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Abstract

The Dirac-Lorentz equation describes the dynamics of a classical point charge in an electromagnetic field, accounting for radiative effects in a manifestly covariant and gauge invariant manner. The validity of this equation is assessed by direct comparison between the Dirac-Lorentz dynamics of an electron subjected to a plane wave in vacuum and the well-known recoil associated with Compton scattering. In the small recoil limit, the classical Dirac-Lorentz is shown to yield the correct momentum transfer. For larger values of the recoil, the quantum scale appears explicitly, and the classical Dirac-Lorentz equation does not properly model this situation, as shown by deriving an exact analytical solution for a monochromatic plane wave of wavenumber k_0 to any order in $k_0 r_0$, where r_0 is the classical electron radius.

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1 Introduction

In 1938, Dirac published an important paper [1] dealing with radiation reaction within the context of classical relativistic electrodynamics, and containing the derivation of a manifestly covariant and gauge invariant equation for the dynamics of a point charge in an electromagnetic field accounting for radiative effects: the Dirac-Lorentz equation [1-5]. The main purpose of that work was to determine which of the divergences arising in QED, if any, had classical counterparts, thus providing physical insight regarding their origin. Interestingly, however, Dirac did not necessarily regard the Dirac-Lorentz equation as representing some classical limit of QED; rather, he considered it as a mathematical extension of the Lorentz equation, possessing both covariance and gauge invariance.

Since then, a rather large number of papers have been published, using the Dirac-Lorentz equation to account for radiation reaction in semi-classical systems. Nevertheless, the question of the domain of applicability of the Dirac-Lorentz equation remains open.

In this paper, we propose a direct comparison between the Dirac-Lorentz dynamics of an electron subjected to a plane wave in vacuum and the well-known recoil associated with Compton scattering. In this manner, the validity of the Dirac-Lorentz equation can be assessed within a simple, well-defined context; furthermore, the problem can be studied analytically and compared in both cases. To our knowledge, the exact plane wave solution for the Dirac-Lorentz equation presented here had not been previously derived.

This paper is organized as follows: to provide the proper background, the Lorentz dynamics of an electron subjected to a plane wave of arbitrary strength are first briefly reviewed, as well as the salient steps of the derivation of the Dirac-Lorentz equation; the plane wave dynamics of an electron are then studied including classical radiation reaction effects, and compared in detail to the well-known Compton scattering kinematics.

2 Lorentz plane wave dynamics

For conciseness, we use electron units, where length, time, mass, and charge are measured in units of the classical electron radius, $r_0 = e^2/4\pi\varepsilon_0 m_0 c^2$, r_0/c, the electron rest mass, m_0, and its absolute charge, e, respectively. In these units, the vacuum permittivity is $\varepsilon_0 = 1/4\pi$, and its permeability is $\mu_0 = 4\pi$; the reduced value of Planck's constant is given by the inverse fine structure constant: $\hbar = 1/\alpha = ƛ_C/r_0$, which is also the ratio between the quantum and classical scales.

The electron normalized 4-velocity and 4-acceleration are defined as $u_\mu = d_\tau x_\mu$, and $a_\mu = d_\tau u_\mu$, where τ is the dimensionless proper time along the dimensionless electron world line, $x_\mu(\tau)$, and where the notation $d_\tau \equiv d/d\tau$ is used. The length of the velocity 4-vector, $u_\mu u^\mu = -1 = u^2 - \gamma^2$ reflects the relation between energy and momentum, while the 4-velocity and 4-acceleration are orthogonal: $d_\tau\left(u_\mu u^\mu\right) = 0 = 2u_\mu a^\mu$.

Within this context, the Lorentz force equation reads: $a_\mu = -F_{\mu\nu}u^\nu = -\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)u^\nu$, where the antisymmetric electromagnetic field tensor $F_{\mu\nu}$ is expressed in terms of the normalized 4-potential, A_μ, and where the standard notation $\partial_\mu \equiv \partial / \partial x^\mu$ is used. For a plane wave with 4-wavenumber k_ν, the 4-potential is only a function of the phase, $\phi = -k_\nu x^\nu$, with $A_\mu\left(x^\nu\right) = A_\mu\left(\phi\right)$, and the partial derivatives reduce to: $\partial_\mu A_\nu\left(\phi\right) = \partial_\mu \phi d_\phi A_\nu = -k_\mu d_\phi A_\nu$. Applying this result to the Lorentz force equation, we have $a_\mu = k_\mu\left(u^\nu d_\phi A_\nu\right) - \left(k_\nu u^\nu\right) d_\phi A_\mu$; now taking the derivative of the phase with respect to the proper time, $d_\tau \phi = \kappa = -d_\tau\left(k_\nu x^\nu\right) = -k_\nu d_\tau x^\nu = -k_\nu u^\nu$, which defines the light-cone variable, κ, we find that:

$$a_\mu = \frac{du_\mu}{d\tau} = k_\mu\left(u^\nu \frac{dA_\nu}{d\phi}\right) + \frac{d\phi}{d\tau}\frac{dA_\mu}{d\phi}, \quad \frac{d}{d\tau}\left(u_\mu - A_\mu\right) = k_\mu\left(u^\nu \frac{dA_\nu}{d\phi}\right). \tag{1}$$

The dynamics of the light-cone variable are described by:

$$\frac{d\kappa}{d\tau} = -k_\mu \frac{du^\mu}{d\tau} = -\left(k_\mu k^\mu\right)\left(\frac{dA^\nu}{d\phi} u_\nu\right) + \left(k^\nu u_\nu\right)\left(k_\mu \frac{dA^\mu}{d\phi}\right). \tag{2}$$

The first term in Eq. (2) corresponds to the dispersion relation in vacuum, or photon mass-shell condition, $k_\mu k^\mu = 0$, while the second term corresponds to the Lorentz gauge condition: $\partial_\mu A^\mu = 0 = \partial_\mu \phi d_\phi A^\mu = -k_\mu d_\phi A^\mu$. The light-cone variable is a constant of the electron motion: $d_\tau \kappa = 0$.

Equation (1) suggest seeking a solution of the form $u_\mu = A_\mu + k_\mu f(\phi)$, where f is a function of the electron phase to be determined. As the radiation pressure force is proportional to $A_\mu A^\mu$ and directed along the incident wave propagation, we consider the linear combination $f(\phi) = \xi A_\mu A^\mu(\phi) + \psi$, where ξ and ψ are constants that are determined by satisfying both Eq. (1) and the condition $u_\mu u^\mu = -1$. Deriving $u_\mu = A_\mu + k_\mu \left[\xi A_\mu A^\mu(\phi) + \psi\right]$ with respect to τ, and inserting the result in Eq. (1), we find that:

$$2\xi A_\nu \frac{dA^\nu}{d\tau} = 2\xi\kappa A_\nu \frac{dA^\nu}{d\phi} = u^\nu \frac{dA_\nu}{d\phi} = \left[A^\nu + k^\nu f(\phi)\right]\frac{dA_\nu}{d\phi} = A_\nu \frac{dA_\nu}{d\phi}, \tag{3}$$

where we have used the gauge condition to eliminate $k^\nu d_\phi A_\nu$. Equation (3) then yields $\xi = 1/2\kappa$. The normalization of the 4-velocity yields ψ:

$$\begin{aligned} u_\mu u^\mu = -1 &= \left(A_\mu + k_\mu f\right)\left(A^\mu + k^\mu f\right) = A_\mu A^\mu + 2fk_\mu A^\mu + f^2 k_\mu k^\mu, \\ -1 &= A_\mu A^\mu + 2fk_\mu A^\mu = A_\mu A^\mu + 2k_\mu A^\mu\left(\xi A_\mu A^\mu + \psi\right) = A_\mu A^\mu\left(1 + \frac{k_\mu A^\mu}{\kappa}\right) + 2k_\mu A^\mu \psi. \end{aligned} \tag{4}$$

The result is:

$$u_\mu\left(x^\nu\right) = A_\mu(\phi) - k_\mu\left[\frac{1 + A_\nu A^\nu(\phi)}{2k_\nu A^\nu(\phi)}\right]. \tag{5}$$

Finally, initial conditions can be matched by regauging the 4-potential by a constant 4-vector: $A^\mu \to A^\mu + u_0^\mu$. Furthermore, because of the photon mass-shell condition, the light-cone variable reduces to $\kappa = -k_\mu u^\mu = -k_\mu A^\mu - \left(k_\mu k^\mu\right) f\left(\phi\right) = -k_\mu A^\mu$, which has the constant value $\kappa = \kappa_0 = -k_\mu u_0^\mu$; as a result, Eq. (5) can be expressed as:

$$u^\mu = u_0^\mu + A^\mu + k^\mu \left(\frac{A_\nu A^\nu + 2A_\nu u_0^\nu}{2k_\nu u_0^\nu} \right), \tag{6}$$

by noting that $\left(A + u_0\right)_\mu \left(A + u_0\right)^\mu = A_\mu A^\mu + 2A_\mu u_0^\mu - 1$. We now have $\lim_{\phi \to \pm\infty} u^\mu\left(\phi\right) = u_0^\mu$, since $\lim_{\phi \to \pm\infty} A^\mu\left(\phi\right) = 0$. This result shows that for a classical electron interacting with a plane wave in vacuum, there is no net energy exchange in the absence of radiative corrections, and is generally known as the Lawson-Woodward theorem. The condition that the 4-potential vanishes at infinity, to within a constant, is quite general; in particular, there are no temporal profiles that will yield electron acceleration for plane waves in vacuum, including chirped pulses. This also confirms that the Lorentz force does not yield radiative recoil: $\Delta\gamma = \gamma_{\phi \to +\infty} - \gamma_{\phi \to -\infty} = \gamma^+ - \gamma_0 = 0$.

3 Dirac-Lorentz equation

The Dirac-Lorentz equation includes such radiative effects; for completeness, the main steps of the derivation are outlined here. The electron 4-current is:

$$j_\mu^s(x_\lambda) = -\int_{-\infty}^{+\infty} u_\mu(x'_\lambda)\delta_4(x_\lambda - x'_\lambda)d\tau', \tag{7}$$

and the corresponding self-electromagnetic field, $F_{\mu\nu}^s = \partial_\mu A_\nu^s - \partial_\nu A_\mu^s$, satisfies the wave equation, $\Box A_\mu^s(x_\lambda) = -4\pi j_\mu^s(x_\lambda)$. Green functions can be used to solve this problem, with $A_\mu^s(x_\lambda) = 4\pi\int_{-\infty}^{+\infty} u_\mu(x'_\lambda)G(x_\lambda - x'_\lambda)d\tau'$.

The self-force is simply given by the Lorentz force in the self-fields:

$$F_\mu^s = -(\partial_\mu A_\nu^s - \partial_\nu A_\mu^s)u^\nu = -\int_{-\infty}^{+\infty} u^\nu(x_\lambda)\left[u_\nu(x'_\lambda)\partial_\mu - u_\nu(x'_\lambda)\partial_\mu\right]G(x_\lambda - x'_\lambda)d\tau'. \tag{8}$$

The advanced and retarded Green functions depend on the spacetime interval $s^2 = (x - x')_\mu (x - x')^\mu$: $G^\pm = -\delta(s^2)\left\{1 \mp \left[(x_0 - x'_0)/\left|x_0 - x'_0\right|\right]\right\}$. As a result, the partial derivatives operate identically to $\partial_\mu \equiv 2(x_\mu - x'_\mu)\partial_{s^2}$:

$$F_\mu^s = -2\int_{-\infty}^{+\infty} u^\nu(x_\lambda)\left[u_\nu(x'_\lambda)(x_\mu - x'_\mu) - u_\nu(x'_\lambda)(x_\mu - x'_\mu)\right]\frac{\partial G}{\partial s^2}d\tau'. \tag{9}$$

Introducing $\tau'' = \tau - \tau'$, and Taylor expanding around the electron, at the singular point $\tau'' = 0$, we have:

$$x_\mu - x'_\mu = \tau'' u_\mu - \frac{1}{2}\tau''^2 a_\mu + \frac{1}{6}\tau''^3 d_\tau a_\mu + \cdots,$$
$$u_\mu\left(x'_\lambda\right) = u_\mu\left(\tau - \tau''\right) = u_\mu - \tau'' a_\mu + \frac{1}{2}\tau''^2 + \cdots, \tag{10}$$

which yields $s^2 \simeq \tau''^2$, and $\partial G / \partial s^2 \simeq -\left(1/2\tau''\right)\left(\partial G / \partial \tau''\right)$. The self-electromagnetic force is:

$$F_\mu^s \simeq \int_{-\infty}^{+\infty} \left\{ -\frac{\tau''}{2} a_\mu + \frac{\tau''^2}{3}\left[\frac{da_\mu}{d\tau} - u_\mu\left(a_\nu a^\nu\right)\right]\right\} \frac{\partial G}{\partial \tau''} d\tau''. \tag{11}$$

This equation can be integrated by parts; following Dirac's procedure and using the time-symmetrical Green function, $G = \left(G^+ - G^-\right)/2$, to renormalize the divergent electromagnetic mass of the point electron, $\int_{-\infty}^{+\infty} \delta\left(\tau''\right) d\tau'' / 2\left|\tau''\right|$, and adding the Lorentz term yields the Dirac-Lorentz equation:

$$a_\mu = -F_{\mu\nu} u^\nu + \tau_0 \left[\frac{da_\mu}{d\tau} - u_\mu\left(a_\nu a^\nu\right)\right]. \tag{12}$$

Here, $\tau_0 = 2/3$ is the time-scale for classical radiative corrections, expressed in units of r_0 / c. A number of conceptual difficulties arise within the context of Eq. (12), including so-called runaway solutions and acausal effects; for more details, see Refs. [1-5].

4 Dirac-Lorentz plane wave dynamics

We now turn our attention to the Dirac-Lorentz dynamics of a point electron in a plane wave. Using the 4-potential, the Dirac-Lorentz equation reads

$$\frac{du_\mu}{d\tau} = -\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)u^\nu + \tau_0\left[\frac{da_\mu}{d\tau} - u_\mu\left(a_\nu a^\nu\right)\right]. \tag{13}$$

As seen in Section 2, in the case of a plane wave, the electron phase is $\phi = -k_\mu x^\mu$, and the partial derivatives of the 4-potential take a simple form:

$$\partial_\mu A_\nu = \frac{\partial A_\nu}{\partial x^\mu} = \frac{\partial \phi}{\partial x^\mu}\frac{dA_\nu}{d\phi} = -k_\mu \frac{dA_\nu}{d\phi} = -k_\mu E_\nu. \tag{14}$$

The Dirac-Lorentz equation now reads

$$\frac{du_\mu}{d\tau} = k_\mu\left(E_\nu u^\nu\right) - E_\mu\left(k_\nu u^\nu\right) + \tau_0\left[\frac{da_\mu}{d\tau} - u_\mu\left(a_\nu a^\nu\right)\right]. \tag{15}$$

Choosing the reference frame so that the incident plane wave propagates along the *z*-axis, with $k_\mu = \left(\varpi_0, 0, 0, \varpi_0\right)$, we have $k_\mu u^\mu = \varpi_0\left(u_z - \gamma\right) = -\kappa$; furthermore, the gauge condition leads to $E_z = E_0$:

$$\partial_\mu A^\mu = 0 = \frac{\partial \phi}{\partial x^\mu}\frac{dA^\mu}{d\phi} = -k_\mu E^\mu = \varpi_0\left(E_0 - E_z\right). \tag{16}$$

The scalar product of the field and 4-velocity is now given by

$$E_\mu u^\mu = \mathbf{E}_\perp \cdot \mathbf{u}_\perp + E_\parallel \left(u_z - \gamma\right), \tag{17}$$

where we have defined $E_\parallel = E_z = E_0$, and Eq. (15) now reads:

$$\frac{du_\mu}{d\tau} = k_\mu \left[\mathbf{E}_\perp \cdot \mathbf{u}_\perp + E_\parallel \left(u_z - \gamma\right)\right] - E_\mu \varpi_0 \left(u_z - \gamma\right) + \tau_0 \left[\frac{da_\mu}{d\tau} - u_\mu \left(a_\nu a^\nu\right)\right]. \tag{18}$$

Since $\mathbf{k}_\perp = \mathbf{0}$, the transverse dynamics are governed by:

$$\frac{d\mathbf{u}_\perp}{d\tau} = -\mathbf{E}_\perp \varpi \left(u_z - \gamma\right) + \tau_0 \left[\frac{d\mathbf{a}_\perp}{d\tau} - \mathbf{u}_\perp \left(a_\nu a^\nu\right)\right] = \kappa \mathbf{E}_\perp + \tau_0 \left[\frac{d\mathbf{a}_\perp}{d\tau} - \mathbf{u}_\perp \left(a_\nu a^\nu\right)\right], \tag{19}$$

while the axial and temporal components of the Dirac-Lorentz equation yield

$$\begin{aligned}
\frac{du_z}{d\tau} &= \varpi \left[\mathbf{E}_\perp \cdot \mathbf{u}_\perp + E_\parallel \left(u_z - \gamma\right)\right] - E_z \varpi \left(u_z - \gamma\right) + \tau_0 \left[\frac{da_z}{d\tau} - u_z \left(a_\nu a^\nu\right)\right], \\
\frac{d\gamma}{d\tau} &= \varpi \left[\mathbf{E}_\perp \cdot \mathbf{u}_\perp + E_\parallel \left(u_z - \gamma\right)\right] - E_0 \varpi \left(u_z - \gamma\right) + \tau_0 \left[\frac{da_0}{d\tau} - \gamma \left(a_\nu a^\nu\right)\right],
\end{aligned} \tag{20a,b}$$

respectively, and reduce to

$$\frac{du_z}{d\tau} = \varpi_0 \mathbf{E}_\perp \cdot \mathbf{u}_\perp + \tau_0 \left[\frac{d^2 u_z}{d\tau^2} - u_z \left(a_\nu a^\nu \right) \right],$$

$$\frac{d\gamma}{d\tau} = \varpi_0 \mathbf{E}_\perp \cdot \mathbf{u}_\perp + \tau_0 \left[\frac{d^2 \gamma}{d\tau^2} - \gamma \left(a_\nu a^\nu \right) \right]. \tag{21a,b}$$

Multiplying Eqs. (21a) and (21b) by ϖ_0, and subtracting the axial from the temporal component, we obtain an equation governing the evolution of the electron light-cone variable:

$$\frac{d\kappa}{d\tau} = \tau_0 \left[\frac{d^2 \kappa}{d\tau^2} - \kappa \left(a_\nu a^\nu \right) \right]. \tag{22}$$

Now using the electron phase as the independent variable, and the fact that $d_\tau \phi = \kappa$, we have:

$$\frac{d\kappa}{d\phi} = \tau_0 \left[\frac{d^2}{d\phi^2} \left(\frac{\kappa^2}{2} \right) - a_\nu a^\nu \right]. \tag{23}$$

4.a First-order recoil

In the limit where radiative corrections are small, one can replace the quantities inside the brackets by their Lorentz dynamics, zeroth-order values:

$$a_\nu a^\nu = \mathbf{a}_\perp^2 + a_z^2 - a_0^2 = \mathbf{a}_\perp^2 = \left(d_\tau \mathbf{u}_\perp \right)^2 = \left(d_\tau \mathbf{A}_\perp \right)^2 = \kappa^2 \left(d_\phi \mathbf{A}_\perp \right)^2, \tag{24}$$

and $d_\phi \kappa = 0$. Equation (23) then reduces to:

$$\frac{d\kappa}{d\tau} \simeq -\kappa^2 \tau_0 \left(\frac{d\mathbf{A}_\perp}{d\phi}\right)^2, \quad \frac{d}{d\phi}\left[\frac{1}{\kappa(\phi)}\right] \simeq \tau_0 \left(\frac{d\mathbf{A}_\perp}{d\phi}\right)^2 = \tau_0 A_0^2 g^2(\phi), \tag{25}$$

where the last equality holds for circularly polarized light, and where $g(\phi)$ is the temporal envelope of the electric field. Equation (25) can be solved to find:

$$\frac{1}{\kappa(\phi)} = \frac{1}{\kappa_0} + \tau_0 A_0^2 \int_{-\infty}^{\phi} g^2(\psi)\, d\psi. \tag{26}$$

Here $\kappa_0 = \lim_{\phi \to -\infty} \kappa(\phi)$ is the initial value of the electron light-cone variable. To calculate the total recoil momentum, we first consider the limit of Eq. (26) for $\phi \to +\infty$:

$$\kappa^+ = \kappa_0 \left[1 + \kappa_0 \tau_0 A_0^2 \int_{-\infty}^{+\infty} g^2(\phi)\, d\phi\right]^{-1} \simeq \kappa_0 - \kappa_0^2 \tau_0 A_0^2 \int_{-\infty}^{+\infty} g^2(\phi)\, d\phi. \tag{27}$$

After the interaction, for small momentum transfer, where $u_z^+ \ll 1$, energy conservation implies that $\gamma^+ \simeq \sqrt{1 + u_z^{+^2}}$. Finally, we use the definition of the light-cone variable: $\kappa = -k_\mu u^\mu = \varpi_0 (\gamma - u_z)$, where $\varpi_0 = \omega_0 r_0 / c$ is the wave frequency measured in electron units. Combining these results, and considering a reference frame where the electron is initially at rest, with $\gamma_0 = 1$, $u_{z0} = 0$, and $\kappa_0 = \varpi_0$, we find that the classical Dirac-Lorentz recoil, Δu_z, is:

$$\begin{aligned} \frac{\kappa^+}{\varpi_0} &\simeq 1 - u_z^+ \simeq \frac{1}{\varpi_0}\left[\kappa_0 - \kappa_0^2 \tau_0 A_0^2 \int_{-\infty}^{+\infty} g^2(\phi)\, d\phi\right], \\ \Delta u_z &= u_z^+ - u_{z0} \simeq \frac{2}{3}\omega_0 \frac{r_0}{c} A_0^2 \int_{-\infty}^{+\infty} g^2(\phi)\, d\phi. \end{aligned} \tag{28}$$

4.b Higher-order perturbation theory

We now consider higher-order terms; proceeding systematically, we first express the square of the 4-acceleration in terms of derivatives with respect to the phase:

$$\begin{aligned} a_\mu a^\mu &= \left(\frac{d\mathbf{u}_\perp}{d\tau}\right)^2 + \left(\frac{du_z}{d\tau}\right)^2 - \left(\frac{d\gamma}{d\tau}\right)^2 \\ &= \kappa^2\left[\left(\frac{d\mathbf{u}_\perp}{d\phi}\right)^2 + \left(\frac{du_z}{d\phi}\right)^2 - \left(\frac{d\gamma}{d\phi}\right)^2\right] \\ &= \kappa^2\left[\left(\frac{d\mathbf{u}_\perp}{d\phi}\right)^2 + \frac{d}{d\phi}(u_z - \gamma)\frac{d}{d\phi}(u_z + \gamma)\right] \\ &= \kappa^2\left[\left(\frac{d\mathbf{u}_\perp}{d\phi}\right)^2 - \frac{\kappa'}{\varpi_0}\frac{d}{d\phi}(u_z + \gamma)\right]. \end{aligned} \tag{29}$$

Here the prime denotes derivation with respect to ϕ.

The transverse dynamics equation is:

$$\kappa \frac{d\mathbf{u}_\perp}{d\phi} = \kappa \mathbf{E}_\perp + \tau_0 \left\{ \kappa \frac{d}{d\phi}\left(\kappa \frac{d\mathbf{u}_\perp}{d\phi} \right) - \mathbf{u}_\perp \kappa^2 \left[\left(\frac{d\mathbf{u}_\perp}{d\phi} \right)^2 - \frac{\kappa'}{\varpi_0} \frac{d}{d\phi}(u_z + \gamma) \right] \right\}, \tag{30}$$

while the evolution of the light-cone variable is governed by

$$\frac{d\kappa}{d\phi} = \tau_0 \left\{ \frac{d^2}{d\phi^2}\left(\frac{\kappa^2}{2} \right) - \kappa^2 \left[\left(\frac{d\mathbf{u}_\perp}{d\phi} \right)^2 - \frac{\kappa'}{\varpi_0} \frac{d}{d\phi}(u_z + \gamma) \right] \right\}. \tag{31}$$

In order to make the perturbation parameter $\varepsilon = \varpi_0 \tau_0$ appear explicitly, we introduce $q = \gamma - u_z = \kappa / \varpi_0$; the light-cone dynamics are now described by:

$$\begin{aligned} \varpi_0 \frac{dq}{d\phi} &= \varpi_0^2 \tau_0 \left\{ \frac{d^2}{d\phi^2}\left(\frac{q^2}{2} \right) - q^2 \left[\left(\frac{d\mathbf{u}_\perp}{d\phi} \right)^2 - q' \frac{d}{d\phi}(u_z + \gamma) \right] \right\}, \\ q' &= \varepsilon \left\{ q'^2 + qq'' - q^2 \left[\left(\frac{d\mathbf{u}_\perp}{d\phi} \right)^2 - q' \frac{d}{d\phi}(u_z + \gamma) \right] \right\} \\ &= \varepsilon \left[q'^2 + qq'' - q^2 \mathbf{u}_\perp'^2 + q^2 q' \left(u_z' + \gamma' \right) \right]. \end{aligned} \tag{32}$$

The transverse dynamics equation reads:

$$\mathbf{u}_\perp' = \mathbf{E}_\perp + \varepsilon \left[q' \mathbf{u}_\perp' + q \mathbf{u}_\perp'' - \mathbf{u}_\perp q \mathbf{u}_\perp'^2 + \mathbf{u}_\perp q q' \left(u_z' + \gamma' \right) \right]. \tag{33}$$

Using the normalization of the 4-velocity, $u_\mu u^\mu = -1 = \mathbf{u}_\perp^2 + u_z^2 - \gamma^2$, and the definition of $q = \gamma - u_z$, the derivative $u_z^{'} + \gamma^{'}$ can be expressed in terms of $\mathbf{u}_\perp$, q, and their derivatives:

$$
\begin{aligned}
&u_z = \frac{1+\mathbf{u}_\perp^2 - q^2}{2q}, \quad \gamma = \frac{1+\mathbf{u}_\perp^2 + q^2}{2q}, \\
&u_z^{'} + \gamma^{'} = \frac{d}{d\phi}\left(\frac{1+\mathbf{u}_\perp^2}{q}\right) = \frac{1}{q^2}\left[2q\mathbf{u}_\perp \cdot \mathbf{u}_\perp^{'} - q^{'}\left(1+\mathbf{u}_\perp^2\right)\right].
\end{aligned} \tag{34}
$$

Using this result in Eqs. (32) and (33), we have:

$$
\begin{aligned}
q^{'} &= \varepsilon\left[q^{'2} + qq^{''} - q^2\mathbf{u}_\perp^{'2} + 2qq^{'}\mathbf{u}_\perp \cdot \mathbf{u}_\perp^{'} - q^{'2}\left(1+\mathbf{u}_\perp^2\right)\right] \\
&= \varepsilon\left[qq^{''} - q^2\mathbf{u}_\perp^{'2} + 2qq^{'}\mathbf{u}_\perp \cdot \mathbf{u}_\perp^{'} - q^{'2}\mathbf{u}_\perp^2\right],
\end{aligned} \tag{35}
$$

and

$$
\mathbf{u}_\perp^{'} = \mathbf{E}_\perp + \varepsilon\left[q^{'}\mathbf{u}_\perp^{'} + q\mathbf{u}_\perp^{''} - \mathbf{u}_\perp q\mathbf{u}_\perp^{'2} + 2\mathbf{u}_\perp q^{'} q\mathbf{u}_\perp \cdot \mathbf{u}_\perp^{'} - \mathbf{u}_\perp \frac{q^{'2}}{q}\left(1+\mathbf{u}_\perp^2\right)\right]. \tag{36}
$$

At this point, we note that a number of terms can be eliminated by taking the limit where the normalized vector potential $A_0^2 \ll 1$: $q^{'}$ and its derivatives are all at least quadratic in A_0, and $\mathbf{u}_\perp$ and its derivatives are all at least linear in A_0, therefore $q^{'}\mathbf{u}_\perp^{'} \propto A_0^{\geq 3}$, $q^{'}\mathbf{u}_\perp \cdot \mathbf{u}_\perp^{'} \propto A_0^{\geq 4}$, and $q^{'2}\mathbf{u}_\perp^2 \propto A_0^{\geq 6}$. This limit is appropriate, since we intend to compare the Dirac-Lorentz recoil to Compton scattering, where the vector potential of the incident photons is vanishingly small. Within this context, Eq. (35) and (36) reduce to:

$$q' = \varepsilon\left[qq'' - q^2\mathbf{u}_\perp'^{\,2} + \mathcal{O}\left(A_0^{\geq 4}\right)\right], \qquad (37)$$

and

$$\mathbf{u}_\perp' = \mathbf{E}_\perp + \varepsilon\left[q\mathbf{u}_\perp'' + \mathcal{O}\left(A_0^{\geq 3}\right)\right]. \qquad (38)$$

Equation (37) shows that $q = q_0 + \mathcal{O}\left(\varepsilon^{\geq 1}\right)\mathcal{O}\left(A_0^{\geq 1}\right)$, therefore, we can recast Eqs. (37) and (38) as

$$q' = \varepsilon q_0\left[q'' - q_0\mathbf{u}_\perp'^{\,2} + \mathcal{O}\left(A_0^{\geq 3}\right)\right], \qquad (39)$$

and

$$\begin{aligned} \mathbf{u}_\perp' &= \mathbf{E}_\perp + \varepsilon q_0\mathbf{u}_\perp'' + \mathcal{O}\left(\varepsilon^{\geq 2}\right) + \varepsilon\left[\mathcal{O}\left(A_0^{\geq 3}\right)\right], \\ \mathbf{u}_\perp' &\simeq \mathbf{E}_\perp + \varepsilon q_0\mathbf{u}_\perp'', \end{aligned} \qquad (40)$$

and solve Eq. (40) by recurrence: assuming that we have, to order n,

$$\mathbf{u}_\perp = \mathbf{A}_\perp + \ldots + \left(\varepsilon q_0\right)^n \frac{d^n\mathbf{A}_\perp}{d\phi^n}, \qquad (41)$$

and deriving twice with respect to ϕ, we find that

$$\mathbf{u}_\perp'' = \mathbf{E}_\perp' + \ldots + \left(\varepsilon q_0\right)^n \frac{d^{n+2}\mathbf{A}_\perp}{d\phi^{n+2}}. \qquad (42)$$

Now replacing $\mathbf{u}_\perp''$ by the above expression in Eq. (40), we have:

$$\begin{aligned}\mathbf{u}_\perp' &\simeq \mathbf{E}_\perp + \varepsilon q_0 \mathbf{u}_\perp'' = \mathbf{E}_\perp + \varepsilon q_0 \left[\mathbf{E}_\perp' + \ldots + (\varepsilon q_0)^n \frac{d^{n+2}\mathbf{A}_\perp}{d\phi^{n+2}} \right] \\ &= \mathbf{E}_\perp + \varepsilon q_0 \mathbf{E}_\perp' + \ldots + (\varepsilon q_0)^{n+1} \frac{d^{n+2}\mathbf{A}_\perp}{d\phi^{n+2}},\end{aligned} \tag{43}$$

which integrates to

$$\mathbf{u}_\perp = \mathbf{A}_\perp + \varepsilon q_0 \mathbf{E}_\perp + \ldots + (\varepsilon q_0)^{n+1} \frac{d^{n+1}\mathbf{A}_\perp}{d\phi^{n+1}}, \tag{44}$$

and proves the recurrence. Eq. (44) can now be generalized to read:

$$\mathbf{u}_\perp = \sum_{n=0}^{\infty} (\varepsilon q_0)^n \frac{d^n \mathbf{A}_\perp}{d\phi^n}. \tag{45}$$

4.c Exact plane wave solution

In the case of a linearly polarized, monochromatic plane wave, where we have,

$$\mathbf{A}_\perp(\phi) = \hat{\mathbf{x}} A_0 \Re\left(e^{i\phi}\right), \quad \frac{d^n \mathbf{A}_\perp}{d\phi^n} = \hat{\mathbf{x}} A_0 \Re\left(i^n e^{i\phi}\right), \tag{46}$$

the summation in Eq. (45) can easily be performed analytically:

$$\mathbf{u}_{\perp} = \sum_{n=0}^{\infty} \left(\varepsilon q_0\right)^n \frac{d^n \mathbf{A}_{\perp}}{d\phi^n} = \hat{\mathbf{x}} A_0 \Re\left[e^{i\phi} \sum_{n=0}^{\infty} \left(i\varepsilon q_0\right)^n\right] = \hat{\mathbf{x}} A_0 \Re\left(\frac{e^{i\phi}}{1 - i\varepsilon q_0}\right) = \hat{\mathbf{x}} A_0 \frac{\cos\phi - \varepsilon q_0 \sin\phi}{1 + \varepsilon^2 q_0^2}. \tag{47}$$

Using this result in the equation governing the dynamics of the light-cone variable leads to a slightly more complicated differential equation:

$$\begin{aligned} \mathbf{u}_{\perp}^{'} &= -\hat{\mathbf{x}} A_0 \frac{\sin\phi + \varepsilon q_0 \cos\phi}{1 + \varepsilon^2 q_0^2}. \\ q^{'} &= \varepsilon q_0 \left[q^{"} - q_0 \left(\frac{A_0}{1 + \varepsilon^2 q_0^2} \right)^2 \left(\sin\phi + \varepsilon q_0 \cos\phi \right)^2 \right], \end{aligned} \tag{48}$$

which can also be solved analytically [6], to obtain

$$q(\phi) = q_0 + \frac{\left(\varepsilon^2 q_0^2 - 2\right)\varepsilon^2 q_0^3 A_0^2}{2\left(4\varepsilon^2 q_0^2 + 1\right)} \left(1 - \cos 2\phi\right) + \frac{\left(5\varepsilon^2 q_0^2 - 1\right)\varepsilon q_0^2 A_0^2}{4\left(4\varepsilon^2 q_0^2 + 1\right)} \sin 2\phi - \frac{\varepsilon q_0^2 A_0^2}{2\left(\varepsilon^2 q_0^2 + 1\right)} \phi. \tag{49}$$

Note that the general solution contains a runaway exponential, of the form $e^{\phi / \varepsilon q_0}$, which is eliminated by choosing the proper initial conditions for $q^{'}$ and $q^{"}$; in addition, $q\left(\phi = 0\right) = q_0$. The second-harmonic oscillatory terms are driven by the ponderomotive force, while radiative recoil accumulates linearly with ϕ.

To determine the momentum transfer over a finite phase interval, $\Delta\phi$, we simply average out the second harmonic motion:

$$\begin{aligned}&\left\langle q(\Delta\phi)\right\rangle - q_0 = -\frac{\varepsilon q_0^2 A_0^2}{2\left(\varepsilon^2 q_0^2+1\right)}\Delta\phi \simeq 1-u_z^+,\\&\Delta u_z \simeq \frac{\varepsilon A_0^2}{2\left(\varepsilon^2+1\right)}\Delta\phi \simeq \varpi_0\tau_0\frac{A_0^2}{2}\Delta\phi = \frac{2}{3}\omega_0\frac{r_0}{c}\frac{A_0^2}{2}\Delta\phi = \frac{2}{3}\omega_0\frac{r_0}{c}\int_0^{\Delta\phi}\mathbf{A}_\perp^2(\psi)\,d\psi.\end{aligned} \tag{50}$$

Here, we have chosen $q_0 = \gamma_0 - u_{z0} = 1$ to model an electron initially at rest; we have also neglected the term in ε^2; finally, we clearly recognize that $\int_0^{\Delta\phi}\mathbf{A}_\perp^2(\psi)\,d\psi = \Delta\phi A_0^2/2$ for a linearly polarized plane wave of constant amplitude over the phase interval $\Delta\phi$. This result is completely analogous to the one derived for circular polarization, and presented in Eq. (28).

The complete result is:

$$\Delta u_z \simeq \frac{2}{3}k_0 r_0 \frac{1}{1+\left(\frac{2}{3}k_0 r_0\right)^2}\int_0^{\Delta\phi}\mathbf{A}_\perp^2(\psi)\,d\psi. \tag{51}$$

To exhibit the higher-order classical radiative corrections, we simply Taylor-expand $\left(1+\varepsilon^2\right)^{-1}$:

$$\Delta u_z \simeq \frac{\varepsilon A_0^2}{2\left(\varepsilon^2+1\right)}\Delta\phi \simeq \frac{2}{3}\omega_0\frac{r_0}{c}\int_0^{\Delta\phi}\mathbf{A}_\perp^2(\psi)\,d\psi\sum_{n=0}^{\infty}\left(-\varepsilon^2\right)^n. \tag{52}$$

Beyond the lowest-order term, the corrections scale as even powers of $\varepsilon = k_0 r_0$; these results are in sharp contrast with the Compton scattering theory, which is presented next.

5 Compton scattering

To assess the validity of the results derived above, we need to compare them with Compton scattering both in the small recoil limit, and for larger values of the momentum transfer. Energy-momentum conservation can be written as $u_\mu^0 + \lambdabar_C k_\mu^0 = u_\mu^+ + \lambdabar_C k_\mu^+$; using the normalization of the 4-velocity and the photon mass-shell condition, one obtains the well-known relation between the initial and final photon states: $k_\mu^+ \left(u_0^\mu + \lambdabar_C k_\mu^0 \right) = k_\mu^0 u_0^\mu$. In the specific frame chosen here, $u_0^\mu = (1,0,0,0)$, and the electron momentum after scattering is simply given by:

$$\mathbf{u}^+(\Omega) = \lambdabar_C k_0 \left\{ \hat{\mathbf{z}} - \frac{\hat{\mathbf{n}}(\Omega)}{1 + \lambdabar_C k_0 \left[1 - \hat{\mathbf{n}}(\Omega) \cdot \hat{\mathbf{z}} \right]} \right\}, \tag{53}$$

where $\hat{\mathbf{n}}(\Omega)$ is the propagation direction of the scattered photon. For direct comparisons with Eqs. (28) and (51), the momentum transfer needs to be averaged over the Compton scattering differential cross-section, which represents the probability of radiating a photon over a small solid angle:

$$\langle \mathbf{u}^+ \rangle = \frac{1}{\sigma} \int \mathbf{u}^+(\Omega) \frac{d\sigma}{d\Omega} d\Omega. \tag{54}$$

5.a Small recoil limit

In the small recoil limit, where $\lambdabar_C k_0 \ll 1$, $\mathbf{u}^+(\Omega) \simeq \lambdabar_C k_0 \left[\hat{\mathbf{z}} - \hat{\mathbf{n}}(\Omega)\right]$; furthermore, in the rest frame of the electron, $d\sigma / d\Omega = r_0^2 \sin^2\theta$, where θ is the angle between the direction of polarization and $\hat{\mathbf{n}}(\Omega)$. Using symmetry arguments, it is easily seen that $\int \hat{\mathbf{n}}(\Omega) \sin^2\theta d\Omega = \mathbf{0}$, and $\langle \mathbf{u}^+ \rangle = \hat{\mathbf{z}} \langle u_z^+ \rangle = \lambdabar_C k_0 \hat{\mathbf{z}}$, as shown in Fig. 1. At this point, to obtain the total momentum transfer we need to evaluate the average number of scattering events between the electron and the incident photons in the plane wave. The electromagnetic energy density in vacuum is $d^3W / dxdydz = E^2 / 4\pi = (\omega_0 m_0 c A / e)^2 / 4\pi$, and the photon density can be written as:

$$n_\lambda = \frac{1}{\hbar\omega_0} \frac{d^3W}{dxdydz} = \frac{1}{2} \frac{A^2}{r_0 \lambdabar_C \lambda_0}. \tag{55}$$

The average number of collisions is then:

$$\begin{aligned} \langle N \rangle &= \sigma \int_{-\infty}^{+\infty} n_\lambda(t) c dt = \frac{8}{3}\pi r_0^2 \frac{1}{2 r_0 \lambdabar_C \lambda_0} \int_{-\infty}^{+\infty} A^2(t) c dt \\ &= \frac{4}{3}\pi \frac{r_0}{\lambdabar_C \lambda_0} A_0^2 c \int_{-\infty}^{+\infty} g^2(f\phi) \frac{d\phi}{\omega_0} = \frac{2}{3} \frac{r_0}{\lambdabar_C} A_0^2 \int_{-\infty}^{+\infty} g^2(\phi) d\phi, \end{aligned} \tag{56}$$

and the average recoil is

$$\Delta u_z = \langle N \rangle \langle u_z^+ \rangle = \frac{2}{3} r_0 k_0 A_0^2 \int_{-\infty}^{+\infty} g^2(\phi) d\phi, \tag{57}$$

which is precisely the result obtained using the classical Dirac-Lorentz equation, shown in Eq. (28).

Before examining the physical meaning of this result, we note that the frequency of the incident plane wave, ω_0, represents an average for a short pulse; however, since the result holds independently of the pulse duration, one can consider arbitrarily long pulses with correspondingly narrow Fourier transform-limited bandwidths.

As expected, the classical Dirac-Lorentz result does not involve Planck's constant, while Compton scattering, for an individual event, clearly reflects the quantum nature of light. Once an average number of collisions are considered, however, Compton scattering yields the same momentum transfer as the classical derivation. This might seem paradoxical, but the averaging clearly yields a continuous momentum transfer value because, while each collision results into a quantized average recoil, the energy density of the incident plane wave itself is partitioned into discrete quanta, thus eliminating Planck's constant from the final result.

This further establishes the well-known fact that, for free electrons, the electrodynamical length scale is the classical electron radius, r_0; indeed, the Compton scattering cross-section is essentially classical, and independent from the Compton wavelength, $\lambda\!\!\!^{-}_C = r_0 / \alpha$: $\sigma = 8\pi r_0^2 / 3$.

5.b Average recoil

Equation (54) can be used to determine the average electron recoil for arbitrary values of $k_0ƛ_C$: using spherical coordinates, with $\hat{\mathbf{n}}(\Omega) = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta\sin\varphi + \hat{\mathbf{z}}\sin\theta\cos\varphi$, we have

$$\left\langle \mathbf{u}_x^+ \right\rangle = -\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \frac{3\sin^2\theta}{8\pi} \frac{\cos\theta}{1+\xi(1-\sin\theta\cos\varphi)} = 0,$$

$$\left\langle \mathbf{u}_y^+ \right\rangle = -\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \frac{3\sin^2\theta}{8\pi} \frac{\sin\theta\sin\varphi}{1+\xi(1-\sin\theta\cos\varphi)} = 0,$$

$$\left\langle \mathbf{u}_z^+ \right\rangle = \xi\left[1 - \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \frac{3\sin^2\theta}{8\pi} \frac{\sin\theta\cos\varphi}{1+\xi(1-\sin\theta\cos\varphi)}\right]$$

$$= \xi + 1 - \frac{3(1+\xi)}{4\xi}\left\{\frac{\left[(1+\xi)^2+\xi^2\right]}{2\xi}\ln\left(\frac{1+\frac{\xi}{1+\xi}}{1-\frac{\xi}{1+\xi}}\right) - 1 - \xi\right\}, \tag{58}$$

where we have defined $\xi = k_0ƛ_C$. For small values of ξ, the recoil is given by

$$\left\langle \mathbf{u}^+ \right\rangle \simeq k_0ƛ_C\left(1 - \frac{2}{5}k_0ƛ_C\right)\hat{\mathbf{z}}, \tag{59}$$

where a quadratic correction term appears; the average momentum transfer is:

$$\Delta u_z = \langle N \rangle \left\langle u_z^+ \right\rangle = \frac{2}{3} r_0 k_0 A_0^2 \int_0^{\Delta\phi} g^2(\phi)\, d\phi \left(1 - \frac{2}{5}\xi\right). \tag{60}$$

This last result is important, as it combines the classical and the quantum scales; the correction term is purely quantum mechanical. By contrast, the Dirac-Lorentz radiative scale is $\varepsilon = k_0 r_0 = \alpha\xi$, and the first correction beyond the lowest-order term is quadratic in ε.

For arbitrary values of $\xi = k_0 \lambda\!\!\!^{-}_C$, the recoil is:

$$\Delta u_z = \langle N \rangle \langle u_z^+ \rangle = \frac{2}{3} r_0 k_0 \left[\!\!\left[1 + \frac{1}{\xi} - \frac{3(1+\xi)}{4\xi^2} \left\{ \frac{\left[(1+\xi)^2 + \xi^2 \right]}{2\xi} \ln \left(\frac{1 + \frac{\xi}{1+\xi}}{1 - \frac{\xi}{1+\xi}} \right) - 1 - \xi \right\} \right]\!\!\right] \int_0^{\Delta\phi} \mathbf{A}_\perp^2(\phi)\, d\phi. \tag{61}$$

Note that the classical scale, $\varepsilon = k_0 r_0 = \alpha\xi$; therefore, perturbation theory still applies for values of $\xi \leq \alpha^{-1} = 137.0359895(61)$, and we can compare the Dirac-Lorentz theory with Compton scattering, as shown in Fig. 2. Clearly, the classical electron theory breaks down beyond the lowest-order value of the momentum transfer, which scales as the classical electron radius; of course, this is not unexpected, as the quantum scale characterizing Compton scattering recoil correction is not present in the classical theory. Therefore, radiative corrections should in most cases be treated via QED, although this becomes difficult in the classical nonlinear regime, where the normalized potential $A_0 \geq 1$. A more detailed inspection of Fig. 2 shows that for $\xi < \alpha$, both theories agree, as recoil remains negligible; the Compton peak is located near $\xi = 2$, where classical recoil is still very small ($\varepsilon = 2\alpha \ll 1$); finally, a crossing point exists at $\varepsilon = 0.22648$, beyond which the two theories predict completely different behaviors: while the

Compton corrections become smaller, the Dirac-Lorentz solution trends toward larger effects before the perturbative approach breaks down.

6 Conclusions

In conclusion, we have presented a direct comparison between the Dirac-Lorentz dynamics of an electron subjected to a plane wave in vacuum and the well-known recoil associated with Compton scattering; in the small recoil limit, the classical Dirac-Lorentz is shown to yield the same momentum transfer as that derived from Compton scattering kinematics. While this further establishes the well-known fact that, for free electrons, the electrodynamical length scale is the classical electron radius, questions remain open about the transition between the classical regime, where Dirac-Lorentz electrodynamics applies, and the quantum electrodynamical regime, where QED concepts, including Delbrück scattering, pair creation, and the Schwinger critical field play a major role. When higher-order corrections are included, an exact analytical solution to the plane wave Dirac-Lorentz equation has been derived, and used to show that the classical electron theory breaks down beyond the lowest-order value of the momentum transfer, which scales as the classical electron radius; of course, this is not unexpected, as the quantum scale characterizing Compton scattering recoil correction is not present in the classical theory. Therefore, radiative corrections should in most cases be treated via QED, although this becomes difficult in the classical nonlinear regime, where the normalized potential $A_0 \geq 1$.

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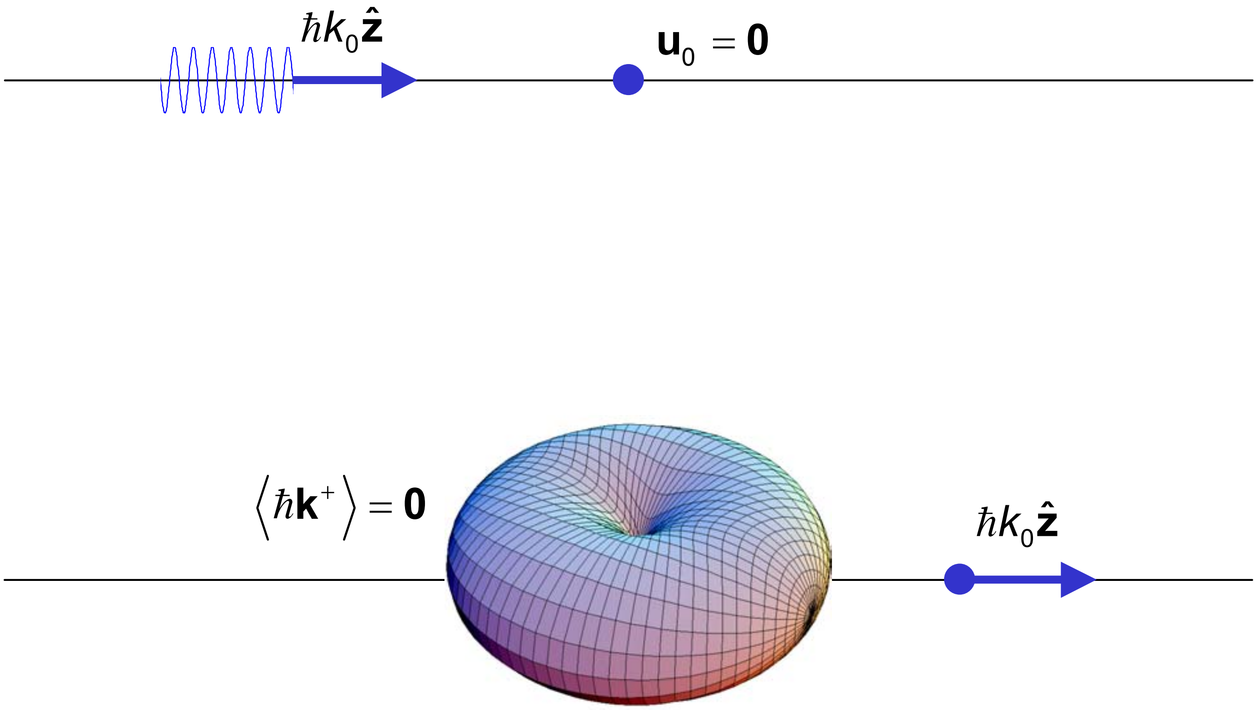
Figure Captions

Fig. 1 Interaction between an electron initially at rest with an incident photon, propagating along the *z*-axis, with momentum $\hbar k_0 \hat{\mathbf{z}}$; after the event, the probability distribution for the scattered photons is given by a dipole radiation pattern, which results in an average null momentum for the scattered radiation: the electron recoil is then equal to $\langle m_0 c \mathbf{u}^+ \rangle = \hbar k_0 \hat{\mathbf{z}}$, on average.

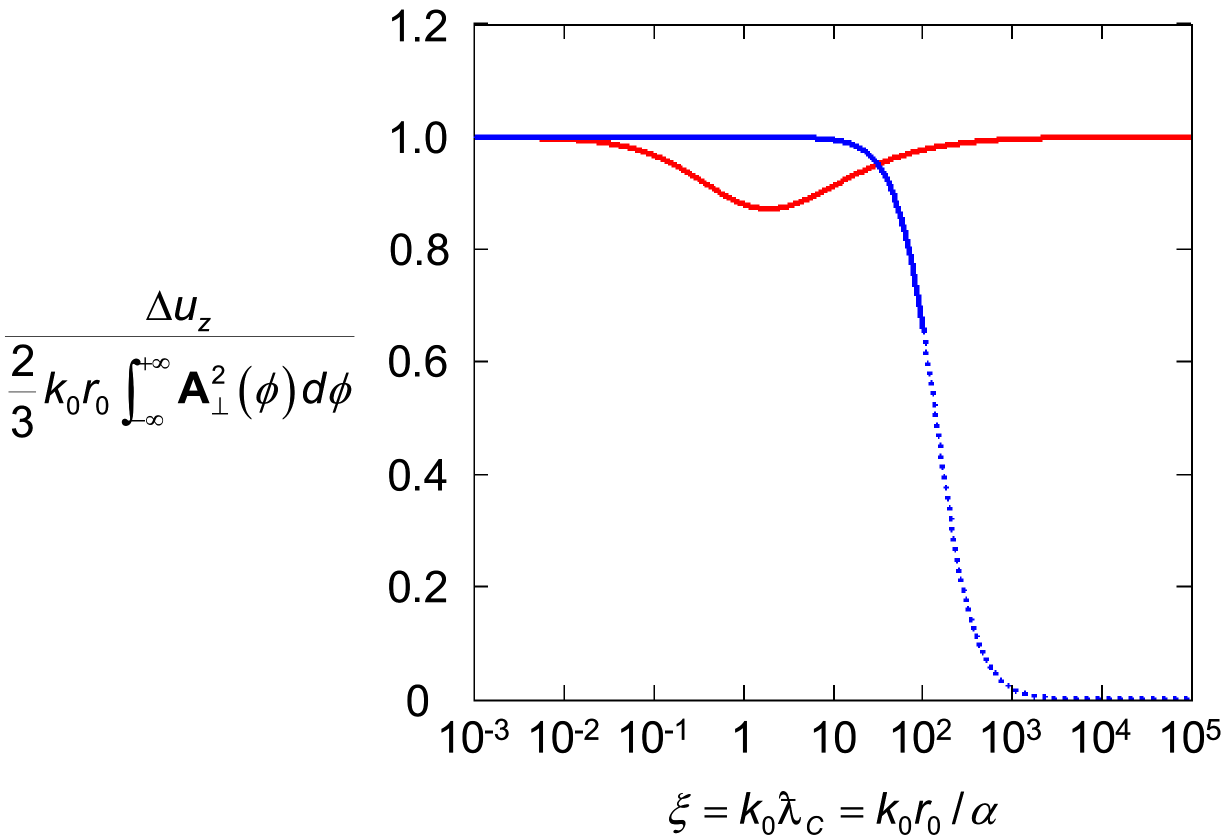
Fig. 2 Comparison between the average axial electron recoil from Compton scattering (red) and the Dirac-Lorentz momentum transfer (blue); the dashed line corresponds to a regime where the perturbation in $\varepsilon = k_0 r_0 = \alpha \xi < 1$ is no longer valid. In both cases, the momentum transfer, Δu_z, is normalized to $\frac{2}{3} k_0 r_0 \int_{-\infty}^{+\infty} \mathbf{A}_\perp^2(\phi) d\phi$.

Fig. 1



$\hbar k_0 \hat{\mathbf{z}}$
$\mathbf{u}_0 = \mathbf{0}$
$\langle \hbar \mathbf{k}^+ \rangle = \mathbf{0}$
$\hbar k_0 \hat{\mathbf{z}}$

Fig. 2


1.2
1.0
0.8
0.6
0.4
0.2
0
$\frac{\Delta u_z}{\frac{2}{3} k_0 r_0 \int_{-\infty}^{+\infty} \mathbf{A}_\perp^2(\phi) d\phi}$
10^{-3} 10^{-2} 10^{-1} 1 10 10^2 10^3 10^4 10^5
$\xi = k_0 \bar{\lambda}_C = k_0 r_0 / \alpha$