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ARGONNE NATIONAL LABORATORY
9700 South Cass Avenue
Argonne, Illinois 60439

NUMERICAL METHODS FOR A POROUS MEDIUM EQUATION

by

Michael E. Rose*

Applied Mathematics Division

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prepared for the degree
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from The University of Chicago

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*Present address: Reservoir Simulation Section, Reservoir Division, Exxon Production Research Co., P. O. Box 2189, Houston, TX 77001.

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ABSTRACT

The degenerate parabolic equation

$$\frac{\partial u}{\partial t} = \nabla \cdot (u^{\nu} \nabla u), \qquad \nu > 0$$

has been used to model the flow of gas through a porous medium. Error estimates for continuous and discrete time finite element procedures to approximate the solution of this equation are proved and a new regularity result is described.

CHAPTER I

A POROUS MEDIUM EQUATION

Introduction

We shall study the porous medium equation

(1.1) $\partial u/\partial t = \nabla\cdot(|u|^{\nu}\nabla u)$ on $\Omega \times (0,T]$,

(1.2) $\partial u/\partial n = 0$ on $\partial\Omega \times [0,T]$,

(1.3) $u(x,0) = u_0(x)$ on Ω ,

where $\nu \geq 1$ is a parameter and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary. The initial function u_0 is assumed to be smooth, nonnegative, and compactly supported in Ω; in particular, the compatibility condition $\partial u_0/\partial n = 0$ holds on $\partial\Omega$.

Our main result is that we shall derive error estimates for numerical approximations to the problem (1.1)-(1.3), which we shall refer to as "the porous medium equation" or "PME".

The PME does not, in general, admit classical solutions. Existence and uniqueness of weak solutions was proved in one space dimension by Oleinik, Kalashnikov, and Yui-Lin ([11],[12]) and in several space dimensions by Lions [10]. These proofs study the PME with different boundary conditions but the arguments carry over to the PME (1.1)-(1.3).

The maximum principle implies that, since u_0 is nonnegative on Ω, $u(x,t)$ is nonnegative for all $(x,t) \in \Omega\times[0,T]$ (see [11],[12]).

We can rewrite (1.1) in the form

$$(1.4) \qquad \partial u/\partial t = \Delta K(u) \quad \text{on} \quad \Omega \times (0,T] ,$$

where $K(\xi) = \int_0^{\xi} k(\tau)d\tau$ and $k(\tau) = |\tau|^{\nu}$. We have defined $k(\tau)$ for the negative reals because, although u is never negative, various numerical approximations to u may take on negative values.

The relations (1.1)-(1.3) are to be understood as a model problem for fluid flow through porous media. The author is interested in numerical methods for (1.1)-(1.3) as a preliminary to the analysis of the more complicated degenerate parabolic equation

$$(1.5) \qquad \frac{\partial}{\partial t} c(x,u) = \mathcal{L}A(x,u) + \nabla\Phi(x,t)\nabla g(x,u) + h(x,u) \quad \text{on} \quad \Omega \times (0,T] ,$$

with nonlinear boundary data

$$(1.6) \qquad \frac{\partial u}{\partial n} = b(x,t,u) \quad \text{on} \quad \partial\Omega \times (0,T] .$$

Here, $\mathcal{L}$ is a linear elliptic operator which is independent of time and $A(x,u)$ satisfies

$$A_u(x,u) = a(x,u) \geq 0 \quad \text{for} \quad u \geq 0$$

with $a(x,\xi_j) = 0$ for certain values $\xi_j \in [0,\infty)$. The spatial dependence of the coefficients is necessary to model the flow of fluids through non-homogeneous media. These equations have been used to model various problems involving the flow of two or more fluids, among them a waterflood problem in petroleum engineering in one space dimension [4]. More generally, one would want to analyze systems of degenerate parabolic equations.

Properties of Solutions of Degenerate Parabolic Equations

The solution of equation (1.1) behaves in a strikingly different way than those of nondegenerate parabolic equations (e.g., the heat equation, $\nu = 0$). Let us consider the PME (1.1)-(1.3) as an initial value problem with $\Omega = \mathbb{R}^1$.

In 1958, Oleinik, Kalashnikov, and Yui-Lin ([11],[12]) proved that if u_0 has compact support, then $u(\cdot,t)$ has compact support for all positive time. In fact, it is possible that the support of $u(\cdot,t)$ may not expand at all for $0 \leq t \leq t_0$, for some $t_0 > 0$. The structure of the interface $\partial \mathrm{Supp}(u(x,t))$ has been studied extensively by B. Knerr in his doctoral dissertation [8].

Another distinction between the porous medium equation and nondegenerate parabolic equations is that smooth or real analytic initial data does not necessarily produce a smooth solution. It is well known that nondegenerate parabolic equations possess a 'smoothing' property whereby L^2 or even distributional initial data yield a smooth solution. Degenerate parabolic equations could be described as having a "roughing" property.

For $\nu > 1$, it has been demonstrated that smooth, compactly supported initial data never yield a C^1 solution ([11],[12]). The space derivative becomes discontinuous at the interface at some positive time. The known regularity results concern smoothness of weighted versions of the gradient of u.

Oleinik and Kalashnikov proved that, in one space dimension,

$$(1.7) \qquad \nabla K(u) = u^{\nu} \nabla u \in L^{\infty}(0,T,L^{\infty}(\Omega)) \ .$$

In fact, $\nabla K(u)$ is continuous. In 1969, Aronson [1] demonstrated that

$$(1.8) \qquad u^{\nu-1}\nabla u \in L^{\infty}(0,T,L^{\infty}(\Omega)) \ .$$

This result is sharp, given his assumptions on the initial data, as shown by the examples cited in Aronson's paper [1]. Further results on the smoothness of $u(x,t)$ and the structure of the interface are contained in [2] and [3].

One can relate these two properties of degenerate parabolic equations. A result of Knerr [8] roughly states that, given smooth initial data with compact support, the support will not expand until $u(x,t)$ becomes nearly vertical at the interface. When the gradient of u becomes discontinuous at the interface, then the support will begin to expand monotonically.

Outline of this Report

The main results of this report are the error estimates we derive for various Galerkin approximations to (1.1)-(1.3). We begin our analysis in Chapter II by studying several perturbations of (1.1)-(1.3) which yield smooth solutions which approximate the solution of (1.1)-(1.3).

In Chapter III we study the regularity theory for (1.1)-(1.3) and a regularized variant of the porous medium equation given by (2.3)-(2.4)-(2.5). Theorem 3.3 is a new regularity result for the porous medium equation in a special (but physically important) case which may be of interest aside from its application to deriving error estimates for numerical approximations.

In Chapter IV we study error estimates for two Galerkin methods to approximate the solution of (1.1)-(1.3). Chapter V contains results for the backward-difference and (in a special case) Crank-Nicolson time discretization of the schemes in the previous chapter.

CHAPTER II

REGULARIZATIONS OF THE POROUS MEDIUM EQUATION

One difficulty in deriving error estimates for degenerate parabolic problems is the roughness of their solutions. In the special case $\nu = 1$, we can establish enough smoothness for u so that it is unnecessary to regularize the porous medium equation (see Theorem 3.3). However, for $\nu > 1$ we must first perturb the problem (1.1)-(1.3) to produce a problem with a smooth solution u_β. There are several ways to do this.

The method we shall discuss is the technique of nondegenerate parabolic approximation. The diffusion coefficient of (1.1) is

$$k(\xi) = |\xi|^\nu, \qquad \nu \geq 1 . \tag{2.1}$$

We shall replace (2.1) with a new diffusion coefficient

$$k_\beta(\xi) \in C^N(\mathbb{R}) \qquad \text{for} \qquad \beta \in (0,1]$$

which satisfies the conditions

$$k_\beta(\xi) = k(\xi) \quad \text{for} \quad \xi \geq \beta , \tag{2.2a}$$

$$k_\beta(\xi) \geq \beta^\nu/2 \quad \text{for} \quad \xi \geq 0 , \tag{2.2b}$$

$$k_\beta(\xi) \geq k(\xi) \quad \text{for} \quad \xi \geq 0 , \tag{2.2c}$$

$$k'_\beta(\xi) \geq 0 \quad \text{for} \quad \xi \geq 0 , \text{ and} \tag{2.2d}$$

$$k_\beta(-\xi) = k_\beta(\xi) . \tag{2.2e}$$

Such a regularization could be produced by taking

$$k_\beta(\xi) = \text{Max}\{\xi^\nu, \tfrac{3}{4}\beta^\nu\} , \qquad \xi \geq 0 ,$$

rounding off the corner, and extending k_β to an even function on the real line. Replacing $k(\xi)$ with $k_\beta(\xi)$ yields the nondegenerate parabolic problem

$$\partial u_\beta/\partial t = \nabla\cdot(k_\beta(u_\beta)\nabla u_\beta) \quad \text{on} \quad \Omega \times (0,T] , \tag{2.3}$$

$$\partial u_\beta/\partial n = 0 \quad \text{on} \quad \partial\Omega \times [0,T] , \tag{2.4}$$

$$u_\beta(x,0) = u_0(x) \quad \text{on} \quad \Omega . \tag{2.5}$$

Since $k_\beta(\xi)$ is $C^N(\mathbb{R})$ and bounded above zero and $u_\beta(x,0) = u_0(x)$ is compactly supported in Ω so that we have compatibility of the initial and boundary data on $\partial\Omega \times \{t=0\}$, (2.3)-(2.5) is a nondegenerate parabolic problem; hence, u_β is C^N on $\overline{\Omega} \times [0,T)$ [9].

Our next task is to show that u_β is close to u in an appropriate sense. Towards this end we rewrite the porous medium equation (1.1) in the form

$$\partial u/\partial t = \Delta\, K(u) , \tag{2.6}$$

where

$$K(\xi) = \int_0^\xi k(\tau)d\tau = \frac{1}{1+\nu}\, |\xi|^\nu \xi . \tag{2.7}$$

We also rewrite the nondegenerate equation (2.3) as

$$\partial u_\beta/\partial t = \Delta\, K_\beta(u_\beta) , \tag{2.8}$$

where

$$K_\beta(\xi) = \int_0^\xi k_\beta(\tau)d\tau . \tag{2.9}$$

Before estimating $u_\beta - u$, we shall need to define an H^{-1} semi-norm on Ω. Let T be the solution operator w = Tf of the Neumann problem

$$(2.10) \qquad -\Delta w = f \quad \text{on} \quad \Omega ,$$

$$(2.11) \qquad \partial w/\partial n = 0 \quad \text{on} \quad \partial\Omega ,$$

where we assume that

$$(2.12) \qquad \frac{1}{|\Omega|} \int_\Omega f \, dx = 0$$

to get existence. Let

$$(2.13) \qquad \frac{1}{|\Omega|} \int_\Omega w \, dx = \frac{1}{|\Omega|} \int_\Omega Tf \, dx = 0$$

for uniqueness.

For a function f(x) on Ω, we define the semi-norm

$$(2.14) \qquad \|f\|^2_{H^{-1}(\Omega)} = (T\tilde{f}, \tilde{f}).$$

where $\tilde{f} = f - \frac{1}{|\Omega|} \int_\Omega f(x)dx.$

This is equivalent to the usual H^{-1} norm on functions of mean value zero. We are now in a position to prove

THEOREM 2.1. Let u be the solution of (1.1)-(1.3) and let u_β be the solution of (2.3)-(2.5). Then

$$(2.15) \qquad \|u_\beta - u\|^2_{L^\infty(0,T,H^{-1}(\Omega))} + C^* \|u_\beta - u\|^{2+\nu}_{L^{2+\nu}(0,T,L^{2+\nu}(\Omega))} \leq C_1 \beta^{2+\nu} ,$$

where $C^* = C^*(\nu)$ and $C_1 = C_1(\nu, |\Omega|)$ are positive constants.

Proof. Using the operator T defined in (2.10)-(2.13), rewrite the PME (2.6) as

$$(2.16) \qquad Tu_t + K(u) = \frac{1}{|\Omega|} \int_\Omega K(u) dx$$

at any time $t > 0$. Similarly, the regularized PME (2.8) is equivalent to

(2.17) $$Tu_{\beta t} + K_\beta(u_\beta) = \frac{1}{|\Omega|}\int_\Omega K_\beta(u_\beta)dx$$

for all $t > 0$. The equations (2.16) and (2.17) are valid because u_t and $u_{\beta t}$ have mean value zero on Ω; this follows trivially from the Neumann boundary data (1.2) and (2.4).

We subtract (2.16) from (2.17) to get

(2.18) $$T(u_{\beta t}-u_t) + (K(u_\beta)-K(u)) = (K(u_\beta)-K_\beta(u_\beta)) + \frac{1}{|\Omega|}\int_\Omega (K_\beta(u_\beta)-K(u))dx$$

at each positive time. Integrate (2.18) against $u_\beta - u$ to get

(2.19) $$(T(u_{\beta t}-u_t), u_\beta - u) + (K(u_\beta)-K(u), u_\beta - u) = (K(u_\beta)-K_\beta(u_\beta), u_\beta - u) \ .$$

Notice that, since $\frac{d}{dt}\int_\Omega (u_\beta - u)dx = \int_\Omega (u_{\beta t}-u_t)dx = 0$, we have $\int_\Omega (u_\beta - u)dx = \int_\Omega (u_\beta(x,0)-u(x,0))dx = \int_\Omega (u_0(x)-u_0(x))dx = 0$. Thus,

$$\left(\frac{1}{|\Omega|}\int_\Omega (K_\beta(u_\beta) - K(u))dx,\ u_\beta - u\right) = 0 \ .$$

The first term on the left side of (2.19) can be written in the form

(2.20) $$\frac{1}{2}\frac{d}{dt}\|u_\beta - u\|^2_{H^{-1}(\Omega)} \ .$$

To bound the second term on the left side of (2.19) we first use the fact [10] that, for any two real numbers a and b,

$$(2.21) \qquad (|a|^{\nu}a-|b|^{\nu}b)\cdot(a-b) \geq C^{*}|a-b|^{2+\nu}, \quad C^{*} = C^{*}(\nu) .$$

Thus,

$$(2.22) \qquad (K(u_\beta)-K(u),u_\beta-u) \geq C^{*}\|u_\beta-u\|^{2+\nu}_{L^{2+\nu}(\Omega)} .$$

Consequently,

$$(2.23) \qquad \frac{1}{2}\frac{d}{dt}\|u_\beta-u\|^2_{H^{-1}(\Omega)} + C^{*}\|u_\beta-u\|^{2+\nu}_{L^{2+\nu}(\Omega)} \leq |(K_\beta(u_\beta)-K(u_\beta),u_\beta-u)| .$$

We bound the right side of (2.23) by

$$(2.24) \qquad C_2\|K_\beta(u_\beta)-K(u_\beta)\|^{\gamma}_{L^{\gamma}(\Omega)} + (C^{*}/2)\|u_\beta-u\|^{2+\nu}_{L^{2+\nu}(\Omega)} ,$$

where $\gamma = 1 + \frac{1}{1+\nu}$ is the exponent conjugate to $2+\nu$, and hide the second term on the right in (2.24) in the second term on the left in (2.23).

Since $k_\beta(\xi) = k(\xi)$ for $|\xi| \geq \beta$, at each $(x,t) \in \Omega\times[0,T]$ we have

$$(2.25) \qquad |K_\beta(u_\beta)-K(u_\beta)| = \left|\int_0^{\min\{u_\beta,\beta\}}(k_\beta(\xi)-k(\xi))d\xi\right| \leq \int_0^{\beta}(\beta^{\nu}-\xi^{\nu})d\xi = \frac{\nu}{1+\nu}\beta^{1+\nu} .$$

We have used the fact that the maximum principle implies that $u_\beta(x,t)$ is nonnegative. Thus,

$$(2.26) \qquad \|K_\beta(u_\beta)-K(u_\beta)\|^{\gamma}_{L^{\gamma}(\Omega)} \leq C_3|\Omega|\beta^{2+\nu} = C_4\beta^{2+\nu} ,$$

$$(2.27) \qquad \frac{1}{2}\frac{d}{dt}\|u_\beta-u\|^2_{H^{-1}(\Omega)} + (C^{*}/2)\|u_\beta-u\|^{2+\nu}_{L^{2+\nu}(\Omega)} \leq C_2C_4\beta^{2+\nu} .$$

Integrating (2.27) in time from 0 to T establishes the theorem with $C_1 = C_2C_4T$. □

There are many other ways to regularize (1.1)-(1.3). One regularization which appears in the literature is equivalent to taking the modified diffusion coefficient to be

$$d_\beta(\tau) = (|\tau|+\beta)^\nu, \qquad \nu \geq 1 . \tag{2.28}$$

However, if $\tilde{u}_\beta$ is the solution of (1.1)-(1.3) with $k(\xi)$ replaced by $d_\beta(\xi)$, we can only prove that

$$\|\tilde{u}_\beta - u\|^2_{L^\infty(0,T,H^{-1}(\Omega))} + C^*\|\tilde{u}_\beta - u\|^{2+\nu}_{L^{2+\nu}(0,T,L^{2+\nu}(\Omega))} \leq C_5\beta^2 , \tag{2.29}$$

which is worse than (2.15). However, we shall need the regularization (2.28) as a theoretical tool in Chapter III for the special case $\nu = 1$. For future reference, in the case $\nu = 1$ $\tilde{u}_\beta$ is defined to be the solution of

$$\frac{\partial}{\partial t}\tilde{u}_\beta = \nabla\cdot((\tilde{u}_\beta+\beta)\nabla\tilde{u}_\beta) \quad \text{on} \quad \Omega \times (0,T] , \tag{2.30}$$

$$\frac{\partial}{\partial n}\tilde{u}_\beta = 0 \quad \text{on } \partial\Omega \times [0,T] , \tag{2.31}$$

$$\tilde{u}_\beta(x,0) = u_0(x) \quad \text{on } \Omega . \tag{2.32}$$

We introduce the notation $\mathcal{D}_\beta(\xi) = \int_0^\xi d_\beta(\tau)d\tau$, so that when $\nu = 1$, $\mathcal{D}_\beta(\xi) = \int_0^\xi (|\tau|+\beta)d\tau = \frac{1}{2}(|\xi|+\beta)\xi$.

Finally, we shall need the result

$$(K_\beta(v)-K_\beta(w), v-w) \geq C^*\int_\Omega |v-w|^{2+\nu}dx \tag{2.33}$$

for any two measurable functions v and w on Ω. This follows from (2.21).

CHAPTER III

SOME REGULARITY RESULTS FOR THE POROUS MEDIUM EQUATION

We begin with an estimate based upon the L^1-contraction property of parabolic equations.

THEOREM 3.1. Let u_β be the solution of (2.3)-(2.5). Then

$$\|\partial u_\beta/\partial t\|_{L^\infty(0,T,L^1(\Omega))} \leq C_6 , \tag{3.1}$$

where $C_6 = \sup_{\beta\in(0,1]} \|\Delta K_\beta(u_0)\|_{L^1(\Omega)} < \infty$.

Proof. Let u_β and v_β be two solutions of (2.3)-(2.5) with initial data u_0 and v_0 respectively. Let $\Omega_+(t) = \{x|u_\beta(x,t) > v_\beta(x,t)\}$ and let $\Omega_-(+) = \{x|v_\beta(x,t) > u_\beta(x,t)\}$. Notice that

$$\int_{\Omega_+(t)} (u_{\beta t}-v_{\beta t})(x,t)dx = \int_{\Omega_+(t)} \Delta(K_\beta(u_\beta)-K_\beta(v_\beta))dx , \tag{3.2}$$

We shall need to show that

$$\int_{\Omega_+(t)} \Delta(K_\beta(u_\beta)-K_\beta(v_\beta))dx \leq 0 . \tag{3.3}$$

Formally, Green's Theorem tells us that

$$\int_{\Omega_+(t)} \Delta(K_\beta(u_\beta)-K_\beta(v_\beta))dx =$$
$$\int_{\partial\Omega_+(t)} \frac{\partial}{\partial n} (K_\beta(u_\beta)-K_\beta(v_\beta))dx$$

where the first term is clearly nonpositive and the second term may be written as

$$\int_{\partial\Omega_+(t)\cap\Omega} \frac{\partial}{\partial n} (K_\beta(u_\beta)-K_\beta(v_\beta))dx + \int_{\partial\Omega_+(t)\cap\partial\Omega} \frac{\partial}{\partial n} (K_\beta(u_\beta)-K_\beta(v_\beta))dx .$$

The first term in the above is nonpositive because $K_\beta(u_\beta)-K_\beta(v_\beta)$ is positive in $\Omega_+(t)$ and zero on $\partial\Omega_+(t) \cap \Omega$. The second term is zero by the Neumann boundary condition (2.4). A rigorous proof of (3.3) which treats the difficulty that $\partial\Omega^+(t)$ is not known to be smooth may be found in Douglas, Dupont, Serrin [5].

We may also use Sard's Theorem to prove (3.3) (see [14], Theorem 3.1). Let $\varepsilon_n \downarrow 0$ be a sequence of positive numbers which are not critical values of $(K_\beta(u_\beta)-K_\beta(v_\beta))(\cdot,t)$ and let $\Omega_+^n(t) = \{x \mid (K_\beta(u_\beta)-K_\beta(v_\beta))(x,t) > \varepsilon_n\}$. Since $\partial\Omega_+^n(t)$ is C^1, we have

$$(3.4) \qquad \int_{\Omega_+(t)} \Delta(K_\beta(u_\beta)-K_\beta(v_\beta))dx = \lim_{n\to\infty} \int_{\Omega_+^n(t)} \Delta(K_\beta(u_\beta)-K_\beta(v_\beta))dx = \lim_{n\to\infty} -\int_{\partial\Omega_+^n(t)} \frac{\partial}{\partial n} (K_\beta(u_\beta)-K_\beta(v_\beta))dx .$$

The integrand of the second term vanishes on $\partial\Omega_+^n(t) \cap \partial\Omega$ by (2.3) and the fact that $K_\beta(u_\beta)-K_\beta(v_\beta)$ is greater than ε_n on $\Omega_+^n(+)$ and equals ε_n on $\partial\Omega_+^n(t) \cap \Omega$ proves that

$$(3.5) \qquad \frac{\partial}{\partial n} (K_\beta(u_\beta)-K_\beta(v_\beta)) \leq 0$$

on $\partial\Omega_+^n(t) \cap \Omega$; this proves that the second term on the right side of (3.4) is nonpositive. We have established the inequality (3.3).

Next, we notice that (3.2) and (3.3) yield

$$\text{(3.6a)} \qquad \int_\Omega \frac{d}{dt} (u_\beta - v_\beta)^+ dx = \int_{\Omega_+(t)} (u_{\beta t} - v_{\beta t}) dx \leq 0 \ .$$

Interchanging the roles of u_β and v_β tells us that

$$\text{(3.6b)} \qquad \int_\Omega \frac{d}{dt} (u_\beta - v_\beta)^- dx = -\int_{\Omega_-(t)} (u_{\beta t} - v_{\beta t}) dx \leq 0 \ .$$

Thus,

$$\text{(3.7)} \qquad \begin{aligned} \frac{d}{dt} \|u_\beta - v_\beta\|_{L^1(\Omega)} &= \frac{d}{dt} \int_\Omega |u_\beta - v_\beta| dx \\ &= \int_\Omega \frac{d}{dt} |u_\beta - v_\beta| dx \\ &= \int_\Omega \frac{d}{dt} (u_\beta - v_\beta)^+ dx - \int_\Omega \frac{d}{dt} (u_\beta - v_\beta)^- dx \leq 0 \ . \end{aligned}$$

Integration in time yields

$$\text{(3.8)} \qquad \|(u_\beta - v_\beta)(t)\|_{L^1(\Omega)} \leq \|(u_\beta - v_\beta)(0)\|_{L^1(\Omega)}$$

for all t, $0 \leq t \leq T$.

Let $v_\beta(t) = u_\beta(t+\Delta t)$ and divide by Δt to obtain

$$\left\| \frac{u_\beta(t+\Delta t) - u_\beta(t)}{\Delta t} \right\|_{L^1(\Omega)} \leq \left\| \frac{u_\beta(\Delta t) - u_\beta(0)}{\Delta t} \right\|_{L^1(\Omega)} \ .$$

Letting $\Delta t \to 0$ establishes the estimate

$$\text{(3.9)} \qquad \left\| \frac{\partial u_\beta}{\partial t} (t) \right\|_{L^1(\Omega)} \leq \left\| \frac{\partial u_\beta}{\partial t} (0) \right\|_{L^1(\Omega)} = \|\Delta K_\beta(u_0)\|_{L^1(\Omega)} \leq C_6 \ ,$$

where C_6 is independent of β, $0 < \beta < 1$. □

We shall need the following estimate to derive our convergence rates later on.

COROLLARY 3.2. Let $\gamma = 1 + \frac{1}{1+\nu}$. Then

$$\int_0^T \int_\Omega |u_{\beta t}|^\gamma dxdt \le C_7 \, \beta^{-\frac{\nu}{1+\nu}} . \tag{3.10}$$

Proof. If (2.3) is integrated against the test function $K_\beta(u_\beta)_t$,

$$(u_{\beta t}, K_\beta(u_\beta)_t) + \frac{1}{2}\frac{d}{dt}\|\nabla K_\beta(u_\beta)\|^2_{L^2(\Omega)} = 0 . \tag{3.11}$$

Thus,

$$\|\sqrt{k_\beta(u_\beta)}\, u_{\beta t}\|^2_{L^2(L^2)} \le \frac{1}{2}\|\nabla K_\beta(u_0)\|^2_{L^2(\Omega)} \le C_8 , \quad 0 < \beta \le 1 . \tag{3.12}$$

Since $\sqrt{k_\beta(u_\beta)} \ge \frac{1}{2}\beta^{\nu/2}$ on $\Omega \times [0,T]$,

$$\|u_{\beta t}\|_{L^2(L^2)} \le C_9 \, \beta^{-\nu/2} . \tag{3.13}$$

By the Riesz-Thorin Theorem

$$\|u_{\beta t}\|_{L^\gamma(\Omega)} \le C_{10}\|u_{\beta t}\|^{\frac{\nu}{2+\nu}}_{L^1(\Omega)} \cdot \|u_{\beta t}\|^{\frac{2}{2+\nu}}_{L^2(\Omega)} , \tag{3.14}$$

so that

$$\|u_{\beta t}\|^\gamma_{L^\gamma(\Omega)} \le C_{11}\|u_{\beta t}\|^{\frac{\nu}{\nu+1}}_{L^1(\Omega)} \|u_{\beta t}\|^{\frac{2}{\nu+1}}_{L^2(\Omega)} . \tag{3.15}$$

Next, integrate in time from 0 to T and use Hölder's inequality to obtain

$$\begin{aligned}\int_0^T \|u_{\beta t}\|^\gamma_{L^\gamma(\Omega)} dt &\le C_{11} \int_0^T \|u_{\beta t}\|^{\frac{\nu}{\nu+1}}_{L^1(\Omega)} \cdot \|u_{\beta t}\|^{\frac{2}{\nu+1}}_{L^2(\Omega)} dt \\ &\le C_{11}\left(\int_0^T \|u_{\beta t}\|_{L^1(\Omega)} dt\right)^{\frac{\nu}{1+\nu}} \left(\int_0^T \|u_{\beta t}\|^2_{L^2(\Omega)} dt\right)^{\frac{1}{1+\nu}} .\end{aligned} \tag{3.16}$$

Then, (3.10) follows from Theorem 3.1, (3.13), and (3.16). □

In the special case $\nu = 1$ a much stronger result can be proved.

THEOREM 3.3. Let $\nu = 1$ and let u be the solution of (1.1)-(1.3). Then,

(3.17) $$\|\partial u/\partial t\|_{L^\infty(0,T,L^3(\Omega))} \leq C_{12} ,$$

where $C_{12} = C_{12}(\tilde{N}, \|\nabla K(u)_t(x,0)\|_{L^2(\Omega)}, \|u_t(x,0)\|_{L^3(\Omega)})$ and $\tilde{N}$ is given in the statement of Lemma 3.5.

It is not known whether an analogue of Theorem 3.3 is valid for $\nu > 1$. Applying the Sobolev imbedding theorem to (3.17) yields the following result.

COROLLARY 3.4. Assume $\nu = 1$. Then

(3.18) if $\dim(\Omega) = 1$ or 2 ,
$$\nabla(u^2) \in L^\infty(0,T,L^\infty(\Omega)) , \text{ and}$$

(3.19) if $\dim(\Omega) = 3$,
$$\nabla(u^2) \in L^\infty(0,T,L^q(\Omega)) \text{ for any } q < \infty .$$

Corollary 3.4 generalizes bounds previously known only for one space dimension when $\nu = 1$. In order to prove Theorem 3.3 we shall need the following lemma.

LEMMA 3.5. Assume that $\nu = 1$ and let $\tilde{u}_\beta$ be the solution of (2.30)-(2.32). Suppose that

$$\operatorname*{Min}_{x\in\Omega} \tilde{u}_{\beta t}(x,0) = \operatorname*{Min}_{x\in\Omega} \Delta\mathcal{D}_\beta(u_0)(x) = -\tilde{N} ,$$

where $\tilde{N}$ is a positive number. Then, (3.20) $$u_{\beta t}(x,t) \geq -\tilde{N}\ , \qquad (x,t) \in \Omega\times(0,T]\ .$$

Proof of Lemma 3.5. If (2.30) is differentiated with respect to time,

(3.21) $$\partial^2/\partial t^2\ \tilde{u}_\beta = \Delta\mathcal{D}_\beta(\tilde{u}_\beta)_t = d_\beta(\tilde{u}_\beta)\Delta\tilde{u}_{\beta t} + 2\nabla d_\beta(\tilde{u}_\beta)\cdot\nabla\tilde{u}_{\beta t} + (\Delta d_\beta(\tilde{u}_\beta))\tilde{u}_{\beta t},$$

where $d_\beta(\tilde{u}_\beta) = (\tilde{u}_\beta+\beta)$ as in (2.28). Let $p = \partial\tilde{u}_\beta/\partial t$ and rewrite (3.21) as

(3.22) $$p_t = (\tilde{u}_\beta+\beta)\Delta p + 2(\nabla\tilde{u}_\beta)\cdot(\nabla p) + p\Delta\tilde{u}_\beta\ .$$

We shall show that p obtains its minimum at $t = 0$. Assume, to the contrary, that p has a negative minimum at $(x_0,t_0) \in \Omega\times(0,T]$. Then at the point (x_0,t_0),

(3.23) $$0 \geq \partial p/\partial t = (\tilde{u}_\beta+\beta)\Delta p + \Delta\tilde{u}_\beta\cdot p\ .$$

Notice that $\tilde{u}_\beta$ is nonnegative by the maximum principle, so that $\tilde{u}_\beta+\beta$ is positive, and that $\Delta p \geq 0$ at (x_0,t_0). Since $p(x_0,t_0) < 0$, we need only show that $\Delta\tilde{u}_\beta$ is negative at (x_0,t_0) to demonstrate that p must attain its minimum on the parabolic boundary of $\Omega\times(0,T]$. At (x_0,t_0) we have

(3.24) $$0 > p = \Delta\mathcal{D}_\beta(\tilde{u}_\beta) = (\tilde{u}_\beta+\beta)\Delta\tilde{u}_\beta + (\nabla\tilde{u}_\beta)^2\ ,$$

so that $\Delta\tilde{u}_\beta$ must be negative. This implies that p must attain its minimum on $\partial\Omega\times(0,T]$ or on $\Omega\times\{t=0\}$. The former possibility is ruled out by the Neumann boundary condition (2.31) and the strong maximum principle for parabolic equations. This proves the lemma. □

Proof of Theorem 3.3. We differentiate (2.30) with respect to time to see that

$$\tilde{u}_{\beta tt} = \Delta \mathcal{D}_\beta(\tilde{u}_\beta)_t \quad \text{on} \quad \Omega\times(0,T] \ . \tag{3.25}$$

Integrating (3.25) against $\mathcal{D}_\beta(\tilde{u}_\beta)_{tt}$ yields

$$(\tilde{u}_{\beta tt}, \mathcal{D}_\beta(\tilde{u}_\beta)_{tt}) + \frac{1}{2}\frac{d}{dt}\|\nabla\mathcal{D}_\beta(\tilde{u}_\beta)_t\|^2_{L^2(\Omega)} = 0 \ , \tag{3.26}$$

where we have used the fact that

$$\frac{\partial}{\partial n}\mathcal{D}_\beta(\tilde{u}_\beta)_t = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial n}\mathcal{D}_\beta(\tilde{u}_\beta)\right) = 0$$

on $\partial\Omega\times[0,T]$.

Since $\nu = 1$, we have

$$(\tilde{u}_{\beta tt}, \ \mathcal{D}_\beta(\tilde{u}_\beta)_{tt}) = \int_\Omega (\tilde{u}_\beta+\beta)(\tilde{u}_{\beta tt})^2 dx + \frac{1}{3}\frac{d}{dt}\int_\Omega (\tilde{u}_{\beta t})^3 dx \ . \tag{3.27}$$

Notice that the first term on the right side of (3.27) is nonnegative. We can integrate (3.26) in time to see that

$$\begin{aligned} \sup_{0\le t\le T} \frac{1}{3}\int_\Omega (\tilde{u}_{\beta t})^3 dx + \|\sqrt{\tilde{u}_\beta+\beta}\ \tilde{u}_{\beta tt}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla\mathcal{D}_\beta(\tilde{u}_\beta)_t\|^2_{L^2(\Omega)} \\ \le \frac{1}{3}\int_\Omega \tilde{u}_{\beta t}(x,0)^3 dx + \frac{1}{2}\|\nabla\mathcal{D}_\beta(\tilde{u}_\beta)_t(x,0)\|^2_{L^2(\Omega)} \\ = \frac{1}{3}\int_\Omega (\Delta\mathcal{D}_\beta(u_0)(x))^3 dx \\ + \frac{1}{2}\|\nabla(d_\beta(u_0)\Delta\mathcal{D}_\beta(u_0))\|^2_{L^2(\Omega)} \\ \le C_{13} \ , \end{aligned} \tag{3.28}$$

where C_{13} is independent of β, $0 < \beta \le 1$. Combining (3.28) with Lemma 3.5, we see that

$$\|\tilde{u}_{\beta t}\|^3_{L^\infty(0,T,L^3)} \le 3C_{13} + 2|\Omega|\tilde{N}^3 \ . \tag{3.29}$$

The estimate (2.29) tells us that $\tilde{u}_\beta \to u$ in $\mathcal{D}'((0,T)\times\Omega)$, so that $\tilde{u}_{\beta t} \to u_t$ distributionally. Since, by (3.29), $\{\tilde{u}_{\beta t}\}_{\beta\in(0,1]}$ is bounded in $L^\infty(0,T,L^3)$, a weak sequential compactness argument will complete the proof of Theorem 3.3. We also may conclude that $\sqrt{u}u_{tt} \in L^2(0,T,L^2)$. $\square$

The proofs of Theorem 3.3 and Lemma 3.4 allow us to conclude that the solution of the problem

(3.30) $\quad \partial v/\partial t = \nabla\cdot(a(v)\nabla v) \quad$ on $\quad \Omega\times(0,T]$,

(3.31) $\quad \partial v/\partial n = 0 \quad$ on $\quad \partial\Omega\times[0,T]$,

(3.32) $\quad v(x,0) = u_0(x) \quad$ on $\quad \Omega$,

where we assume that $a(\xi) \in C^2([0,\infty))$ and

(3.33a) $\quad a(0) = 0, \quad a(\xi) > 0 \quad$ for $\quad \xi > 0$,

(3.33b) $\quad a'(\xi) \geq \delta, \quad \forall\xi \in (0,\xi_0], \quad$ some $\delta > 0$ and $\xi_0 > 0$, and

(3.33c) $\quad a''(\xi) \leq 0 \quad \forall\xi \in (0,\xi_0]$,

has the regularity property

(3.34) $\quad \partial v/\partial t \in L^\infty(0,T,L^3(\Omega))$.

The assumptions (3.33b)-(3.33c) are satisfied in some physical gas flow models. The Sobolev imbedding theorem would imply that

(3.35) $\quad a(v)\nabla v \in L^\infty(0,T,L^\infty) \quad$ for $\quad \dim(\Omega) = 1$ or 2 ,

and

(3.36) $\quad a(v)\nabla v \in L^\infty(0,T,L^q) \quad$ for $\quad \dim(\Omega) = 3$,

for any $q < \infty$. Since $a(v)\nabla v$ represents the flux, these results should be of some physical interest.

CHAPTER IV

SOME CONTINUOUS-TIME GALERKIN SCHEMES

In this chapter we shall derive error estimates for several continuous-time Galerkin schemes which approximate u_β (or u directly). We begin with a scheme which yields a continuous piecewise-linear approximation V_h to $v_\beta = K_\beta(u_\beta)$.

Let $\{\mathcal{M}_h\}_{0<h<1}$ be a family of finite-dimensional subspaces of $H^1(\Omega)$ which consist of continuous piecewise-linear functions on elements of maximum diameter h. It is well-known that $\{\mathcal{M}_h\}$ has the approximation property

(4.1a) $$\inf_{\chi\in\mathcal{M}_h} \|f-\chi\|_{L^p(\Omega)} \le C_{14}h^2\|f\|_{W^{2,p}(\Omega)}$$

for all $f \in W^{2,p}(\Omega)$, $1 \le p \le \infty$. We shall also assume that the underlying triangulation of $\{\mathcal{M}_h\}$ satisfies the quasi-uniformity condition in [6], so that the following inverse property holds

(4.1b) $$\|\chi\|_{H^1(\Omega)} \le C_{15}h^{-1}\|\chi\|_{L^2(\Omega)}$$

for $\chi \in \mathcal{M}_h$. A simple duality argument shows that (4.1b) implies

(4.1c) $$\|\chi\|_{L^2(\Omega)} \le C_{15}h^{-1}\|\chi\|_{H^{-1}(\Omega)} .$$

We shall only use spaces of continuous piecewise-linear elements in our work because the roughness of the solution of (1.1)-(1.3) implies that very little accuracy would be gained by using spaces with higher order approximation properties than (4.1a).

In this chapter, we present a global convergence rate analysis. In practice, one would want to refine the space mesh near the interface where the solution of (1.1)-(1.3) is rough. The local refinement properties of Galerkin methods recommend their use for problems such as (1.1)-(1.3).

We shall define V_h: $[0,T] \to \mathcal{M}_h$ by the ordinary differential equation

$$(4.2) \qquad \left(\frac{\partial}{\partial t} H_\beta(V_h),\chi\right) + (\nabla V_h,\nabla\chi) = 0$$

for $\chi \in \mathcal{M}_h$ and $0 < t \leq T$, where $H_\beta(\xi) = (K_\beta)^{-1}(\xi)$. We define U_h to be $H_\beta(V_h)$, so that (4.2) can be rewritten as

$$(4.3) \qquad \left(\frac{\partial}{\partial t} U_h,\chi\right) + (\nabla K_\beta(U_h),\nabla\chi) = 0 \ .$$

In (4.3), $K_\beta(U_h) \in \mathcal{M}_h$, but U_h itself is not piecewise-linear.

We construct our initial data by letting $V_h(0) \in \mathcal{M}_h$ satisfy

$$(4.4) \qquad P_h H_\beta(V_h(0)) = P_h u_0 \ ,$$

where P_h is the L^2 projection onto $\mathcal{M}_h$. The existence and uniqueness of $V_h(0)$ in (4.4) follows from the fact that $P_h \circ H_\beta$ is a continuous coercive monotone operator on $\mathcal{M}_h$. By standard monotone operator theory, this tells us that $P_h \circ H_\beta$ is one-to-one and onto as a map from $\mathcal{M}_h$ to itself [15]. This is proved in the remarks at the end of this chapter. Notice that $P_h U_h(0) = P_h u_0$.

Since $1 \in \mathcal{M}_h$, the definition of the L^2 projection tells us that

$$\frac{1}{|\Omega|}\int_\Omega P_h u_0 dx = \frac{1}{|\Omega|}\int_\Omega u_0 dx$$

and (4.3) implies

$$\frac{1}{|\Omega|}\int_\Omega \frac{\partial}{\partial t} U_h dx = 0 \ .$$

These equalities imply that $u_\beta - U_h$ and therefore $P_h(u_\beta - U_h)$ have mean value zero on Ω for all $t \geq 0$.

We now introduce a discrete version of the solution operator T that we discussed in Chapter II. Let $T_h: L^2(\Omega) \to \mathcal{M}_h$ be defined by $T_h f = W_h$, where

$$(4.5) \qquad (\nabla W_h, \nabla \chi) = (f, \chi), \qquad \forall \chi \in \mathcal{M}_h \ .$$

We assume that

$$(4.6) \qquad \frac{1}{|\Omega|}\int_\Omega f dx = \frac{1}{|\Omega|}\int_\Omega W_h dx = 0$$

to get a well-posed problem.

We rewrite (4.2) in the form

$$(4.7) \qquad T_h \frac{\partial}{\partial t} H_\beta(V_h) + V_h = \frac{1}{|\Omega|}\int_\Omega V_h dx \ ,$$

which is equivalent to

$$(4.8) \qquad T_h \frac{\partial}{\partial t} U_h + K_\beta(U_h) = \frac{1}{|\Omega|}\int_\Omega K_\beta(U_h) dx \ .$$

We subtract (4.8) from (2.17) and get

$$(4.9) \qquad T_h\left(\frac{\partial}{\partial t} u_\beta - \frac{\partial}{\partial t} U_h\right) + (K_\beta(u_\beta) - K_\beta(U_h))$$
$$= (T_h - T)\frac{\partial u_\beta}{\partial t} + \frac{1}{|\Omega|}\int_\Omega (K_\beta(u_\beta) - K_\beta(U_h)) dx \ .$$

Notice that

$$T_h\left(\frac{\partial}{\partial t} u_\beta - \frac{\partial}{\partial t} U_h\right) = T_h \frac{\partial}{\partial t} P_h(u_\beta - U_h) \ , \tag{4.10}$$

since $T_h(I-P_h) = 0$. Next, we integrate (4.9) against $u_\beta - U_h$, which has mean value zero. This yields

$$\frac{1}{2}\frac{d}{dt}(T_hP_h(u_\beta-U_h), P_h(u_\beta-U_h)) + (K_\beta(u_\beta) - K_\beta(U_h), u_\beta-U_h) = ((T_h-T)\ \partial u_\beta/\partial t, u_\beta-U_h) \ . \tag{4.11}$$

The first term on the left side of (4.11) is

$$\frac{1}{2}\frac{d}{dt}\|P_h(u_\beta-U_h)\|^2_{\tilde{H}^{-1}(\Omega)} \tag{4.12}$$

where the $\tilde{H}^{-1}$ semi-norm on $\mathcal{M}_h$ is defined by

$$\|\chi\|^2_{\tilde{H}^{-1}(\Omega)} = (T_h\tilde{\chi}, \tilde{\chi}), \qquad \chi \in \mathcal{M}_h \ , \tag{4.13}$$

where $\tilde{\chi} = \chi - \frac{1}{|\Omega|}\int_\Omega \chi(x)dx$. This semi-norm is equivalent to the previous H^{-1} semi-norm defined by (2.14) on $\mathcal{M}_h$, i.e. there exist positive constants δ and Λ such that

$$\delta(T\chi,\chi) \leq (T_h\chi,\chi) \leq \Lambda(T\chi,\chi) \tag{4.14}$$

for $\chi \in \mathcal{M}_h$ with mean value zero. We shall identify the two semi-norms in this report.

To verify (4.14), notice that $T_h = E_hT$, where E_h is the elliptic projection (4.54)-(4.55) and

$$\|\chi\|^2_{\tilde{H}^{-1}(\Omega)} = (T_h\chi,\chi) = \|\nabla T_h\chi\|^2_{L^2(\Omega)} = \|\nabla E_hT\chi\|^2_{L^2(\Omega)}$$
$$\leq \|\nabla T\chi\|^2_{L^2(\Omega)} = (T\chi,\chi) = \|\chi\|^2_{H^{-1}(\Omega)}$$

because $\|\nabla E_h f\|_{L^2(\Omega)} \le \|\nabla f\|_{L^2(\Omega)}$ for all $f \in H^1(\Omega)$. Next, by (4.19) for $p=2$ and (4.1c) for the $\tilde{H}^{-1}$ norm,

$$\|\chi\|^2_{H^{-1}(\Omega)} = (T\chi,\chi) = (T_h\chi,\chi) + ((T-T_h),\chi,\chi)$$
$$\le (T_h\chi,\chi) + C_{16}h^2\|\chi\|^2_{L^2(\Omega)}$$
$$\le (T_h\chi,\chi) + C_{16}\cdot C_{15}\|\chi\|^2_{\tilde{H}^{-1}(\Omega)}$$
$$= (1+C_{16}C_{15})\|\chi\|^2_{\tilde{H}^{-1}(\Omega)} .$$

We recall the coercivity result

(4.15) $$(K_\beta(u_\beta) - K_\beta(U_h),\ u_\beta-U_h) \ge C^*\|u_\beta-U_h\|^{2+\nu}_{L^{2+\nu}(\Omega)} .$$

Combining (4.11), (4.12), (4.14), and (4.15) yields

(4.16) $$\frac{1}{2}\frac{d}{dt}\|P_h(u_\beta-U_h)\|^2_{H^{-1}(\Omega)} + C^*\|u_\beta-U_h\|^{2+\nu}_{L^{2+\nu}(\Omega)}$$
$$\le \left|\left((T_h-T)\ \frac{\partial u_\beta}{\partial t},\ u_\beta-U_h\right)\right| .$$

We use the elementary inequality

(4.17) $$ab \le \frac{1}{p}\,a^p + \frac{1}{q}\,b^q$$

for $a,b \ge 0$, $\frac{1}{p}+\frac{1}{q} = 1$, and $1 < p,q < \infty$ to bound the right side of (4.16) by

(4.18) $$C_{15}\|(T-T_h)\ \partial u_\beta/\partial t\|^\gamma_{L^\gamma(\Omega)} + (C^*/2)\|u_\beta-U_h\|^{2+\nu}_{L^{2+\nu}(\Omega)} .$$

The symbol γ denotes $1+\frac{1}{1+\nu}$, the conjugate exponent of $2+\nu$. We shall hide the last term of (4.18) in the second term on the left side of (4.16).

We shall need the result

(4.19) $$\|(T-T_h)f\|_{L^p(\Omega)} \le C_{16}h^2\|f\|_{L^p(\Omega)} ,$$

$\forall f \in L^p(\Omega)$, $1 < p < \infty$, in one space dimension. This is proved by interpolating the L^∞ estimates in [7] and the standard L^2 result and then using a duality argument to account for $p \in (1,2]$. In two space dimensions, R. Scott [13] has shown that

$$(4.19a) \qquad \|(T-T_h)f\|_{L^p(\Omega)} \leq C_{16}'\left(\log(\tfrac{1}{h})\right)^{\left|1-\frac{2}{p}\right|} h^2 \|f\|_{L^p(\Omega)} \; .$$

We shall use the one-dimensional result (4.19) throughout our analysis. The logarithmic term in (4.19a) is computationally insignificant so that our error bounds are essentially valid in two space dimensions.

We now have the bound

$$(4.20) \qquad \int_0^T \|(T_h-T)\ \partial u_\beta/\partial t\|^\gamma_{L^\gamma(\Omega)}\ dt \leq C_{16} h^{2\gamma} \int_0^T \|\partial u_\beta/\partial t\|^\gamma_{L^\gamma(\Omega)}\ dt \; .$$

Theorem 3.3 and Corollary 3.2 tell us that

$$(4.21) \qquad \int_0^T \|\partial u_\beta/\partial t\|^\gamma_{L^\gamma(\Omega)} \leq \begin{cases} C_4 & \nu = 1 \; , \\ C_5 \beta^{-\frac{\nu}{1+\nu}}, & \nu > 1 \; . \end{cases}$$

Integrating (4.16) in time and using (4.18), (4.20), and (4.21) yield the estimate

$$(4.22) \qquad \|P_h(u_\beta-U_h)\|^2_{L^\infty(0,T,H^{-1}(\Omega))} + \frac{C^*}{2} \|u_\beta-U_h\|^{2+\nu}_{L^{2+\nu}(0,T,L^{2+\nu}(\Omega))}$$
$$\leq \begin{cases} C_{17} h^3 \; , & \nu = 1 \; , \\ C_{18} h^{2\gamma} \beta^{-\frac{\nu}{1+\nu}}, & \nu > 1 \; . \end{cases}$$

Notice that, in the special case $\nu = 1$, we can use (4.2) with $\beta = 0$ to compute V_h and U_h. However, (4.22) does not allow us to do this for $\nu > 1$.

THEOREM 4.1. Assume that $\nu = 1$ and let V_h be the solution of (4.2) with $\beta = 0$ with initial data (4.4). Let $U_h = H_0(V_h)$. Then for $\dim(\Omega) = 1$,

$$(4.23) \qquad \|u-U_h\|_{L^\infty(0,T,H^{-1})} \le C_{19}h^{3/2} ,$$

and

$$(4.24) \qquad \|u-U_h\|_{L^2(0,T,L^2)} \le C_{20}\|u-U_h\|_{L^3(0,T,L^3)} \le C_{21}h .$$

Furthermore, the inverse hypothesis (4.1c) on $\mathcal{M}_h$ implies that

$$(4.25) \qquad \|u-U_h\|_{L^\infty(0,T,L^2)} \le C_{22}h^{1/2} .$$

We remark that (4.23)-(4.25) are still valid for any $\beta \in [0,h]$. This follows from Theorem 2.1.

Proof of Theorem 4.1. The bound (4.22) for $\beta = 0$ and $\nu = 1$ tells us that

$$(4.26) \qquad \|P_h(u-U_h)\|^2_{L^\infty(H^{-1})} + C^*\|u-U_h\|^3_{L^3(L^3)} \le C_{17}h^3$$

which establishes (4.24). Next, we note that

$$(4.27) \qquad \|(I-P_h)(u-U_h)\|_{L^\infty(H^{-1})} \le C_{23}h\|(I-P_h)(u-U_h)\|_{L^\infty(L^2)} .$$

Suppose $\Omega_\varepsilon(t) = \{x | u(x,t) \ge \varepsilon\}$ and $\Omega^c_\varepsilon(t) = \Omega \sim \Omega_\varepsilon(t)$. Let $u^\varepsilon = u$ on $\Omega_\varepsilon(t)$ and $u^\varepsilon = \varepsilon$ on $\Omega^c_\varepsilon(t)$. Since $|u^\varepsilon - u| \le \varepsilon$, we have

$$(4.28) \qquad \begin{aligned} \|(I-P_h)(u)\|_{L^\infty(L^2)} &\le \|(I-P_h)(u^\varepsilon)\|_{L^\infty(L^2)} + \|(I-P_h)(u^\varepsilon-u)\|_{L^\infty(L^2)} \\ &\le C_{24}h\|u^\varepsilon\|_{L^\infty(H^1)} + C_{25}\varepsilon \le \tilde{C}_{24}h\|u^\varepsilon\|_{L^\infty(H^1(\Omega_\varepsilon(t)))} + \tilde{C}_{25}\varepsilon \\ &= \tilde{C}_{24}h\|u\|_{L^\infty(H^1(\Omega_\varepsilon(t)))} + \tilde{C}_{25}\varepsilon . \end{aligned}$$

where we have used the continuity of P_h on $L^\infty(\Omega)$ [6]. In both (4.27) and (4.28) we have used the estimate

$$\|(I-P_h)f\|_{H^j(\Omega)} \leq C_{26}h\|f\|_{H^{j+1}(\Omega)}$$

for j = -1 or 0.

Integrating (1.1) against $K(u)_t$ yields

(4.29) $$(u_t, K(u)_t) + \frac{1}{2}\frac{d}{dt}\|\nabla K(u)\|^2_{L^2(\Omega)} = 0 ,$$

so that $\nabla K(u) = u\nabla u \in L^\infty(L^2)$. Since $u \geq \varepsilon$ on $\Omega_\varepsilon(t)$, it follows that

(4.30) $$\|u\|_{L^\infty(H^1(\Omega_\varepsilon(t)))} \leq C_{27}\varepsilon^{-1} .$$

Thus,

(4.31) $$\|(I-P_h)u\|_{L^\infty(L^2)} \leq C_{28}\left(\varepsilon + \frac{h}{\varepsilon}\right) .$$

Choosing $\varepsilon = \sqrt{h}$ proves that

(4.32a) $$\|(I-P_h)u\|_{L^\infty(L^2)} \leq C_{28}h^{1/2} , \text{ and}$$

(4.32b) $$\|(I-P_h)u\|_{L^\infty(H^{-1})} \leq C_{28}h^{3/2} .$$

Similar bounds holds for U_h in place of u; the proof is identical. Thus

(4.33) $$\|(I-P_h)(u-U_h)\|_{L^\infty(L^2)} \leq \|(I-P_h)(u)\|_{L^\infty(L^2)} + \|(I-P_h)(U_h)\|_{L^\infty(L^2)} \leq C_{29}h^{1/2}$$

and

(4.34) $$\|(I-P_h)(u-U_h)\|_{L^\infty(H^{-1})} \leq C_{26}h\|(I-P_h)(u-U_h)\|_{L^\infty(L^2)} \leq C_{30}h^{3/2} .$$

The bounds (4.26) and (4.34) yield the estimate (4.23).

Finally, the inverse hypothesis (4.1c) for $\mathcal{M}_h$ tells us that

$$\|P_h(u-U_h)\|_{L^\infty(L^2)} \leq C_{30}\cdot\frac{1}{h}\cdot\|P_h(u-U_h)\|_{L^\infty(H^{-1})} \leq C_{31}h^{1/2} . \tag{4.35}$$

Next, the bound (4.33) and the corresponding bound for U_h imply that

$$\|(I-P_h)(u-U_h)\|_{L^\infty(L^2)} \leq C_{32}\ h^{1/2} . \tag{4.36}$$

Combining (4.35) and (4.36) yields (4.25). □

Next, we prove a similar result for the more general case $\nu \geq 1$.

THEOREM 4.2. Let V_h be the solution of (4.2)-(4.4) with $\dim(\Omega) = 1$ and

$$\beta = \beta(h) = h^\sigma,\ \sigma = \frac{2\gamma}{2+\nu+\frac{\nu}{1+\nu}} . \tag{4.37}$$

Then,

$$\|u-U_h\|_{L^\infty(H^{-1})} \leq C_{33}\ h^{\left(\frac{\nu+2}{2}\right)\sigma} , \tag{4.38}$$

$$\|u-U_h\|_{L^2(L^2)} \leq C_{20}\|u-U_h\|_{L^{2+\nu}(L^{2+\nu})} \leq C_{34}\ h^\sigma , \tag{4.39}$$

and

$$\|u-U_h\|_{L^\infty(L^2)} \leq C_{35}\ h^{\left[\sigma\left(\frac{\nu+2}{2}\right) - 1\right]} . \tag{4.40}$$

We remark that, for $\nu = 1$, Theorem 4.2 produces $L^\infty(H^{-1})$ and $L^3(L^3)$ error estimates of $O(h^{9/7})$ and $O(h^{6/7})$ which are worse than the conclusions of Theorem 4.1 in this special case.

Proof of Theorem 4.2. Recall the estimate (4.22) for $\nu \geq 1$. By using the techniques of (4.27)-(4.34) for $\nu > 1$ with $\varepsilon = h^{\frac{1}{1+\nu}}$, we can prove that

$$\|(I-P_h)(u_\beta-U_h)\|_{L^\infty(0,T,H^{-1}(\Omega))} \le C_{23}h\|(I-P_h)(u_\beta-U_h)\|_{L^\infty(0,T,L^2(\Omega))} \tag{4.41}$$
$$\le C_{36}\cdot h\cdot\left(\varepsilon+\frac{h}{\varepsilon^\nu}\right) = C_{36}h^\gamma \ .$$

Thus, we can strengthen the first term in (4.22) and write

$$\|u_\beta-U_h\|^2_{L^\infty(0,T,H^{-1})} + (C^*/2)\|u_\beta-U_h\|^{2+\nu}_{L^{2+\nu}(0,T,L^{2+\nu})} \tag{4.42}$$
$$\le C_{37}\ h^{2\gamma}\ \beta^{-\frac{\nu}{1+\nu}} \ .$$

If we choose $\beta = h^\sigma$ in (2.15) so that

$$h^{2\gamma}/\beta^{\frac{\nu}{1+\nu}} = \beta^{2+\nu} \ , \tag{4.43}$$

we have established bounds (4.38) and (4.39). Finally, we use the inverse hypothesis (4.1c), the bound (4.22) on $\|P_h(u_\beta-U_h)\|_{L^\infty(H^{-1})}$, and the bound (4.41) to establish (4.40). □

Here are some convergence rates for 'small' values of ν.

(4.44)

ν	$L^\infty(H^{-1})$	$L^2(L^2)$	$L^\infty(L^2)$
1	9/7	6/7	2/7
2	8/7	4/7	1/7
3	25/23	10/23	2/23

Notice that, although β is not explicitly present in the bounds (4.38)-(4.40), the Galerkin approximation U_h to u_β was computed with the diffusion coefficient k_β, not k.

We conjecture that the following regularity result is true:

$$\|\partial u/\partial t\|_{L^\gamma(0,T,L^\gamma)} \le C_{38} \ , \tag{4.45}$$

for all $\nu \ge 1$. If (4.45) is true, we can set $\beta = 0$ in (4.2)-(4.4) and we could prove the error estimates

(4.46) $$\|u-U_h\|_{L^\infty(H^{-1})} \le C_{39}\, h^\gamma ,$$

(4.47) $$\|u-U_h\|_{L^2(L^2)} \le C_{20}\|u-U_h\|_{L^{2+\nu}(L^{2+\nu})} \le C_{40}\, h^{2/(1+\nu)} ,$$

and

(4.48) $$\|u-U_h\|_{L^\infty(L^2)} \le C_{41}\, h^{1/(1+\nu)} .$$

This would yield the rates

(4.49)

ν	$L^\infty(H^{-1})$	$L^2(L^2)$	$L^\infty(L^2)$
1	3/2	1	1/2
2	4/3	2/3	1/3
3	5/4	1/2	1/4

We repeat that (4.49) has been proved for = 1. The same rates are valid for any $\beta \in [0, h^{2/1+\nu}]$.

Another Continuous-Time Scheme

The reader may wonder why we considered a continuous piecewise-linear approximation V_h to $K_\beta(u_\beta)$ instead of approximating u_β directly by a piecewise-linear function $\tilde{U}_h$. The reason is that the convergence seems to be slower for the second scheme.

However, in some generalizations of the porous medium equation, we may be interested in u, not K(u), and it may be difficult to recover u from K(u). Thus, one may want to compute $\tilde{U}_h \approx u_\beta$ directly.

We shall study the following approximation $\tilde{U}_h$: $[0,T] \to \mathcal{M}_h$ to u_β:

(4.50) $$\left(\frac{\partial}{\partial t}\tilde{U}_h, \chi\right) + (\nabla K_\beta(\tilde{U}_h), \nabla\chi) = 0 ,$$

$\forall \chi \in \mathcal{M}_h$, $0 < t \le T$, and

(4.51) $$\tilde{U}_h(0) = P_h u_0 .$$

With T and T_h defined by (2.10)-(2.13) and (4.5)-(4.6), respectively, we have

$$(4.52)\qquad T_h\left(\frac{\partial}{\partial t} u_\beta - \frac{\partial}{\partial t} \tilde{U}_h\right) + (K_\beta(u_\beta) - K_\beta(\tilde{U}_h))$$

$$= (T_h - T)\frac{\partial}{\partial t} u_\beta + (E_h - I)K_\beta(\tilde{U}_h)$$

$$+ \frac{1}{|\Omega|}\int_\Omega (K_\beta(u_\beta) - K_\beta(\tilde{U}_h))dx \ .$$

Here $E_h\colon H^1(\Omega) \to \mathcal{M}_h$ is the elliptic projection onto $\mathcal{M}_h$, i.e.

$$(4.53)\qquad (\nabla f - \nabla E_h f, \nabla\chi) = 0 \ , \qquad \forall\chi \in \mathcal{M}_h \ ,$$

for all $f \in H^1(\Omega)$, and

$$(4.54)\qquad \frac{1}{|\Omega|}\int_\Omega E_h f dx = \frac{1}{|\Omega|}\int_\Omega f dx \ .$$

The appearance of the term $(E_h - I)K_\beta(\tilde{U}_h)$ in (4.52) complicates the analysis and slows the convergence rate. It cannot be omitted because $K_\beta(\tilde{U}_h) \notin \mathcal{M}_h$.

We integrate (4.52) against $u_\beta - \tilde{U}_h$, which has mean value zero, to get

$$(4.55)\qquad \frac{1}{2}\frac{d}{dt}\|P_h u_\beta - \tilde{U}_h\|^2_{H^{-1}(\Omega)} + C^*\|u_\beta - \tilde{U}_h\|^{2+\nu}_{L^{2+\nu}(\Omega)}$$

$$\leq C_{42}\|(T_h - T)\ \partial u_\beta/\partial t\|^{\gamma}_{L^\gamma(\Omega)} + C_{42}\|(E_h - I)K_\beta(\tilde{U}_h)\|^{\gamma}_{L^\gamma(\Omega)}$$

$$+ (C^*/2)\|u_\beta - \tilde{U}_h\|^{2+\nu}_{L^{2+\nu}(\Omega)} \ .$$

Hide the last term on the left side as before. Integrating in time from 0 to T, we get

$$(4.56)\qquad \frac{1}{2}\|P_h u_\beta - \tilde{U}_h\|^2_{L^\infty(H^{-1})} + (C^*/2)\|u_\beta - \tilde{U}_h\|^{2+\nu}_{L^{2+\nu}(L^{2+\nu})}$$
$$\le C_{42}\int_0^T \|(T_h - T)\ \partial u_\beta/\partial t\|^\gamma_{L^\gamma(\Omega)}$$
$$+ C_{42}\int_0^T \|(E_h - I)K_\beta(\tilde{U}_h)\|^\gamma_{L^\gamma(\Omega)}\ dt\ .$$

The first term on the right side of (4.56) can be bounded as in (4.20)-(4.21). To bound the second one, we use the estimate

$$(4.57)\qquad \|(E_h - I)f\|_{L^p(\Omega)} \le C_{43}h^1\|\nabla f\|_{L^p(\Omega)}$$

for $1 < p < \infty$ and $f \in W^{1,p}(\Omega)$ to see that

$$(4.58)\qquad \int_0^T \|(E_h - I)K_\beta(\tilde{U}_h)\|^\gamma_{L^\gamma(\Omega)}\ dt \le C_{43}h^\gamma \int_0^T \|\nabla K_\beta(\tilde{U}_h)\|^\gamma_{L^\gamma(\Omega)}\ dt\ .$$

In order to complete our analysis, we notice that (4.17) yields

$$(4.59)\qquad \int_0^T \|\nabla K_\beta(\tilde{U}_h)\|^\gamma_{L^\gamma(\Omega)}\ dt \le \frac{2-\gamma}{2}\int_0^T\int_\Omega k_\beta(\tilde{U}_h)^{\frac{\gamma}{2-\gamma}}dxdt$$
$$+ \frac{\gamma}{2}\int_0^T\int_\Omega k_\beta(\tilde{U}_h)(\nabla\tilde{U}_h)^2 dxdt\ .$$

We can bound the second term on the right side of (4.59) by using the test function $\tilde{U}_h$ in (4.50) to get

$$(4.60)\qquad \frac{1}{2}\frac{d}{dt}\|\tilde{U}_h\|^2_{L^2(\Omega)} + (k_\beta(\tilde{U}_h)\nabla\tilde{U}_h,\ \nabla\tilde{U}_h) = 0\ .$$

Integration in time shows that $\sqrt{k_\beta(\tilde{U}_h)}\ \nabla\tilde{U}_h = \nabla\int_0^{\tilde{U}_h}\sqrt{k_\beta(\tau)}\ d\tau \in L^2(L^2)$, which takes care of the first term on the right side of (4.59). Since $k_\beta(\xi) = |\xi|^\nu$ for $|\xi| \ge \beta$, $\nabla(\tilde{U}_h^{1+\nu/2}) \in L^2(L^2)$. Since we know that the mean value of $\tilde{U}_h$ is independent of β and h (it equals the mean value of u_0), Poincare's inequality tells us that $\tilde{U}_h^{1+\nu/2} \in L^2(L^2)$. Since $\frac{\gamma}{2(2-\gamma)} = \frac{1}{2} + \frac{1}{\nu} \le 1 + \frac{\nu}{2}$ for $\nu \ge 1$, we know that $\tilde{U}_h^{\frac{\gamma}{2(2-\gamma)}} \in L^2(L^2)$, which takes care of the first term on the right side of (4.59). Thus,

(4.61) $$\int_0^T \|\nabla K_\beta(U_h)\|^\gamma_{L^\gamma(\Omega)}\, dt \le C_{44} \, .$$

We remark that this argument could be replaced by proving an L^∞ stability result for $\tilde{U}_h$. Another method would be to modify the definition (2.2) of $k_\beta(\xi)$ by adding the requirement that $k_\beta(\xi) \le M+1$ for all real ξ while retaining (2.2a) in the form

$$k_\beta(\xi) = k(\xi) \quad \text{for} \quad \beta \le |\xi| \le M \, .$$

This would require minor changes in our coercivity estimates.

We could then bound the second term on the right side of (4.56) by using (4.60) and the L^∞ bound on $\sqrt{k_\beta(\tilde{U}_h)}$.

It may be possible to get better bounds by estimating $\Delta K_\beta(\tilde{U}_h)$, which equals $k'_\beta(\tilde{U}_h)(\nabla\tilde{U}_h)^2$ on the interiors of our elements because $\tilde{U}_h$ is piecewise-linear.

We have proved

THEOREM 4.3. Assume $\nu \ge 1$ and let $\tilde{U}_h$ be the solution of (4.51)-(4.52) with β given by (4.38). Then when $\dim(\Omega) = 1$,

(4.63) $$\|u-\tilde{U}_h\|_{L^\infty(H^{-1})} \le C_{46}\, h^{\gamma/2} \, , \quad \text{and}$$

(4.64) $$\|u-\tilde{U}_h\|_{L^2(L^2)} \le C_{20}\|u-\tilde{U}_h\|_{L^{2+\nu}(L^{2+\nu})} \le C_{47}\, h^{1/(2+\nu)} \, .$$

Remarks on the Initial Data

We need to show that (4.4) has a unique solution $V_h(0) \in \mathcal{M}_h$. We begin by showing that $P_h \circ H_\beta$ is a monotone operator on $\mathcal{M}_h$. For V^1 and V^2 in $\mathcal{M}_h$,

$$(4.65) \qquad (P_h H_\beta(V^1) - P_h H_\beta(V^2),\ V^1 - V^2) = (V^1 - V^2,\ H_\beta(V^1) - H_\beta(V^2))$$
$$= (K_\beta(H_\beta(V^1)) - K_\beta(H_\beta(V^2)),\ H_\beta(V^1) - H_\beta(V^2))$$
$$\geq C^* \| H_\beta(V^1) - H_\beta(V^2) \|^{2+\nu}_{L^{2+\nu}(\Omega)} \geq 0 ,$$

where we have used (2.33) for $\beta \geq 0$. If V^1 and V^2 both satisfy (4.4), then $H_\beta(V^1) = H_\beta(V^2)$ by (4.65), so that $V^1 = V^2$. The continuity of the operator $P_h \circ H_\beta$ on $\mathcal{M}_h$ follows from the continuity of $H_\beta(\xi)$ on the real line.

To demonstrate coercivity, set $V^2 = 0$ in (4.65) to get

$$(4.66) \qquad (P_h H_\beta(V^1), V^1) \geq C^* \| H_\beta(V^1) \|^{2+\nu}_{L^{2+\nu}} .$$

Notice that

$$(4.67) \qquad |V^1| = |K_\beta(H_\beta(V^1))| \leq |K(H_\beta(V^1))| + |(K_\beta - K)(H_\beta(V^1))|$$
$$\leq \frac{1}{1+\nu} |H_\beta(V^1)|^{1+\nu} + O(\beta^{1+\nu}) ,$$

where we have used (2.25). Thus, using (4.67) and the equivalence of norms on the finite-dimensional space $\mathcal{M}_h$,

$$(4.68) \qquad \| H_\beta(V^1) \|^{2+\nu}_{L^{2+\nu}} \geq C' \| V^1 \|^{\gamma}_{L^\gamma} - C'' \beta^{2+\nu} \geq C''' \| V^1 \|^{\gamma} - C'' \beta^{2+\nu} .$$

We have proved that

$$(4.69) \qquad \frac{(P_h H_\beta(V^1), V^1)}{\| V^1 \|} \geq \frac{C''' \| V^1 \|^{\gamma} - C'' \beta^{2+\nu}}{\| V^1 \|} \geq \frac{C'''}{2} \| V^1 \|^{\frac{1}{1+\nu}} \to \infty$$

as $\| V^1 \| \to \infty$, i.e. we have proved that $P_h \circ H_\beta$ is coercive on $\mathcal{M}_h$. Since $P_h \circ H_\beta$ is a continuous coercive monotone operator on $\mathcal{M}_h$ with respect to the Hilbert space structure given by the L^2 inner product, $P_h \circ H_\beta$ is surjective [15]. This proves that (4.4) has a solution.

Although our choice of initial data in (4.4) is convenient for our analysis, it is not trivial to compute. However, it is not known whether the same convergence rates can be attained using explicitly defined initial data such as $V_h(0) = P_h K_\beta(u_0)$ or $V_h(0) = E_h K_\beta(u_0)$. Slower rates can be demonstrated for these choices of the initial data.

Remarks on Numerical Approximation in Two or Three Space Dimensions

For $\dim(\Omega) = 2$, we must use the estimate (4.19a) instead of (4.19) in our analysis. This will result in slightly slower rates of convergence. In Theorem 4.1, we must replace the estimates (4.23), (4.24), and (4.25) with

$$(4.23)' \qquad \|u-U_h\|_{L^\infty(H^{-1})} \le C'_{19}\left(\log\left(\frac{1}{h}\right)\right)^{\frac{1}{4}} h^{3/2} ,$$

$$(4.24)' \qquad \|u-U_h\|_{L^2(L^2)} \le C_{20}\|u-U_h\|_{L^3(L^3)} \le C'_{21}\left(\log\left(\frac{1}{h}\right)\right)^{\frac{1}{6}} h ,$$

and

$$(4.25)' \qquad \|u-U_h\|_{L^\infty(L^2)} \le C'_{22}\left(\log\left(\frac{1}{h}\right)\right)^{\frac{1}{4}} h^{1/2} ,$$

respectively. In Theorem 4.2, we must replace (4.38), (4.39), and (4.40) with

$$(4.38)' \qquad \|u-U_h\|_{L^\infty(H^{-1})} \le C'_{33}\left(\log\left(\frac{1}{h}\right)\right)^{\frac{\nu}{2+2\nu}} h^{\left(\frac{2+\nu}{2}\right)\cdot\sigma} ,$$

$$(4.39)' \qquad \|u-U_h\|_{L^2(L^2)} \le C_{20}\|u-U_h\|_{L^{2+\nu}(L^{2+\nu})}$$

$$\le C'_{34}\left(\log\left(\frac{1}{h}\right)\right)^{\frac{\nu}{(1+\nu)(2+\nu)}} h^\sigma ,$$

and

$$(4.40)' \qquad \|u-U_h\|_{L^\infty(L^2)} \le C'_{35}\left(\log\left(\frac{1}{h}\right)\right)^{\frac{\nu}{2+2\nu}} \cdot h^{\left[\sigma\left(\frac{2+\nu}{2}\right)-1\right]} .$$

Finally, our conjectured estimates (4.46), (4.47), and (4.48) must be replaced by the following in two space dimensions:

$$(4.46)' \qquad \|u-U_h\|_{L^\infty(H^{-1})} \le C'_{39}\left(\log\left(\frac{1}{h}\right)\right)^{\frac{\nu}{2+2\nu}} h^\gamma ,$$

$$(4.47)' \qquad \|u-U_h\|_{L^2(L^2)} \le C^{20}\|u-U_h\|_{L^3(L^3)} \le C'_{40}\left(\log\left(\frac{1}{h}\right)\right)^{\frac{\nu}{(1+\nu)(2+\nu)}} h^{\frac{2}{1+\nu}} ,$$

and

$$(4.48)' \qquad \|u-U_h\|_{L^\infty(L^2)} \le C'_{41}\left(\log\left(\frac{1}{h}\right)\right)^{\frac{\nu}{2+2\nu}} h^{\frac{1}{1+\nu}} .$$

In three space dimensions, L^p bounds of the form (4.19) or (4.19a) are unknown for $p \neq 2$. A cruder version of our arguments using the estimate

$$(4.70) \qquad \|(T-T_h)f\|_{L^2(\Omega)} \le C''_{16} \|f\|_{L^2(\Omega)}$$

for $f \in L^2(\Omega)$ and $\dim(\Omega) = 3$ will yield slower rates of convergence than our methods for $\dim(\Omega) = 1$ or 2.

CHAPTER V

BACKWARD-DIFFERENCE AND CRANK-NICOLSON SCHEMES

We shall first consider a backward-difference scheme corresponding to (4.2)-(4.4). Let $\Delta t = T/M$, where M is a positive integer. We define $\{V_h^n\}_{n=0}^{M} \subset \mathcal{M}_h$ by the equations

$$(5.1) \qquad \left(\frac{H_\beta(V_h^{n+1}) - H_\beta(V_h^n)}{\Delta t}, \chi\right) + (\nabla V_h^{n+1}, \nabla\chi) = 0 , \quad \chi \in \mathcal{M}_h ,$$

$n = 0,1,\ldots,M-1$, and the initial data $V_h^0 \in \mathcal{M}_h$ defined by

$$(5.2) \qquad P_h H_\beta(V_h^0) = P_h u_0 .$$

Writing U_h^n for $H_\beta(V_h^n)$, we can rewrite (5.1) in the form

$$(5.3) \qquad \left(\frac{U_h^{n+1} - U_h^n}{\Delta t}, \chi\right) + (\nabla K_\beta(U_h^n), \nabla\chi) = 0 , \quad \chi \in \mathcal{M}_h ,$$

$n = 0,1,\ldots,M-1$, and

$$(5.4) \qquad P_h U_h^0 = P_h u_0 .$$

We emphasize that $K_\beta(U_h^n) \in \mathcal{M}_h$, whereas $U_h^n \notin \mathcal{M}_h$. We should point out that the coercivity of the 'elliptic' term in (5.1) ensures the existence of the V_h^n's.

It follows from (4.5)-(4.6) that

$$(5.5) \qquad T_h\left\{\frac{U_h^{n+1} - U_h^n}{\Delta t}\right\} + K_\beta(U_h^{n+1}) = \frac{1}{|\Omega|}\int_\Omega K_\beta(U_h^{n+1})dx .$$

Let us introduce the notation

(5.6) $$(\partial^+ U_h)^n \equiv \frac{U_h^{n+1} - U_h^n}{\Delta t}, \quad \text{and}$$

(5.7) $$u_\beta^n = u_\beta(\cdot, t_n), \qquad t_n = n\Delta t .$$

Notice that, since $1 \in \mathcal{M}_h$, $(\partial^+ U_h)^n$ has mean value zero. This implies that $u_\beta^n - U_h^n$ has mean value zero for all n.

The porous medium equation (2.3)-(2.5) can be put in the form

(5.8) $$T(\partial^+ u_\beta)^n + K_\beta(u_\beta^{n+1}) = T\left\{(\partial^+ u_\beta)^n - \frac{\partial u_\beta}{\partial t}(t_{n+1})\right\} + \frac{1}{|\Omega|}\int_\Omega K_\beta(u_\beta^{n+1})\,dx .$$

Subtract (5.8) from (5.5) to see that

(5.9) $$T_h\Big((\partial^+ u_\beta)^n - (\partial^+ U_h)^n\Big) + \Big(K_\beta(u_\beta^{n+1}) - K_\beta(U_h^{n+1})\Big) = T\left((\partial^+ u_\beta)^n - \frac{\partial u_\beta^{n+1}}{\partial t}\right) + (T_h - T)(\partial^+ u_\beta)^n + \frac{1}{|\Omega|}\int_\Omega \Big(K_\beta(u_\beta^{n+1}) - K_\beta(U_h^{n+1})\Big)\,dx .$$

Next, integrate (5.9) against $e_h^{n+1} = u_\beta^{n+1} - U_h^{n+1}$ to see that

(5.10) $$\frac{1}{2\Delta t}\left\{(T_h e_h^{n+1}, e_h^{n+1}) - (T_h e_h^n, e_h^n)\right\} + \Big(K_\beta(u_\beta^{n+1}) - K_\beta(U_h^{n+1}), u_\beta^{n+1} - U_h^{n+1}\Big)$$
$$\le \Big(T_h(\partial^+ e_h)^n, e_h^{n+1}\Big) + \Big(K_\beta(u_\beta^{n+1}) - K_\beta(U_h^{n+1}), u_\beta^{n+1} - U_h^{n+1}\Big)$$
$$= \left(T\left\{(\partial^+ u_\beta)^n - \frac{\partial u_\beta^{n+1}}{\partial t}\right\}, e_h^{n+1}\right) + \Big((T_h - T)(\partial^+ u_\beta)^n, e_h^{n+1}\Big).$$

Let us use the coercivity estimate (4.15), the inequality (4.17), and the identification of H^{-1} semi-norms (4.14) to express (5.10) as

$$(5.11)\qquad \frac{1}{2\Delta t}\left\{\|P_h e_n^{n+1}\|^2_{H^{-1}(\Omega)} - \|P_h e_h^n\|^2_{H^{-1}(\Omega)}\right\} + (C^*/2)\|e_h^{n+1}\|^{2+\nu}_{L^{2+\nu}(\Omega)}$$
$$\le C_{48}\|T\{(\partial^+ u_\beta) - \partial u_\beta^{n+1}/\partial t\}\|^\gamma_{L^\gamma(\Omega)} + C_{48}\|(T_h - T)(\partial^+ u_\beta)^n\|^\gamma_{L^\gamma(\Omega)} ,$$

where $\gamma = \frac{\nu+2}{\nu+1}$.

We begin our analysis of the first term on the right side of (5.11) by noticing that

$$(5.12)\qquad -T\{(\partial^+ u_\beta) - \partial u_\beta^{n+1}/\partial t\}$$
$$= -T\left\{\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (u_{\beta t}(t_{n+1}) - u_{\beta t}(s))\,ds\right\}$$
$$= -T\left\{\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\int_s^{t_{n+1}} u_{\beta tt}(\tau)\,d\tau ds\right\}$$
$$= \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\int_s^{t_{n+1}} (-Tu_{\beta tt})(\tau)\,d\tau ds .$$

Since $-Tu_{\beta tt} = K_\beta(u_\beta)_t - \frac{1}{|\Omega|}\int_\Omega K_\beta(u_\beta)_t dx = \widetilde{K_\beta(u_\beta)_t}$, where $\tilde{f}$ denotes $f - \frac{1}{|\Omega|}\int_\Omega f dx$ for measurable functions f, we have the inequality

$$(5.13)\qquad \|T\{(\partial^+u_\beta)^n - \partial u_\beta^{n+1}/\partial t\}\|_{L^\gamma(\Omega)}$$

$$\leq \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\int_s^{t_{n+1}} \|\widetilde{K_\beta(u_\beta)_t}\|_{L^\gamma(\Omega)} d\tau ds$$

$$\leq \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t_{n+1}-s)^{\frac{1}{2+\nu}} \|\widetilde{K_\beta(u_\beta)_t}\|_{L^\gamma(t_n,t_{n+1},L^\gamma)} ds$$

$$\leq C_{49}(\Delta t)^{\frac{1}{2+\nu}} \|\widetilde{K_\beta(u_\beta)_t}\|_{L^\gamma(t_n,t_{n+1},L^\gamma)} .$$

Raising (5.13) to the power γ and summing on $n = 0,1,\ldots,M-1$ yield

$$(5.14)\qquad \sum_{n=0}^{M-1} \|T\{(\partial^+u_\beta)^n - \partial u_\beta^{n+1}/\partial t\}\|^\gamma_{L^\gamma(\Omega)}$$

$$\leq C_{50}(\Delta t)^{\frac{1}{1+\nu}} \|\widetilde{K_\beta(u_\beta)_t}\|^\gamma_{L^\gamma(0,T,L^\gamma)} .$$

We claim that

$$(5.15)\qquad \|\widetilde{K_\beta(u_\beta)_t}\|_{L^\gamma(L^\gamma)} \leq C_{51} ,$$

where C_{51} is independent of β. Since

$$(5.16)\qquad \|\widetilde{K_\beta(u_\beta)_t}\|_{L^\gamma(L^\gamma)} \leq \|K_\beta(u_\beta)_t\|_{L^\gamma(L^\gamma)}$$

$$\leq C_{20}\|K_\beta(u_\beta)_t\|_{L^2(L^2)}$$

because $1\leq\gamma<2$, it is enough to get an $L^2(L^2)$ bound on $K_\beta(u_\beta)_t$. This follows from the equality

$$(5.17)\qquad (u_{\beta t},K_\beta(u_\beta)_t) + \frac{1}{2}\frac{d}{dt}\|\nabla K_\beta(u_\beta)\|^2_{L^2(\Omega)} = 0$$

and the estimate

$$\|u_\beta\|_{L^\infty(L^\infty)} \leq \|u_0\|_{L^\infty} = M$$

because

(5.18) $$\|K_\beta(u_\beta)_t\|_{L^2(L^2)} \le \|\sqrt{k_\beta(u_\beta)}\|_{L^\infty(L^\infty)} \|\sqrt{k_\beta(u_\beta)}\, u_{\beta t}\|_{L^2(L^2)}$$
$$\le \frac{1}{2}\sqrt{k_\beta(M)}\, \|\nabla K_\beta(u_0)\|_{L^2(\Omega)} \le C_{52} .$$

Thus, we have proved that

(5.19) $$\sum_{n=0}^{M-1} \|T\{(\partial^+ u_\beta)^n - \frac{\partial}{\partial t} u_\beta^{n+1}\}\|^\gamma_{L^\gamma(\Omega)} \le C_{52}(\Delta t)^{\frac{1}{1+\nu}} .$$

Our next task is to analyze the second term on the right side of (5.11). We use the estimate (4.19) to see that

(5.20) $$\|(T-T_h)(\partial^+ u_\beta)^n\|^\gamma_{L^\gamma(\Omega)} \le C_{53}\, h^{2\gamma}\|(\partial^+ u_\beta)^n\|^\gamma_{L^\gamma(\Omega)} .$$

Next, Hölder's inequality implies that

(5.21) $$\|(\partial^+ u_\beta)\|_{L^\gamma(\Omega)} \le \frac{1}{\Delta t} \cdot (\Delta t)^{\frac{1}{2+\nu}} \cdot$$
$$\|\frac{\partial}{\partial t} u_\beta\|_{L^\gamma(t_n, t_{n+1}, L^\gamma(\Omega))} .$$

Raising (5.21) to the power γ, using it in (5.20), and summing on n yield

(5.22) $$\sum_{n=0}^{M-1} \|(T-T_h)(\partial^+ u_\beta)^n\|^\gamma_{L^\gamma(\Omega)} \le C_{54}\, h^{2\gamma} \cdot \frac{1}{\Delta t} \|u_{\beta t}\|^\gamma_{L^\gamma(0,T,L^\gamma)} .$$

We now apply the bound (4.22) to see that (5.22) is bounded by

(5.23) $$C_{55}\, h^3 \frac{1}{\Delta t}, \quad \text{when} \quad \nu = 1 ,$$

and by

(5.24) $$C_{56}\, h^{2\gamma} \cdot \frac{1}{\Delta t} \cdot \beta^{-\frac{\nu}{1+\nu}}, \quad \text{when} \quad \nu > 1 .$$

We can multiply the inequality (5.10) by $2\Delta t$ and sum on n to see that

(5.25) $$\operatorname*{Max}_n \|P_h(u_\beta - U_h)^n\|^2_{H^{-1}(\Omega)} + C^* \sum_{n=0}^{M-1} \|(u_\beta - U_h)^{n+1}\|^{2+\nu}_{L^{2+\nu}(\Omega)} \Delta t$$

$$\le C_{53}(\Delta t)^\gamma + C_{54}\, h^3 ,$$

if $\nu = 1$, and

$$\text{(5.26)} \qquad \max_n \|P_h(u_\beta - U_h)^n\|^2_{H^{-1}(\Omega)} + C^* \sum_{n=0}^{M-1} \|(u_\beta - U_h)^n\|^{2+\nu}_{L^{2+\nu}(\Omega)} \Delta t$$

$$\leq C_{53}(\Delta t)^\gamma + C_{54}\, h^{2\gamma}\beta^{-\frac{\nu}{1+\nu}}$$

for $\nu > 1$.

We have proved the following theorem.

THEOREM 5.1. Assume that $\nu = 1$ and let $\{V_h^n\}_{n=0}^M$ be the solution of (5.1)-(5.2) with $\beta = 0$. Then, when $\dim(\Omega) = 1$,

$$\text{(5.27)} \qquad \max\|(u - U_h)^n\|_{H^{-1}(\Omega)} \leq C_{55}(h^{3/2} + (\Delta t)^{3/4}) ,$$

and

$$\text{(5.28)} \qquad \left(\sum_{n=0}^{M-1} \|(u - U_h)^{n+1}\|^2_{L^2(\Omega)} \Delta t\right)^{1/2}$$

$$\leq C_{56}\left(\sum_{n=0}^{M-1} \|(u - U_h)^{n+1}\|^3_{L^3(\Omega)} \Delta t\right)^{1/3}$$

$$\leq C_{57}(h + (\Delta t)^{1/2}) .$$

By the inverse hypothesis (4.1c) on $\mathcal{M}_h$, (5.27) implies

$$\text{(5.29)} \qquad \max_n \|(u - U_h)^n\|_{L^2(\Omega)} \leq C_{58}(h^{1/2} + h^{-1}(\Delta t)^{3/4})$$

$$= C_{58}(h^{1/2} + (\Delta t)^{1/4}) \quad \text{for} \quad \Delta t = h^2 .$$

We remark that (5.27)-(5.29) remain valid for any $\beta \in [0,h]$ by Theorem 2.1.

THEOREM 5.2. Let $\nu > 1$ and let $\{V_h^n\}_{n=0}^M$ be the solution of (5.1)-(5.2) with $\beta = \beta(h) = h^\sigma$ defined by (4.38). Then, when $\dim(\Omega) = 1$,

$$\text{(5.30)} \qquad \max_n \|(u - U_h)^n\|_{H^{-1}(\Omega)} \leq C_{59}\left(h^{\frac{2+\nu}{2}\cdot\sigma} + (\Delta t)^{\gamma/2}\right), \quad \text{and}$$

(5.31)
$$\left(\sum_{n=0}^{M-1} \|(u-U_h)^{n+1}\|^2_{L^2(\Omega)} \Delta t\right)^{1/2} \le C_{56}\left(\sum_{n=0}^{M-1} \|(u-U_h)^{n+1}\|^{2+\nu}_{L^{2+\nu}(\Omega)} \Delta t\right)^{\frac{1}{2+\nu}} \le C_{60}\,(h^{\sigma} + (\Delta t)^{1/(1+\nu)}) ,$$

where $\gamma = 1 + \frac{1}{1+\nu}$ and $\sigma = \sigma(\nu) = 2\gamma/(2+\nu + \frac{\nu}{1+\nu})$. Using the inverse hypothesis (4.1c) we have

(5.32)
$$\max_n \|(u-U_h)^n\|_{L^2(\Omega)} \le C_{59}\, h^{-1}\left(h^{\frac{2+\nu}{2}\cdot\sigma} + (\Delta t)^{\gamma/2}\right) .$$

We should choose $\Delta t = h^{\sigma(1+\nu)}$ to match up the space and time convergence rates.

We remark that the two-to-one ratio of the space convergence rate to the time rate is well-known for backward-difference approximation to nondegenerate parabolic equations and has been proved for the special case $\nu = 1$ of the porous medium equation. We conjecture that this is true for $\nu > 1$ and that the right side of (5.30) can be replaced by

(5.33)
$$C_{61}(h^{\gamma} + (\Delta t)^{\gamma/2})$$

and the bound (5.31) can be replaced by

(5.34)
$$C_{62}(h^{2/(1+\nu)} + (\Delta t)^{1/(1+\nu)}) .$$

Another Backward-Difference Scheme

We shall consider the analogue of (4.51)-(4.52). Let $\{\tilde{U}_h^n\}_{n=0}^{M} \subset \mathcal{M}_h$ be defined by

$$(5.35) \qquad \left(\frac{\tilde{U}_h^{n+1} - \tilde{U}_h^n}{\Delta t}, \chi\right) + (\nabla K_\beta(\tilde{U}_h^{n+1}), \nabla\chi) = 0 , \qquad \forall\chi \in \mathcal{M}_h$$

and $n = 0,1,\ldots,M$, with initial data

$$(5.36) \qquad \tilde{U}_h^0 = P_h u_0 .$$

The analysis in (4.53)-(4.66) and (5.5)-(5.22) can be combined to prove the following estimates for $\dim(\Omega) = 1$

$$(5.37) \qquad \max_n \|u - \tilde{U}_h^n\|_{H^{-1}(\Omega)} \le C_{63}(h^{\gamma/2} + (\Delta t)^{\gamma/2})$$

and

$$(5.38) \qquad \left(\sum_{n=0}^{M-1} \|u^n - \tilde{U}_h^n\|_{L^2(\Omega)}^2 \Delta t\right)^{1/2} \le C_{64}\left(h^{\frac{1}{(1+\nu)}} + (\Delta t)^{\frac{1}{(1+\nu)}}\right).$$

<u>A Crank-Nicolson Scheme for the Porous Medium Equation</u>

Let $\{V_h^n\}_{n=0}^M \subset \mathcal{M}_h$ satisfy

$$(5.39) \qquad ((\partial^+ H_\beta(V_h))^n, \chi) + (\nabla V_h^{n+1/2}, \nabla\chi) = 0 , \qquad \forall\chi \in \mathcal{M}_h;$$

$n = 0,1,\ldots,M-1$, and let

$$(5.40) \qquad P_h H_\beta(V_h^0) = P_h u_0 .$$

Here, $V_h^{n+1/2}$ denotes $\frac{1}{2}(V_h^n + V_h^{n+1})$.

We shall consider the special case $\nu = 1$, $\beta = 0$. An extension of Theorem 3.3 which would enable us to prove that

$$(5.41) \qquad K(u)_{tt} \in L^\gamma(0,T,L^\gamma)$$

for $\nu > 1$ would be required to extend our analysis to (5.39)-(5.40) with $\nu > 1$.

We rewrite (5.39) as

$$((\partial^+ U_h)^n, \chi) + (\nabla K_\beta (U_h)^{n+1/2}, \nabla\chi) = 0 , \quad \forall\chi \in \mathcal{M}_h \tag{5.42}$$

and $n = 0,1,\ldots,M-1$, where

$$P_h U_h^0 = P_h u_0 \tag{5.43}$$

defines the initial data.

For any four real numbers A, B, a, and b, we have the inequality

$$\left(\frac{K_\beta(A) + K_\beta(B)}{2} - \frac{K_\beta(a) - K_\beta(b)}{2}\right)\left(\frac{A+B}{2} - \frac{a+b}{2}\right) \geq \frac{1}{1+\nu}\left|\frac{A+B}{2} - \frac{a+b}{2}\right|^{2+\nu}. \tag{5.44}$$

To see this, first assume that $A+B \geq a+b$. Notice that

$$K_\beta(A) + K_\beta(B) - K_\beta(a) - K_\beta(b) = \left(\int_a^A + \int_b^B\right) k_\beta(\tau)\,d\tau \geq \int_{-\frac{1}{2}((A+B)-(a+b))}^{+\frac{1}{2}((A+B)-(a+b))} k_\beta(\tau)\,d\tau = 2\, K_\beta\left(\frac{1}{2}((A+B)-(a+b))\right), \tag{5.45}$$

because the left side represents the integral of $k_\beta(\tau)$ over an interval of length $(A+B)-(a+b)$, and this integral is bounded from below by the integral of $k_\beta(\tau)$ over an interval of the same length centered about the origin. Thus,

$$(5.46) \qquad \left(\frac{K_\beta(A) + K_\beta(B)}{2} - \frac{K_\beta(a) + K_\beta(b)}{2}\right)\left(\frac{A+B}{2} - \frac{a+b}{2}\right)$$

$$\geq K_\beta\left(\frac{A+B}{2} - \frac{a+b}{2}\right)\left(\frac{A+B}{2} - \frac{a+b}{2}\right) \geq \frac{1}{1+\nu}\left|\frac{A+B}{2} - \frac{a+b}{2}\right|^{2+\nu},$$

when $A+B \geq a+b$. The bound (5.44) follows by symmetry. Letting $A = u^{n+1}$, $B = u^n$, $a = U_h^{n+1}$, and $b = U_h^n$ and integrating (5.44) over Ω, we see that

$$(5.47) \qquad \left(K_\beta(u_\beta)^{n+1/2} - K_\beta(U_h)^{n+1/2}, (u_\beta - U_h)^{n+1/2}\right)$$

$$\geq \frac{1}{1+\nu}\int_\Omega \left|(u_\beta - U_h)^{n+1/2}\right|^{2+\nu} dx .$$

We are now in a position to analyze the Crank-Nicolson scheme rewritten in the form

$$(5.48) \qquad T_h((\partial^+ U_h)^n) + K_\beta(U_h)^{n+1/2} = \frac{1}{|\Omega|}\int_\Omega K_\beta(U_h)^{n+1/2}\, dx .$$

It follows from (2.17) that

$$(5.49) \qquad T(\partial^+ u_\beta)^n + K_\beta(u_\beta)^{n+1/2} = T\{(\partial^+ u_\beta)^n - (\partial u_\beta/\partial t)^{n+1/2}\}$$

$$+ \frac{1}{|\Omega|}\int_\Omega K_\beta(u_\beta)^{n+1/2} dx .$$

Subtracting (5.48) from (5.49) yields

$$(5.50) \qquad T_h\{(\partial^+ u_\beta)^n - (\partial^+ U_h)^n\} + (K_\beta(u_\beta)^{n+1/2} - K_\beta(U_h)^{n+1/2})$$

$$= T\{(\partial^+ u_\beta)^n - (\partial u_\beta/\partial t)^{n+1/2}\} + (T_h - T)(\partial^+ u_\beta)^n$$

$$+ \frac{1}{|\Omega|}\int_\Omega \left(K_\beta(u_\beta)^{n+1/2} - K_\beta(U_h)^{n+1/2}\right) dx .$$

Integrate (5.50) against $(u_\beta - U_h)^{n+1/2}$ and apply (5.47) to see that, with $e_h^n = (u_\beta - U_h)^n$,

$$(5.51)\qquad \frac{1}{2\Delta t}\left\{\|P_h e_h^{n+1}\|^2_{H^{-1}(\Omega)} - \|P_h e_h^n\|^2_{H^{-1}(\Omega)}\right\} + \frac{1}{1+\nu}\|e_h^{n+1/2}\|^{2+\nu}_{L^{2+\nu}(\Omega)}$$
$$= \left(T\{(\partial^+ u_\beta)^n - (\partial u_\beta/\partial t)^{n+1/2}\},\ e_h^{n+1/2}\right) + \left((T_h - T)(\partial^+ u_\beta)^n,\ e_h^{n+1/2}\right).$$

Applying (4.17) yields

$$(5.52)\qquad \frac{1}{2\Delta t}\left\{\|P_h e_h^{n+1}\|^2_{H^{-1}(\Omega)} - \|P_h e_h^n\|^2_{H^{-1}(\Omega)}\right\} + \frac{1}{2(1+\nu)}\|e_h^{n+1/2}\|^{2+\nu}_{L^{2+\nu}(\Omega)}$$
$$\le C_{64}\|T\{(\partial^+ u_\beta)^n - (\partial u_\beta/\partial t)^{n+1/2}\}\|^{\gamma}_{L^{\gamma}(\Omega)} + C_{64}\|(T - T_h)(\partial^+ u_\beta)^n\|^{\gamma}_{L^{\gamma}(\Omega)}.$$

The analysis of the second term on the right side of (5.52) has already been done in the proofs of Theorems 5.1 and 5.2.

To analyze the first term on the right side of (5.52), we notice that

$$(5.53)\qquad (\partial^+ u_\beta)^n - (\partial u_\beta/\partial t)^{n+1/2} = -\frac{1}{2\Delta t}\int_{t_n}^{t_{n+1}} (t - t_n)(t_{n+1} - t)\, u_{\beta ttt}\, dt.$$

Thus,

$$(5.54)\qquad \| T\{(\partial^+ u_\beta)^n - (\partial u_\beta/\partial t)^{n+1/2}\}\|_{L^\gamma(\Omega)}$$

$$\leq \frac{1}{2\Delta t}\int_{t_n}^{t_{n+1}} (t-t_n)(t_{n+1}-t)\|-Tu_{\beta ttt}\|_{L^\gamma(\Omega)}\,dt$$

$$\leq C_{65}\,\frac{1}{\Delta t}\,(\Delta t)^{2+\frac{1}{2+\nu}}\|-Tu_{\beta ttt}\|_{L^\gamma(t_n,t_{n+1},L^\gamma)}$$

$$= C_{65}\,(\Delta t)^{1+\frac{1}{2+\nu}}\|-Tu_{\beta ttt}\|_{L^\gamma(t_n,t_{n+1},L^\gamma)}\;.$$

Thus,

$$(5.55)\qquad \sum_{n=0}^{M-1}\| T\{(\partial^+ u_\beta)^n - (\partial u_\beta/\partial t)^{n+1/2}\|^\gamma_{L^\gamma(\Omega)}$$

$$\leq C_{65}\,(\Delta t)^{\gamma+\frac{1}{1+\nu}}\|-Tu_{\beta ttt}\|^\gamma_{L^\gamma(0,T,L^\gamma)}\;.$$

We must now derive $L^\gamma(L^\gamma)$ bounds for $-Tu_{\beta ttt} = \widetilde{K_\beta(u_\beta)_{tt}}$, where $\widetilde{K_\beta(u_\beta)_{tt}} = K_\beta(u_\beta)_{tt} - \frac{1}{|\Omega|}\int_\Omega K_\beta(u_\beta)_{tt}$. Unfortunately, we have been able to do this only in the special case $\nu = 1$. Notice that, when $\nu = 1$ and $\beta = 0$,

$$(5.56)\qquad K(u)_{tt} = (u_t)^2 + uu_{tt}\;.$$

The proof of Theorem 3.3 tells us that

$$(5.57)\qquad \|(u_t)^2\|_{L^\gamma(L^\gamma)} = \|(u_t)^2\|_{L^{3/2}(L^{3/2})}$$

$$= \|u_t\|_{L^3(L^3)} \leq \|u_t\|_{L^\infty(L^3)} \leq C_{67}\;,$$

and

$$(5.58)\qquad \|uu_{tt}\|_{L^\gamma(L^\gamma)} = \|uu_{tt}\|_{L^{3/2}(L^{3/2})} \leq C_{20}\,\|uu_{tt}\|_{L^2(L^2)}$$

$$\leq C_{68}\|\sqrt{u}\|_{L^\infty(L^\infty)}\|\sqrt{u}\,u_{tt}\|_{L^2(L^2)} \leq C_{69}\;,$$

so that

$$\|\widetilde{K(u)_{tt}}\|_{L^\gamma(L^\gamma)} \le \|K(u)_{tt}\|_{L^\gamma(L^\gamma)} \tag{5.59}$$

$$\le \|(u_t)^2\|_{L^\gamma(L^\gamma)} + \|uu_{tt}\|_{L^\gamma(L^\gamma)}$$

$$\le C_{67} + C_{69} = C_{70} \; .$$

Multiplying (5.45) by Δt and summing on n yield the bound

$$\frac{1}{2} \operatorname*{Max}_n \|P_h(u-U_h)\|^2_{H^{-1}(\Omega)} + \frac{1}{2(1+\nu)} \sum_{n=0}^{M-1} \|(u-U_h)^{n+1/2}\|^{2+\nu}_{L^{2+\nu}(\Omega)} dt \tag{5.60}$$

$$\le C_{71}(h^{2\gamma} + (\Delta t)^{2\gamma}),$$

in the case $\nu = 1$, $\beta = 0$. We have proved the following result, modulo the required bounds on $(I-P_h)(e_h^n)$ in $H^{-1}(\Omega)$ which we have derived in Chapter IV.

<u>THEOREM 5.3</u>. Assume $\nu = 1$ and let $\{V_h^n\}_{n=0}^{M}$ be the solution of (5.39)-(5.40). Let $U_h^n = H(V_h^n)$. Then, for $\dim(\Omega) = 1$,

$$\operatorname*{Max}_n \|(u-U_h)^n\|_{H^{-1}(\Omega)} \le C_{72}(h^{3/2} + (\Delta t)^{3/2}) \tag{5.61}$$

and

$$\left(\sum_{n=0}^{M-1} \|(u-U_h)^{n+1/2}\|^2_{L^2(\Omega)} \Delta t\right)^{1/2} \tag{5.62}$$

$$\le C_{56}\left(\sum_{n=0}^{M-1} \|(u-U_h)^{n+1/2}\|^3_{L^3(\Omega)} \Delta t\right)^{1/3} \le C_{73}(h + (\Delta t)) \; ,$$

and, with $\Delta t = h$,

$$\operatorname*{Max}_n \|(u-U_h)^n\|_{L^2(\Omega)} \le C_{74}(h^{1/2} + (\Delta t)^{1/2}) \; . \tag{5.63}$$

We conjecture that our analysis of the Crank-Nicolson procedure (5.39)-(5.40) can be extended to treat the case $\nu > 1$. If we assume the regularity results (4.46) and (5.41) then we can establish the bounds

$$\max_n \|(u-U_h)^n\|_{H^{-1}(\Omega)} \le C_{75}(h^\gamma + (\Delta t)^\gamma) \,, \tag{5.64}$$

and

$$\left(\sum_{n=0}^{M} \|(u-U_h)^n\|^2_{L^2(\Omega)} \Delta t\right)^{1/2} \le C_{75} \left(h^{\frac{2}{1+\nu}} + (\Delta t)^{\frac{2}{1+\nu}}\right). \tag{5.65}$$

If we choose $\Delta t = h$, we would have

$$\max_n \|(u-U_h)^n\|_{L^2(\Omega)} \le C_{74} \left(h^{\frac{1}{1+\nu}} + (\Delta t)^{\frac{1}{1+\nu}}\right). \tag{5.66}$$

Any $\beta \in [0, h^{\frac{2}{1+\nu}}]$ would produce the same order of approximation of U_h^n to u.

Discrete-time estimates in two and three space dimensions can be obtained by our methods. The rates for the time step Δt are unchanged and the rates for the space mesh size h must be modified as in the remarks at the end of the previous chapter.

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An ANL Report on a much more general class of porous medium equations will appear in the near future. Those on the distribution list will receive a copy.

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