FERMILAB-PUB-06-071-AD

Two-Beam Instability in Electron Cooling

A. Burov
Fermi National Accelerator Laboratory, Batavia, IL 60510

May 2, 2006

Abstract

The drift motion of cooling electrons makes them able to respond to transverse perturbations of a cooled ion beam. This response may lead to dipole or quadrupole transverse instabilities at specific longitudinal wave numbers. While the dipole instabilities can be suppressed by a combination of the Landau damping, machine impedance, and the active damper, the quadrupole and higher order modes can lead to either emittance growth, or a lifetime degradation, or both. The growth rates of these instabilities are strongly determined by the machine $x-y$ coupling. Thus, tuning out of the coupling resonance and / or reduction of the machine coupling can be an efficient remedy for these instabilities.

1 Introduction

Electron cooling is an effective method to increase phase space density of hadron beams. Since its invention by G. I. Budker [1], and experimental proof at NAP-M [2], it has been successfully used at many non-relativistic ion storage rings (see e. g. [3], [4]), and recently 8.9 GeV/c antiprotons have been electron-cooled at Recycler ring, Fermilab [5]. To cool the ion beam in a storage ring, it is merged with the a comoving electron beam in a part of the ion orbit. Cooling results then from a thermal flux from the hot ions to the cold electrons. Normally, the ion beam does not need focusing elements inside the cooler, while the electron beam is focused by a homogeneous solenoidal magnetic field.

Being able to make beams brighter, electron cooling brings specific problems for the cooled beams as well. First of all, electron cooling, as any cooling, makes the cooled beam less stable against any kind of coherent motion [6], would it be driven by an impedance of the chamber, or by a stochastic cooling system, or by a structure resonance of space-charge shifted envelope modes [7]. All these issues are out of scope of this paper, devoted to analysis of coherent interaction of the cooled (ion) beam and the cooling electron beam.

The cooling effect is caused by interactions of ions with single-particle, or microscopic fields of the electron beam. However, macroscopic fields of the electron beam act on the ions too, and this interaction can cause damage to the cooled beam. In principle, the situation here is similar to beam-beam effects in colliders, where the beam-beam effects are classified as either incoherent (weak-strong) or coherent (strong-strong) ones.

A tune shift by the space charge of the electron beam drives high-order resonances in the ion motion, and this incoherent effect may reduce a lifetime of the ions. This effect was first pointed out by D. Reistad [8], who suggested that as an explanation for reduction of the ion life time with the electron current, observed at CELSIUS ring [9], and called such kind of phenomena as "electron heating". Theoretical and numerical analysis of this incoherent beam-beam effect for CELSIUS parameters was given in Ref. [10]. Another possibility for incoherent beam-beam interaction could be an excitation of the ion oscillations by a noise of the electron beam at the betatron side-bands of the ions; influence of a noise of the electron current is estimated in Ref. [11]. Some coherent ion-electron interaction seems to be responsible for the lifetime reduction observed at several coolers, according to V. Parkhomchuk [12], [13]. Recent empirical data on various beam-beam effects can be found in Ref. [14].

Coherent ion-electron interaction was theoretically considered in a model of transversely immobile, totally magnetized electrons [15]; the longitudinal and the transverse impedances introduced by the electron beam were calculated. These impedances appeared to be too low for realistic parameters of electron coolers, indicating that the electron drift mobility should be taken into account. That approach was first attempted in two simultaneous and somewhat similar papers, [16] and [17]. A transverse offset of the ion beam causes a dipole electric field, forcing electrons to drift in the orthogonal transverse direction. This drift gives its own electric field, acting back on the ions. Being linear and local, this electron response can be described as a perturbation of the ion's revolution matrix. At first order, this non-symplectic perturbation matrix is proportional to a product of the electron and ion currents. The perturbation of the cooler matrix was calculated in Ref. [16] for arbitrary ion-electron and electron-ion phase advances, and neglecting the Larmor phase advance of ions. The stability analysis was then limited by a consideration of the determinant of the perturbed cooler matrix. It was found that the determinant is slightly decreased by the small perturbation. From here, it was concluded that ion-electron instabilities are impossible for realistic parameters in a framework of that model. In Ref. [17], the perturbed cooler matrix was calculated assuming the electron-ion, ion-electron and ion Larmor phase advances being small. Based on analysis of the perturbed revolution matrix, it was shown that horizontal-vertical coupling of the unperturbed ion motion is essential for the ion-electron instability. Without coupling, the growth rate was proved to be zero at lowest order over a beam-beam interaction parameter. Slightly later, in Ref. [20], the eigenvalue analysis for the perturbed revolution matrix was performed in a case when the zero-current revolution matrix is block-diagonal, with identical blocks, assuming also a beam waist in a middle of the cooler. The instability growth rate was found analytically for these conditions in the same order as in Ref. [17]. This rate, being non-zero, seemed to contradict to the conclusion of Ref. [17] about coupling. It is shown below how this seeming contradiction is untangled.

In what follows, first, the perturbed cooler matrix is re-derived; all the significant terms are kept at lowest order over the small phases. Then, a general perturbation formalism for coupled optics is described and applied for the coherent ion-electron interaction. The instability growth rate is analytically calculated in terms of general 4D Twiss parameters. In case when a single source of coupling is the cooler's solenoidal field, a simple expression for the growth rate is presented. Then, a suppression of the dipole ion-electron instability by the Landau damping, and its suppression or overshadowing by the chamber impedance are discussed. It is found that it is quite possible for this dipole instability to be either suppressed or overshadowed. The quadrupole ion-electron instabilities are discussed, their growth rates are presented. The quadrupole instability is concluded to be able being more effective than the dipole one, leading either to an emittance growth, or life-time degradation, or both. An important conclusion is that a straightforward remedy for all orders coherent ion-electron instabilities is tuning out of the coupling resonance and / or suppression of the machine coupling. This conclusion is supported by the recent Recycler experience, where the emittance growth essentially disappeared after these measures were taken [19].

2 Beam-Beam Dipole Interaction In the Cooler

The cooling electron beam is usually round and approximately of a constant density within its radius. As for the cooled ion beam, it can be shown that its transverse density distribution does not influence its dipole interaction with electrons, as long as the ions are mainly inside the electron beam, and the electron motion reduces to a drift in a permanent magnetic field of the cooler. This statement is proved here in two steps. First, it is shown that the electron beam drifts as an incompressible liquid, so its dipole motion is insensitive to the ion distribution inside the electron beam, as soon as the ion dipole moment is fixed. And second, it is used that the ion response to the electron beam offset does not depend on the specific ion position inside the electron beam.

For the electrons, the continuity equation gives the total time derivative of the beam density: $dn_e/dt = -n_e \boldsymbol{\nabla} \cdot \mathbf{v}_e(\mathbf{r})$. For the drift motion, the local velocity $\mathbf{v}_e(\mathbf{r})$ is proportional to a vector product of the quasi-static electric field $\mathbf{E}(\mathbf{r})$ and the constant magnetic field $\mathbf{B}$, i. e. $\mathbf{v}_e \propto \mathbf{E}(\mathbf{r}) \times \mathbf{B}$. Substituting this into the continuity equation gives $dn_e/dt \propto \boldsymbol{\nabla} \cdot [\mathbf{E}(\mathbf{r}) \times \mathbf{B}] = \mathbf{B} \cdot [\boldsymbol{\nabla} \times \mathbf{E}(\mathbf{r})] = \mathbf{0}$. Thus, in an arbitrary electrostatic field, the electron beam drifts as an incompressible liquid. The ion dipole perturbation moves the electron beam as a whole, and this motion is determined by the ion dipole moment only, being independent of the other details of the ion beam distribution. For the constant-density electron beam with a radius a_e, this means that all the density perturbations are at the beam surface, $\partial n_e/\partial t \propto \delta(r - a_e)$.

Turning to the back reaction of the electron beam on the ions, it is sufficient to note that the dipole electric field inside the displaced incompressible electron beam is homogeneous. Thus, wherever an inner ion is, its response to the electron dipole motion is independent of its position. Thus, both electron response to the ion dipole moment and the ion response to the electron dipole moment are independent of the ion distribution inside the electron beam. Whatever the real ion distribution is, the dipole beam-beam interaction goes in the same way, as if the ions were homogeneously distributed within the cross-section of the electron beam. That is why the parameter of the ion density n_i has to be taken here as the linear ion density λ_i, divided by the cross-section of the electron beam, $n_i \equiv \lambda_i/(\pi a_e^2)$.

Equations of motion for the electron and ion centroids $\xi_{e,i} \equiv x_{e,i} + iy_{e,i}$ can be presented as

$$\begin{aligned} \xi_i'' + k_{ie}^2(\xi_i - \xi_e) + ik_{iL}\xi_i' &= 0 \; ; \\ \xi_e'' + k_{ei}^2(\xi_e - \xi_i) - ik_{eL}\xi_e' &= 0 \; . \end{aligned} \tag{1}$$

Here k_{ie} and k_{ei} are space charge wave numbers, $k_{ie}^2 = 2\pi n_e Z_i r_p/(\gamma^3\beta^2 A_i)$, $k_{ei}^2 = 2\pi n_i Z_i r_e/(\gamma^3\beta^2)$ with $n_{i,e}$ as electron and ion densities (particles per unit volume of the electron beam), Z_i and A_i as the ion charge and mass numbers ($Z_i = -1$ for antiprotons), $r_{p,e}$ are the proton and electron classical radii, γ and β are the relativistic factors, $k_{iL} = Z_i eB/(p_i c)$, $k_{eL} = eB/(p_e c)$ are Larmor wave numbers for the ions and electrons in the magnetic field B of the cooler.

Integration of the equations (1) depends on the phase advances over the cooler length l for all the four wave numbers, $\psi_{ie} = k_{ie}l$, $\psi_{ei} = k_{ei}l$, $\psi_{iL} = k_{iL}l$, $\psi_{eL} = k_{eL}l$. The ion phases are always small, $\psi_{ie}, \psi_{iL} \ll 1$. The electron Larmor phase, on the contrary, is usually big: $\psi_{eL} \gg 1$. First, this allows to use drift approximation for electrons, which assumes that the second derivative term ξ_e'' is averaged to zero. Second, a term $k_{ie}^2\xi_i$ in the ion's equation takes into account small focusing perturbation from the electron beam, and can be neglected as well. As a result, the equations of motion are reduced to the following set:

$$\begin{aligned} \xi_i'' - k_{ie}^2\xi_e + ik_{iL}\xi_i' &= 0 \; ; \\ \xi_e' - ik_{ed}(\xi_i - \xi_e) &= 0 \; . \end{aligned} \tag{2}$$

with $k_{ed} = k_{ei}^2/k_{eL}$ as the electron drift wave number. Usually, the electron drift phase $\psi_{ed} = k_{ed}l$ is small, $\psi_{ed} \ll 1$.Thus, the electron response to the ion offset is only slightly modified by the ion focusing term $k_{ed}\xi_e$. However, this term and the ion Larmor term $ik_{iL}\xi_i'$ still could play a role if the ion coupling is strongly suppressed, as it is shown below.

Electron response to the ion offset acts back on the ions; in other words, ions interact with themselves by means of electron medium. In the cooler, electron and ion beams move with the same velocity; thus, every ion sample sees only itself, the coherent interaction is local. This local linear response can be described by means of a matrix. When there is no electron beam, the cooler's matrix is just one of a drift, slightly modified by the magnetic field of the cooler. With the electron beam, this matrix gets a small perturbation, proportional to an ion-electron interaction parameter. This parameter, as it is clear from Eqs. (2), can be expressed as a product of electron-ion and ion-electron phase advances

$$\alpha \equiv \psi_{ie}^2\psi_{ed} \; . \tag{3}$$

For all coolers, this parameter is small, $\alpha \simeq 10^{-3} - 10^{-5}$; thus, a first-order perturbation approach over that small value is sufficient. The resulting cooler matrix $\mathbf{C}$, perturbed by the beam-beam interaction, can be presented as follows:

$$\mathbf{C} = \mathbf{S}_f^{-1}(\mathbf{S}_0 + \alpha\mathbf{M})\mathbf{S}_f \equiv \mathbf{C}_0 + \alpha\mathbf{S}_f^{-1}\mathbf{M}\mathbf{S}_f \; , \tag{4}$$

where $\mathbf{S}_f$ and $\mathbf{S}_0$ are the fringe (entrance) and the inner parts of the solenoid matrix, in first order over the magnetic field substituted as

$$\mathbf{S}_f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k_{iL} & 0 \\ 0 & 0 & 1 & 0 \\ -k_{iL} & 0 & 0 & 1 \end{pmatrix} \; ; \; \mathbf{S}_0 = \begin{pmatrix} 1 & l & 0 & l\psi_{iL}/2 \\ 0 & 1 & 0 & \psi_{iL} \\ 0 & -l\psi_{iL}/2 & 1 & l \\ 0 & -\psi_{iL} & 0 & 1 \end{pmatrix} \; ; \tag{5}$$

$$\mathbf{C}_0 \approx \begin{pmatrix} 1 & l & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \psi_{iL}/2 & 0 \\ 0 & 1 & 0 & \psi_{iL}/2 \\ -\psi_{iL}/2 & 0 & 1 & 0 \\ 0 & -\psi_{iL}/2 & 0 & 1 \end{pmatrix} \; ; \tag{6}$$

and the matrix $\mathbf{M}$ describes the ion-electron perturbation. According to Ref. [17], the beam-beam matrix in a block 2×2 form looks as

$$\begin{aligned} \mathbf{M} &= \begin{pmatrix} \mathcal{M}_{\mathrm{d}} & -\mathcal{M}_{\mathrm{c}} \\ \mathcal{M}_{\mathrm{c}} & \mathcal{M}_{\mathrm{d}} \end{pmatrix} \equiv \mathbf{M}_{\mathrm{d}} + \mathbf{M}_{\mathrm{c}} \\ \mathcal{M}_{\mathrm{d}} &\equiv \frac{\psi_{iL}}{6} \begin{pmatrix} 1/4 & l/10 \\ 1/l & 1/2 \end{pmatrix} + \frac{\psi_{ed}}{6} \begin{pmatrix} 1/4 & l/20 \\ 1/l & 1/2 \end{pmatrix} \; ; \; \mathcal{M}_{\mathrm{c}} \equiv \frac{1}{2} \begin{pmatrix} 1/3 & l/12 \\ 1/l & 1/3 \end{pmatrix} \; . \end{aligned} \tag{7}$$

Note that the diagonal part $\mathcal{M}_{\mathrm{d}}$ of the beam-beam matrix $\mathbf{M}$ is small compared to its coupling part $\mathcal{M}_{\mathrm{c}}$; on first glance, the diagonal part $\mathcal{M}_{\mathrm{d}}$ might look unnecessary to keep. The reason it is still kept is that the coupling part $\mathcal{M}_{\mathrm{c}}$ gives zero growth rates at first order over the interaction parameter α, when the unperturbed ion optics is uncoupled, as was pointed out in Ref. [17]. The relatively small diagonal part $\mathcal{M}_{\mathrm{d}}$ contains two terms. One is proportional to the local coupling due to the small Larmor phase ψ_{iL}, and another is the second-order effect over the electron drift phase ψ_{ed}, resulted from the small ion focusing term $k_{ed}\xi_e$ in the electron motion. This last contribution was neglected in Ref. [17], and is kept here for more generality. Note that the diagonal-coupling block structure of the matrix $\mathbf{M}$ is common for all rotation-invariant matrices.

The beam-beam perturbation matrix $\mathbf{M}$ was obtained in Ref. [16] in drift approximation, with arbitrary drift and ion-electron phases $\psi_{ed}, \ \psi_{ie}$, and neglecting the ion Larmor phase, $\psi_{iL} = 0$. At the common region of zero ion Larmor phase and small phases $\psi_{ed}, \ \psi_{ie}$, the matrices (7) and the corresponding matrix of Ref. [16] are not quite the same: while their coupling blocks $\mathcal{M}_{\mathrm{c}}$ are identical, the diagonal blocks $\mathcal{M}_{\mathrm{d}}$ differ. This difference of the diagonal blocks relates to the correction of ion focusing by the electron space charge, the term $k_{ie}^2\xi_i$, omitted in this analysis and kept in Ref. [16]. According to what was mentioned above, this term does not change the growth rate when the ion-electron phase is small, $\psi_{ie} \ll 1$. Thus, although matrices $\mathcal{M}_{\mathrm{d}}$ are not the same, they have to lead to the same growth rates at that common region, if both Eq. (7) and the corresponding result of Ref. [16], namely Eq. (15), are correct. According to what is shown below, this requirement is satisfied; so Eq. (7) agrees with Ref. [16].

3 Perturbation Theory for Arbitrary Coupling

Let it be assumed that a symplectic 4×4 revolution matrix $\mathbf{R}^{(0)}$ of a ring is affected by a small, but not necessarily symplectic perturbation $\mathbf{P}$. As a result, eigenvalues λ_i of the new revolution matrix

$$\mathbf{R} \equiv \mathbf{R}^{(0)} + \mathbf{P} \cdot \mathbf{R}^{(0)} \equiv (\mathbf{I} + \mathbf{P}) \cdot \mathbf{R}^{(0)}, \tag{8}$$

with 4×4 identity matrix $\mathbf{I}$, are shifted from their unperturbed values $\lambda_i^{(0)}$. The perturbation matrix $\mathbf{P}$ is referred hereafter as the normalized perturbation. The shifts of the eigenvalues can be calculated at first order of the perturbation theory, essentially in the same way as the similar problem is solved in Quantum Mechanics. Indeed, while the quantum eigen-states are described by ortho-normalized eigenvectors of Hermitian operators, the eigenvectors of symplectic matrices are symplectic-orthogonal. With a proper definition of the scalar product, the algebraic statements are absolutely identical.

According to Ref. [21], eigenvectors of the unperturbed revolution matrix $\mathbf{V}_1^{(0)}$, $\mathbf{V}_{-1}^{(0)} \equiv \mathbf{V}_1^{(0)*}$, $\mathbf{V}_2^{(0)}$, $\mathbf{V}_{-2}^{(0)} \equiv \mathbf{V}_2^{(0)*}$ can be presented as:

$$\mathbf{V}_1^{(0)} = \left(\sqrt{\beta_{1x}} \ , \quad -\frac{i(1-u)+\alpha_{1x}}{\sqrt{\beta_{1x}}} \ , \quad \sqrt{\beta_{1y}} e^{i\nu_1} \ , \quad -\frac{iu+\alpha_{1y}}{\sqrt{\beta_{1y}}} e^{i\nu_1} \right)^{\mathrm{T}} \ ; \tag{9}$$

$$\mathbf{V}_2^{(0)} = \left(\sqrt{\beta_{2x}} e^{i\nu_2} \ , \quad -\frac{iu+\alpha_{2x}}{\sqrt{\beta_{2x}}} e^{i\nu_2} \ , \quad \sqrt{\beta_{2y}} \ , \quad -\frac{i(1-u)+\alpha_{2y}}{\sqrt{\beta_{2y}}} \right)^{\mathrm{T}} \ ; \tag{10}$$

where the superscript $^{\mathrm{T}}$ stands for the transposed form, and $\mathbf{R}^{(0)} \cdot \mathbf{V}_m^{(0)} = \exp(-i\mu_m)\mathbf{V}_m^{(0)}$. Eigenvector parameters β_{1x}, β_{2y}, etc. are determined by the machine optics. In Ref. [21] they are referred as 4D Twiss functions. The symplectic orthogonality can be described then as

$$\mathbf{V}_m^{(0)+} \cdot \mathbf{U} \cdot \mathbf{V}_n^{(0)} = -2i\delta_{mn}\mathrm{sgn}(m) \ ; \tag{11}$$

with the superscript $^{+}$ meaning Hermit-conjugation, δ_{mn} is the Kronecker symbol, $\mathrm{sgn}(m)$ is the sign function, and

$$\mathbf{U} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{12}$$

is the symplectic unit matrix.

The four perturbed eigenvectors $\mathbf{V}_n$ can be expanded over the unperturbed set $\mathbf{V}_n^{(0)}$ as $\mathbf{V}_n = \sum_m D_{nm}\mathbf{V}_m^{(0)}$ with unknown coefficients D_{nm}.Substitution of this expansion in the definition of the new eigen-system, $\mathbf{R} \cdot \mathbf{V}_n = \lambda_n \mathbf{V}_n$, yields $(\lambda_n - \lambda_n^{(0)}) \sum_m D_{nm}\mathbf{V}_m^{(0)} = \lambda_n^{(0)} \mathbf{P} \sum_m D_{nm}\mathbf{V}_m^{(0)}$. Using the symplectic orthogonality, and neglecting the second-order terms, i. e. substituting $D_{nm} = \delta_{nm}$, the perturbations of the eigenvalues are found:

$$\lambda_n - \lambda_n^{(0)} = \frac{i}{2} \lambda_n^{(0)} \mathbf{V}_n^{(0)+} \cdot \mathbf{U} \cdot \mathbf{P} \cdot \mathbf{V}_n^{(0)} \ ; \quad (n > 0) \tag{13}$$

The eigenvalues can be also expressed in terms of the complex phase advances, $\lambda_n \equiv \exp(-i\mu_n)$. Shifts of the phase advances $\delta\mu_n \equiv \mu_n - \mu_n^{(0)}$ then follow:

$$\delta\mu_n = -\frac{1}{2} \mathbf{V}_n^{(0)+} \cdot \mathbf{U} \cdot \mathbf{P} \cdot \mathbf{V}_n^{(0)} \ . \tag{14}$$

This leads to the growth rates

$$\Lambda_n = \mathrm{Im}(\delta\mu_n)/T_0 = -\,\mathrm{Im} \left(\mathbf{V}_n^{(0)+} \cdot \mathbf{U} \cdot \mathbf{P} \cdot \mathbf{V}_n^{(0)} \right) / \left(2T_0\right) \ , \tag{15}$$

with T_0 as the revolution time.

One more useful relation follows immediately from a definition of the perturbation (8):

$$2T_0 \left(\Lambda_1 + \Lambda_2\right) = \mathrm{Det}\left(\mathbf{R}\right) - 1 = \mathrm{Tr}\left(\mathbf{P}\right) \ , \tag{16}$$

where $\mathrm{Det}\left(\mathbf{R}\right)$ and $\mathrm{Tr}\left(\mathbf{P}\right)$ are the determinant and the trace of the matrices. From this last property, it follows the off-diagonal block B of the beam-beam matrix M gives rise to growth rates with opposite signs, so that their sum is zero.

A perturbation approach to the coupled beam optics was recently examined in Ref. [22]; a resulting tune shift formula, essentially equivalent to Eq. (14), was derived there.

The perturbation formalism is approved, if the tune shifts (14) are smaller than the unperturbed tune separation,

$$|\delta\mu_{1,2}| \ll |\mu_1 - \mu_2|. \tag{17}$$

If this condition is not satisfied, an unconditioned application of the perturbation formalism generally leads to incorrect results. In the degenerated case, $\mu_1 = \mu_2$, the unperturbed basis is not unique; in fact, there is a one-parametric family of the symplectic bases (9), all being eigenvectors of the same revolution matrix. Similar to the case in Quantum Mechanics, it can be shown that the perturbation formula (15) is still valid for the degenerate eigenvalues, if the correct basis is chosen. Namely, the correct basis makes the perturbation diagonal on its sub-space, $\mathbf{V}_1^{(0)+} \cdot \mathbf{U} \cdot \mathbf{P} \cdot \mathbf{V}_2^{(0)} = \mathbf{V}_2^{(0)+} \cdot \mathbf{U} \cdot \mathbf{P} \cdot \mathbf{V}_1^{(0)} = 0$. An eigen-system problem can be treated in this way if an opposite to (17) condition is satisfied, $|\mu_1 - \mu_2| \ll |\delta\mu| \ll 1$.

4 Beam-Beam Growth Rates

For symmetry' sake, the reference point of the revolution matrix is taken here in the middle of the cooling solenoid. Then the beam-beam matrix $\mathbf{M}$, Eq. (7), leads to the normalized perturbation of Eq. (8)

$$\mathbf{P} = \alpha \mathbf{C}_0^{-1/2} \cdot \mathbf{S}_f^{-1} \cdot \mathbf{M} \cdot \mathbf{S}_f \cdot \mathbf{C}_0^{-1/2} \equiv \begin{pmatrix} \mathcal{P}_\mathrm{d} & -\mathcal{P}_\mathrm{c} \\ \mathcal{P}_\mathrm{c} & \mathcal{P}_\mathrm{d} \end{pmatrix} \equiv \mathbf{P}_\mathrm{d} + \mathbf{P}_\mathrm{c} \;, \tag{18}$$

where $\mathbf{C}_0^{-1/2}$ is an inverse matrix for one-half of the solenoid, edges are included.

It is convenient to treat the diagonal and coupled contributions to the perturbation matrix (18), $\mathcal{P}_\mathrm{d}$ and $\mathcal{P}_\mathrm{c}$ separately. The diagonal block $\mathcal{P}_\mathrm{d}$ contains additional small parameters, the ion Larmor and the electron drift phases ψ_{iL} and ψ_{ed}, so this block can be neglected for the beginning. In that order, the block $\mathcal{M}_\mathrm{d}$ and the edge matrices $\mathbf{S}_f$ have to be omitted, $\mathbf{S}_f \to \mathbf{I}$, and the inverse half-solenoid matrix $\mathbf{C}_0^{-1/2}$ has to be taken as the inverse half-drift matrix, $\mathbf{C}_0^{-1/2} \to \mathbf{D}^{-1/2}$, leading to

$$\mathbf{P} \to \mathbf{P}_\mathrm{c} = \alpha \mathbf{D}^{-1/2} \cdot \mathbf{M}_\mathrm{c} \cdot \mathbf{D}^{-1/2} \;. \tag{19}$$

From here, two consequences follow. First, the trace of the normalized perturbation $\mathbf{P}_\mathrm{a}$ is zero; thus, the two transverse modes get growth rates of opposite signs, according to Eq. (16). And second, the rates are equal to zero if the unperturbed optics is uncoupled, as this was pointed out in Ref. [17]. Indeed, the skew matrix $\mathbf{P}_\mathrm{c}$ transforms the uncoupled eigenvector $\mathbf{V}_n^{(0)}$ purely into the alien plane, giving a zero scalar product with the original vector. Since electrons are drifting in a direction orthogonal to the original direction of ions, the field of the electron beam is orthogonal to a velocity of the uncoupled ion beam; thus, work done by this force is equal to zero, so the rate is zero as well. Finally, direct analytical calculation (by means of *Mathematica* [18] for me) leads to a following growth rates, caused by the coupled part of the perturbation:

$$\mathbf{\Lambda}_\mathrm{c} = \pm \frac{\alpha \; \kappa_{xy}}{2T_0} \;\; ; \;\; \kappa_{xy} \equiv \frac{\sqrt{\beta_{1x}\beta_{1y}}}{l} \sin\nu_1 \tag{20}$$

where the coupling parameter $\kappa_{xy} \equiv \sqrt{\beta_{1x}\beta_{1y}/l^2}\sin\nu_1 = \sqrt{\beta_{2x}\beta_{2y}/l^2}\sin\nu_2$ was introduced. By a general property of the Twiss parameters $\sqrt{\beta_{1x}\beta_{1y}}\sin\nu_1 = \sqrt{\beta_{2x}\beta_{2y}}\sin\nu_2$ (see Ref [21]). If the mode offsets (x, y) seen turn-by-turn in the reference point belong to a so thin ellipse, that it is almost a line, i. e. $\sin\nu_1 \approx \sin\nu_2 \approx 0$, this would lead to a strong suppression of the growth rate, even if other coupling parameters are not small. Note that such sort of coupling would only mean that the mode remains to be planar, it just does not coincide with original horizontal or vertical direction. That sort of coupling is excluded by a rotation of the coordinate system; that is why it does not yield any growth rate.

For the Fermilab Recycler Ring, where antiprotons are cooled by 4.35 MeV DC electron beam [5], [19], assuming $300 \cdot 10^{10}$ antiprotons, evenly distributed over 50% of the circumference, and cooled with 0.5 A electron beam of 3.5 mm radius inside of a 20 m long cooler with 100 G field, taking the coupling parameter $\kappa_{xy} = 0.3$ (until recently, the working point was located at the coupling resonance), the rate is calculated as $\mathbf{\Lambda}_\mathrm{c}^{-1} = 1.5$ s. For more bunching, this value grows as the local density; thus, it could easily be 10-20 times higher while compressed bunches are prepared for extraction.

The diagonal contribution to the perturbation matrix, $\mathbf{P}_\mathrm{d}$, compared with the alien part $\mathbf{P}_\mathrm{c}$, contains additional small phases ψ_{iL} and ψ_{ed}. That is why it is reasonable to limit its consideration by an uncoupled optics, when otherwise dominating coupled-part contribution is zeroed. In this case, the two uncoupled modes get the same growth rate

$$\mathbf{\Lambda}_\mathrm{d} = \frac{\alpha}{2T_0}\left(\frac{\psi_{iL}}{24} - \frac{\psi_{ed}}{12}\right) \;. \tag{21}$$

There are several noticeable features of this result. First, it does not depend on details of the ion optics. Second, it has an odd dependence on the ion's charge, $\mathbf{\Lambda}_\mathrm{d} \propto Z_i^3$. Depending on which phase dominates, this term either drives an instability for positively charged ions and stabilization for antiprotons or vice versa. And third, usually the phases are small, $\psi_{iL}, \psi_{ed} \lesssim 0.1-0.01$; thus, the coupling-driven rates $\mathbf{\Lambda}_\mathrm{c}$ overshadow the uncoupled rates $\mathbf{\Lambda}_\mathrm{d}$, unless the coupling is extremely small, $|\kappa_{xy}| \lesssim 10^{-2}-10^{-3}$. The second-order term over the electron drift phase in Eq. (21) was found in Ref. [16], and the ion Larmor term was derived in Ref. [17].

In Ref. [16], the analysis was based on consideration of the determinant of the cooler matrix. Because of that, the only rate found there is the second term in Eq. (21), yielding damping for positively charged ions. Neither coupling, nor the ion Larmor phase were taken into account. Slightly later, in Ref. [20], the two-beam problem was treated for a block-diagonal revolution matrix

$$\mathbf{R}^{(0)} = \begin{pmatrix} \mathcal{R}_\mathrm{o} & 0 \\ 0 & \mathcal{R}_\mathrm{o} \end{pmatrix} \; ; \; \mathcal{R}_\mathrm{o} = \begin{pmatrix} \cos\mu & \beta_\mathrm{o}\sin\mu \\ -\sin\mu\;/\beta_\mathrm{o} & \cos\mu \end{pmatrix} \tag{22}$$

with identical x and y blocks, and $\alpha_x = \alpha_y = 0$. For that uncoupled machine optics, the growth rate was derived as

$$\mathbf{\Lambda}_{1,2} = \pm\frac{\alpha\;\beta_\mathrm{o}}{4T_0 l} \tag{23}$$

This result might seem to contradict to a conclusion of Ref. [17], claiming that the rate's linear term over the interaction parameter α is zero for uncoupled cases. This also seems to contradict to Eq. (20) above. The contradiction disappears, however, when the degeneration of the revolution matrix (22) is taken into account. Indeed, with that degenerate matrix, the eigenvectors are not unique. For instance, they can be taken as uncoupled:

$$\mathbf{V}_1^{(0)} = \mathbf{V}_\mathrm{x} \equiv \left(\sqrt{\beta_\mathrm{o}}\;, \quad -i/\sqrt{\beta_\mathrm{o}}\;, \quad 0\;, \quad 0\right)^\mathrm{T} ; \;\; \mathbf{V}_2^{(0)} = \mathbf{V}_\mathrm{y} \equiv \left(0\;, \quad 0\;, \quad \sqrt{\beta_\mathrm{o}}\;, \quad -i/\sqrt{\beta_\mathrm{o}}\right)^\mathrm{T} . \tag{24}$$

These eigenvectors, being applied to the growth rate formula, yield zero. The eigenvectors can also be taken as totally coupled circular modes:

$$\mathbf{V}_1^{(0)} = (\mathbf{V}_\mathrm{x} + i\mathbf{V}_\mathrm{y})/\sqrt{2}\;; \;\mathbf{V}_2^{(0)} = (i\mathbf{V}_\mathrm{x} + \mathbf{V}_\mathrm{y})/\sqrt{2}\;, \tag{25}$$

leading exactly to the rates of Eq. (23). The question is, what set of the eigenvectors is correct: the uncoupled, the circular, or none of them? According to conclusion of the previous section, at the coupling resonance, the correct set of the eigenvectors diagonalizes the normalized perturbation inside the degenerated sub-spaces. The circular modes, being the eigenvectors for rotations, diagonalize the rotation-invariant beam-beam perturbation. Thus, the normalized circular basis (25) constitutes the proper linear combinations for this degenerate case, and the result (23) just follows from .Eq. (20).

This example shows that, without machine coupling, the growth rate at the coupling resonance, $\mu_1 = \mu_2$, is the same as it would be with strong coupling. However, if the tunes are separated from each other, the growth rate would go down, since a role of the coupling gets smaller. In the tune space, there is a size of the coupling area, conventionally associated with the minimal tune split. The growth rate, plotted as a function of the tune separation, forms a resonance curve with a width equal to the width of the coupling area. In case of no machine coupling, the width of the growth rate resonance plotted against the tune separation, is determined by the ion-electron interaction itself. The phase split induced by the ion-electron interaction is about $\Lambda T_0 = \alpha\beta_\mathrm{o}/(4l)$, and that tiny width would be just undetectable for many practical cases. Thus, without machine coupling, the growth rate is zero everywhere except a punctured point of the coupling resonance. Therefore, being computationally correct for that special point, the consideration and the result of Ref. [20] are misleading: the crucial dependence on coupling, stressed in Ref. [17], is not even questioned in Ref. [20].

Dependence of the ion-electron growth rate on the tune separation was studied numerically in Ref. [23]: the rate was calculated for parameters of Recycler (FNAL) and ACR (RIKEN). The revolution matrix

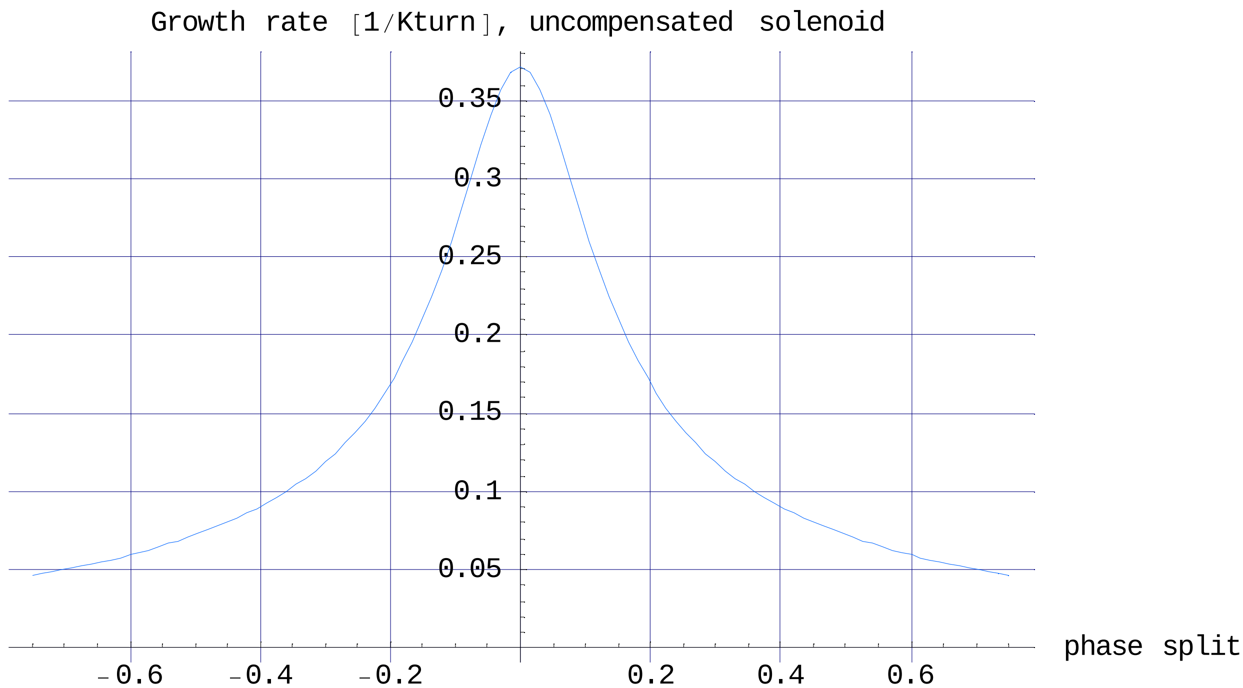

Figure 1: Ion-electron growth rate, in inverse Kturns ($10^{-3}/T_0$) as a function of the phase split $\mu_{\rm x} - \mu_{\rm y}$, reproducing Fig. 1 of Ref. [23].

perturbed by the cooler was presented in the following form:

$$\begin{aligned} \mathbf{R} &= \mathbf{D}^{-1/2} \cdot \mathbf{R}_{\rm D} \cdot \mathbf{D}^{-1/2} \mathbf{C} \; ; \\ \mathbf{R}_{\rm D} &= \begin{pmatrix} \mathcal{R}_{\rm x} & 0 \\ 0 & \mathcal{R}_{\rm y} \end{pmatrix} \; ; \; \mathcal{R}_{\rm x,y} = \begin{pmatrix} \cos\mu_{\rm x,y} & \beta_{\rm o} \sin\mu_{\rm x,y} \\ -\sin\mu_{\rm x,y}/\beta_{\rm o} & \cos\mu_{\rm x,y} \end{pmatrix} \; , \end{aligned} \tag{26}$$

where the cooler matrix $\mathbf{C}$ was taken as in Eq. (4), and $\mathbf{D}^{-1/2}$ is the inverse half-drift matrix. Note that this form of the revolution matrix assumes that there is some machine coupling. Indeed, while the outer matrix $\mathbf{D}^{-1/2} \cdot \mathbf{R}_{\rm D} \cdot \mathbf{D}^{-1/2}$ is block-diagonal, the cooler matrix $\mathbf{C}$ is not. Without the electron current, the cooler matrix reduces to a solenoid matrix, causing some coupling, proportional to the Larmor phase advance of the ions. In this case, it can be checked that the growth rate resonance behavior is described by a following compact formula:

$$\mathbf{\Lambda}_{\rm c} = \pm \frac{\alpha \; \beta_{\rm o}}{4 T_0 l} \frac{1}{\sqrt{1 + \left(\mu_{\rm x} - \mu_{\rm y}\right)^2 / \psi_{iL}^2}} \; . \tag{27}$$

The growth rate as a function of the phase split $\mu_{\rm x} - \mu_{\rm y}$ for ACR parameters was numerically calculated in Ref. [23] from the eigenvalues of the perturbed revolution matrix, and it is reproduced here in Fig. 1 for the same parameters: $\alpha = 5.4 \cdot 10^{-4}$, $\psi_{iL} = 0.10$, $\beta_{\rm o}/l = 2.8$.

The resonance of Fig. 1 changes dramatically, if solenoid-driven coupling is compensated. It can be done, for instance, by a following or a mirror-symmetric anti-solenoid. In that case, the revolution matrix is represented slightly differently, than in Eq. (26):

$$\mathbf{R} = \mathbf{C}_0^{-1/2} \cdot \mathbf{R}_{\rm D} \cdot \mathbf{C}_0^{-1/2} \cdot \mathbf{C} \; . \tag{28}$$

As a result of that compensation, the width of the growth rate reduces by several hundreds, as it is seen from Fig. 2, generated for the same parameters as Fig. 1 above.

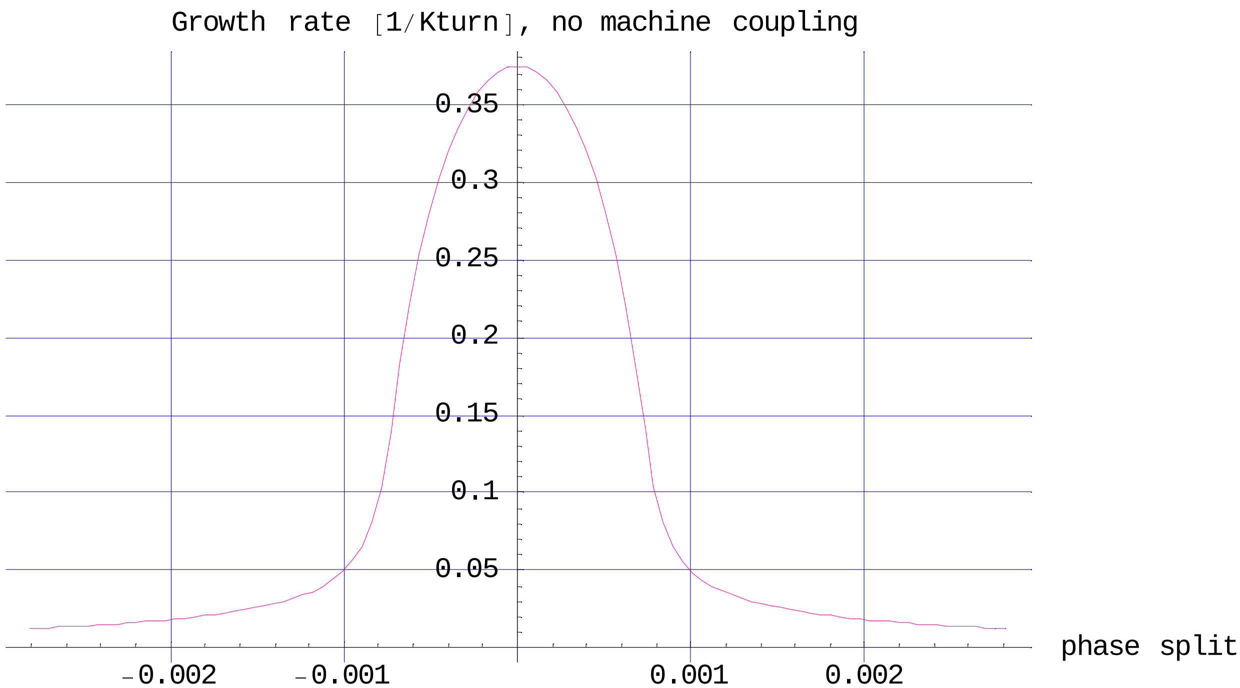

Figure 2: Growth rate for the same parameters as in Fig. (1) above, except the solenoid coupling is compensated. The peak maximum is the same, but the width is reduced more than two orders of magnitude.

To conclude, it can be stated that compensation of the machine coupling, and staying further away from the coupling resonance are essential tools for the instability suppression.

Finally, it is interesting to calculate the ion-electron growth rate if there is no magnetic field at all in the cooler, as it is discussed for RHIC [24]. In this case, the beam-beam interaction is uncoupled, and the beam-beam matrix $\mathbf{M}$ is block-diagonal:

$$\mathbf{M} = \mathbf{M}_{\rm d} = \begin{pmatrix} \mathcal{M}_{\rm d} & 0 \\ 0 & \mathcal{M}_{\rm d} \end{pmatrix} \ ; \ \mathcal{M}_{\rm d} = \frac{\psi_{ei}^2 \, \psi_{ie}^2}{6} \begin{pmatrix} 1/4 & l/20 \\ 1/l & 1/4 \end{pmatrix} \tag{29}$$

Application of Eqs. (15) and (18) leads to the growth rate

$$\Lambda_{\rm d} = -\frac{\psi_{ei}^2 \, \psi_{ie}^2}{24 T_0} \propto -Z_i^2 < 0 \ . \tag{30}$$

This growth rate is always negative; in other words, the beam-beam interaction damps the coherent motion. The value of this rate for that case is rather small though. Assuming 10^9 fully stripped gold ions at 15 mm mrad of the normalized 95% emittance, inside a bunch with 30 cm of the rms length, $\gamma = 100$, cooled with identical electron bunches with $6 \cdot 10^{10}$ electrons per bunch, both the cooler length and beta-function of 60 m, this gives $\Lambda_{\rm d}^{-1} = 5$ hr. That low rate cannot play any role.

5 Stabilization by Impedance

So far, the dipole ion-electron motion has been considered, and the growth rates have been found and discussed. However, the instability does not reveal itself if it is either suppressed by Landau damping, or overshadowed by the ring impedance. The growth rate calculated above for the Recycler, does not look high, it is at the range of ˜0.1 − 1 s. For the same antiproton beam, the resistive wall instability is about 100 times faster for the lowest frequency mode. Does it mean that here the two-beam interaction is too weak to make the ion beam unstable?

To answer that question, the beam modes have to be considered in more details. First of all, being caused by a local interaction, the ion-electron rates do not depend on the longitudinal number. So, for high wave numbers the beam-beam rates could be comparable with the impedance-driven rates. Ion-electron rates do not depend neither on the value, nor on the sign of the longitudinal wave-number. The rates are identical for the fast modes ($n > 0$) and slow modes ($n < 0$). That makes a difference with the conventional impedance-driven oscillations, where only the slow modes can be unstable (see e. g. Ref. [25], pp. 81, 364).

The Landau damping $\Lambda_L(n)$ at the longitudinal wave number n is proportional to the longitudinal distribution function at $\Delta p/p \simeq \Delta\nu_{\rm sc}/|\xi - n\eta|$, where ξ is the chromaticity, $\eta = 1/\gamma_t^2 - 1/\gamma^2$ is the slippage factor, and $\Delta\nu_{\rm sc}$ is the incoherent space charge tune shift. There is an interval of the longitudinal numbers $n_* - \Delta n \lesssim n \lesssim n_* + \Delta n$ with $n_* \equiv \xi/\eta$ and $\Delta n \simeq (0.2 - 0.3)\Delta\nu_{\rm sc}/(|\eta|\Delta p/p)$, where there is no Landau damping, as soon as $|\eta|\Delta p/p \ll \Delta\nu_{\rm sc}$. To avoid impedance-driven instabilities at these wave numbers, a sign of the chromaticity is usually kept the same as the sign of the slippage factor. If so, then $n_* > 0$, and the modes $n \approx n_*$ are damped due to the conventional impedance itself. If, for the critical mode numbers $n \approx n_*$, the impedance-related damping exceeds the electron-ion growth, and for the higher numbers, $n \gg n_*$, the growth is suppressed by the Landau damping, the ion-electron dipole instability would not be seen at all. For the Recycler, $n_* \approx 800$, and with the parameters above, the impedance-related damping time is 0.1 s. This time is definitely shorter than the beam-beam growth time for the cooling regime with a long bunch, and can be comparable with the beam-beam growth time for compressed bunches while "mining" [19].

6 Quadrupole Instability

Let it be assumed that the dipole ion-electron instability is somehow suppressed - either by a combination of the Landau damping and the ring impedance, or by an active damper. If so, could the ion-electron interaction give rise to instabilities for quadrupole and higher order modes?

Normally, if the dipole modes are stable, the quadrupole modes are even more so. On the contrary, when there is an unstable dipole mode, the beam would be lost most likely, whatever happens with quadrupole modes. That is why the quadrupole stability is almost never an issue. The quadrupole growth rate scales as $2\,(a_i/b)^2$, compared with the dipole rate (see Ref. [25], p. 196), where a_i and b are the ion beam rms radius and the aperture radius respectfully. With $a_i/b \simeq 10 - 20$, this gives something like two orders of magnitude for additional suppression for the quadrupole mode. For the ion-electron interaction, though, this suppression is not so strong. A role of the aperture radius in this case is played by the electron beam radius a_e, which value in many cases is comparable with the ion radius. Applying the quadrupole form-factor $2\,(a_i/a_e)^2$, the ion-electron growth rate of the quadrupole modes follows:

$$\mathbf{\Lambda}_{1,2}^{(2)} = \pm \frac{\alpha\ \kappa_{xy}}{T_0} \left(\frac{a_i}{a_e}\right)^2 . \tag{31}$$

If the electron radius is much smaller than the machine aperture, the impedance stabilization, discussed in the previous section, could easily not work for the quadrupole and higher order modes. For the above Recycler example, the quadrupole resistive growth/damping time is calculated as 10 s, which is much longer than the antiproton-electron growth time for the presented parameters. To stabilize the fast microwave $n_* - \Delta n \lesssim n \lesssim n_* + \Delta n$ quadrupole modes against the ion-electron instability, the coupling has to be probably suppressed to a level where the growth is slower than the electron cooling. If the coupling is not weak enough, the quadrupole instability develops until it is stopped by its own non-linearity. After that, the persistent quadrupole oscillations of the cooled beam will just stay at that self-stabilized level. At a center of these wave numbers, $n = n_*$, there is no Landau damping, but edges of this instability interval, $n \simeq n_* \pm \Delta n$ are just determined by an equilibrium between the growth rate and the Landau damping. Thus, for these edges, the coherent quadrupole oscillations are transferred into the incoherent transverse motion, and so the instability drives either emittance growth, or the life-time degradation, or both of them.

Note that the quadrupole mode rate (31) is proportional to the beam emittance; the formula is applicable till ions are mostly inside of the electron beam.

7 Summary

Coherent beam-beam interaction between a cooled hadron beam and a cooling electron beam has been analytically considered here. A previous conclusion of the author [17] about a crucial role of the machine coupling for the beam-beam stability is confirmed and detailed. A compact expression for the growth rate is found in terms of 4D Twiss parameters of an arbitrary coupled lattice. A resonant shape of the growth rate is found to be described by a simple formula when a single source of the machine coupling is the cooler's solenoid. In many cases, the dipole beam-beam rate is either suppressed or overshadowed by a combination of the Landau damping, the chamber impedance and the active damper, while the quadrupole beam-beam instability can still develop. The quadrupole fast modes within a specific interval of the longitudinal numbers are way less likely to be suppressed, and can be excited near the coupling resonance, leading either to the emittance growth or the lifetime degradation, or to the both of them. Tuning out of the coupling resonance or / and reduction of the machine coupling give an efficient remedy for the ion-electron coherent instabilities of all orders.

This conclusion was tested recently at the Recycler. Until not long ago, the Recycler stayed just at the coupling resonance, having the coupling parameter κ_{xy} at about its maximum. At that working point, there was observed an emittance growth, associated with the electron current and the antiproton peak current. According to what is described in this paper, the growth rate has to be significantly reduced by more tune separation, and this measure was suggested by the author. When the tune separation was increased to ˜0.01 and the coupling parameter dropped by about a factor of 10, the emittance growth dropped by about the same factor [19].

The author is thankful to Martin Hu, Sergei Nagaitsev, Lionel Prost, and Alexander Shemyakin for numerous details of the observations in the Recycler. Meiqin Xiao and Alexander Valishev helped me with the optical file for the Recycler. My special thanks are to Valeri Lebedev for his multiple remarks related to various aspects of this paper.

References

[1] G. I. Budker, "Method for Damping Particle Oscillations in the Proton and Antiproton Storage Rings", Atomnaya Energia, Vol. **22**, No 5, p.246 (1967).

[2] G. I. Budker et al., IEEE Trans. Nucl. Sci. NS-22, p. 2093 (1975).

[3] I. Meshkov, A. Sidorin, "Electron Cooling", NIM-A, Vol. **532**, Nos. 1-2, p.19 (2004).

[4] I. Meshkov, "Overview of Recent Trends in Beam Cooling", Proc. COOL'05, Galena, IL (2005).

[5] S. Nagaitsev et al., "Experimental Demonstration of Relativistic Electron Cooling", Phys. Rev. Lett. **96**, 044801 (2006).

[6] J. Bosser et al., "Stability of Cooled Beams", NIM-A, Vol. **441**, Nos. 1-2, p.1 (2000).

[7] A. Burov and S. Nagaitsev, "Envelope Instability as a Source of Diffusion", ibid, p. 18

[8] D. Reistad, private communication, 1993.

[9] D. Reistad et al., "Measurements of electron cooling and electron heating at CELSIUS", CERN 94-03, p.183 (1994).

[10] V. Ziemann, "Resonances Driven by the Electric Field of the Electron Cooler", TSL Note 98-43, Uppsala, Sweden (1998).

[11] S. Konstantinov, V. Parkhomchuk, and V. Reva, "Space Charge Oscillation in Electron Cooling Equipment", Tech. Phys. Vol. **48**, No 1, p.85 (2003).

[12] V. Parkhomchuk, D. Pestrikov, "Coherent Instabilities at Electron Cooling", CERN 94-03, p. 327 (1994).

[13] V. Parkhomchuk, "New insights in the theory of electron cooling", NIM-A, Vol. **441**, Nos. 1-2, p. 9 (2000).

[14] Yu. Korotaev, I. Meshkov et al., "Electron Cooling of Intense Ion Beams" Proc. COOL'05, Galena, IL (2005).

[15] A. Burov, "Electron Cooler Impedances", Part. Acc., Vol. **57**, p.131 (1997).

[16] P. Zenkevich and A. Bolshakov, "Influence of electron cooler on dipole ion oscillations in high current storage ring", NIM-A, Vol. **441**, Nos. 1-2, p.36 (2000).

[17] A. Burov, "Electron drift instability in storage rings with electron cooling", ibid, p. 23.

[18] *Mathematica 5.1*, Wolfram Research, Inc. (2005)

[19] L. Prost et al., Proc. HB2006, Tsukuba, Japan (2006) (*to be presented*).

[20] V. Parkhomchuk and V. Reva, "Coherent Interaction of Ion and Electron Beams in Systems with Electron Cooling" JETP Vol. **91**, No. 5, p. 975 (2000).

[21] V. Lebedev and S. Bogacz, "Betatron Motion with Coupling", JLAB-ACC-99-19 (1999).

[22] A. Wolski, Phys. Rev. ST Accel. Beams **9**, 024001 (2006).

[23] P. Zenkevich, "Coherent dipole instability due to ... electron cooler" ITEP 26-01 (2001).

[24] A. Burov, "Electron cooling against IBS for high energy colliders", FERMILAB-TM-2213 (2003).

[25] A. Chao, "Physics of Collective Beam Instabilities in High Energy Accelerators", J. Wiley & Sons Inc. (1993).