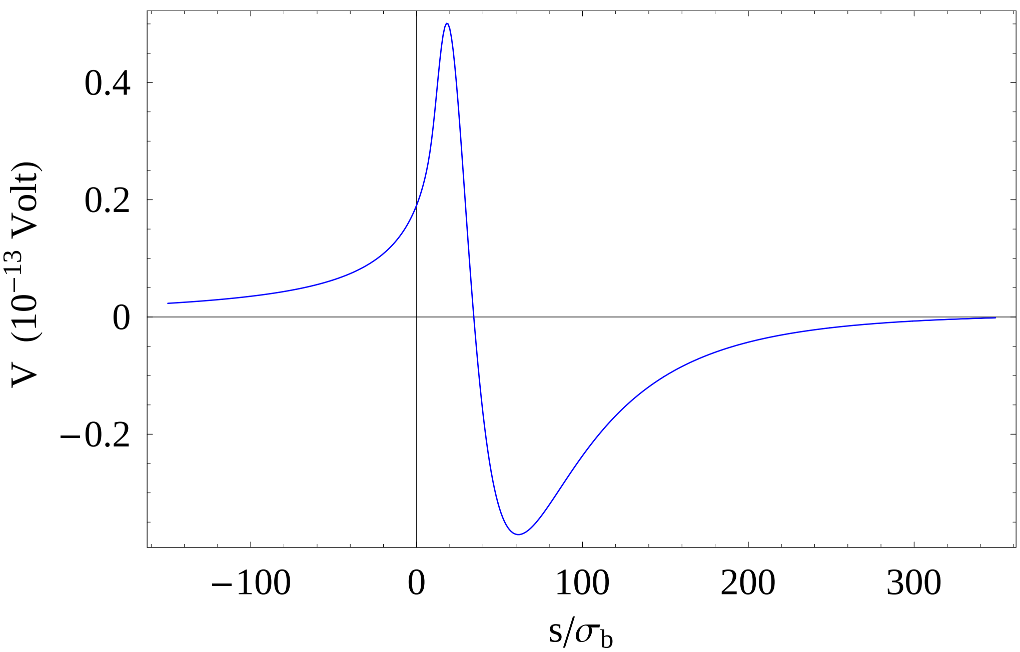

Figure 8: Voltage induced in the wire loop by the dipole harmonics of the azimuthal B^{ϕ} field of a single beam particle. $\Delta = 0.01$ cm, other parameters are the same as in Fig. (6).