Modeling quasi-static poroelastic propagation using an asymptotic approach

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SUMMARY

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1 INTRODUCTION

Since the formulation of poroelasticity (Biot(1941)) and its reformulation (Rice & Cleary(1976)), there have been many efforts to solve the coupled system of equations. Perhaps because of the complexity of the governing equations, most of the work has been directed towards finding numerical solutions. For example, Lewis and co-workers published early papers (Lewis & Schrefler(1978); Lewis et al.(1991)Lewis, Schrefler, & Simoni) concerned with finite-element methods for computing consolidation, subsidence, and examining the importance of coupling. Other early work dealt with flow in a deformable fractured medium (Narasimhan & Witherspoon(1976); Noorishad et al.(1984)Noorishad, Tsang, & Witherspoon). This effort eventually evolved into a general numerical approach for modeling fluid flow and deformation (Rutqvist et al.(2002)Rutqvist, Wu, Tsang, & Bodvarsson). As a result of this and other work, numerous coupled, computer-based algorithms have emerged, typically falling into one of three categories: one-way coupling, loose coupling, and full coupling (Minkoff et al.(2003)Minkoff, Stone, Bryant, Peszynska, & Wheeler). In one-way coupling the fluid flow is modeled using a conventional numerical simulator and the resulting change in fluid pressures simply drives the deformation. In loosely coupled modeling distinct geomechanical and fluid flow simulators are run for a sequence of time steps and at the conclusion of each step information is passed between the simulators. In full coupling, the fluid flow and geomechanics equations are solved simultaneously at each time step (Lewis & Sukirman(1993); Lewis & Ghafouri(1997); Gutierrez & Lewis(2002)).

One disadvantage of a purely numerical approach to solving the governing equations of poroelasticity is that it is not clear how the various parameters interact and influence the solution. Analytic solutions have an advantage in that respect; the relationship between the medium and fluid properties is clear from the form of the solution. Unfortunately, analytic solutions are only available for highly idealized conditions, such as a uniform (Rudnicki(1986)) or one-dimensional (Simon et al.(1984)Simon, Zienkiewicz, & Paul; Gajo & Mongiovi(1995); Wang & Kumpel(2003)) medium. In this paper I derive an asymptotic, semi-analytic solution for coupled deformation and flow. The approach is similar to trajectory- or ray-based methods used to model elastic and electromagnetic wave propagation (Aki & Richards(1980); Kline & Kay(1979); Kravtsov & Orlov(1990); Keller & Lewis(1995)) and, more recently, diffusive propagation (Virieux et al.(1994)Virieux, Flores-Luna, & Gibert; Vasco et al.(2000)Vasco, Karasaki, & Keers; Shapiro et al.(2002)Shapiro, Rothert, Rath, & Rindschwentner; Vasco(2007)). The asymptotic solution is valid in the presence of smoothly-varying, heterogeneous flow properties. The situation I am modeling is that of a formation with heterogeneous flow properties and uniform mechanical properties. The boundaries of the layer may vary arbitrary and can define discontinuities in both flow and mechanical properties. Thus, using the techniques presented here, it is possible to model a stack of irregular layers with differing mechanical properties. Within each layer the hydraulic conductivity and porosity can vary smoothly but with an arbitrarily large magnitude. The advantages of this approach are that it produces explicit, semi-analytic expressions for the arrival time and amplitude of the Biot slow and fast waves, expressions which are valid in a medium with heterogeneous properties. As shown here, the semi-analytic expressions provide insight into the nature of pressure and deformation signals recorded at an observation point. Finally, the technique requires considerably fewer computer resources than does a fully numerical treatment.

2 METHODOLOGY

2.1 The Governing Equations

There are many situations in which the mechanical properties of a formation are assumed to be described by an average value. That is, the mechanical properties are thought to relatively constant within a given formation. It is often true that, when compared to the variation in hydraulic conductivity which can change by orders of magnitude, elastic properties are much less variable within a given formation, of the order of a few tens of percent or so. In addition, it is often necessary to characterize the elastic properties by a restricted number of parameters. Typically, there are only a few measurements of the mechanical properties of any given formation. I will therefore assume that the mechanical properties are constant for each geologic unit while allowing the flow properties, such as porosity and hydraulic conductivity, to vary within the unit.

The equations governing the space ($\mathbf{x}$) and time (t) variation of the solid matrix displacement vector $\mathbf{u}(\mathbf{x}, t)$ and the pore fluid pressure $p_f(\mathbf{x}, t)$ in a medium with constant elastic moduli and spatially varying flow properties are (Biot(1956); Wang(2000); Showalter(2000); Pride(2005))

$$(\lambda + 2\mu)\nabla\nabla \cdot \mathbf{u} - \mu\nabla \times \nabla \times \mathbf{u} - \alpha\nabla p_f = \mathbf{0}$$

$$C\frac{\partial p_f}{\partial t} + \alpha\frac{\partial(\nabla \cdot \mathbf{u})}{\partial t} - \nabla \cdot (k\nabla p_f) = q \quad (1)$$

where λ and μ are Lamé coefficients (Aki & Richards(1980)), in particular μ is the shear modulus, α is the dimensionless coefficient of effective stress, C is the bulk compressibility introduced by (Biot(1941)), k is the hydraulic conductivity, and q is the fluid volume injection rate (de Marsily(1986)). Taking the Laplace transform of these equations, I may work in the s-domain, where s is a complex variable and the Laplace transforms of $\mathbf{u}(\mathbf{x}, t)$ and $p_f(\mathbf{x}, t)$ are denoted by $\mathbf{U}(\mathbf{x}, s)$ and $P(\mathbf{x}, s)$, respectively. In the s-domain the system of equations (1) becomes

$$(\lambda + 2\mu)\nabla\nabla \cdot \mathbf{U} - \mu\nabla \times \nabla \times \mathbf{U} - \alpha\nabla P = \mathbf{0}$$

$$CP + \alpha\nabla \cdot \mathbf{U} - \nabla \cdot (K\nabla P) = Q. \quad (2)$$

where

$$K = -\frac{k}{s} \quad (3)$$

and Q is the Laplace transform of q divided by s. The first equation provides an expression for the pressure gradient, ∇P, in terms of the displacement components

$$\nabla P = \Upsilon\nabla\nabla \cdot \mathbf{U} - \Psi\nabla \times \nabla \times \mathbf{U} \quad (4)$$

where I have defined the coefficients

$$\Upsilon = \frac{\lambda + 2\mu}{\alpha}, \quad (5)$$

and

$$\Psi = \frac{\mu}{\alpha}. \quad (6)$$

Taking the gradient of the second equation of the system (2), I arrive at an equation in terms of the pressure gradient and the components of displacement

$$\nabla P + \nabla\left(C^{-1}\alpha\nabla \cdot \mathbf{U}\right) - \nabla\left[C^{-1}\nabla \cdot (K\nabla P)\right] = \mathbf{Q} \quad (7)$$

where $\mathbf{Q} = \nabla(C^{-1}Q)$. Substituting the expression for the pressure gradient ∇P, equation (4), into equation (7) results in a set of three equations for the three components of displacement

$$\Upsilon\nabla\nabla \cdot \mathbf{U} - \Psi\nabla \times \nabla \times \mathbf{U} + \nabla\left(C^{-1}\alpha\nabla \cdot \mathbf{U}\right)$$

$$-\nabla\left\{C^{-1}\nabla \cdot K\left[\Upsilon\nabla\nabla \cdot \mathbf{U} - \Psi\nabla \times \nabla \times \mathbf{U}\right]\right\} = \mathbf{Q}. \quad (8)$$

This is the basic equation that I shall treat using an asymptotic approach known as the method of multiple scales (Anile et al.(1993)Anile, Hunter, Pantano, & Russo). The results of this analysis are presented in the following sub-sections. The details of the derivation are contained in the Appendices of this paper.

2.2 An Asymptotic Solution for the Displacement of the Solid

In this study I am assuming that the heterogeneity is smoothly-varying between given boundaries. That is, the asymptotic technique described below requires that the flow properties K and C vary over a scale-length which is large compared to the scale-length of a propagating elastic displacement. That is, if the length-scale of the variation in flow properties is L, and the length-scale over which the elastic displacement increases from zero to observable value is l, then the scale-length ratio $\epsilon = l/L$ is much smaller than 1. This assumption is compatible with solving the inverse problem in which one is trying to infer smoothly-varying heterogeneity from a limited number of measurements (Parker(1994)). I should also point out that the methodology does allow for discrete changes at known interfaces, as do general ray methods (Kravtsov & Orlov(1990)). Some of the issues associated with propagating the displacement field across a known interface are discribed in the final sub-section.

If I consider equation (8) in slow coordinates $\mathbf{X}$, which are defined in reference to the scale ratio ϵ, $\mathbf{X} = \epsilon\mathbf{x}$, an expression containing terms of various orders in ϵ results [equation (A7) in Appendix A]. An asymptotic solution is a power series in ϵ

$$\mathbf{U}(\mathbf{X}, \varphi) = e^{-\varphi}\sum_{n=0}^{\infty}\epsilon^n\mathbf{U}_n(\mathbf{X}), \quad (9)$$

where $\varphi(\mathbf{X})$ is the local phase (Kline & Kay(1979)). Substituting the power series (9) into equation (A7) results in an expression with an infinite number of terms of various orders in ϵ. Because I am assuming that the heterogeneity is smoothly varying with respect to the variation across the displacement front, $\epsilon \ll 1$, and the lowest order terms in ϵ are the most important. In the following two sub-sections I consider terms of the two most significant orders: ϵ^0 and ϵ^1.

2.2.1 *Terms of Order $\epsilon^0 \sim 1$: An Equation for the 'Phase'*

Considering the terms of lowest order, ϵ^0, produces the equation

$$\left(\Gamma + C^{-1}K\Upsilon p^2\right)\mathbf{pp} \cdot \mathbf{U}_0 - \Psi\mathbf{p} \times \mathbf{p} \times \mathbf{U}_0 = 0 \quad (10)$$

where I have defined the vector

$$\mathbf{p} = \nabla\varphi, \quad (11)$$

the coefficient

$$\Gamma = \Upsilon + C^{-1}\alpha, \quad (12)$$

and used the fact that

$$\frac{\partial^l\mathbf{U}}{\partial\varphi^l} = (-1)^l\mathbf{U} \quad (13)$$

which follows from the particular form of the solution (9). I can write equation (10) as a matrix equation

$$\left[\left(\Gamma + C^{-1}K\Upsilon p^2\right)\mathbf{pp} \cdot \mathbf{I} - \Psi\mathbf{p} \times (\mathbf{p} \times \mathbf{I})\right]\mathbf{U}_0 = 0 \quad (14)$$

which, upon defining

$$\eta = \frac{\Gamma}{\Psi} \quad (15)$$

and

$$\Omega = \frac{C^{-1}K\Upsilon}{\Psi}, \quad (16)$$

I may write as

$$\left[\left(\eta + \Omega p^2\right) \mathbf{p}(\mathbf{p} \cdot \mathbf{I}) - \mathbf{p} \times (\mathbf{p} \times \mathbf{I})\right] \mathbf{U}_0 = 0. \qquad (17)$$

Noting that

$$\mathbf{p}(\mathbf{p} \cdot \mathbf{I}) = \mathbf{p} \times (\mathbf{p} \times \mathbf{I}) + p^2 \mathbf{I} \qquad (18)$$

and after some re-arranging, I may write equation (17) as

$$\left[(p^2 - \Lambda)\mathbf{I} - \mathbf{p}\mathbf{p}^T\right] \mathbf{U}_0 = 0 \qquad (19)$$

where

$$\Lambda = \frac{(\Omega p^2 + \eta)p^2}{\eta - \Omega p^2 - 1}. \qquad (20)$$

For a non-zero vector $\mathbf{U}_0$ equation (19) can only be satisfied if the determinant of the coefficient matrix vanishes. Expanding the determinant produces the polynomial (Kravtsov & Orlov(1990))

$$\Lambda(p^2 - \Lambda)^2 = 0 \qquad (21)$$

which has solutions if $\Lambda = 0$ or if $p^2 - \Lambda = 0$. From equation (20) I find that the condition, $\Lambda = 0$, implies that either

$$\Omega p^2 + \eta = 0 \qquad (22)$$

or $p^2 = 0$. The first root implies that

$$\mathbf{p} \cdot \mathbf{p} = -\frac{\eta}{\Omega}. \qquad (23a)$$

while the second root is equivalent to

$$\mathbf{p} \cdot \mathbf{p} = 0 \qquad (23b)$$

As discussed below, the second root implies an infinite propagation velocity from the source to the observation point, an artifact from neglecting second derivatives in Biot's equations (Biot(1962)). This root corresponds to the 'fast' seismic response, in this case an instantaneous response, associated with Biot's equations (Biot(1956); Biot(1962)). The root corresponding to equation (23a) is the Biot slow wave which is associated with fluid pressure diffusion (Pride(2005)).

The root $\Omega p^2 + \eta = 0$ and the Biot slow wave

In what follows I will derive an expression for the Biot slow wave, later I discuss the nature of the fast wave, also know as the 'P-wave' response (Pride(2005)). After some manipulation, and taking note of the definition (11), equation (23a) may be written

$$\nabla\varphi \cdot \nabla\varphi = \left(\frac{C}{k} + \frac{\alpha^2}{\lambda + 2\mu}\frac{1}{k}\right) s, \qquad (24)$$

a partial differential equation for the function $\varphi(\mathbf{X})$, the equivalent of the Eikonal equation (Aki & Richards(1980); Kravtsov & Orlov(1990); Anile et al.(1993)Anile, Hunter, Pantano, & Russo). For hyperbolic wave propagation, the right-hand-side of the Eikonal equation represents the square of the slowness (the inverse of the magnitude of velocity). The same is also true for diffusive propagation (Virieux et al.(1994)Virieux, Flores-Luna, & Gibert; Vasco et al.(2000)Vasco, Karasaki, & Keers; Shapiro et al.(2002)Shapiro, Rothert, Rath, & Rindschwentner), where the arrival time is carefully defined, as discussed below. If I denote the total velocity magnitude by v, equation (24) provides an expression for the total velocity

$$\frac{1}{v^2} = \frac{C}{k}s + \frac{\alpha^2}{\lambda + 2\mu}\frac{s}{k}. \qquad (25)$$

Vasco et al. (2000) noted that the velocity associated with the propagation of a fluid pressure disturbance in a non-deformable medium, v_f, satisfies

$$\frac{1}{{v_f}^2} = \frac{C}{k}s. \qquad (26)$$

Thus, the right-hand-side of equation (25) represents a modified propagation velocity, with the modification induced by the deformation of the medium. Indeed, the modification of the velocity due to the deformation of the solid matrix, v_s, given by,

$$\frac{1}{{v_s}^2} = \frac{\alpha^2}{\lambda + 2\mu}\frac{s}{k} \qquad (27)$$

depends upon both the elastic parameters and the Darcy conductivity. If I denote the fluid, solid, and total slownesses (inverse velocities), by ς_f, ς_s, and ς, respectively, then the total slowness is the sum

$$\varsigma^2 = \frac{C}{k}s + \frac{\alpha^2}{\lambda + 2\mu}\frac{s}{k} = {\varsigma_f}^2 + {\varsigma_s}^2. \qquad (28)$$

Thus, the deformation slows down the propagation of a disturbance in a porous medium as compared to propagation in a non-deformable medium. It can be shown directly that

$$\frac{C}{k} + \frac{\alpha^2}{\lambda + 2\mu}\frac{1}{k} = \frac{1}{D} \qquad (29)$$

where D is the hydraulic diffusivity (Wang & Kumpel(2003)), which is related to the Darcy or hydraulic conductivity k

$$D = \frac{2}{9}\frac{(1-\nu)(1+\nu_u)^2 \mu B^2}{(1-\nu_u)(\nu_u - \nu)}k \qquad (30)$$

where ν is Poisson's ratio and ν_u is the undrained Poisson's ratio, B is Skemption's ratio (Rice & Cleary(1976); Wang & Kumpel(2003)).

The nonlinear, scalar partial differential equation (24), the Eikonal equation, is equivalent to a system of ordinary differential equations, the bi-characteristic equations (Courant & Hilbert(1962); Kravtsov & Orlov(1990)). The bi-characteristic equations define the solution along a trajectory through the medium, the ray path $\mathbf{X}(r)$, which is defined by

$$\frac{d\mathbf{X}}{dr} = \frac{\mathbf{p}}{\varsigma} \qquad (31)$$

$$\frac{d\mathbf{p}}{dr} = \nabla\varsigma$$

where ς is the total slowness, defined in equation (28), and r signifies the distance along the ray path, The system of ordinary differential equations (31) can be solved using a numerical technique such as a shooting method coupled to a globally convergent Newton-Raphson algorithm (Press et al.(1992)Press, Teukolsky, Vetterling, & Flannery). Writing equation (24) in ray coordinates I have

$$\frac{d\varphi}{dr} = \varsigma \qquad (32)$$

which may be integrated to give

$$\varphi = \int_{\mathbf{X}} \varsigma dr \qquad (33)$$

or

$$\varphi = \sqrt{s}\int_{\mathbf{X}} \sqrt{\frac{C}{k} + \frac{\alpha^2}{\lambda + 2\mu}\frac{1}{k}}dr, \qquad (34)$$

where the integral is over the raypath from a given source to an observation point. If I define the integral

$$\tau = \int_{\mathbf{X}} \sqrt{\frac{C}{k} + \frac{\alpha^2}{\lambda + 2\mu}\frac{1}{k}} dr, \tag{35}$$

equation (34) may be written in a more compact fashion

$$\varphi = \sqrt{s}\tau. \tag{36}$$

As will be shown below the quantity τ is related to the 'arrival time' of the displacement disturbance. I will refer to it as the 'phase' associated with the propagating displacement front. Others call τ the 'pseudo-phase' because it is not completely equivalent to the phase of a propagating wave (He et al.(2006)He, Datta-Gupta, & Vasco).

As an illustration of the computation of trajectories consider a quadratic depth variation of hydraulic conductivity (Figure 1). The trajectories are computed using equations (31) for disturbances leaving the source at various directions or take-off angles. The ray paths curve in response to the spatial variation in hydraulic conductivity, bending towards regions of higher conductivity. A the trajectories corresponding to a two-dimensional conductivity distribution are shown in Figure 2. Again, the ray paths curve in response to the lateral variations in k. It may be shown that the geometry of the ray path is such that the total integral expression for φ, equation (34), is minimized (Kline & Kay(1979); Kravtsov & Orlov(1990)).

Equation (22), the condition for the vanishing of the determinant, can be used to better understand the nature of the displacement vector $\mathbf{U}_0$ associated with the root $\Lambda = 0$. Using equation (22) in the vector-matrix expression (17) results in

$$[\mathbf{p} \times (\mathbf{p} \times \mathbf{I})]\, \mathbf{U}_0 = 0. \tag{37}$$

This system of equations is satisfied if $\mathbf{p} \times \mathbf{U}_0$ vanishes which is equivalent to

$$\mathbf{U}_0(\mathbf{X}) = U_0(\mathbf{X})\hat{\mathbf{p}} \tag{38}$$

where $U_0(\mathbf{X})$ is a scalar magnitude which depends upon position and $\hat{\mathbf{p}}$ is a unit vector in the direction of $\mathbf{p}$. Thus, the vector $\mathbf{U}_0$ is proportional to slowness vector $\mathbf{p}$. From equation (38) I find that the displacement vector lies along the tangent to the trajectory $\mathbf{X}(r)$. In other words, the vector $\mathbf{U}_0$ corresponding to the condition $\Lambda = 0$, is a longitudinal displacement along the trajectory.

The root $p^2 = 0$ and the Biot fast wave

The root $p^2 = 0$, a consequence of the vanishing of Λ in equation (21), is an artifact due to the fact that second derivatives are neglected in equation (1). That is, a complete formulation of Biot's equations would include second derivatives (Pride(2005)). They are neglected in the case of quasi-static poroelastic deformation because they are small for slow displacements and pressure changes. However, neglecting the second derivatives results in some non-physical behaviour, such as an infinite propagation velocity for the fast wave. In reality, the fast wave travels at the speed of a wave in the elastic solid (Pride(2005)). The most satisfactory remedy to the difficulties associated with the root $p^2 = 0$ is to include the second derivatives in the formulation. This will be the topic of a future study of asymptotic solutions for poroelastic propagation.

Another approach is to attack the equation

$$\mathbf{p} \cdot \mathbf{p} = 0 \tag{39}$$

directly which requires consideration of a complex vector $\mathbf{p}$

$$\mathbf{p} = \mathbf{p}_r + i\mathbf{p}_i \tag{40}$$

where $\mathbf{p}_r$ and $\mathbf{p}_i$ are the real and imaginary components of $\mathbf{p}$. This is equivalent to complex ray tracing which is used to model the propagation of evanesant waves (Choudhary & Felsen(1973); Felsen(1976)). The complex equation (39) is equivalent to the two real equations

$$\mathbf{p}_r \cdot \mathbf{p}_r - \mathbf{p}_i \cdot \mathbf{p}_i = 0 \tag{41}$$

$$\mathbf{p}_r \cdot \mathbf{p}_i = 0 \tag{42}$$

which implies that $|\mathbf{p}_r| = |\mathbf{p}_i|$ and that $\mathbf{p}_r$ is perpendicular to $\mathbf{p}_i$, respectively. Correspondingly, I can write the function φ as a complex number

$$\varphi = \varphi_r + i\varphi_i \tag{43}$$

where $\mathbf{p}_r = \nabla\varphi_r$ and and $\mathbf{p}_i = \nabla\varphi_i$. The exponential factor in the asymptotic expression (9) is given by

$$e^{-\varphi} = e^{-\varphi_r}e^{-i\varphi_i} \tag{44}$$

and decays the least when φ_r vanishes and φ_i increases along the path from the source to the observation point.

The approach I adopt involves completing the quasi-static equations by including a second derivative term

$$\alpha\epsilon\frac{\partial^2 \mathbf{u}}{\partial t^2} + (\lambda + 2\mu)\nabla\nabla \cdot \mathbf{u} - \mu\nabla \times \nabla \times \mathbf{u} - \alpha\nabla p_f = 0 \tag{45}$$

where ϵ is a small number. This approach has the advantage that it does not require computing a complex eikonal φ nor does it involve the full complexity of the complete Biot's equations (Pride(2005)). It is an approximate technique that will provide some insight into the nature of the Biot fast wave. The addition of the second derivative term may be thought of as considering the equation in a projective space where the zero of the equation lies at infinity (Kendig(1977)). Utilizing a complex projective space allows one the unify hyperbolic, parabolic, and elliptic equations. Due to the modification of the governing equation (45), the expression for the pressure gradient, equation (4), becomes

$$\nabla P = \Upsilon\nabla\nabla \cdot \mathbf{U} - \Psi\nabla \times \nabla \times \mathbf{U} + \epsilon s^2\mathbf{U} \tag{46}$$

with corresponding modifications to the displacement equation (8).

Applying the method of multiple scales to the modified displacement equation leads to perturbed asymptotic expressions. For example, to lowest order, ε^0 I arrive at the matrix equation

$$\epsilon s^2\mathbf{U}_0 + \left(\Gamma + C^{-1}K\epsilon s^2 + C^{-1}K\Upsilon p^2\right)\mathbf{pp} \cdot \mathbf{U}_0$$

$$-\Psi\mathbf{p} \times \mathbf{p} \times \mathbf{U}_0 = 0. \tag{47}$$

If I consider the displacement in the longitudinal direction, as in equation (38), and I define

$$\eta' = \frac{\Gamma + C^{-1}\epsilon s^2}{\Psi} \tag{48}$$

the matrix equation (47) can be written as

$$\left[\epsilon s^2 + (\eta' + \Omega p^2)p^2\right]\mathbf{I} \cdot U_0\hat{\mathbf{p}} = 0. \tag{49}$$

The equation has a non-trivial solution when the determinant of the matrix vanishes, or when

$$\epsilon s^2 + (\eta' + \Omega p^2)p^2 = 0, \tag{50}$$

a quadratic equation in p^2. The solutions of the quadratic equation (50) are given by

$$p^2 = \frac{-\eta' \pm \sqrt{(\eta')^2 - 4\Omega\epsilon s^2}}{2\Omega} \tag{51}$$

or, if I define,

$$\gamma = \frac{\eta'}{\Omega} \tag{52}$$

and

$$\epsilon' = \frac{4\epsilon}{\Omega\gamma^2} \tag{53}$$

equation (51) may be written as

$$\mathbf{p}\cdot\mathbf{p} = \gamma\frac{1 \pm \sqrt{1-\epsilon' s^2}}{2}. \tag{54}$$

Note that when ϵ' vanishes equation (54) gives the two solutions, $p^2 = 0$ and $p^2 = -\eta/\omega$ associated with the vanishing of Λ [see equation (20)].

Because ϵ is assumed to be small, I can expand the square root in equation (54) as a power series and retain terms of zero- and first-order in ϵ. The result is the expression

$$\mathbf{p}\cdot\mathbf{p} = s\left[g^{\pm}{}_1 - \epsilon g^{\pm}{}_2 s - \epsilon g^{\pm}{}_3 s^2\right] \tag{55}$$

where

$$g^{\pm}{}_1 = \frac{1}{k}\left(C + \frac{\alpha^2}{\lambda+2\mu}\right)\left(\frac{1\pm 1}{2}\right), \tag{56}$$

$$g^{\pm}{}_2 = \pm\frac{\mu}{\alpha}, \tag{57}$$

and

$$g^{\pm}{}_3 = -\frac{1}{k}\left(\frac{\alpha}{\lambda+2\mu}\right)\left(\frac{1\pm 1}{2}\right) \tag{58}$$

where the plus corresponds to the diffusive Biot slow wave, and the minus corresponds to the elastic Biot fast wave. As in the previous sub-section, I can consider equation (55) in ray coordinates, given by the ordinary differential equations (31). Expressions similar to equations (32) and (33) result and I may write the phase φ as

$$\varphi = \sqrt{s}\int_{\mathbf{X}} \sqrt{g^{\pm}{}_1 - \epsilon g^{\pm}{}_2 s - \epsilon g^{\pm}{}_3 s^2}\,dr. \tag{59}$$

Because I am interested in the Biot fast wave, I consider the negative root and the terms $g^{\pm}{}_1$ and $g^{\pm}{}_3 s^2$ vanish [see equations (56) and (58)]. Thus, the phase function φ is given by

$$\varphi = s\int_{\mathbf{X}} \sqrt{\frac{\epsilon\mu}{\alpha}}\,dr \tag{60}$$

which I may write as

$$\varphi = s\tau_f \tag{61}$$

where

$$\tau_f = \int_{\mathbf{X}} \sqrt{\frac{\epsilon\mu}{\alpha}}\,dr. \tag{62}$$

Note that the phase field for the fast wave depends upon the ratio of μ to α and is not a function of the hydraulic conductivity k. Thus, the trajectories associated with the fast wave only depend on the elastic properties. Because I am assuming homogeneous elastic properties within the region of interest, the trajectories are straight lines from the source to the observation point. As shown below, the straight line trajectories may be deflected at an interface between media with different elastic properties.

2.2.2 *Terms of Order ϵ: An Equation for the Amplitude of the Biot Slow Wave*

In this sub-section I consider the amplitude equation associated with the Biot slow wave. As noted above, there is an additional mode of propagation, the Biot fast wave which travels in the manner of an elastic wave. I shall not present a derivation of this mode of propagation in this paper. Because the treatment of the fast wave, encapsulated in equation (45), is approximate, a perturbation of the quasi-static equation, a detailed derivation is not warrented in this paper. Rather, I shall present a more detailed account in a future paper in which the full Biot equations are considered. For the calculations presented in the Applications section below, the fast wave was treated simply as an elastic wave propagating in a uniform medium.

As shown in Appendix A, considering terms of order ϵ produces the expression

$$\begin{gathered}
\Gamma\nabla(\mathbf{p}\cdot\mathbf{U}_0) + \Gamma\mathbf{p}(\nabla\cdot\mathbf{U}_0) \\
-\Psi\nabla\times(\mathbf{p}\times\mathbf{U}_0) - \Psi\mathbf{p}\times(\nabla\times\mathbf{U}_0) \\
+\nabla C^{-1}\alpha\mathbf{p}\cdot\mathbf{U}_0 + \nabla C^{-1}K\Upsilon p^2(\mathbf{p}\cdot\mathbf{U}_0) \\
+C^{-1}\Upsilon\mathbf{p}\mathbf{p}\cdot\nabla K(\mathbf{p}\cdot\mathbf{U}_0) \\
-C^{-1}\Psi\mathbf{p}\nabla K\cdot(\mathbf{p}\times\mathbf{p}\times\mathbf{U}_0) \\
+C^{-1}\Upsilon\nabla K p^2(\mathbf{p}\cdot\mathbf{U}_0) \\
+C^{-1}\Upsilon K\nabla\left(p^2\mathbf{p}\cdot\mathbf{U}_0\right) \\
+C^{-1}\Upsilon K\mathbf{p}\nabla\cdot(\mathbf{p}\mathbf{p}\cdot\mathbf{U}_0) \\
+C^{-1}\Upsilon K\mathbf{p}\mathbf{p}\cdot\nabla(\mathbf{p}\cdot\mathbf{U}_0) \\
+C^{-1}\Upsilon K\mathbf{p}p^2(\nabla\cdot\mathbf{U}_0) = 0.
\end{gathered} \tag{63}$$

This is a rather complicated equation for the components of the displacement vector $\mathbf{U}_0$. Some simplifications follow if I consider the longitudinal mode of propagation associated with the root $\Lambda = 0$ which was described in the previous sub-section. This particular form of the displacement vector represents deformation in the direction of the vector $\mathbf{p}$, as given by equation (38). As shown in Appendix B, if I assume the form (38) for $\mathbf{U}_0(\mathbf{X})$ and project the vector equation onto the unit vector in the direction of $\mathbf{p}$, a single equation for scalar quantity $U_0(\mathbf{X})$ results. Because of the first set of ray equations (31), projection of the gradient operator may be written as a derivative along the trajectory $\mathbf{X}(r)$. Thus, I can write equation (63) as an ordinary differential equation for the scalar amplitude $U_0(\mathbf{X})$ as a function of distance along the trajectory $\mathbf{X}(r)$

$$\frac{dU_0}{dr} + \varrho U_0 = 0 \tag{64}$$

where

$$\varrho = R_{\nabla}\nabla\cdot\hat{\mathbf{p}} + R_{C^{-1}}\frac{dC^{-1}}{dr} + R_k\frac{dk}{dr}, \tag{65}$$

$$R_{\nabla} = \frac{\Gamma + 2k\Pi^2\Upsilon C^{-1}}{2\Gamma - 3k\Pi^2\Upsilon C^{-1}}, \tag{66}$$

$$R_{C^{-1}} = \frac{\alpha\Pi + k\Pi^3\Upsilon - \frac{1}{2}C^2\Pi^{-1}k^{-1}}{2\Gamma\Pi - 3k\Pi^3\Upsilon C^{-1}}, \tag{67}$$

$$R_k = \frac{2\Pi^2\Upsilon C^{-1} - \frac{1}{2}k^{-1}}{2\Gamma - 3k\Pi^2\Upsilon C^{-1}}, \tag{68}$$

and

$$\Pi = \sqrt{\frac{C}{k} + \frac{\alpha^2}{\lambda + 2\mu}\frac{1}{k}}. \tag{69}$$

Note that the first term in (65) is related to the divergence of the trajectories, geometrical spreading, while the latter two terms are related to the heterogeneity. In particular, the derivatives of C^{-1} and k vanish in a medium with constant flow properties.

Because the coefficient ϱ in equation (64) can be thought of as a function of the distance along the trajectory r, the differential equation can be solved directly. The explicit solution of this ordinary differential equation is

$$U_0(\mathbf{X}) = A_0 \exp\left[-\int_{\mathbf{X}(r)} \varrho dr\right] \tag{70}$$

and A_0 is a constant of integration which is determined by the source amplitude. Defining

$$\chi = \int_{\mathbf{X}(r)} \varrho dr \tag{71}$$

I can write equation (70) as

$$U_0(\mathbf{X}) = A_0 e^{-\chi}. \tag{72}$$

The solution (72) signifies exponential decay along the trajectory in which the rate of decay depends upon the geometrical spreading of the trajectories and upon the rate of change of the flow properties along $\mathbf{X}(r)$. The exact dependence of this decay upon the properties of the medium is encapsulated in the coefficients R_∇, $R_{C^{-1}}$, and R_k.

2.3 Construction and Interpretation of the Zeroth-Order Displacement Field Associated with the Biot Slow Wave

A physical interpretation of the function φ, a quantity sometimes refered to as the pseudo-phase (Virieux et al.(1994)Virieux, Flores-Luna, & Gibert), follows if I consider the zeroth-order solution for the displacement of the solid phase

$$\mathbf{U}(\mathbf{X}, \varphi) = e^{-\varphi}\mathbf{U}_0(\mathbf{X}) \tag{73}$$

[see equation (9)]. Making use of equations (34) and (38), equation (73) becomes

$$\mathbf{U}(\mathbf{X}, \varphi) = e^{-\sqrt{s}\tau} U_0(\mathbf{X})\hat{\mathbf{p}}. \tag{74}$$

Substituting the expression for $U_0(\mathbf{X})$, equation (72), I arrive at the solution in the s domain

$$\mathbf{U}(\mathbf{X}, \varphi) = A_0 e^{-\chi} e^{-\sqrt{s}\tau}\hat{\mathbf{p}}. \tag{75}$$

The inverse Laplace transform of (75) gives the time-domain expression (Spiegel(1990))

$$\mathbf{u}(\mathbf{X}, t) = A_0 \hat{\mathbf{p}} e^{-\chi} \frac{\tau}{2\sqrt{\pi t^3}} e^{-\tau^2/4t} H(t) \tag{76}$$

where $H(t)$ is the step-function, which is zero for negative values and unity for positive values. A similar solution was described by (Virieux et al.(1994)Virieux, Flores-Luna, & Gibert) in relation to diffusive electromagnetic imaging and by (Vasco et al.(2000)Vasco, Karasaki, & Keers; Shapiro et al.(2002)Shapiro, Rothert, Rath, & Rindschwentner) for pressure propagation. For a general time-varying source the solution will be a convolution of equation (76) with the source-time function, which I will denote by $q(t)$

$$\mathbf{u}(\mathbf{X}, t) = A_0 \hat{\mathbf{p}} e^{-\chi} \frac{\tau}{2\sqrt{\pi t^3}} e^{-\tau^2/4t} H(t) * q(t) \tag{77}$$

where $*$ denotes a temporal convolution (Bracewell(1978)).

Note that (77) is the solution associated with a delta-function source. For diffusive processes it is more common to employ a step-function source in which energy or mass is introduced into the subsurface at a constant rate following the activation of the source. Because the step-function results from integrating a delta-function source, I can derive the solution associated with a step-function source by simply integrating (77) with respect to time. The resulting solution is

$$\mathbf{u}(\mathbf{X}, t) = A_0 \hat{\mathbf{p}} e^{-\chi} \int_0^t \frac{\tau}{2\sqrt{\pi u^3}} e^{-\tau^2/4u} du \tag{78}$$

or

$$\mathbf{u}(\mathbf{X}, t) = A_0 \hat{\mathbf{p}} e^{-\chi} \mathrm{erfc}\left(\frac{\tau}{2\sqrt{t}}\right) \tag{79}$$

where erfc is the complimentary error function (Spiegel(1990)). Note that in a homogeneous medium, equation (79) agrees with the results of (Rudnicki(1986)) for early times. It is similar in form a generalization of the (Theis(1935)) solution for transient pressure variations due to fluid withdrawal (Vasco et al.(2000)Vasco, Karasaki, & Keers).

As shown by (Virieux et al.(1994)Virieux, Flores-Luna, & Gibert), one may interpret the quantity τ in terms of an 'arrival time'. Specifically, consider the time associated with the arrival of the peak of the pulse in equation (77). Alternatively, if working with a step-function source, consider the time associated with the arrival peak of the time derivative of the expression (79), the steepest slope (Vasco et al.(2000)Vasco, Karasaki, & Keers). Differentiate the expression (77) for $u(\mathbf{X}, t)$ with respect to t

$$\frac{\partial \mathbf{u}(\mathbf{X}, t)}{\partial t} = A_0 \hat{\mathbf{p}} e^{-\chi} \frac{\tau}{2\sqrt{\pi}} e^{-\tau^2/4t} \left[-\frac{3}{2\sqrt{t^5}} + \frac{\tau^2}{4\sqrt{t^7}}\right] \tag{80}$$

which has a zero when the quantity in square brackets vanishes, i.e. when

$$t = \frac{\tau^2}{6}. \tag{81}$$

Thus, the phase τ is given by

$$\tau = \sqrt{6T_{peak}} \tag{82}$$

where T_{peak} denotes the time at which (80) vanishes. This coincides with the peak of the derivative of the longitudinal solid displacement. Thus, the 'phase', $\tau(\mathbf{X})$, is proportional to the square root of the time at which the longitudinal deformation is a maximum.

2.4 Construction of the Zeroth-Order Displacement Field Associated with the Biot Fast Wave

Using the asymptotic representation, equation (9), I can also construct the displacement field associated with the Biot fast wave. The zeroth-order solution takes the form

$$\mathbf{U}(\mathbf{X}, \varphi) = e^{-s\tau_f} U_0(\mathbf{X})\hat{\mathbf{p}} \tag{83}$$

where τ_f is given by the integral (62). Taking the inverse Laplace transform (Spiegel(1990)) produces the time-domain expression

$$\mathbf{u}(\mathbf{X}, t) = U_0(\mathbf{X})\delta(t - \tau_f)\hat{\mathbf{p}} \tag{84}$$

which is a time-shifted delta function. Thus, unlike the slow wave, the Biot fast wave is not of a diffusive nature. Rather, it has the character of a propagating elastic wave. Notice that, due to the nature of the source term Q in equation (2), which is the input pressure source divided by s, the time integral of the original source-time pressure function is used to calculate the displacement.

2.5 On the Use of a Numerical Simulator to Construct the Trajectory $\mathbf{X}(r)$ Associated with the Biot Slow Wave

The preceding considerations provide a method for constructing an asymptotic solution from the output of a numerical simulator. Such constructions prove useful in solving inverse problems, as outlined in (Vasco et al.(2000)Vasco, Karasaki, & Keers) and (Vasco & Finsterle(2004)). That is, the asymptotic solution provide semi-analytic expressions for model parameter sensitivities required for the inversion of field data. The basic idea is to use the pressure-deformation history from a numerical simulator to compute τ and hence $\mathbf{p}$, and then numerically integrate the first set of differential equations in (31) to find the trajectories $\mathbf{X}(r)$. Given the trajectories and the phase, one can construct the displacement field from equations (77) or (79), depending on the nature of the source-time function. The sensitivities to changes in flow properties can be obtained from either a perturbation approach (He et al.(2006)He, Datta-Gupta, & Vasco) or a Born technique (Vasco et al.(2000)Vasco, Karasaki, & Keers).

In detail, I first model the pressure change and deformation using a numerical simulator. From the simulator output I compute the arrival time T_{peak} by either estimating the arrival time of the peak [equation (77)] or the time at which the slope is greatest [equation (79)]. From the distribution of T_{peak} over the simulation grid I compute $\mathbf{p}$ from the gradient of τ or equivalently φ [see equations (35) and (36)]. Given $\mathbf{p}$, the trajectories are found by integrating the first set of equations (31) numerically. The trajectories are obtained numerically by stepping down the gradient of $\tau(\mathbf{X})$, starting at an observation point. The formal procedure we use, a second-order Runge-Kutta technique, is known as Heun's method. Heun's method is quite simple and can be implemented in approximately twenty to thirty lines of computer code. In essence, Heun's method improves upon an Euler iteration by computing the gradient at an intermediate point. That is, after the i-th step along the trajectory we take an intermediate step based upon equation (31)

$$\hat{\mathbf{X}}_i = \mathbf{X}_i - \frac{\mathbf{p}(\mathbf{X}_i)}{\varsigma(\mathbf{X}_i)}\delta r \tag{85}$$

where $\mathbf{p} = \nabla\tau$. The (i+1)-th step is simply the average of the gradients at $\mathbf{X}_i$ and the intermediate point $\hat{\mathbf{X}}_i$

$$\mathbf{X}_{i+1} = \mathbf{X}_i - \frac{\delta r}{2}\left[\frac{\mathbf{p}(\mathbf{X}_i)}{\varsigma(\mathbf{X}_i)} + \frac{\mathbf{p}(\hat{\mathbf{X}}_i)}{\varsigma(\hat{\mathbf{X}}_i)}\right]. \tag{86}$$

I should emphasize that, while I am computing the trajectory numerically, based upon equation (31), I do not solve the two-point boundary value problem itself. Rather, I let the reservoir simulator determine the distribution of $\tau(\mathbf{x})$ which satisfies the initial and boundary conditions. If I were to actually determine both $\tau(r)$ and $\mathbf{X}(r)$ I must solve either the coupled set of differential equations (31) by numerical integration or solve the Eikonal equation using a technique such as the fast marching method (Sethian(1999)). Both procedures require more extensive coding, although there are packages of programs and efficient algorithms to accomplish these tasks (Press et al.(1992)Press, Teukolsky, Vetterling, & Flannery; Sethian(1999)).

2.6 Propagation Across a Discontinuity

The asymptotic solution is based upon the assumption that the medium is smoothly varying, in a sense made precise in Appendix A. However, discontinuities are often present in the subsurface and will influence the fluid pressure and deformation fields at depth. As in ray methods, (Aki & Richards(1980); Kravtsov & Orlov(1990)), it is possible to account for discontinuities in material properties by including them as boundary conditions (Keller & Lewis(1995)). The boundary conditions I consider apply to the transient quantities in the pressure and deformation fields and do not apply to steady-state properties, as modeled using streamlines in fluid flow (Bear(1972); de Marsily(1986)).

The boundary conditions are based upon the continuity of the transient disturbance and its normal derivative across the discontinuity (Keller & Lewis(1995)). For the derivation that follows I shall work in the transform domain, where the displacement in the solid is given by equation (75). I shall consider two media, labeled by the subscripts 1 and 2, separated by an interface $\mathcal{I}$. The boundary conditions across $\mathcal{I}$ are

$$\mathbf{U}_1(\mathbf{X}, \varphi) = \mathbf{U}_2(\mathbf{X}, \varphi) \tag{87}$$

and

$$\frac{\partial \mathbf{U}_1}{\partial n} = \frac{\partial \mathbf{U}_2}{\partial n} \tag{88}$$

where n is in the direction of the normal vector to the interface, $\hat{\mathbf{n}}$, and the equations only hold on the interface. In general, in addition to the incident front there will be a reflected and a transmitted front. The amplitudes of the incident, transmitted, and reflected front are denoted by $\mathbf{U}_i$, $\mathbf{U}_t$, and $\mathbf{U}_r$, respectively. From equation (87) I have

$$\mathbf{U}_t(\mathbf{X}, \varphi) = \mathbf{U}_i(\mathbf{X}, \varphi) + \mathbf{U}_r(\mathbf{X}, \varphi). \tag{89}$$

Substituting the asymptotic form of the solution (75) for each field quantity in equation (89) I arrive at the constraint

$$\tau_i(\mathbf{X}) = \tau_r(\mathbf{X}) = \tau_t(\mathbf{X})$$

for $\mathbf{X}$ on the interface $\mathcal{I}$. It follows that the derivative of the functions τ_i, τ_r, and τ_t in any direction of the tangent plane of the surface are equal. Because of equation (36) I can relate the spatical derivatives of τ to derivatives of φ and thus to the components of $\mathbf{p}$, because of equation (11). Thus, the continuity of the derivatives of $\mathbf{p}$ in the tangent directions of $\mathcal{I}$ may be written as

$$\mathbf{p}_r \times \hat{\mathbf{n}} = \mathbf{p}_i \times \hat{\mathbf{n}} \tag{90}$$

and

$$\mathbf{p}_t \times \hat{\mathbf{n}} = \mathbf{p}_i \times \hat{\mathbf{n}} \tag{91}$$

where $\hat{\mathbf{n}}$ is the normal to the interface $\mathcal{I}$. Taking the norm of equations (90) and (91), using the definition of the magnitude of the vector cross product, and making use of the Eikonal equation (23a) or (24) gives

$$\sin\theta_r = \sin\theta_i \tag{92}$$

and

$$\sqrt{\gamma_1}\sin\theta_i = \sqrt{\gamma_2}\sin\theta_t \tag{93}$$

where

$$\gamma_l = \frac{C_l}{k_l} + \frac{{\alpha_l}^2}{\lambda_l + 2\mu_l}\frac{1}{k_l} \tag{94}$$

for $l = 1, 2$. Given the incidence angle θ_i I can use equation (93) to find the angle associated with the refracted front. An example of the bending of a trajectory across a sharp interface is given in Figure 3. In response to a decrease in hydraulic conductivity, the trajectory bends. The change in angle, governed by equation (93), is such that the total phase integral, equation (33), is minimized (Kravtsov & Orlov(1990)).

The preceding relationship between incident, reflected, and transmitted take-off angle, equations (92) and (93) can be combined with a two-point ray-tracing technique (Press et al.(1992)Press, Teukolsky, Vetterling, & Flannery) in order to compute trajectories in a medium with discontinuities. First, the ray is traced through the upper-most medium until it encounters an interface. When the interface is encountered I use equation (93) along with the value of θ^i and the properties above (γ_1) and below (γ_2) the discontinuity to refract the trajectory as it crosses the boundary. Thus, I compute the angle, θ^t, at which the transmitted field trajectory leaves the discontinuity. Then the raytracing proceeds in the second medium using a ray-tracing technique and the new angle θ^t. The procedure continues across all relevant interfaces until the observation point is reached. Thus, a raytracing routine may be used to compute $\mathbf{X}(r)$ with a finite number of adjustments, one for each discontinuity.

Once the ray is found, the amplitude is calculated using the solution of the amplitude equation (72), and accounting for the continuity of the components of $\mathbf{U}$ across the interface. For example, the continuity of the displacement field across the interface leads to the equation

$$\mathbf{U}^i + \mathbf{U}^r = \mathbf{U}^t \tag{95}$$

(Aki & Richards(1980)). This equation, coupled with relationships between $\mathbf{p}^i$, $\mathbf{p}^r$, $\mathbf{p}^t$, ,$\mathbf{U}^i$, $\mathbf{U}^r$, $\mathbf{U}^t$, and the properties on each side of the interface lead to reflection and transmission coefficients associated with the discontinuity (Aki & Richards(1980)). The reflection and transmission coefficients are used to modify the field amplitudes as the waves encounter the interface. That is, they are multiplication factors which are applied to the incident field amplitude in order to calculate the transmitted and reflected field amplitudes.

In summary, the presence of a discontinuity does not fundamentally alter the process of computing the trajectory in a medium with continuous changes in flow properties. I still employ a standard technique, such as a Newton-Raphson routine (Press et al.(1992)Press, Teukolsky, Vetterling, & Flannery), to conduct two-point raytracing. The discontinuity induces a refraction of the ray as it crosses the interface. This refraction is computed using (93) and incorporated into the routine for computing the trajectory. Similarly, a trajectory associated with the reflected wave may be calculated using equation (92) and the two-point raytracing routine. The amplitudes are still determined by solving the amplitude equation (64), in the regions separated by the interface. The discontinuity itself gives rise to a partitioning of energy which is governed by equation (95) and described by reflection and transmission coefficients. The computation of reflection and transmission coefficients is straight-forward and has been discussed elsewhere (Aki & Richards(1980)). Because the main focus of this paper is the propagation of pressure and displacement wavefields in smoothly varying media, I shall not discuss the computation of reflection and transmission coefficients any further.

3 APPLICATIONS

In this section I implement the expressions presented above in order to calculate pressure and displacement fields due to fluid injection in homogeneous and heterogeneous whole spaces. The results are compared to both analytic (Rudnicki(1986); Wang & Kumpel(2003)) and numerical computations (Masson et al.(2006)Masson, Pride, & Nihei) in order to assess the utility and accuracy of the asymptotic approach. I begin this section with some observations concerning the calculation of the full poroelastic response due to fluid injection.

3.1 Calculation of the Full Transient Disturbance Induced by Fluid Injection

As noted in the Methodology section, there are two modes of propagation in a poroelastic medium, the Biot fast and slow waves. Both modes are longitudinal displacements of the solid matrix but with differing velocities. A question arises as to how these two modes are generated and synthesized to form the response at a receiver. The answer to this question requires consideration of the mechanics of poroelastic deformation due to the injection of fluid. As fluid is injected at a well, the fluid pressure increases, deforming the solid matrix near the wellbore. This deformation propagates outward as an elastic disturbance from the region of greatest pressure change to the surrounding medium. That is, the elastic medium responds to the fluid volume change due to injection. According to the quasi-static governing equations (1), the response of the elastic medium is instantaneous. However, for the full poroelastic governing equations, containing second order partial derivatives in time, the response propagates with the elastic wave speed of the medium (Pride(2005)). With time, the region of greatest pressure change migrates away from the wellbore and propagates through the fluid component of the medium. As the fluid pressure disturbance propagates through the medium it induces deformation within the elastic matrix.

The relationship of these physical considerations to the mathematical expressions derived above may be summarized by the diagram in Figure 4. A slow wave, signified by the solid line in Figure 4, propagates from the wellbore source to a particular receiver. For a delta-function source-time function, the propagation from the source location to a point a distance r along the trajectory, $\mathbf{X}(r)$, is given by equation (77). At the point $\mathbf{X}(r)$ the pressure-induced displacement is propagated outward to the receiver location as an elastic disturbance. In essence, the pore-pressure induced displacement 'sheds' elastic deformation as it propagates. Because the elastic properties of the formation are uniform the elastic trajectories are straight-lines, as shown in Figure 4. Thus, the total solid phase displacement at the receiver is a summation over all the pressure-induced deformation along the trajectory, accounting for the elastic propagation from point of generation, $\mathbf{X}(r)$, to the receiver. The effect of the elastic propagation is contained in the time-domain expression for the Biot fast wave, equation (84). The total solid phase displacement is an integral over the Biot slow wave path from the source at $\mathbf{X}_0$ to the receiver at $\mathbf{X}_f$

$$\mathbf{u}(\mathbf{X}_f, t) = \int_{\mathbf{X}_0}^{\mathbf{X}_f} U^e(\mathbf{X}_r, \mathbf{X}_f) A_0 \hat{\mathbf{p}} e^{-\chi_r} \frac{\tau_r}{2\sqrt{\pi t^3}} e^{-{\tau_r}^2/4t} H(t) dr. \tag{96}$$

In this expression I have assumed instantaneous propagation of the fast wave from a position on the trajectory $\mathbf{X}_r$ to the receiver and the effects of the fast propagation are contained in the amplitude term $U^e(\mathbf{X}_r, \mathbf{X}_f)$. The quantities τ_r and χ_r are the diffusive phase and amplitude terms for propagation from $\mathbf{X}_0$ to $\mathbf{X}_r$. For point r of the trajectory of the Biot slow wave there will be a corresponding total trajectory which travels along the slow wave trajectory to the point $\mathbf{X}(r)$ and then travels along the Biot fast wave trajectory from $\mathbf{X}(r)$ to $\mathbf{X}_f$, as in Figure 4.

3.2 Homogeneous Whole Space Solution

As a first application I consider the displacement and pressure changes due to injection of a fluid volume into a homogeneous poroelastic whole space. The source-time function $q(t)$ is a step-function, as shown in Figure 5. The medium is characterized by the elastic parameters λ =5.91 GPa, μ =0.45 GPa, α =0.83 and by a constant hydraulic conductivity of 1.00×10^{-12} and porosity of 0.33.

The asymptotic solution was calculated using a two-point ray-tracing code, the numerical implementation of a solver for the system of equations (31) (Press et al.(1992)Press, Teukolsky, Vetterling, & Flannery). In addition, the phase and amplitude terms τ and χ were computed using the expressions (35) and (71). The complete solution for the Biot slow wave is given by equation (79). This quantity gives the pressure induced deformation associated with the slow wave and was used to compute the pressure response. The displacement was computed using the temporal integral of the expression (96). The time integration is necessary because the source is a step function and equation (96) is the response due to a delta function source.

For comparison, I implemented the analytic expressions derived by (Rudnicki(1986)), as presented in (Wang & Kumpel(2003)) for the displacement

$$\mathbf{u}(\mathbf{X}, t) = \frac{q_0(1+\nu_u)B}{24\pi(1-\nu_u)D} \frac{\mathbf{X}-\mathbf{X}_s}{R} F(\xi), \qquad (97)$$

and the pressure

$$p(\mathbf{X}, t) = \frac{q_0}{4\pi k} \frac{1}{R} \mathrm{erfc}(\frac{\xi}{2}) \qquad (98)$$

induced by a step function source located at point $\mathbf{X}_s$, where

$$\xi = \frac{R}{\sqrt{Dt}} \qquad (99)$$

and

$$F(\xi) = \mathrm{erfc}(\frac{\xi}{2}) + \frac{2}{\xi^2}\mathrm{erf}(\frac{\xi}{2}) - \frac{2}{\xi\sqrt{\pi}} \exp(-\frac{\xi^2}{4}). \qquad (100)$$

In these expressions q_0 is the flow rate, ν_u is the undrained Poisson's ratio, B is Skempton's coefficient (Skempton(1954)), D is the hydraulic diffusivity, R is the distance from the source to the receiver, and k is the hydraulic conductivity. These parameters can be derived from λ, μ, α, C, and k, as shown in (Wang & Kumpel(2003)).

In addition to the analytic solution of (Rudnicki(1986)), I also calculated pressure changes and solid matrix displacements using a finite-difference code (Masson et al.(2006)Masson, Pride, & Nihei). The code uses an explicit time stepping, staggered-grid finite difference method for solving Biot's equations in the low-frequency limit (for frequencies less then 10 kHz). The formulation is not entirely equivalent to the quasi-static solution treated in (Rudnicki(1986)) and presented here because (Masson et al.(2006)Masson, Pride, & Nihei) consider the low-frequency 'seismic limit' of the poroelastic governing equations. Thus, their approach incorporates inertial terms, in particular they include second-order time derivatives. To this end, additional parameters, such as fluid viscosity, and fluid and solid densities, are present. These parameters are not found in the quasi-static governing equations (1) treated here. The finite-difference code incorporates rigorous stability conditions and has been compared to exact analytical solutions for both fast and slow waves. Three pressure-difference fields from the numerical calculation are shown in Figure 6. Because the source is a step function in time, by differencing the pressure I can generated a pressure field which is similar to that generated by a delta function source. The pressure difference is more closely related to a propagating disturbance or wave front, as is evident in Figure 6. As noted above, from the pressure history I can compute the quantity τ, using equation (82) and the distribution of the arrival time of the peak slope of the transient pressure curve. The arrival time of the peak slope is shown in Figure 7, along with the trajectory obtained by solving the ray equations (31).

In Figure 8 I compare the asymptotic, the analytic, and the numeric solutions for a receiver located five meters from the source, indicated by the unfilled star in Figures 6 and 7. The solutions have been normalized so that the peak amplitudes of the curves correspond to a value of 1. In general, there is excellent agreement between the different methods. This is encouraging because these are very different approaches for calculating pressures and displacements. Note that the pressure change at the receiver is close to zero for early times and gradually increases after about 0.5 seconds. In contrast, the longitudinal displacement of the solid matrix increases roughly linearly from the start of injection and gradually approaches a constant vallue. This difference is explained by the generation of elastic displacement at the injection point due to the pressure change. As noted above, the induced displacement propagates from the region of greatest pressure change to the observation point as an elastic wave, a Biot fast wave. As indicated by equation (84), this mode of propagation is non-diffusive and only decays in amplitude because of geometrical spreading. Thus, deformation associated with early injection is observed at the receiver almost instantaneously, with an amplitude decrease characteristic of elastic wave propagation. Over time the displacement increases as the pressure change propagates away from the well and deformation accumulates in the whole space. The rate of increase slows over time because the amplitude of the diffusive wave decreases significantly away from the well (Figure 6). Thus, the asymptotic solution provides insight into the nature of the displacement variations of the solid phase.

3.3 Heterogeneous Whole Space Solution

I also consider a smoothly-varying, heterogeneous distribution of hydraulic conductivity, as shown in Figure 9. The variation in k is characterized by the formula

$$k = \frac{50k_0}{1+Y} \qquad (101)$$

where $Y = X + Z$ is sum of the distance along the X and Z axes. The pressure differences for three different times, 1s, 3s and 10s, are shown in Figure 10. The outwardly propagating disturbance is no longer rotationally symmetric, though it is symmetric about a line 45^o from the X-axis, due to the symmetry of the conductivity distribution. The resulting travel time distribution is asymmetric as well, though the phase deviations are more subtle (Figure 11) than are the wavefield amplitudes. The trajectory from the source to the receiver does deviate from a straight line, due to the heterogeneous distribution of k. Because of the heterogeneity, an analytic solution is not possible and I only compare the trajectory-based asymptotic solution to a numerical finite-difference solution (Masson et al.(2006)Masson, Pride, & Nihei). The numeric and asymptotic pressure and solid phase displacements are shown in Figure 12. The general characteristics are similar to the solutions in the homogeneous medium. That is, the pressure at the observation point increases very slowly at first while the solid displacement seems to vary almost linearly after the onset of fluid injection. As mentioned above, the difference behaviour is best explained by the differences in the modes of propagation. That is, the pressure propagates as a diffusive disturbance while the displacement of the solid phase travels in the manner of an elastic waves, arriving at the observation point almost instantaneously and with much less amplitude decay with distance. Though there is good overall agreement between the numeric and the asymptotic calculations there are differences in detail. These deviations are most likely due to the differences in formulation. For example, the numeric solution contains additional parameters, such as fluid viscosity, fluid density, and solid density associated with the second-order time derivatives of the fluid and solid displacements (Masson et al.(2006)Masson, Pride, & Nihei). Also, there are approximations in the numeric solution due to discretization and the presence of grid with boundaries at a finite difference from the observation point. Given these differences the agreement between the two solutions is deemed acceptable.

4 CONCLUSIONS

The asymptotic approach provides explicit, semi-analytic expressions for the phase and amplitude of both the diffusive Biot slow wave and the hyperbolic Biot fast wave, two solutions of the equations governing the response of a poroelastic medium to a pressure source [equations (1)]. The diffusive slow wave decays rapidly as it propagates while the fast wave decays in the manner of an elastic wave. Thus, the fast wave generated by a rapid change in pressure, will produce the most significant early time displacements at a remote observation point. As the Biot slow wave propagates it generates or sheds elastic deformation in the form of Biot fast waves. The total response is a summation over the Biot slow wave and the accompanying Biot fast waves. This superposition of slow and fast waves accounts for the rapid increase in longitudinal displacement directly after the onset of pumping at a source well. In contrast, the pressure response, which is primarily due to the Biot slow wave, is gradual and begins after some time delay due to propagation from the source to the receiver. I should note that the fast wave will also generate a slow wave due to the coupling of pressure and deformation. However, in the situations considered here the accompanying slow wave appears to be small enough to be neglected. In some cases a propagating fast wave, a wave of elastic deformation, does give rise to fluid pressure changes which are observable. For example, deformation tied to a distance earthquake, propagating as an elastic wave, has been observed to produce fluid pressure changes and changes in micro-seismicity in quasi-stable geothermal regions (Jónsson et al.(2003)Jónsson, Segall, Pedersen, & Björnsson).

The analysis and applications in this paper illustrate the utility of asymptotic techniques in the study of coupled fields in complicated settings. Typically, in geophysics asymptotic methods have been limited to modeling high-frequency seismic and electromagnetic wave progatation (Aki & Richards(1980)). It is not generally appreciated that asymptotic methods are applicable to a wide range of equations, and can model behaviour ranging from diffusive propagation to hyperbolic, wave-like propagation. The generality of asymptotic techniques has been pointed out by others, particular in the study of diffusive, dispersive, and non-linear systems (Jeffrey & Taniuti(1964); Whitham(1974); Taniuti & K.(1983); Anile et al.(1993)Anile, Hunter, Pantano, & Russo; Sachdev(2000)).

In geophysical and hydrological applications, asymptotic methods have been used in a more general sense by a limited number of investigators. One of the earliest known applications was to electromagnetic propagation within the Earth (Virieux et al.(1994)Virieux, Flores-Luna, & Gibert). This was followed by applications to transient fluid pressure propagation in the subsurface (Vasco et al.(2000)Vasco, Karasaki, & Keers; Shapiro et al.(2002)Shapiro, Rothert, Rath, & Rindschwentner). Asymptotic methods have also been applied to governing equations of mixed character, in which the propagation cannot necessarily be classified as strictly wave-like or diffusive. For example, the method of multiple scales has been used to model tracer transport which can vary from hyperbolic to diffusive propagation, depending on nature of the tracer and the flow conditions (Vasco & Finsterle(2004)). This technique has also be used to model two-phase flow in the sub-surface, a type of non-linear front propagation which can vary in character from hyperbolic to diffusive (Vasco(2004)). Recently, the method of multiple scales has been used to model broadband electromagnetic wave propagation in the Earth (Vasco(2007)). In that case complex raytracing was used to model propagation that had characteristics of both hyperbolic and diffusive propagation.

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5 APPENDIX A: THE METHOD OF MULTIPLE SCALES

In this Appendix I use the method of multiple scales to derive an asymptotic solution to the governing equations. The solution is valid when the heterogeneity is smoothly varying. If I denote the scale of variation of the background properties by L and the mean wavelength of propagating disturbance by l, then $L \gg l$. The ratio of scales is characterized by the parameter $\epsilon = l/L$ which is assumed to be much smaller than 1.

I shall investigate disturbances for which the amplitude, wavenumber and frequency vary slowly, at the scale-length of the variation in background properties L while the phase varies rapidly, at the scale-length l. In the method of multiple scales one defines slow variables $\mathbf{X}$ in terms of the physical variables $\mathbf{x}$ by

$$X^i = \epsilon x^i, \qquad (A1)$$

where $\epsilon = l/L$. The displacement field is then written as a formal power series expansion in ϵ

$$\mathbf{U}(\mathbf{X}, \varphi) = e^{-\varphi} \sum_{p=0}^{\infty} \epsilon^p \mathbf{U}_p(\mathbf{X}), \qquad (A2)$$

where $\varphi(\mathbf{X})$ is the local phase. The series (A2) may be thought of as a local plane wave expansion in the Laplace s domain.

Now $\mathbf{U}(\mathbf{X}, \varphi)$ depends on $\mathbf{x}$ through the dependence of $\mathbf{X}$ and φ. Thus, the spatial derivatives which appear in the vector differential equation (8) may be rewritten in terms of the new variables. Specifically, the partial derivative with respect to x_i may be written

$$\frac{\partial}{\partial x_i} = \frac{\partial X_i}{\partial x_i}\frac{\partial}{\partial X_i} + \frac{\partial \varphi}{\partial X_i}\frac{\partial}{\partial \varphi}$$

$$= \epsilon \frac{\partial}{\partial X_i} + \frac{\partial \varphi}{\partial X_i}\frac{\partial}{\partial \varphi}. \qquad (A3)$$

The first step involves writing the differential operators in equation (8) in terms of the slow variables, as in equations (A3). Thus, the differential operators $\nabla\cdot$ and $\nabla\times$ in equation (8) become

$$\nabla_{\mathbf{x}} \cdot = \epsilon \nabla \cdot + \nabla\varphi \cdot \frac{\partial}{\partial \varphi} \qquad (A4a)$$

$$\nabla_{\mathbf{x}} \times = \epsilon \nabla \times + \nabla\varphi \times \frac{\partial}{\partial \varphi}, \qquad (A4b)$$

respectively, where the subscript $\mathbf{x}$ signifies that the gradient is computed with respect to the original variables x_i.. Expressions such as these are substituted into equation (8), which is re-written in terms of $\mathbf{U}(\mathbf{X}, \varphi)$. For example, consider the first term in equation (8), $\nabla_{\mathbf{x}} \nabla_{\mathbf{x}} \cdot \mathbf{U}$,

$$\nabla_{\mathbf{x}} \nabla_{\mathbf{x}} \cdot \mathbf{U} = \nabla_{\mathbf{x}} \left[\epsilon \nabla \cdot \mathbf{U} + \nabla\varphi \cdot \frac{\partial \mathbf{U}}{\partial \varphi} \right]. \qquad (A5)$$

Expanding the differential operator $\nabla_{\mathbf{x}}$ further, results in the expression

$$\nabla_{\mathbf{x}} \nabla_{\mathbf{x}} \cdot \mathbf{U} = \epsilon^2 \nabla\nabla \cdot \mathbf{U} + \epsilon \nabla \left(\nabla\varphi \cdot \frac{\partial \mathbf{U}}{\partial \varphi} \right)$$

$$+ \epsilon \nabla\varphi \left(\nabla \cdot \frac{\partial \mathbf{U}}{\partial \varphi} \right) + \nabla\varphi \left(\nabla\varphi \cdot \frac{\partial^2 \mathbf{U}}{\partial^2 \varphi} \right) \qquad (A6)$$

which contains terms of various orders in ϵ. Carrying out the procedure for all the differential operators in equation (8), retaining only terms of order ϵ^0 and ϵ^1, results in the following expression

$$\Gamma \nabla\varphi \left(\nabla\varphi \cdot \frac{\partial^2 \mathbf{U}}{\partial \varphi^2} \right) - \Psi \nabla\varphi \times \left(\nabla\varphi \times \frac{\partial^2 \mathbf{U}}{\partial \varphi^2} \right)$$

$$-C^{-1} \Upsilon K \nabla\varphi \nabla\varphi \cdot \nabla\varphi \left(\nabla\varphi \cdot \frac{\partial^4 \mathbf{U}}{\partial \varphi^4} \right)$$

$$+\epsilon \Gamma \nabla (\nabla\varphi \cdot \frac{\partial \mathbf{U}}{\partial \varphi}) + \epsilon \Gamma \nabla\varphi \left(\nabla \cdot \frac{\partial \mathbf{U}}{\partial \varphi} \right)$$

$$-\epsilon \Psi \nabla \times (\nabla\varphi \times \frac{\partial \mathbf{U}}{\partial \varphi}) - \epsilon \Psi \nabla\varphi \times (\nabla \times \frac{\partial \mathbf{U}}{\partial \varphi})$$

$$+\epsilon \nabla C^{-1} \alpha \nabla\varphi \cdot \frac{\partial \mathbf{U}}{\partial \varphi}$$

$$-\epsilon \nabla C^{-1} K \Upsilon \nabla\varphi \cdot \nabla\varphi \left(\nabla\varphi \cdot \frac{\partial^3 \mathbf{U}}{\partial \varphi^3} \right)$$

$$-\epsilon C^{-1} \nabla\varphi \Upsilon \nabla K \cdot \nabla\varphi \left(\nabla\varphi \cdot \frac{\partial^3 \mathbf{U}}{\partial \varphi^3} \right)$$

$$+\epsilon C^{-1} \Psi \nabla\varphi \nabla K \cdot \left(\nabla\varphi \times \nabla\varphi \times \frac{\partial^3 \mathbf{U}}{\partial \varphi^3} \right) \qquad (A7)$$

$$-\epsilon C^{-1} \nabla K \Upsilon \nabla\varphi \cdot \nabla\varphi \left(\nabla\varphi \cdot \frac{\partial^3 \mathbf{U}}{\partial \varphi^3} \right)$$

$$-\epsilon C^{-1} K \Upsilon \nabla \left(\nabla\varphi \cdot \nabla\varphi \nabla\varphi \cdot \frac{\partial^3 \mathbf{U}}{\partial \varphi^3} \right)$$

$$-\epsilon C^{-1} K \Upsilon \nabla\varphi \nabla \cdot \left(\nabla\varphi \nabla\varphi \cdot \frac{\partial^3 \mathbf{U}}{\partial \varphi^3} \right)$$

$$-\epsilon C^{-1} K \Upsilon \nabla\varphi \nabla\varphi \cdot \nabla \left(\nabla\varphi \cdot \frac{\partial^3 \mathbf{U}}{\partial \varphi^3} \right)$$

$$-\epsilon C^{-1} K \Upsilon \nabla\varphi \nabla\varphi \cdot \nabla\varphi \left(\nabla \cdot \frac{\partial^3 \mathbf{U}}{\partial \varphi^3} \right) = 0.$$

Before I examine terms of order ϵ^0 and ϵ^1 in more detail, I rewrite equation (A7) in terms of the slowness vector

$$\mathbf{p} = \nabla\varphi \qquad (A8)$$

and make use of the fact that the particular form of $\mathbf{U}(\mathbf{X}, \varphi)$ in equation (A2) leads to an expression for the partial derivatives of $\mathbf{U}$ with respect to φ

$$\frac{\partial^l \mathbf{U}}{\partial \varphi^l} = (-1)^l \mathbf{U}. \qquad (A9)$$

Thus, equation (A7) becomes

$$-\Gamma \mathbf{p} \left(\mathbf{p} \cdot \mathbf{U} \right) + \Psi \mathbf{p} \times \left(\mathbf{p} \times \mathbf{U} \right)$$

$$-C^{-1} \Upsilon K p^2 \mathbf{p} \left(\mathbf{p} \cdot \mathbf{U} \right)$$

$$+\epsilon \Gamma \nabla (\mathbf{p} \cdot \mathbf{U}) + \epsilon \Gamma \mathbf{p} (\nabla \cdot \mathbf{U})$$

$$-\epsilon\Psi\nabla\times(\mathbf{p}\times\mathbf{U})-\epsilon\Psi\mathbf{p}\times(\nabla\times\mathbf{U})$$

$$+\epsilon\nabla C^{-1}\alpha\mathbf{p}\cdot\mathbf{U}+\epsilon\nabla C^{-1}K\Upsilon p^2(\mathbf{p}\cdot\mathbf{U})$$

$$+\epsilon C^{-1}\Upsilon\mathbf{pp}\cdot\nabla K(\mathbf{p}\cdot\mathbf{U}) \qquad (A10)$$

$$-\epsilon C^{-1}\Psi\mathbf{p}\nabla K\cdot(\mathbf{p}\times\mathbf{p}\times\mathbf{U})$$

$$+\epsilon C^{-1}\Upsilon\nabla Kp^2(\mathbf{p}\cdot\mathbf{U})$$

$$+\epsilon C^{-1}\Upsilon K\nabla\left(p^2\mathbf{p}\cdot\mathbf{U}\right)$$

$$+\epsilon C^{-1}\Upsilon K\mathbf{p}\nabla\cdot(\mathbf{pp}\cdot\mathbf{U})$$

$$+\epsilon C^{-1}\Upsilon K\mathbf{pp}\cdot\nabla(\mathbf{p}\cdot\mathbf{U})$$

$$+\epsilon C^{-1}\Upsilon K\mathbf{p}p^2(\nabla\cdot\mathbf{U})=0.$$

5.1 Terms of Order ϵ^0

Taking the form of $\mathbf{U}(\mathbf{X},\varphi)$, equation (A2), into account, one sees that equation (A10) contains an infinite number of terms, each of some particular order in ϵ. Because I am assuming that $\epsilon \ll 1$, only terms of the lowest order in ϵ are important. After substituting the power series representation (A2) I find that the terms of lowest order in ϵ, those of order $\epsilon^0 \sim 1$, are

$$\left(\Gamma+C^{-1}\Upsilon Kp^2\right)\mathbf{pp}\cdot\mathbf{U}_0-\Psi\mathbf{p}\times(\mathbf{p}\times\mathbf{U}_0)=0 \qquad (A11)$$

a linear equation for the components of $\mathbf{U}_0$.

5.2 Terms of Order ϵ^1

Next, I consider terms of order ϵ, which are obtained by substituting the series representation (A2) into equation (A10). I assume that the vector $\mathbf{U}_1$ is parallel to $\mathbf{U}_0$ so that the terms containing $\mathbf{U}_1$ vanish, due to equation (A11). Then, I am left with the equation

$$\Gamma\nabla(\mathbf{p}\cdot\mathbf{U}_0)+\Gamma\mathbf{p}(\nabla\cdot\mathbf{U}_0)$$

$$-\Psi\nabla\times(\mathbf{p}\times\mathbf{U}_0)-\Psi\mathbf{p}\times(\nabla\times\mathbf{U}_0)$$

$$+\nabla C^{-1}\alpha\mathbf{p}\cdot\mathbf{U}_0+\nabla C^{-1}K\Upsilon p^2(\mathbf{p}\cdot\mathbf{U}_0)$$

$$+C^{-1}\Upsilon\mathbf{pp}\cdot\nabla K(\mathbf{p}\cdot\mathbf{U}_0) \qquad (A12)$$

$$-C^{-1}\Psi\mathbf{p}\nabla K\cdot(\mathbf{p}\times\mathbf{p}\times\mathbf{U}_0)$$

$$+C^{-1}\Upsilon\nabla Kp^2(\mathbf{p}\cdot\mathbf{U}_0)$$

$$+C^{-1}\Upsilon K\nabla\left(p^2\mathbf{p}\cdot\mathbf{U}_0\right)$$

$$+C^{-1}\Upsilon K\mathbf{p}\nabla\cdot(\mathbf{pp}\cdot\mathbf{U}_0)$$

$$+C^{-1}\Upsilon K\mathbf{pp}\cdot\nabla(\mathbf{p}\cdot\mathbf{U}_0)$$

$$+C^{-1}\Upsilon K\mathbf{p}p^2(\nabla\cdot\mathbf{U}_0)=0.$$

6 APPENDIX B: CALCULATION OF THE AMPLITUDE FUNCTION $U_0(\mathbf{X})$

In this Appendix I derive the amplitude function for longitudinal displacement, $U_0(\mathbf{X})$, presented in equation (38),

$$\mathbf{U}_0(\mathbf{X})=U_0(\mathbf{X})\hat{\mathbf{p}}.$$

Substituting this form into the terms of order ϵ, equation (A12), produces the equation

$$\Gamma\nabla(pU_0)+\Gamma\mathbf{p}\nabla\cdot(U_0\hat{\mathbf{p}})$$

$$-\Psi\mathbf{p}\times\nabla\times(U_0\hat{\mathbf{p}})$$

$$+\nabla C^{-1}\alpha pU_0+\nabla C^{-1}K\Upsilon p^3U_0$$

$$+C^{-1}\Upsilon\mathbf{pp}\cdot\nabla KpU_0 \qquad (B1)$$

$$+C^{-1}\Upsilon\nabla Kp^3U_0$$

$$+C^{-1}\Upsilon K\nabla\left(p^3U_0\right)$$

$$+C^{-1}\Upsilon K\mathbf{p}\nabla\cdot(\mathbf{p}pU_0)$$

$$+C^{-1}\Upsilon K\mathbf{pp}\cdot\nabla(pU_0)$$

$$+C^{-1}\Upsilon K\mathbf{p}p^2\nabla\cdot(U_0\hat{\mathbf{p}})=0.$$

Because the vector $\mathbf{p}$ is determined from equation (24) or the system (31), equation (B1) consists of three equations in a single unknown, $U_0(\mathbf{X})$. I can reduce equation (B1) to a single equation along the trajectory $\mathbf{X}(r)$ by taking the dot product with the unit vector $\hat{\mathbf{p}}$. Upon projecting onto the vector $\mathbf{p}$, Equation (B1) becomes

$$\Gamma\hat{\mathbf{p}}\cdot\nabla(pU_0)+\Gamma p\nabla\cdot(U_0\hat{\mathbf{p}})$$

$$+\hat{\mathbf{p}}\cdot\nabla C^{-1}\alpha pU_0+\hat{\mathbf{p}}\cdot\nabla C^{-1}K\Upsilon p^3U_0$$

$$+2C^{-1}\Upsilon\hat{\mathbf{p}}\cdot\nabla Kp^3U_0$$

$$+C^{-1}\Upsilon K\hat{\mathbf{p}}\cdot\nabla\left(p^3U_0\right)$$

$$+C^{-1}\Upsilon Kp\nabla\cdot\left(p^2U_0\hat{\mathbf{p}}\right) \qquad (B2)$$

$$+C^{-1}\Upsilon Kp^2\hat{\mathbf{p}}\cdot\nabla(pU_0)$$

$$+C^{-1}\Upsilon Kp^3\nabla\cdot(U_0\hat{\mathbf{p}})=0.$$

Note that because of the first set of ray equations (31), the projections of the gradient operator may be written as a derivative along the trajectory $\mathbf{X}(r)$. Thus, I can write equation (B2) as an ordinary differential equation

$$M\frac{dU_0}{dr}+NU_0=0 \qquad (B3)$$

where

$$M=\left(2+3C^{-1}Kp^2\right)p\Upsilon+2\alpha C^{-1}p \qquad (B4)$$

and

$$N=N_\nabla\nabla\cdot\hat{\mathbf{p}}+N_{p'}\frac{dp}{dr}+N_{C^{-1}}\frac{dC^{-1}}{dr}+N_K\frac{dK}{dr}. \qquad (B5)$$

where

$$N_\nabla = \alpha C^{-1} p + \left(1 + 2KpC^{-1}\right) p\Upsilon \qquad (B6)$$

$$N_{p'} = \alpha C^{-1} + \left(1 + 6Kp^2 C^{-1}\right) \Upsilon \qquad (B7)$$

$$N_{C^{-1}} = \left(\alpha + \Upsilon K p^2\right) p \qquad (B8)$$

$$N_K = 2C^{-1}\Upsilon p^3 \qquad (B9)$$

The amplitude function appears to depend upon the transform variable s because it appears in the expression for p, as given in equations (23a) and (24) and in the definition of K, equation (3). I examine this dependence in more detail by substituting the expression (3) for K and an expression for p

$$p = \Pi\sqrt{s} \qquad (B10)$$

where

$$\Pi = \sqrt{\frac{C}{k} + \frac{\alpha^2}{\lambda + 2\mu}\frac{1}{k}}. \qquad (B11)$$

From this representation I can write

$$\frac{dp}{dr} = \frac{d\Pi}{dr}\sqrt{s} \qquad (B12)$$

and substitute for K in the coefficients, (B6), (B7), (B8), and (B9), rewriting N as the sum

$$N = N'_\nabla \nabla \cdot \hat{\mathbf{p}} + N'_{C^{-1}} \frac{dC^{-1}}{dr} + N'_k \frac{dk}{dr}. \qquad (B13)$$

where

$$N'_\nabla = \left(\Gamma\Pi + 2k\Upsilon\Pi^3 C^{-1}\right)\sqrt{s} \qquad (B14)$$

$$N'_{C^{-1}} = \left(\alpha\Pi + \Upsilon k \Pi^3 - \frac{1}{2}C^2\Pi^{-1}k^{-1}\right)\sqrt{s} \qquad (B15)$$

$$N'_k = \left(2\Upsilon\Pi^3 C^{-1} - \frac{1}{2}\Pi k^{-1}\right)\sqrt{s}. \qquad (B16)$$

Similarly, I can rewrite M so that the dependence upon s is explicit

$$M = \left(2\Gamma\Pi - 3k\Pi^3\Upsilon C^{-1}\right)\sqrt{s}. \qquad (B17)$$

Thus, all coefficients are proportional to $\sqrt{s}$ and I can factor it out of the differential equation (B3), eliminating the dependence of $U_0(\mathbf{X})$ upon s. The explicit solution of the ordinary differential equation (B3) is

$$U_0(\mathbf{X}) = A_0 \exp\left[-\int_{\mathbf{X}} \varrho dr\right] \qquad (B18)$$

where

$$\varrho = R_\nabla \nabla \cdot \hat{\mathbf{p}} + R_{C^{-1}} \frac{dC^{-1}}{dr} + R_k \frac{dk}{dr}. \qquad (B19)$$

and A is the initial amplitude, a constant of integration which is determined by the source and

$$R_\nabla = \frac{\Gamma + 2k\Upsilon\Pi^2 C^{-1}}{2\Gamma - 3k\Pi^2\Upsilon C^{-1}} \qquad (B20)$$

$$R_{C^{-1}} = \frac{\alpha\Pi + \Upsilon k\Pi^3 - \frac{1}{2}C^2\Pi^{-1}k^{-1}}{2\Gamma\Pi - 3k\Pi^3\Upsilon C^{-1}} \qquad (B21)$$

$$R_k = \frac{2\Upsilon\Pi^2 C^{-1} - \frac{1}{2}k^{-1}}{2\Gamma - 3k\Pi^2\Upsilon C^{-1}}. \qquad (B22)$$

Note that the first term in (B19) is related to the divergence of the trajectories, the geometrical spreading, while the latter two terms are related to the heterogeneity. In particular, the derivatives of C^{-1} and k vanish in a medium with constant flow properties.

Figure 1. Two-dimensional illustration of the method used to quantify trajectory divergence for computing amplitudes.