PROBABILITY DISTRIBUTIONS FOR MAGNETOTELLURICS

by

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ABSTRACT

Estimates of the magnetotelluric transfer functions can be viewed as ratios of two complex random variables. It is assumed that the numerator and denominator are governed approximately by a joint complex normal distribution. Under this assumption, probability distributions are obtained for the magnitude, squared magnitude, logarithm of the squared magnitude, and the phase of the estimates. Normal approximations to the distributions are obtained by calculating mean values and variances from error propagation, and the distributions are plotted with their normal approximations for different percentage errors in the numerator and denominator of the estimates, ranging from 10% to 75%.

The distribution of the phase is approximated well by a normal distribution for the range of errors considered, while the distribution of the logarithm of the squared magnitude is approximated by a normal distribution for a much larger range of errors than is the distribution of the squared magnitude. The distribution of the squared magnitude is most sensitive to the presence of noise in the denominator of the estimate, in which case the true distribution deviates significantly from normal behavior as the percentage errors exceed 10%. In contrast, the normal approximation to the distribution of the logarithm of the magnitude is useful for errors as large as 75%.

INTRODUCTION

In the frequency domain, the equations governing the magnetotelluric relations between the signal components (subscript s) of the electric and magnetic fields at the earth's surface are

$$E_{si} = Z_{ix}H_{sx} + Z_{iy}H_{sy}, \quad i = x \text{ or } y, \quad (1)$$

and
$$H_{sz} = T_{zx}H_{sx} + T_{zy}H_{sy}. \quad (2)$$

The tensor impedances, Z_{ix} and Z_{iy} , and the tipper functions, T_{zx} and T_{zy} , are functions of frequency and conductivity structure. Equations (1) and (2) can be written in the general form

$$O_{si} = G_{ix}I_{sx} + G_{iy}I_{sy}, \quad i = x, y, \text{ or } z, \quad (3)$$

where, from the viewpoint of linear system theory, G_{ix} and G_{iy} are transfer functions of a dual input, single output linear system through which the inputs, I_{sx} and I_{sy} , are related deterministically to the output, O_{si} .

The transfer functions must be estimated in some fashion from field measurements which are invariably contaminated by noise from various sources. We assume the field measurements consist of signals and additive random noises, e.g., $I_x = I_{sx} + i_x$, where lower case letters represent noise components. By definition, the signal components satisfy equation (3) while the noise components do not. We assume the signals and noises are statistically independent.

The transfer functions G_{ix} and G_{iy} are usually computed from noisy field data as the least squares estimates (subscript m)

$$G_{mij} = W_{mij}/V_m, \quad (4)$$

where

$$W_{mix} = \overline{O_i A^*} \overline{I_y B^*} - \overline{I_y A^*} \overline{O_i B^*},$$

$$W_{miy} = \overline{I_x A^*} \overline{O_i B^*} - \overline{O_i A^*} \overline{I_x B^*},$$

and

$$V_m = \overline{I_x A^*} \overline{I_y B^*} - \overline{I_y A^*} \overline{I_x B^*}.$$

The quantities A and B are two reference fields whose signal components, A_s and B_s , are related linearly to the input signals, I_{sx} and I_{sy} , i.e., $[A_s \ B_s]^T = \underline{M} [I_{sx} \ I_{sy}]^T$, where $\underline{M}$ is a nonsingular transfer matrix. In equation (4), W_{mij} and V_m are functions of various spectral estimates, $\overline{XY^*}$, where the bar indicates an average over N data sets and * indicates complex conjugate. We assume that the spectral estimates are resolved sufficiently so that G_{mij} gives G_{ij} when I_x , I_y , and O_i are noise-free. Hereafter, the subscripts ij in G_{mij} and W_{mij} are suppressed except where required to distinguish between estimates.

The estimates can be written as functions of the expected values and random errors in W_m and V_m , i.e.,

$$G_m = \frac{W_m}{V_m} = \frac{W_b + w}{V_b + v}, \quad (5)$$

where the random errors, w and v, have zero mean. The subscript b distinguishes between the true values, W and V, and the expected values, W_b and V_b , which may contain bias errors. We distinguish between conventional estimates calculated using local references (e.g. Sims et al, 1971) and remote reference estimates calculated using

references from a separate site (Gamble et al, 1979a). Remote references are used to avoid the bias errors introduced in W_m and V_m when noises in A or B correlate with noises in I_x , I_y , or O_i . Expectation is defined with respect to the conditional distribution of the noises in the field measurements, as in Stodt (1982a,b). The conditional distribution of the noises is the joint distribution of the noise components when the signals and noises are statistically independent.

Random errors are usually assessed by computing confidence intervals, and such computations require that probability distributions be specified. However, surprisingly little work has been reported with regard to specifying probability distributions for the functions of the G_m which are of primary interest in MT, i.e., the squared magnitude, $|G_m|^2$, the logarithm of the squared magnitude, $\log|G_m|^2$, and the phase, ϕ_{G_m} . Bentley (1973) and Fournier and Febrer (1976) present empirical evidence which suggests that normal approximations to the distributions of $\log|G_m|$ and ϕ_{G_m} are appropriate. However, they did not attempt to separate the effects of random and bias errors in their work. Gamble et al (1979b) discuss the distribution of errors in various nonlinear functions of the G_m calculated using remote references, assuming V_m is noise-free and W_m is normally distributed. They argue that the distribution of errors in nonlinear functions of the G_m should be approximately normal when the errors in W_m are small enough that they propagate linearly. However, they do not address the case where V_m is noisy, or explore the range of validity of linear error propagation.

It is, in fact, possible to use complex distribution theory to

compute certain theoretical probability distributions which are useful for MT. In this paper, we compute the distributions of $|G_m|^2$, $\log|G_m|^2$, and ϕ_{G_m} under the assumption that random errors in W_m and V_m are described adequately by a joint complex Gaussian distribution. The validity of this assumption is examined with regard to conventional and remote reference estimates. In obtaining the distributions, we consider separately the cases where either W_m is noisy or V_m is noisy. We also consider the general case where both W_m and V_m are noisy. In an Appendix, normal approximations to the true distributions are obtained by calculating mean values and variances using error propagation, and a general discussion of nonlinear error propagation sheds light on the ranges of validity of the approximations. The true distributions are compared graphically with their normal approximations for different percentage errors in W_m and V_m , and the estimation of confidence intervals is discussed.

PROBABILITY DISTRIBUTIONS

In this section we obtain probability distributions for $|G_m|^2$, $\log|G_m|^2$, and ϕ_{G_m} , by treating G_m as the ratio of two complex random variables, as defined in equation (5). Our starting point is the joint Gaussian distribution of the complex random variables, $Z_i = C_i + z_i$, $i = 1, 2, \dots, n$, given by (e.g. Miller, 1974)

$$g(\underline{Z}) = \frac{1}{\pi^n \det \underline{R}} \exp [-(\underline{Z}^* - \underline{C}^*)^T \underline{R}^{-1} (\underline{Z} - \underline{C})] , \quad (6)$$

where $\underline{Z}$ and $\underline{C}$ are n -dimensional column vectors, and C_i is the expected value, $E[Z_i]$, of Z_i . The matrix $\underline{R}$ is the Hermitian $n \times n$ covariance matrix

$$\underline{R} = [\text{Cov}(Z_i, Z_j^*)] \quad i, j = 1, 2, \dots, n,$$

where
$$\text{Cov}(Z_i, Z_j^*) = E[Z_i Z_j^*] - E[Z_i] E[Z_j^*] . \quad (7)$$

The diagonal elements of $\underline{R}$ are the variances of the complex random variables Z_i , i.e.

$$R_{ii} = \text{Var } Z_i = E[|Z_i|^2] - |E[Z_i]|^2 . \quad (8)$$

Note that by definition, $\text{Var } Z_i$ is the sum of $\text{Var } X_i$ and $\text{Var } Y_i$, where $Z_i = X_i + jY_i$. Equation (6) has meaning as a compact way to write a $2n$ -dimensional Gaussian distribution of the real and imaginary parts, X_i and Y_i , of the Z_i . Subsequent analyses are simplified by using $g(\underline{Z})$ to represent the joint distribution of $\underline{X}$ and $\underline{Y}$, but equation (6)

is not completely general. It can be shown that the representation is valid only if the conditions

$$\text{Cov}(Z_i, Z_j) = 0, \quad i, j = 1, 2, \dots, n \quad (9)$$

are satisfied. Equation (9) restricts the form of the covariance matrix of the real and imaginary parts.

In general, the real and imaginary parts of W_m and V_m can be treated as normally distributed as N increases because of the central limit theorem. If we make the identifications, $V_m = Z_1$ and $W_m = Z_2$, the question then arises whether or not the covariance matrix of real and imaginary parts is such that their joint distribution can be written as a complex Gaussian distribution. From equation (9), it must be shown that $\text{Cov}(W_m, W_m)$, $\text{Cov}(V_m, V_m)$, and $\text{Cov}(W_m, V_m)$ are zero for equation (6) to apply. It is shown in Appendix A that these covariances are zero if the base field noises are uncorrelated with the reference field noises. However, they are not all necessarily zero when the base and reference field noises correlate. The probability distributions presented here are derived under the assumption that the joint distribution of W_m and V_m is given by equation (6), and thus apply strictly only to remote reference MT data. However, they should be good approximations to the true distributions for conventional MT data as well, provided that $|\text{Cov}(W_m, W_m)|$, $|\text{Cov}(V_m, V_m)|$, and $|\text{Cov}(W_m, V_m)|$ are much smaller than $\text{Var } W_m$, $\text{Var } V_m$, and $|\text{Cov}(W_m, V_m^*)|$ respectively.

Before deriving the distributions of the squared magnitude, logarithm of the squared magnitude, and the phase of G_m , we discuss briefly the general approach, which is discussed thoroughly in texts

on probability theory. Suppose $y_i = f_i(x_1, x_2, \dots, x_n), i=1, 2, \dots, n$ are functions of the random variables x_i , which have the joint density function $p(x_1, x_2, \dots, x_n)$. Suppose further that unique solutions for the x_i in terms of the $y_i, x_i = f_i^{-1}(y_1, y_2, \dots, y_n)$, exist over the ranges of definition of the x_i . Then we can write

$$p^*(y_1, y_2, \dots, y_n) = |J| p(x_1, x_2, \dots, x_n), \quad (10)$$

where $J = \det [\partial x_i / \partial y_j]$, is the determinant of the Jacobian of the transformation. It is understood that the f_i^{-1} are substituted for the x_i to give p^* in equation (10). The region of definition of $p^*(y_1, y_2, \dots, y_n)$ is obtained by substituting the limits of the ranges of definition of the x_i into the equations, $x_i = f_i^{-1}(y_1, y_2, \dots, y_n)$ and solving them for the limits of the y_i .

In practice, individual or joint distributions of particular subsets of the y_i may be required. These are obtained by integrating over the range of the undesired variables, that is, by calculating the appropriate marginal distribution from the joint distribution, $p^*(y_1, y_2, \dots, y_n)$. Success in obtaining explicit formulas thus depends on one's ability to carry out the required integrations.

The transformation of $g(Z)$, as given by equation (6), to polar coordinates provides a specific example of the application of equation (10). Noting that $J = \prod_{i=1}^n r_i$ is the determinant of the Jacobian of the transformation to polar form of the real and imaginary parts of the Z_i , then

$$q^*(r_1, r_2, \dots, r_n, \phi_1, \phi_2, \dots, \phi_n) = \prod_{i=1}^n r_i q(Z_1, Z_2, \dots, Z_n), \quad (11)$$

where the Z_i are replaced by $r_i e^{j\phi_i}$.

We now proceed to obtain specific formulas. Given a single complex random variable Z whose distribution is given by equation (6), then the distributions of the magnitude and phase of Z can be derived by changing to polar coordinates, $g^*(r, \phi) = rg(Z)$, and calculating the marginal distributions (Miller, 1974, p. 87-88),

$$\begin{aligned} p(r) &= \int_0^{2\pi} g^*(r, \phi) d\phi \\ &= \frac{2r}{R} \exp \left[-\frac{(r^2 + |C|^2)}{R} \right] I_0 \left(\frac{2|C|r}{R} \right), r \geq 0 \end{aligned} \quad (12)$$

and

$$\begin{aligned} q(\phi) &= \int_0^{\infty} g^*(r, \phi) dr \\ &= \frac{1}{2\pi} \exp \left[-\frac{|C|^2 [1 + \sin^2(\phi - \gamma)]}{2R} \right] D_{-2} \left[-|C| \left(\frac{2}{R} \right)^{1/2} \cos(\phi - \gamma) \right], \\ &\quad \theta \leq \phi < \theta + 2\pi \end{aligned} \quad (13)$$

where θ is fixed but arbitrary, the expected value of Z is given by $E[Z] = |C|e^{j\gamma}$, and R is the variance of Z . The functions $I_n(\cdot)$ and $D_{-n}(\cdot)$ are the modified Bessel function and parabolic cylinder function respectively, and $p(r)$ and $q(\phi)$ are zero outside their intervals of definition. Equations (12) and (13) both simplify when $|C|/\sqrt{R}$ is zero since $I_0(0)$ and $D_{-2}(0)$ are both unity. In particular, $q(\phi)$ approaches a uniform distribution as $|C|/\sqrt{R}$ approaches zero. Note that by making appropriate identifications for Z , $p(r)$ and $q(\phi)$ give the distributions of the magnitudes and phases of W_m and V_m , and also of G_m when V_m is noise-free.

The distributions of the squared magnitude and its logarithm are of interest in MT; they can each be obtained from the distribution of the magnitude, $p(r)$, through a simple transformation of variables.

Letting $w = |Z|^2$ and $v = \log |Z|^2$, then we find

$$p^*(w) = p(r)/2\sqrt{w}, \quad 0 \leq w < \infty, \quad (14)$$

and

$$p^*(v) = p(r)10^{v/2} \ln 10^{1/2}, \quad -\infty < v < \infty, \quad (15)$$

where $r = \sqrt{w}$ in equation (14) and $r = 10^{v/2}$ in equation (15).

Equations (12) and (13) apply when $Z = G_m$ only if the denominator V_m is noise-free. Defining $W_m = Z_2 = r_2 e^{j\phi_2}$ and $V_m = Z_1 = r_1 e^{j\phi_1}$, then when both Z_1 and Z_2 are noisy the distributions of the magnitude, $|G_m| = r_2/r_1$, and the phase, $\phi_{G_m} = \phi_2 - \phi_1$, must be derived from the joint distribution, $g(Z_1, Z_2)$. Transforming equation (6) to polar coordinates, $g^*(r_1, r_2, \phi_1, \phi_2) = r_1 r_2 g(Z_1, Z_2)$, then the joint distributions, $p(r_1, r_2)$ and $q(\phi_1, \phi_2)$, are given by

$$p(r_1, r_2) = \int_0^{2\pi} \int_0^{2\pi} g^*(r_1, r_2, \phi_1, \phi_2) d\phi_1 d\phi_2, \quad (16)$$

and

$$q(\phi_1, \phi_2) = \int_0^\infty \int_0^\infty g^*(r_1, r_2, \phi_1, \phi_2) dr_1 dr_2. \quad (17)$$

Turning attention to equation (16) first and carrying out the integrations (see Appendix B) gives

$$p(r_1, r_2) = 4r_1 r_2 \exp[-(S_{11}r_1^2 + S_{22}r_2^2 + Q)] \det \underline{S} \quad (18)$$

$$\cdot \sum_{n=0}^{\infty} (-)^n \epsilon_n I_n(2r_1|U_1|) I_n(2r_2|U_2|) I_n(2r_1 r_2 |S_{12}|) \cos n(\psi_{12} + \alpha_2 - \alpha_1),$$

$$0 \leq r_1, r_2 < \infty,$$

where

$$Q = S_{11}|C_1|^2 + S_{22}|C_2|^2 + 2|S_{12}||C_1||C_2| \cos(\gamma_1 - \gamma_2 - \psi_{12}),$$

$$U_1 = C_1 S_{11} + C_2 S_{12} = |U_1| e^{j\alpha_1},$$

$$U_2 = C_2 S_{22} + C_1 S_{12}^* = |U_2| e^{j\alpha_2},$$

$$C_i = E[Z_i] = |C_i| e^{j\gamma_i}, \quad i = 1, 2,$$

$$S_{12} = |S_{12}| e^{j\psi_{12}},$$

$$\epsilon_0 = 1; \quad \epsilon_n = 2, \quad n = 1, 2, \dots$$

and $\underline{S} = [S_{ij}]$ is the inverse of the covariance matrix, $\underline{R}$, defined in equation (6). The functions $I_n(\cdot)$ are modified Bessel functions.

The distribution of the ratio r_2/r_1 gives the distribution of $|G_m|$. It can be derived from equation (18) by making the change of variables, $r = r_2/r_1$ and $s = r_1$, so that $p^*(r, s) = sp(r_1, r_2)$. Then the distribution of r is given by

$$h(r) = \int_0^\infty sp(s, rs) ds, \quad 0 \leq r < \infty. \quad (19)$$

The integral is evaluated in Appendix B. In the special case when Z_1 and Z_2 are uncorrelated, $h(r)$ reduces to the relatively simple expression

$$h(r) = \frac{2re^{(Y/T - Q)}}{R_{11}R_{22}T^2} \left[\left(1 + \frac{Y}{T}\right) I_0(X) + X I_1(X) \right] \quad (20)$$

where

$$Y = |U_1|^2 + r^2 |U_2|^2,$$

$$X = \frac{2|U_1||U_2|r}{T},$$

$$T = S_{11} + r^2 S_{22},$$

and the expressions for Q and U_i , $i = 1$ or 2 , simplify, since S_{12} is zero. The distributions of $w = r^2$ and $v = \log r^2$ are obtained from equations (16) and (17) by replacing $p(r)$ with $h(r)$.

We now turn our attention to the distribution of the phase,

$\phi_{G_m} = \phi_2 - \phi_1$. After expanding the quadratic form in the exponent of $g(Z_1, Z_2)$ and changing to polar coordinates, the joint distribution, $q(\phi_1, \phi_2)$, given by equation (17), can be written as

$$q(\phi_1, \phi_2) = \frac{e^{-Q_{\det S}}}{\pi^2} \int_0^\infty \int_0^\infty r_1 r_2 \exp[-S_{11}r_1^2 - S_{22}r_2^2 - 2Cr_1r_2 + 2Ar_1 + 2Br_2] dr_1 dr_2, \quad (21)$$

where

$$A = S_{11}|C_1| \cos(\phi_1 - \gamma_1) + |S_{12}||C_2| \cos(\phi_1 - \gamma_2 - \psi_{12}),$$

$$B = S_{22}|C_2| \cos(\phi_2 - \gamma_2) + |S_{12}||C_1| \cos(\phi_2 - \gamma_1 + \psi_{12}),$$

and

$$C = |S_{12}| \cos(\phi_1 - \phi_2 - \psi_{12}).$$

Making the transformation of variables, $\theta = \phi_1$, $\phi = \phi_2 - \phi_1$, then the distribution of the phase difference, ϕ , is

$$q^*(\phi) = \int_{a(\phi)}^{b(\phi)} q(\theta, \phi + \theta) d\theta. \quad (22)$$

The limits of integration, $a(\phi)$ and $b(\phi)$, are defined in Figure 1.

Miller (1974, p. 89) calculates $q(\phi_1, \phi_2)$ in the special case when the mean values, C_1 and C_2 , are both zero, and $q^*(\phi)$ can be computed

easily from his result. However, with MT data, C_1 , C_2 , and S_{12} are generally all nonzero, and the evaluation of the integrals in equations (21) and (22) is difficult. When S_{12} is zero, (i.e., when the real parts of Z_1 and Z_2 are uncorrelated with their imaginary parts), the marginal distributions of ϕ_1 and ϕ_2 are both obtained from equation (13) and $q(\phi_1, \phi_2)$ factors into their product, but $q^*(\phi)$ is still difficult to calculate explicitly from equation (22). Note however, that in this case equation (22) becomes

$$q^*(\phi) = \frac{b(\phi)}{a(\phi)} \int q_{\phi_1}(\theta) q_{\phi_2}(\phi + \theta) d\theta, \quad (23)$$

showing that when ϕ_1 and ϕ_2 are statistically independent, the distribution of their difference, $\phi_2 - \phi_1$, is given by the cross correlation of the marginal distributions, $q_{\phi_1}(\phi_1)$ and $q_{\phi_2}(\phi_2)$. We shall see that the individual distributions of ϕ_1 and ϕ_2 are approximated well by normal distributions for noise levels where $0 \leq \sqrt{R}/|C| < 0.75$. Thus the distribution of $\phi_{G_m} = \phi_2 - \phi_1$ will also be approximated well by a normal distribution, since ϕ_{G_m} is a linear combination of ϕ_2 and ϕ_1 , and linear combinations of Gaussian random variables are themselves Gaussian.

In practice, we desire to approximate the distributions presented in this section with normal distributions for simplicity in computing confidence intervals. Mean values and variances of approximating normal distributions can be calculated using error propagation techniques. The general procedure is discussed and expressions are derived in Appendix C.

GRAPHICAL PRESENTATION OF THE DISTRIBUTIONS

In this section, probability distributions for the squared magnitude, logarithm of the squared magnitude, and the phase of G_m are plotted with their normal approximations for a range of random errors in W_m and V_m , where G_m is the ratio, W_m/V_m . Our goal in each case is to study the accuracy of approximation by normal distributions as the noises in W_m and V_m increase. In this regard, it is straightforward to show that each of the distributions can be written as functions of the dimensionless parameters, $\sqrt{R_i}/|C_i|$, where we have defined $C_i = E[Z_i]$ and $R_i = \text{Var } Z_i$. Having made the identifications $W_m = Z_2$ and $V_m = Z_1$, we note that these parameters are the ratios of the standard deviations of W_m and V_m to the magnitudes of their expected values. We use these parameters as a measure of the percentage errors in W_m and V_m .

The format for all the distributions presented in this section is similar. The different theoretical distributions are plotted with their normal approximations for four values of the ratios $\sqrt{R_i}/|C_i|$ (0.1, 0.25, 0.5, and 0.75) in order to assess behavior as the percentage of error in W_m or V_m increases. In every case, both the theoretical distribution and the approximating normal distribution are normalized by the peak value of the normal distribution for convenience. In addition, plots of the logarithm of the ratio of the theoretical to the approximating normal distributions are provided as

an aid for comparisons. All distributions are plotted for the values $|C_i| = 1$. In each case, however, the distributions when $|C_i| \neq 1$ can be deduced by a simple transformation of variables, as noted in the Figure captions. The abscissas of all plots are incremented in units of one standard deviation of the approximating normal distribution.

We consider first the case where V_m is noise-free so that the expected value of G_m is W_b/V . Making the identification, $Z = G_m$, then equations (12) and (13) give the distributions of the magnitude and phase of G_m . Figures 2 and 3 show the distribution of the squared magnitude of Z as calculated from equations (12) and (14). The normal approximation represents the theoretical distribution very well within two standard deviations of the mean of the squared magnitude, $w = |Z|^2$, when the error in Z is less than 25%. However, the approximation deteriorates rapidly as the errors in Z increase above 25%. In particular, the normal approximation allows a significant probability of observing a negative value of w , which is impossible. The large discrepancies which are observed in the tails of the two distributions when the errors in Z are less than 25% are not significant from a practical viewpoint, since the probability of ever observing a value so far removed from the mean in a finite collection of data is negligible.

Figures 4 and 5 show the distribution of the logarithm of the squared magnitude, $v = \log|Z|^2$, as calculated from equations (12) and (15). The normal approximation is an excellent representation of the theoretical distribution within two standard deviations of the mean of v when the error in Z is less than 25%. In addition, the approximation is acceptable even for errors in Z as large as 75%.

Comparing Figures 4 and 5 with Figures 2 and 3 shows that normal approximations to the distribution of $\log|Z|^2$ are useful over a much larger range of errors in Z than are normal approximations for the distribution of $|Z|^2$.

Figures 6 and 7 show the distribution of the phase of Z , as calculated from equation (13). The normal approximation is an excellent representation of the theoretical distribution when the error in Z is less than 75%.

We now examine the distributions of the squared magnitude, the logarithm of the squared magnitude, and the phase of G_m , where G_m is the ratio, W_m/V_m , and V_m is noisy. We consider first the case where W_m is noise-free. Figures 8 and 9 show the distribution of the squared magnitude of the ratio, $w = |G_m|^2 = r_2^2/r_1^2$, calculated from equations (20) and (14), where we have defined $r_2 = |W_m|$, $r_1 = |V_m|$, and the noise in W_m is negligible. The normal approximation represents the true distribution well only when the errors in V_m are 10% or less. As noise increases, note that the peak of the distribution is shifted strongly towards zero, while the falloff in the tail becomes more gradual. Comparing Figures 8 and 9 with Figures 2 and 3 shows that the behavior when only V_m is noisy becomes very different from the behavior when only W_m is noisy as the percentage errors increase.

The distributions of the logarithm of the squared magnitude and of the phase of G_m when only V_m is noisy are given by the mirror images of the distributions plotted in Figures 4 and 5 and in Figures 6 and 7 respectively. To show this, note that the logarithm of the squared magnitude and the phase of the ratio, $G_m = W_m/V_m$, are given by

the differences, $\log|G_m|^2 = \log|W_m|^2 - \log|V_m|^2$ and $\phi_{G_m} = \phi_{W_m} - \phi_{V_m}$. By applying equation (10) to make the change of variable, $Y = -X$, we obtain the distributions when V_m is noisy from those when W_m is noisy. Thus, the ranges of usefulness of the normal approximations to the distributions of $\log|G_m|^2$ and ϕ_{G_m} are the same, regardless of whether W_m or V_m is noisy.

We now consider the case where the numerator and denominator, W_m and V_m , are both noisy. As noted in a previous section, the theoretical distribution of the phase of G_m is difficult to obtain explicitly in the general case. However, the distributions of the phases of W_m and V_m individually are approximated very well by normal distributions when the percentage errors are less than 75%. We therefore expect the distribution of the phase of G_m , $\phi_{G_m} = \phi_{W_m} - \phi_{V_m}$, to be approximated well by a normal distribution with mean value given by $\phi_{W_b} - \phi_{V_b}$, and variance given by equation (C-24).

The distribution of the magnitude of G_m is given in the general case by equation (B-5). However, equation (20) is easier to compute so we present the distributions of $|G_m|^2$ and $\log|G_m|^2$ in the special case when the covariance, $\text{Cov}(W_m, V_m^*)$, is zero. From the expressions for the variances given by equations (C-14), (C-17), and (C-19), we expect that the primary effect of a nonzero covariance will be to narrow or widen the distributions according to whether the quantity $\text{Re}[G_b^* \text{Cov}(W_m, V_m^*)]$ is positive or negative.

Figures 10 and 11 show the distribution of the squared magnitude, $w = |G_m|^2$, calculated from equations (20) and (14), and Figures 12 and 13 show the distribution of the logarithm of the squared magnitude, $v = \log|G_m|^2$, calculated from equations (20) and (15). The percentage

errors in W_m and V_m are equal in both cases. Note that normal approximations to the distribution of $|G_m|^2$ are accurate only when the percentage errors are less than about 10%. In contrast, a normal distribution approximates the theoretical distribution of $\log|G_m|^2$ quite well for percentage errors as large as 75%. We showed in equations (22) and (23) that the distribution of the difference of two independent random variables is given by the cross correlation of the individual distributions. This result explains the symmetry of the distribution of $\log|G_m|^2$ when the percentage errors in W_m and V_m are equal and allows us to predict the behavior when the percentage errors are different by using Figures 4 and 5.

CONFIDENCE INTERVALS

Examination of the Figures presented in the last section shows that the distributions of the squared magnitude, r^2 , the logarithm of the squared magnitude, $\log r^2$, and the phase, ϕ , of a ratio, Z_2/Z_1 , are approximated well by normal distributions with means and variances calculated using linear error propagation when the relative noise levels are sufficiently small. Approximate confidence intervals are then obtained easily from the normal distributions. The validity of the approximations at various confidence levels can be assessed for different relative noise levels from the plotted distributions. Clearly, the normal approximation to the distribution of $\log r^2$ is valid over a much larger range of relative errors than is the normal approximation to the distribution of r^2 . Considering the distribution of $\log r^2$, the widths of the approximate and true confidence intervals are comparable up through the 95% confidence level, for the relative noise levels considered here. However, when the relative noises in r_2 and r_1 are different, Figures 4 and 5 show that the one standard deviation (68%) approximate confidence interval represents closely an interval for the expected value, $E[\log (r_2/r_1)^2]$, while the two standard deviation (96%) interval represents more closely an interval for $\log |C_2/C_1|^2$. The difference is insignificant for small errors.

Confidence intervals calculated from a normal distribution using estimated variances, s^2 , calculated from the expressions in Appendix C

do not account for the uncertainty introduced by substituting estimated quantities for their expected values in the theoretical expressions for the variances. For example, an estimate of $\text{Var } G_{mij}$ is obtained by replacing expected values by measured quantities in equation (C-15). In the special case when only the output measurement, O_i , is noisy, then R_j and V_m in equation (C-15) are noise-free and an estimate, $\overline{|N_{mi}|^2}$, of $E[|N_i|^2]$ is obtained by replacing G_{ix} and G_{iy} with their estimates, G_{mix} and G_{miy} , in the expression for $\overline{|N_i|^2}$. The statistic, $2N\overline{|N_{mi}|^2}/E[|N_i|^2]$, is distributed as chi-square with $2N-4$ degrees of freedom when the noise in O_i is stationary and normally distributed. Thus, as Gamble et al (1979b) point out, an unbiased estimate of $E[|N_i|^2]$ is given by $\overline{|N_{mi}|^2}/(N-2)$, since the expected value of a chi-square random variable is its degrees of freedom, and an unbiased estimate, s^2 , of $\text{Var } G_{mij}$ is given by

$$s^2 = \frac{\overline{|N_{mi}|^2} \overline{|R_j|^2}}{(N-2) \overline{|V|^2}}.$$

Let X_m be the real or imaginary part of G_m and let X be the expected value of X_m . Noting that G_m and X_m are normally distributed and that $2\text{Var } X_m = \text{Var } G_m$, the statistic $t = (X_m - X) \sqrt{2}/s$ has a Student's t distribution with $2N-4$ degrees of freedom, and a $100(1-\alpha)$ percent confidence interval for X is given by $|X_m - X| \leq st_{2N-4, \alpha}/\sqrt{2}$. The quantity $t_{2N-4, \alpha}$ is the two sided percentage point of Student's t distribution with $2N-4$ degrees of freedom at the α significance level, that is, the probability that the inequality $|t| \leq t_{2N-4, \alpha}$ is true is $1 - \alpha$. The increase in width of these intervals over those obtained using a normal distribution decreases rapidly with increasing N . For

example, if $N = 3$ then the 96% and 68% confidence intervals calculated from Student's t are approximately 120% and 35% larger than those from a normal distribution, while if $N = 5$ they are only 25% and 10% larger respectively.

In the general case when all field measurements are noisy, the distribution of $2N\overline{|N_{mi}|^2}/E[|N_i|^2]$ is no longer exactly chi-square, and in addition, the calculated quantities, $\overline{|R_i|^2}$ and $|V_m|^2$, contain errors. The confidence intervals will be modified significantly when the percentage errors in these calculated quantities are large, but the exact effect cannot be analyzed using elementary distributions. In addition, when N is small, the $O(N^{-1})$ errors which are neglected in deriving equation (C-15) (Stodt, 1982b) can become significant.

DISCUSSION AND CONCLUSIONS

The expected values, $E[r_1^m r_2^n]$, are the moments of the joint distribution, $p(r_1, r_2)$, of the magnitudes of two complex random variables, Z_1 and Z_2 . When the distribution of Z_1 and Z_2 is given by equation (6), it is interesting to note that the moments are finite only if $m, n > -2$. This result is deduced easily by examining the behavior of the integrand of

$$E[r_1^m r_2^n] = \int_0^\infty \int_0^\infty r_1^m r_2^n p(r_1, r_2) dr_1 dr_2$$

as r_1 and r_2 approach zero, where $p(r_1, r_2)$ is given by equation (B-5). Turning attention to the ratio, $G_m = W_m/V_m$, we see that when V_m is noisy, $E[G_m]$ exists, but $E[|G_m|^2]$ does not. Thus, $\text{Var } G_m$ and $\text{Var } |G_m|^2$ do not exist either. In contrast, the moments

$$E[(\log(r_2/r_1))^2] = 2^m \int_0^\infty \int_0^\infty (\log r_2 - \log r_1)^m p(r_1, r_2) dr_1 dr_2$$

exist for all $m > 0$ in view of the identity, $\lim_{r \rightarrow 0} r^x \log r = 0$; $x > 0$, and the moments of the phase, $E[\phi^m]$, exist for all $m > 0$ since the distribution, $q(\phi)$, is bounded and defined nonzero over a finite range.

Physically, the divergence of the expected value $E[|G_m|^2]$ when the denominator V_m is noisy indicates a slow rate of falloff in the tail of the distribution of $|G_m|^2$. In spite of this behavior, we note that the variance expressions given by equations (C-15), (C-17),

(C-19) and (C-23) are useful estimates of the dispersions of the respective distributions, provided the errors are small enough that the normal approximations are valid in the region where the probability of obtaining an estimate is significant.

The interval of definition of the distribution of the phase, $q(\phi)$, deserves further attention. Note from equation (13) that $q(\phi)$ is symmetric about the point $\phi = \gamma$ when $q(\phi)$ is defined nonzero in the interval $\gamma - \pi \leq \phi < \gamma + \pi$, and consequently we have $E[\phi] = \gamma$. In practice however, γ is unknown and ϕ typically is evaluated in the interval $-\pi \leq \phi < \pi$. Then $q(\phi)$ is defined nonzero in this same interval and is no longer symmetric about the point $\phi = \gamma$ unless γ is zero. Individual phase calculations from real and imaginary components will at most be shifted by 2π to bring them from the interval $\gamma - \pi \leq \phi < \gamma + \pi$ into the interval $-\pi \leq \phi < \pi$, and hence can be treated as unbiased. However, an average of individual phase determinations will be biased when γ is nonzero, since then we have $E[\phi] \neq \gamma$. Miller (1975, p. 94) calculates $E[\phi']$, $0 \leq \phi' < 2\pi$. Making the change of variable, $\phi = \phi' - \pi$, then there results

$$E[\phi] = 2 \sum_{n=1}^{\infty} \frac{\Gamma(1 + \frac{n}{2})}{n\Gamma(1 + n)} \left(\frac{|C|}{\sqrt{R}}\right)^n {}_1F_1\left(\frac{n}{2}; 1 + n; -\frac{|C|^2}{R}\right) \sin n\gamma, \quad (24)$$

where $\Gamma(\cdot)$ and ${}_1F_1(\cdot)$ are the gamma and confluent hypergeometric functions respectively. The bias from averaging phases is such that the inequality $0 \leq E[\phi]/\gamma \leq 1$ holds when ϕ is defined in the interval $-\pi \leq \phi < \pi$, and is significant only when there is a significant probability that the individual phases can lie in either the second or third quadrants of the complex plane. This conclusion is not obvious from equation (24), but becomes clear upon examination of Figure 14.

The most extreme case occurs when γ is $-\pi$. Then $q(\phi)$ is bimodal and symmetric; equation (24) verifies that $E[\phi]$ is zero, so the bias error is π . This type of phase bias error is avoided by averaging real and imaginary parts, rather than magnitudes and phases, from different determinations of a complex random variable, Z .

When the errors in W_m and V_m are governed by a complex Gaussian distribution, the distribution of the phase, ϕ_{G_m} , of G_m is approximated well by a normal distribution for the range of errors considered in this paper. In addition, the distribution of $\log|G_m|^2$ is approximated by a normal distribution over a much larger range of errors than is the distribution of $|G_m|^2$. The distribution of $|G_m|^2$ is particularly sensitive to noise in V_m . These observations suggest that symmetric confidence intervals from normal approximations should be calculated for ϕ_{G_m} in linear space, and for $|G_m|^2$ in log space. Note that if the squared magnitudes of the tipper functions are plotted on a linear scale, then symmetric confidence intervals should be computed for $|T_{zx}|^2$ and $|T_{zy}|^2$ in log space. These intervals are then transformed to the linear scale, where they become asymmetric as the errors in T_{zx} and T_{zy} become large.

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LIST OF FIGURES

1. Transformed region $R(\phi_1, \phi_2) \rightarrow R(\theta, \phi)$, where $\theta = \phi_1$,
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 $b(\phi) = \gamma_2 + \pi - \phi$ when $\gamma_2 - \gamma_1 \leq \phi < \gamma_2 - \gamma_1 + 2\pi$, and $a(\phi) = \gamma_2 - \pi - \phi$, $b = \gamma_1 + \pi$ when $\gamma_2 - \gamma_1 - 2\pi < \phi \leq \gamma_2 - \gamma_1$.

2. Probability distribution, $p^*(w)$, of the squared magnitude, $w = |Z|^2$ (solid curve) when the distribution of Z is given by equation (6), with parameters, $\sqrt{R}/|C|=0.1$ and 0.25 , where R is the variance and C is the expected value of Z . The dashed curve is an approximating normal distribution, $n(w)$, with mean and variance given by equations (C-16) and (C-17). When $|C| \neq 1$, the plots represent $p^{**}(\hat{w}) = |C|^2 p^*(\hat{w}|C|^2)$ and $n^*(\hat{w}) = |C|^2 n(\hat{w}|C|^2)$ where the abscissa is the scaled variable, $\hat{w} = w/|C|^2$. The abscissa is incremented in units of one standard deviation of $n^*(\hat{w})$, measured with respect to the expected value, $E[\hat{w}]$, represented by the negative tick mark.

3. Same as Figure 4 except that $\sqrt{R}/|C| = 0.5, 0.75$.

4. Probability distribution, $p^*(v)$, of the logarithm of the squared magnitude, $v = \log |Z|^2$ (solid curve) when the distribution of Z is given by equation (6), with parameters $\sqrt{R}/|C| = 0.1$ and 0.25 , where R is the variance and C is the expected value of Z . The dashed curve is an approximating normal distribution, $n(v)$, with mean and variance given by equations (C-18) and the first term of (C-19). When $|C| \neq 1$, the plots represent $p^{**}(v) = |C|^2 p^*(\hat{v} + v')$ and $n^*(\hat{v}) = |C|^2 n(\hat{v} + v')$ where the abscissa is the shifted variable, $\hat{v} = v - v'$, and $v' = \log |C|^2$. The abscissa is incremented in units of one standard deviation of $n^*(\hat{v})$, measured with respect to the expected value, $E[\hat{v}]$, which is represented by the negative tick mark.
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6. Probability distribution, $q(\phi)$, of the phase, $\phi = \arg(Z)$ (solid curve) when the distribution of Z is given by equation (6) with parameters $\sqrt{R}/|C| = 0.1$ and 0.25 , where R is the variance and C is the expected value of Z . The dashed curve is an approximating normal distribution, $n(\phi)$, with mean value γ , where $C = |C|e^{j\gamma}$, and variance given by $\text{Var } \phi = (\text{Var } Z)/2|C|^2$. The plots represent $q(\phi)$ and $n(\phi)$ for any value of $|C|$. The abscissa is incremented in units of one standard deviation of $n(\phi)$, measured with respect to the expected value, $E[\phi] = \gamma$, which is chosen as zero for plotting purposes.
7. Same as Figure 6 except that $\sqrt{R}/|C| = 0.5, 0.75$.

8. Probability distribution, $h^*(w)$, of the squared magnitude, $w = |Z|^2 = |Z_2|^2/|Z_1|^2$ (solid curve) when the joint distribution of Z_1 and Z_2 is given by equation (6) with parameters, $\text{Cov}(Z_1, Z_2^*) = \sqrt{R_2}/|C_1| = 0$, and $\sqrt{R_1}/|C_1| = 0.1, 0.25$, where R_i is the variance and C_i is the expected value of Z_i . The dashed curve is an approximating normal distribution, $n(w)$, with variance given by equation (C-20) and mean value given by the expected value, $E[w] = |Z_b|^2 + \text{Var } Z$, where $Z_b = [1 - \exp(-|C_1|^2/R_1)]$ and $\text{Var } Z$ is given by equation (C-14). When $|C| = |C_2|/|C_1| \neq 1$, the plots represent $h^{**}(\hat{w}) = |C|^2 h^*(\hat{w}|C|^2)$ and $n^*(\hat{w}) = |C|^2 n(\hat{w}|C|)$, where the abscissa is the scaled variable, $\hat{w} = w/|C|^2$. The abscissa is incremented in units of one standard deviation of $n^*(\hat{w})$, measured with respect to the expected value, $E[\hat{w}]$, which is represented by the negative tick mark.
9. Same as Figure 8, except that $\sqrt{R_1}/|C_1| = 0.5, 0.75$.
10. Same as Figure 8, except that $\sqrt{R_1}/|C_1| = \sqrt{R_2}/|C_2| = 0.1, 0.25$.
11. Same as Figure 8, except that $\sqrt{R_1}/|C_1| = \sqrt{R_2}/|C_2| = 0.5, 0.75$.
12. Probability distribution, $h^*(v)$, of the logarithm of the squared magnitude, $v = \log |Z|^2 = \log |Z_2/Z_1|^2$ when the joint distribution of Z_1 and Z_2 is given by equation (6), with parameters, $\text{Cov}(Z_1, Z_2^*) = 0$ and $\sqrt{R_2}/|C_2| = \sqrt{R_1}/|C_1| = 0.1, 0.25$, where R_i is the variance and C_i is the expected value of Z_i . The dashed curve is an approximating normal distribution, $n(v)$,

with mean and variance given by equations (C-21) and (C-22).

When $|C| = |C_2|/|C_1| \neq 1$, the plots represent $h^{**}(\hat{v}) = |C|^2 h^*(\hat{v} + v')$ and $n^*(\hat{v}) = |C|^2 n(\hat{v} + v')$, where the abscissa is the shifted variable, $\hat{v} = v - v'$, and $v' = \log|C|^2$. The abscissa is incremented in units of one standard deviation of $n^*(\hat{v})$ measured with respect to the expected value, $E[\hat{v}]$, which is represented by the negative tick mark.

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14. Sketches of the distribution of the phase, $q(\phi)$; $-\pi \leq \phi < \pi$, with corresponding plots in the complex plane of the circular region of radius s centered at the true value, $|C|e^{j\gamma}$, where the probability of obtaining a measured value is significant. Figure (a) illustrates a situation where $E[\phi] = \gamma$, while Figures (b) and (c) illustrate two different situations where $E[\phi] \neq \gamma$ so that an arithmetic average of independent phase determinations will be biased significantly. Note from the Figures that the value of the integral in the expression for $E[\phi]$ is bounded between $-\gamma$ and 0 so that $0 \leq E[\phi]/\gamma \leq 1$.
15. Comparison of two different approximations for the variances of $|G_m|^2$ and $\log |G_m|^2$ when V_m is noisy, where $|G_m| = |W_m/V_m| = r_2/r_1$, $\text{Cov}(W_m, V_m^*) = 0$, and $(\text{Var } W_m)/|W_b|^2 = (\text{Var } V_m)/|V_b|^2$. The ratios, $\text{Var } |G_m|^2 / \text{Var } (r_2/r_1)^2$ and $\text{Var } \log |G_m|^2 / \text{Var } \log (r_2/r_1)^2$, are plotted v.s. $(\text{Var } V_m)/|V_b|^2$, where $V_b = E[V_m]$. $\text{Var } (r_2/r_1)^2$ and $\text{Var } \log (r_2/r_1)^2$ are calculated from equations

(C-20) and (C-22), and are the values used to calculate the normal approximations, $n(w)$ and $n(v)$, in Figures 10 through 13. $\text{Var } |G_m|^2$ and $\text{Var } \log |G_m|^2$ are calculated from equation (C-17) and the first term of equation (C-19) and are the approximations most easily computed from MT data.

Figure 1

Figure 2

Figure 3

Figure 4

Figure 5

Figure 6

Figure 7

Figure 8

Figure 9

Figure 10

Figure 11

Figure 12

Figure 13

Figure 14

$$E[\phi] = \gamma + \int_{-\pi}^{\pi} (\phi - \gamma) q(\phi) d\phi$$

Figure 15

APPENDIX A

EXPRESSIONS FOR $\text{Cov}(W_m, W_m)$, $\text{Cov}(V_m, V_m)$, AND $\text{Cov}(W_m, V_m)$

Expressions for $\text{Cov}(W_m, W_m)$, $\text{Cov}(V_m, V_m)$, and $\text{Cov}(W_m, V_m)$ can be derived once W_m and V_m are expressed in suitable form. We assume that the field measurements are sums of signals and noises, $I_x = I_{sx} + i_x$, $I_y = I_{sy} + i_y$, $O_i = O_{si} + o_i$, $A = A_s + a$, and $B = B_s + b$, where the signals and noises are independent. The signal components are treated as parameters by defining expectation with respect to the distribution of the noises as in Stodt (1982b), so that the expected values of the measured fields are their signal components.

It is straightforward to show that W_m and V_m can be written in the forms

$$W_{mij} = N^{-1} \sum_{k=1}^N [O_{sik} R_{sjk}^* + o_{ik} R_{sjk}^* + i_{lk} R_{ok}^*], \quad (\text{A-1})$$

where i_{lk} is i_{yk} or i_{xk} when j is x or y respectively, and

$$V_m = N^{-1} \sum_{k=1}^N [I_{sxx} R_{sxx}^* + i_{xk} R_{sxx}^* + i_{yk} R_{yyk}^*], \quad (\text{A-2})$$

where

$$R_{sxx} = A_k \overline{I_{sy}^* B} - B_k \overline{I_{sy}^* A},$$

$$R_{syk} = B_k \overline{I_{sx}^* A} - A_k \overline{I_{sx}^* B},$$

$$R_{yyk} = B_k \overline{I_x^* A} - A_k \overline{I_x^* B},$$

and

$$R_{ok} = A_k \overline{O_i^* B} - B_k \overline{O_i^* A} \quad .$$

Expressions for the covariances can be derived by computing the covariances of the sums given by equations (A-1) and (A-2). For example, making the identifications,

$$W_m = W_1 + W_2 + W_3$$

and

$$V_m = V_1 + V_2 + V_3 \quad ,$$

we can write the expression

$$\begin{aligned} \text{Cov}(V_m, V_m) &= \text{Cov}(V_1, V_1) + \text{Cov}(V_2, V_2) + \text{Cov}(V_3, V_3) \\ &\quad + 2\text{Cov}(V_1, V_2) + 2\text{Cov}(V_1, V_3) + 2\text{Cov}(V_2, V_3), \quad (\text{A-3}) \end{aligned}$$

and similar expressions can be written for $\text{Cov}(W_m, W_m)$ and $\text{Cov}(W_m, V_m)$. The individual terms in equation (A-3) are all evaluated in similar fashion. For example, applying equation (7) we obtain

$$\text{Cov}(V_2, V_3) = N^{-2} \sum_{k=1}^N \sum_{l=1}^N E[i_{xk} R_{s x k}^* i_{yl} R_{y l}^*] - E[i_{xk} R_{s x k}^*] E[i_{yl} R_{y l}^*].$$

If the distribution of the noises in the field measurements is given by equation (6), then the fourth order moment is given by (Stodt, 1982b)

$$\begin{aligned} E[i_{xk} R_{s x k}^* i_{yl} R_{y l}^*] &= E[i_{xk} R_{s x k}^*] E[E i_{yl} R_{y l}^*] \\ &\quad + E[i_{xk} R_{y l}^*] E[i_{yl} R_{s x k}^*] + O(N^{-1}) \quad , \end{aligned}$$

where $O(N^{-1})$ error is introduced because the crosspowers in $R_{s x k}$ and $R_{y l}$ are treated as constants in the expansion. If we assume further that individual harmonics of the discrete Fourier transforms of the

field measurements are independent when $k \neq 1$, then we can write

$$\begin{aligned} \text{Cov}(V_2, V_3) &\approx N^{-2} \sum_{k=1}^N \sum_{l=1}^N E[i_{xk} R_{yl}^*] E[i_{yl} R_{sxx}^*] \\ &\approx N^{-2} \sum_{k=1}^N E[i_{xk} R_{yk}^*] E[i_{yk} R_{sxx}^*]. \end{aligned}$$

Expanding R_{sxx} and R_{yk} , we find

$$\begin{aligned} \text{Cov}(V_2, V_3) &\approx N^{-2} \sum_{k=1}^N \{ E[i_{xk} b_k^*] \overline{I_x A^*} - E[i_{xk} a_k^*] \overline{I_x B^*} \} \\ &\quad \cdot \{ E[i_{yk} a_k^*] \overline{I_{sy} B^*} - E[i_{yk} b_k^*] \overline{I_{sy} A^*} \}. \end{aligned} \quad (\text{A-4})$$

Notice that each term of equation (A-4) contains the expected value of the product of base and conjugate reference field noise terms. In fact, with the exception of $\text{Cov}(V_1, V_1)$, $\text{Cov}(W_1, W_1)$, and $\text{Cov}(W_1, V_1)$, all the individual terms of $\text{Cov}(V_m, V_m)$, and also of $\text{Cov}(W_m, W_m)$ and $\text{Cov}(W_m, V_m)$, share this property. Therefore, they are all approximately zero when W_m and V_m are computed from remote reference data, assuming base and reference field noises are uncorrelated. However, if W_m and V_m are computed from conventional data, these covariances are generally nonzero. The terms, $\text{Cov}(V_1, V_1)$, $\text{Cov}(W_1, W_1)$ and $\text{Cov}(W_1, V_1)$ are always approximately zero because the expressions like equation (A-4) which are derived for these quantities contain the terms, $E[a^* a^*]$, $E[b^* b^*]$, or $E[a^* b^*]$, which are zero in view of equation (9) in the text.

APPENDIX B

DERIVATIONS OF $p(r_1, r_2)$ AND $p(r_2/r_1)$

This Appendix is devoted to the derivations of the joint distribution $p(r_1, r_2)$ and to the distribution of the ratio, $r = r_2/r_1$.

Distribution of (r_1, r_2)

After expanding the quadratic form, $[\underline{Z}^* - \underline{C}^*]^T \underline{R}^{-1} [\underline{Z} - \underline{C}]$, in the exponent of $g(Z_1, Z_2)$ and substituting $Z_i = r_i e^{j\phi_i}$ and $C_i = |C_i| e^{j\gamma_i}$, equation (11) can be written as

$$q^*(r_1, r_2, \phi_1, \phi_2) = \frac{r_1 r_2 \det \underline{S}}{\pi^2} \exp[2(Mr_2 + Nr_1 - Q)],$$

where

$$Q = S_{11}[r_1^2 + |C_1|^2] + S_{22}[r_2^2 + |C_2|^2] \\ + 2|S_{12}||C_1||C_2| \cos(\gamma_1 - \gamma_2 - \psi_{12}),$$

$$M = S_{22}|C_2| \cos(\phi_2 - \gamma_2) + |S_{12}||C_1| \cos(\phi_2 - \gamma_1 + \psi_2),$$

$$N = S_{11}|C_1| \cos(\phi_1 - \gamma_1) + |S_{12}||C_2| \cos(\phi_1 - \gamma_2 - \psi_{12})$$

$$-r_2|S_{12}| \cos(\phi_1 - \phi_2 - \psi_{12})$$

and $\underline{S} = \underline{R}^{-1}$ is the inverse of the covariance matrix of (Z_1, Z_2) , with off diagonal elements $S_{12} = |S_{12}| e^{j\psi_{12}} = S_{21}^*$. Thus $p(r_1, r_2)$ is the marginal distribution

$$p(r_1, r_2) = \frac{r_1 r_2 e^{-Q_{\text{det}} S}}{\pi^2} \int_0^{2\pi} e^{2Mr_2} \int_0^{2\pi} e^{2Nr_1} d\phi_1 d\phi_2.$$

After rearrangement, the integral over ϕ_1 can be evaluated as (Gradshteyn and Ryzhik, 1980, p. 488, 3.937-2)

$$\int_0^{2\pi} \exp(A \cos \phi_1 + B \sin \phi_1) d\phi_1 = 2\pi I_0[(A^2 + B^2)^{1/2}], \quad (\text{B-1})$$

where

$$A = 2r_1[S_{11}|C_1| \cos \gamma_1 + |S_{12}||C_2| \cos(\gamma_2 + \psi_{12}) - r_2|S_{12}| \cos(\phi_2 + \psi_{12})]$$

and B is obtained from A by replacing $\cos(\cdot)$ with $\sin(\cdot)$. The rearrangement of the exponent $2Nr_1$ is accomplished by expanding the cosine terms in N using the relation

$$\cos(a - b) = \cos a \cos b + \sin a \sin b, \quad (\text{B-2})$$

where $a = \phi_1$, and regrouping terms. The integral over ϕ_2 remains to be evaluated. Towards this end, we rewrite $A^2 + B^2$ as

$$A^2 + B^2 = |V|^2 + |W|^2 - 2\text{Re}(VW^*),$$

where

$$V = 2Z_1 Z_2 S_{12},$$

and

$$W = 2Z_1[C_1 S_{11} + C_2 S_{12}] \equiv 2Z_1 U_1.$$

Noting that $2\text{Re}(VW^*) = |V||W| \cos(\phi_2 + \psi_{12} - \alpha_1)$, where

$U_1 \equiv |U_1| e^{j\alpha_1}$, then from the Neumann addition theorem (Watson, 1966, p. 365) we find

$$I_0[(|V|^2 + |W|^2 - 2|V||W| \cos(\phi_2 + \psi_{12} - \alpha_1))^{1/2}]$$

$$= \sum_{n=0}^{\infty} (-)^n \epsilon_n I_n(|V|) I_n(|W|) \cos n(\phi_2 + \psi_{12} + \alpha_1). \quad (B-3)$$

Collecting results from equations (B-1) and (B-3) gives

$$p(r_1, r_2) = \frac{2r_1 r_2 e^{-Q \det S}}{\pi} \sum_{n=0}^{\infty} (-)^n \epsilon_n I_n(|V|) I_n(|W|) \cdot \int_0^{2\pi} e^{2Mr_2} \cos n(\phi_2 + \psi_{12} - \alpha_1) d\phi_2, \quad (B-4)$$

where the order of integration and summation has been interchanged.

After setting $U_2 = C_2 S_{22} + C_1 S_{12}^* \equiv |U_2| e^{j\alpha_2}$, the exponent $2Mr_2$ can be rewritten as $2\text{Re}(Z_2 U_2^*) = 2r_2 |U_2| \cos(\phi_2 - \alpha_2)$. Now expand $\cos n(\phi_2 + \psi_{12} - \alpha_1)$ using equation (B-2), after making the identifications $a = n(\phi_2 - \alpha_2)$ and $b = n(\psi_{12} + \alpha_2 - \alpha_1)$. The integral in equation (B-4) is thus separated into two parts. Noting that both integrands are periodic with period 2π , the integral involving $\sin n(\phi_2 - \alpha_2)$ is zero, while in view of the identity

$$\int_0^\pi e^{Z \cos \theta} \cos n\theta d\theta = \pi I_n(Z),$$

the integral involving $\cos n(\phi_2 - \alpha_2)$ is

$$\cos b \int_0^{2\pi} e^{Z \cos(\phi_2 - \alpha_2)} \cos n(\phi_2 - \alpha_2) d\phi_2 = 2\pi I_n(Z) \cos b,$$

where $Z = 2r_2 |U_2|$. Substituting this result for the integral in equation (B-4) gives equation (18). When the real parts of Z_1 and Z_2 are independent of their imaginary parts, then ψ_{12} is zero (Stodt, 1982b) and in this special case, equation (18) can be shown to be equivalent to a result derived by Miller, et al (1958).

Distribution of $r = r_2/r_1$

After making the transformation $s = r_1$, $r = r_2/r_1$, where the determinant of the Jacobian is $J = s$, the distribution of r is given

formally by equation (19). The integral can be evaluated by expanding certain of the modified Bessel functions in $p(r_1, r_2)$ in the power series

$$I_n(2Z) = \sum_{k=0}^{\infty} \frac{Z^{2k+n}}{k!(n+k)!}$$

and integrating term by term. Two approaches are considered. First, expand $I_n(2r_1|U_1|)$ and $I_n(2r_2|U_2|)$ in series. Then equation (19) becomes

$$\begin{aligned} h(r) = & 4re^{-Q} \det S \sum_{n=0}^{\infty} (-)^n \epsilon_n \cos n(\psi_{12} + \alpha_2 - \alpha_1) \\ & \cdot \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{|U_1|^{2k+n} (r|U_2|)^{2l+n}}{k!l!(n+k)!(n+l)!} \\ & \cdot \int_0^{\infty} s^{2(k+1+n)} e^{-(S_{11} + S_{22}r^2)s^2} I_n(2r|S_{12}|s^2) ds. \end{aligned}$$

Miller (1964, p. 115) gives the identity

$$\begin{aligned} \int_0^{\infty} s^a e^{-bs^2} I_n(cs^2) ds &= \frac{c^n \Gamma[(a+2n+1)/2]}{2^{n+1} b^{(a+2n+1)/2} \Gamma(n+1)} \\ &\cdot {}_2F_1[a+2n+1)/4, (a+2n+3)/4; n+1; \frac{c^2}{b}], \end{aligned}$$

which is valid provided $b > |c|$ and $\text{Re}(a+2n) > -1$. The second condition is obviously satisfied, while the first requires that $S_{22}r^2 - 2|S_{12}|r + S_{11} > 0$. The polynomial in r has complex roots and $r > 0$, so the first condition is also satisfied. Carrying out the integration and simplifying terms gives the result

$$h(r) = 2re^{-Q} \det S \sum_{n=0}^{\infty} \frac{(-)^n \epsilon_n r^n |S_{12}|^n \cos n(\psi_{12} + \alpha_2 - \alpha_1)}{n!}$$

$$\begin{aligned}
& \cdot \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(2n+k+l+1)!}{k! l! (n+k)! (n+l)!} \cdot \frac{|U_1|^{2k+n} (r|U_2|)^{2l+n}}{T^{2n+k+l+2}} \\
& \cdot {}_2F_1 \left(\frac{2n+k+l+2}{2}, \frac{2n+k+l+3}{2}; n+1; \frac{4r^2 |S_{12}|^2}{T^2} \right), \quad (B-5)
\end{aligned}$$

where $T = [S_{11} + r^2 S_{22}]$ and ${}_2F_1(\cdot)$ is the Gauss hypergeometric function. Equation (B-5) is valid in general, but is difficult to compute.

An alternative representation for $h(r)$ can be obtained by expanding $I_n(2r_1 r_2 |S_{12}|)$ in $p(r_1, r_2)$ in its power series and integrating term by term. Following this strategy, equation (19) becomes

$$\begin{aligned}
h(r) &= 4re^{-Q \det S} \sum_{n=0}^{\infty} (-)^n \epsilon_n \cos n(\psi_{12} + \alpha_2 - \alpha_1) \sum_{k=0}^{\infty} \frac{(r|S_{12}|)^{2k+n}}{k! (n+k)!} \\
&\cdot \int_0^{\infty} s^{2(2k+n+1)+1} e^{-(S_{11} + S_{22}r^2)s^2} I_n(2|U_1|s) I_n(2|U_2|rs) ds.
\end{aligned} \quad (B-6)$$

Miller (1964, p. 53) gives the identity

$$\int_0^{\infty} s e^{-cs^2} I_n(as) I_n(bs) ds = \frac{1}{2c} e^{(a^2 + b^2)/4c} I_n\left(\frac{ab}{2c}\right),$$

where $\text{Re } c > 0$, $\text{Re } n > -1$, and a and b are arbitrary complex numbers. Differentiating m times with respect to c , then we obtain

$$\int_0^{\infty} s^{2m+1} e^{-cs^2} I_n(as) I_n(bs) ds = (-)^m \frac{d^m}{dc^m} \left[\frac{1}{2c} e^{(a^2 + b^2)/4c} I_n\left(\frac{ab}{2c}\right) \right].$$

This expression has the same form as the integral in equation (B-6).

Carrying out the integration gives

$$h(r) = 2re^{-Q} \det \underline{S} \sum_{n=0}^{\infty} \epsilon_n \cos n(\psi_{12} + \alpha_2 - \alpha_1) \\ \cdot \sum_{k=0}^{\infty} \frac{-(r/|S_{12}|)^{2k+n}}{k! (n+k)!} \cdot \frac{d^{2k+n+1}}{dT^{2k+n+1}} [F(T)],$$

(B-7)

where

$$F(T) = \frac{1}{2T} \exp [(|U_1|^2 + r^2|U_2|^2)/T] I_n \left(\frac{2r|U_1||U_2|}{T} \right).$$

The derivative in equation (B-7) presents a serious obstacle to evaluation in the general case. However, in the special case when $\text{Cov}(Z_1, Z_2^*)$ is zero, equation (B-7) reduces to the relatively simple expression

$$h(r) = -2re^{-Q} S_{11}S_{22} \frac{dF(T)}{dT},$$

where $\det \underline{S} = S_{11}S_{22}$, $S_{ii} = R_{ii}^{-1}$, and the expressions for Q and U_i , $i = 1$ or 2 simplify, since S_{12} is zero. Carrying out the differentiation gives equation (20) in the text, which is equivalent to a result derived by Miller (1964, p. 53).

APPENDIX C

ERROR PROPAGATION AND NORMAL APPROXIMATIONS

In this Appendix we explore the behavior of nonlinear error propagation and calculate expected values and variances for normal approximations to the distributions derived in the text.

A nonlinear function $f(Z)$ of a complex random variable Z can be written as

$$f(Z) = f(Z_b) + F(Z_b, z)z, \quad (C-1)$$

where $Z = Z_b + z$ and $Z_b = E[Z]$. Then the expected value and variance of $f(Z)$ are given by

$$E[f(Z)] = f(Z_b) + \text{Cov}(F, z), \quad (C-2)$$

and
$$\text{Var } f(Z) = \text{Var } Fz, \quad (C-3)$$

when they exist, where the functional dependence of F is suppressed for clarity. The variance of a product XY can be derived by successive applications of the defining equations (7) and (8). The general result is

$$\begin{aligned} \text{Var } XY = & \text{Var } X \text{Var } Y + |X_b|^2 \text{Var } Y + |Y_b|^2 \text{Var } X \\ & + \text{Cov}(|X|^2, |Y|^2) - |\text{Cov}(X, Y)|^2 - 2\text{Re}[X_b^* Y_b^* \text{Cov}(X, Y)], \end{aligned} \quad (C-4)$$

where $X_b = E[X]$ and $Y_b = E[Y]$. Applying this result to equation (C-3)

and noting that $E[z]$ is zero by definition, then

$$\begin{aligned} \text{Var } f(Z) = & |F_b|^2 \text{Var } z + [1 - R_{F,z}^2] \text{Var } F \text{Var } z \\ & + R_{|F|^2, |z|^2}^2 [\text{Var } |F|^2 \text{Var } |z|^2]^{1/2}, \end{aligned} \quad (\text{C-5})$$

where $R_{A,B}^2 = |\text{Cov}(A,B)|^2 / \text{Var } A \text{Var } B$ is the squared correlation coefficient satisfying the inequality $0 \leq R_{A,B}^2 \leq 1$, and $F_b = E[F(Z_b, z)]$. The last two terms on the right are zero when $f(Z)$ is linear. When $f(Z)$ is nonlinear, their contribution relative to that of the first term increases as the ratio $\text{Var } F / |F_b|^2$ increases.

In order to proceed further, $F(Z_b, z)$ must be specified. If $f(Z)$ is an analytic function over the entire range of probable values of Z , then $F(Z_b, z)$ can be approximated uniformly by a polynomial in z . One approach is to use the Taylor expansion of $f(Z)$, from which

$$F(Z_b, z) = \sum_{n=1}^{\infty} \frac{{}^n f(Z_b)}{n!} z^{n-1},$$

where ${}^n f(Z_b)$ is the n th derivative of $f(Z)$ evaluated at Z_b . If the range of z is sufficiently small,

$$F(Z_b, z) \approx {}^1 f(Z_b) + \frac{{}^2 f(Z_b)}{2} z, \quad (\text{C-6})$$

and hence

$$\frac{\text{Var } F}{|F_b|^2} \approx \frac{|{}^2 f(Z_b)|^2}{4 |{}^1 f(Z_b)|^2} \text{Var } z.$$

When this ratio is $\ll 1$, the first term of equation (C-5) approximates $\text{Var } f(Z)$. Additionally, from equations (C-1) and (C-6),

$$E[f(Z)] \approx f(Z_b) + \frac{{}^2 f(Z_b)}{2} \text{Cov}(z, z), \quad (\text{C-7})$$

and the second term on the right approximates the bias error

introduced because of the nonlinearity of $f(Z)$.

The analysis can be extended to functions of more than one complex variable. Defining $f(Z_i) = f(Z_1, Z_2, \dots, Z_k)$, then the k -dimensional Taylor expansion of $f(Z_i)$ about the points $E[Z_i] = Z_{bi}$ can be written as

$$f(Z_i) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(Z_i) |_{Z_i = Z_{bi}}, \quad (C-8)$$

where D^n is the operator $D^n = \left(\sum_{i=1}^k z_i \frac{\partial}{\partial z_i} \right)^n$.

In equation (C-8), D^n operates on $f(Z_i)$ and the result is evaluated at the points $Z_i = Z_{bi}$ to give the expansion in terms of the z_i .

Equation (C-8) can be rewritten in a form similar to equation (C-1) as

$$f(Z_i) = f(Z_{bi}) + \underline{F} \circ \underline{z},$$

where $\underline{z}$ is the vector $[z_1, \dots, z_k]$. The vector $\underline{F}$ is given by

$$\underline{F} = \sum_{n=1}^{\infty} \frac{1}{n!} \nabla [D^{n-1} f(Z_i)] |_{Z_i = Z_{bi}},$$

where it is understood that the gradient is with respect to the Z_i for fixed z_i . Thus, the expected value and variance of $f(Z_i)$ are

$$E[f(Z_i)] = f(Z_{bi}) + E[\underline{F} \circ \underline{z}],$$

and

$$\text{Var } f(Z_i) = \text{Var } \underline{F} \circ \underline{z},$$

where, e.g.,

$$E[\underline{A} \circ \underline{B}] = \sum_i E[A_i B_i],$$

and

$$\text{Var } \underline{A} \circ \underline{B} = \sum_i \text{Var } A_i B_i + \sum_{i \neq j} \text{Cov}(A_i B_i, A_j B_j).$$

When the range of $\underline{z}$ is sufficiently small, then

$$\underline{F} \approx \underline{\nabla}(f + \frac{1}{2} Df) |_{Z_i = Z_{bi}} ,$$

whence $E[\underline{F}] \equiv \underline{F}_b \approx \underline{\nabla}f ,$

and $\text{Var } \underline{F} \circ \underline{U} \approx \frac{1}{4} \text{Var } \underline{\nabla}(Df) \circ \underline{U} ,$

where $\underline{U}$ is a vector whose components are unity. Thus, extending the analysis for functions of a single variable, if the ratio $(\text{Var } \underline{F} \circ \underline{U})/|\underline{F}_b|^2 \ll 1$, then

$$E[f(Z_i)] \approx f(Z_{bi}) + \frac{1}{2} \text{Cov}(\underline{\nabla}(Df) \circ \underline{z}) \quad (C-9)$$

and $\text{Var } f(Z_i) \approx \text{Var } \underline{\nabla}f \circ \underline{z} . \quad (C-10)$

With regard to MT, the functions of interest are real and can be written in the form $f(Z_i, Z_i^*)$. The conjugate values are treated as separate variables when applying the above analysis. Gamble et al (1979b) obtain approximate variance expressions from equation (C-10) for various functions of the G_{mij} .

The validity of the analysis just presented depends on the behavior of $f(Z)$ in the vicinity of Z_b . Suppose $f(Z)$ is only analytic in the disk centered at Z_b with radius r_c . When the probability that $|z| > r_c$ is zero, then the expectation and summation operators can be interchanged in view of the uniform convergence of the Taylor series, and the existence of $E[f(Z)]$ and $\text{Var } f(Z)$ depends only on the existence of the moments $E[z^n z^{*m}]$, where the maximum values of m and n are determined by the highest order nonzero derivative, ${}^n f(Z_b)$. This statement is invalid if $|z|$ can exceed r_c , and in particular it can happen that $\text{Var } f(Z)$ or $E[f(Z)]$ do not exist, even though the moments of z do. This behavior can occur because the Taylor expansion of $f(Z)$

is invalid outside the disk. In fact, if $|z|$ can exceed r_c , then $f(Z)$ cannot be approximated uniformly over the entire range of Z by any polynomial in z (Walsh, 1965 p. 47-48). However, if the probability is small that $|z| > r_c$ so that a truncated distribution with zero probability that $|z| > r_c$ is a good approximation to the true distribution of Z , the analysis presented above is still meaningful. Then the calculated values, $E[f(Z)]$ and $\text{Var } f(Z)$, give good estimates of the location of $f(Z_b)$ and the dispersion of the distribution of $f(Z)$ when the dispersion of the distribution of Z is sufficiently small, even though the theoretical means and variances may not exist.

Functions $f(X)$ of a real random variable $X = X_b + x$ can be approximated uniformly over a finite interval by a polynomial in x if $f(X)$ is continuous, but not necessarily analytic over the entire interval (Walsh, 1965 p. 47-48). However, if the probability is significant that $|x| > r_c$, then the expected value and variance calculated from equations (C-2), (C-3), and (C-6) may not be meaningful, even if the first and second derivatives of $f(X)$ exist at the point $X = X_b$. When equations (C-1) and (C-6) provide a good approximation to $f(X)$, then from the distribution $p(x)$ of the errors in X , the distribution of $Y = f(X)$ is obtained approximately from equation (10) as

$$p^*(Y) = \frac{1}{|a|} p\left(\frac{Y - b}{a}\right), \quad (C-11)$$

where

$$a = f'(X_b)$$

and

$$b = f(X_b) + \frac{f''(X_b)}{2} \text{Var } x.$$

It often happens that $f(Z_i)$ can be expressed as a composite function, i.e., $f(Z_i) = g(h(Z_i))$ where $h(Z_i)$ can be real or complex.

In such cases, the approximate mean and variance of $f(Z_i)$ can be calculated in one step using equations (C-9) and (C-10).

Alternatively, the error propagation can be carried out in two stages by calculating the mean and variance of $h(Z_i)$ first and then using the results to calculate the mean and variance of $g(h)$. For the functions considered in this paper, the latter approach gives better approximations as the uncertainties in the Z_i increase. When more than one composite representation is possible, the most accurate results are obtained by utilizing the representation requiring the least approximation when carrying out the nonlinear error propagation.

The remainder of this section is devoted to a listing of approximate expected values and variances for the MT functions considered in this paper. These are the values used in the normal approximations to the distributions which were derived in the previous section.

$$G_m = W_m/V_m$$

The expected value and variance of a ratio can be obtained approximately from equations (C-9) and (C-10). However, the following derivations lead to results which are accurate over a larger range of errors.

The relation $E[AB] = E[A]E[B] + \text{Cov}(A,B)$ holds by definition, so that the expected value of the ratio, $G_m = W_m/V_m$, can be written as

$$E[W_m/V_m] = W_b E[1/V_m] + \text{Cov}(W_m, 1/V_m) . \quad (C-12)$$

Miller (1974, p. 93) has derived the result

$$E[1/V_m] = V_b^{-1} [1 - \exp(-|V_b|^2 / \text{Var } V_m)] ,$$

when the distribution of V_m is given by equation (6). Hence, the expected value of G_m is given approximately by

$$E[G_m] = G_b \approx \frac{W_b}{V_b} [1 - \exp(-|V_b|/\text{Var } V_m)], \quad (C-13)$$

where the last term in equation (C-12) is neglected.

The ratio G_m is analytic when the error in V_m satisfies the inequality $|v| < |V_b|$. When the range of v is suitably restricted, the variance of G_m can be derived as follows. First, rewrite the expression for the errors, $g = G_m - G_b$, in the alternate form, $V_m g = W_m - G_b V_m$. Applying equation (C-4), the variance of the left side is

$$\text{Var } V_m g = [|V_b|^2 + \text{Var } V_m] \text{Var } g + B,$$

where $B = \text{Cov}(|g|^2, |V_m|^2) - |\text{Cov}(V_m, g)|^2,$

while the variance of the right side is given by

$$\text{Var } V_m g = \text{Var } W_m + |G_b|^2 \text{Var } V_m - 2\text{Re}[G_b^* \text{Cov}(W_m, V_m^*)] \equiv A.$$

Equating these expressions and solving for $\text{Var } g$ gives

$$\text{Var } g = \frac{A - B}{|V_b|^2 + \text{Var } V_m}. \quad (C-14)$$

The term B is usually neglected as an approximation. In actual computations, the denominator is estimated from the measured value $|V_m|^2$, where $E[|V_m|^2] = |V_b|^2 + \text{Var } V_m$. This approximation introduces additional $O(N^{-1})$ error. It can be shown (Stodt 1982b) that the expression

$$\text{Var } G_{mij} = \frac{E[|N_i|^2] E[|R_j|^2]}{N E[|V_m|^2]}, \quad (C-15)$$

where

$$N_i = O_i - G_{ix} I_x - G_{iy} I_y ,$$

$$R_x = A \overline{I_y^*} B - B \overline{I_y^*} A ,$$

and

$$R_y = B \overline{I_x^*} A - A \overline{I_x^*} B ,$$

which was derived by Gamble et al (1979b), is equivalent to equation (C-14) with B neglected. Equation (C-15) is easier to compute than is equation (C-14).

$$\underline{|Z|^2 = ZZ^*}$$

The expected value of $|Z|^2$ is deduced from the definition of variance, i.e.

$$E[|Z|^2] = |Z_b|^2 + \text{Var } Z. \quad (\text{C-16})$$

The variance of the product ZZ^* is obtained in general from equation (C-4).

In the special case when the distribution of Z is given by equation (6), then equation (C-4) reduces to

$$\begin{aligned} \text{Var } |Z|^2 &= 2|Z_b|^2 \text{Var } Z + (\text{Var } Z)^2 \\ &= 2E[|Z|^2] \text{Var } Z - (\text{Var } Z)^2, \end{aligned} \quad (\text{C-17})$$

in view of equation (9).

$$\underline{\log |Z|^2}$$

Letting $r^2 = |Z|^2$, we rewrite $\log r^2$ as

$$\log r^2 = \log e \cdot [\ln E[r^2] + \ln (1 + \frac{\Delta r^2}{E[r^2]})] ,$$

where $\Delta r^2 = r^2 - E[r^2]$ and $\ln e = 1$. The function $\ln(1 + X)$ is

analytic over the range $|X| < 1$. When the probability is small that Δr^2 exceeds $E[r^2]$, the expected value and variance estimated from the first term of the Taylor expansion, $\ln(1 + X) \approx X$, are

$$E[\log r^2] \approx \log E[r^2], \quad (C-18)$$

and

$$\text{Var } \log r^2 \approx (\log e)^2 \frac{\text{Var } r^2}{E^2[r^2]}.$$

Substituting equation (44) for $\text{Var } r^2$, we find

$$\text{Var } \log r^2 \approx \frac{2(\log e)^2 \text{Var } Z}{E[r^2]} \left[1 - \frac{\text{Var } Z}{2E[r^2]} \right]. \quad (C-19)$$

The variance is estimated conservatively by neglecting the second term in the brackets.

$$|G_m|^2 = |W_m/V_m|^2, \text{ and } \log |G_m|^2$$

When the range of errors in V_m are suitably restricted, the expected value and variance of $|G_m|^2$ and $\log |G_m|^2$ are given approximately as functions of G_b and $\text{Var } G_m$ by substituting $Z = G_m$ and $Z_b = G_b$ in equations (C-16) and (C-17) and in equations (C-18) and (C-19) respectively. These are the approximations which are usually made in practice. However, as the noise in V_m increases, $\text{Var } |G_m|^2$, as given by equation (C-17), overestimates the dispersion of the distribution of $|G_m|^2$. A better estimate is obtained from the variance of the ratio $U = U_2/U_1$, where $U_2 = |W_m|^2$ and $U_1 = |V_m|^2$. From equation (C-14), we deduce the result

$$\text{Var } \frac{U_2}{U_1} \approx \frac{\text{Var } U_2 + E^2[U] \text{Var } U_1 - 2E[U] \text{Cov}(U_2, U_1)}{E^2[U_1] + \text{Var } U_1}, \quad (C-20)$$

where, it can be shown that

$$\text{Cov}(U_2, U_1) = |\text{Cov}(W_m, V_m^*)|^2 + 2\text{Re}[W_b^* V_b \text{Cov}(W_m, V_m^*)]$$

when the distribution of W_m and V_m is given by equation (6) (Miller, 1974, p. 86). It is best to approximate $E[U]$ with $E[U_2]/E[U_1]$ in equation (C-20).

It has also been found that the expected value and variance of $\log |G_m|^2$ are estimated more accurately from the difference, $\log r_2^2 - \log r_1^2$, i.e.,

$$E[\log(r_2/r_1)^2] = E[\log r_2^2] - E[\log r_1^2] \quad (\text{C-21})$$

and

$$\begin{aligned} \text{Var } \log(r_2/r_1)^2 &= \text{Var } \log r_2^2 + \text{Var } \log r_1^2 \\ &\quad - 2 \text{Cov}(\log r_2^2, \log r_1^2). \end{aligned} \quad (\text{C-22})$$

The covariance term in equation (C-22) is approximately

$$\text{Cov}(\log r_2^2, \log r_1^2) = \text{Cov}(r_2^2, r_1^2) = \text{Cov}(U_2, U_1).$$

The different approximations to the variances are compared in Figure 15, where the ratios, $\text{Var } |G_m|^2 / \text{Var } (r_2/r_1)^2$ and $\text{Var } \log |G_m|^2 / \text{Var } \log (r_2/r_1)^2$ are plotted as functions of $(\text{Var } V_m)/|V_b|^2$ for the case where W_m and V_m are uncorrelated and $(\text{Var } W_m)/|W_b|^2 = (\text{Var } V_m)/|V_b|^2$.

ϕ_{G_m}

The variance of the phase of G_m is determined approximately from the phase error, $\Delta\phi = \phi_{G_m} - \phi_{G_b}$, which can be expressed as

$$\Delta\phi = \tan^{-1}[\dot{\text{Im}}(G_m)/\dot{\text{Re}}(G_m)] ,$$

where

$$\begin{bmatrix} \dot{\text{Re}}(G_m) \\ \dot{\text{Im}}(G_m) \end{bmatrix} = \begin{bmatrix} \cos\phi_{G_b} & \sin\phi_{G_b} \\ -\sin\phi_{G_b} & \cos\phi_{G_b} \end{bmatrix} \begin{bmatrix} \text{Re}(G_m) \\ \text{Im}(G_m) \end{bmatrix} ,$$

and

$$\phi_{G_b} = \tan^{-1}[\text{Im}(G_b)/\text{Re}(G_b)] .$$

Note that $E[\dot{\text{Im}}(G_m)]$ is zero and $E[\dot{\text{Re}}(G_m)]$ is $|G_b|$ by definition.

Since $\tan^{-1}x = x - x^3/3 + x^5/5 - \dots$ when $x^2 < 1$, we have $\Delta\phi \approx$

$\text{Im}(G_m)/\text{Re}(G_m)$ for small errors, and we find

$$\text{Var } \phi_{G_m} \approx \text{Var } \frac{\dot{\text{Im}}(G_m)}{\dot{\text{Re}}(G_m)} \approx \frac{\text{Var } G_m}{2E[|V_m|^2]} . \quad (\text{C-23})$$

The variance of the ratio is deduced from equation (C-14), recognizing that $\text{Cov}[\dot{\text{Re}}(G_m), \dot{\text{Im}}(G_m)]$ and $E[\dot{\text{Im}}(G_m)]$ are both zero. The approximation is good provided the probability is small that $\Delta\phi$ will be large enough to invalidate the approximation $\Delta\phi \approx \dot{\text{Im}}(G_m)/\dot{\text{Re}}(G_m)$.