Transient eddy current response due to a subsurface crack
in a conductive plate

by

Fangwei Fu

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY

Major: Electrical Engineering

Program of Study Committee:
John Bowler, Major Professor
Aleksandar Dogandžić
Mani Mina
Ronald Roberts
Jiming Song

Iowa State University

Ames, Iowa

2006

Copyright © Fangwei Fu, 2006. All rights reserved.

Graduate College
Iowa State University

This is to certify that the doctoral dissertation of
Fangwei Fu
has met the dissertation requirements of Iowa State University

Major Professor

For the Major Program

DEDICATION

I would like to dedicate this thesis to my wife Lingli without whose support I would not have been able to complete this work. I would also like to thank my family and friends for their loving guidance and financial assistance during the writing of this work.

TABLE OF CONTENTS

LIST OF TABLES	viii
LIST OF FIGURES	ix
ABSTRACT	xv
CHAPTER 1. INTRODUCTION	1
1.1 Transient Eddy Current Nondestructive Evaluation	1
1.2 Transient Eddy Current Signal	1
1.2.1 Data Category	1
1.2.2 Signal Interpretation	2
1.2.3 Transient Eddy Current Theory	3
1.2.4 Transient Eddy Current Experiment	7
1.3 Modified Dodd and Deeds's Formulae	8
1.3.1 Magnetic Vector Potential	9
1.3.2 Electromagnetic Fields	13
1.3.3 Fields Due to a Conductive Plate	15
1.4 Evaluation of Partial Fractions	16
1.4.1 First Order Poles	16
1.4.2 Higher Order Poles	17
1.5 Scope of the Dissertation	20
CHAPTER 2. TRANSIENT EDDY CURRENT DRIVER PICK-UP	
PROBE RESPONSE DUE TO A CONDUCTIVE PLATE	22

2.1	Introduction	22
2.2	Field Formula	24
2.3	Short Time Approximation	27
2.3.1	Step Current	28
2.3.2	Exponential Current	30
2.4	Long Time Approximation	32
2.4.1	Step Current	32
2.5	Numerical Results	35
2.6	Conclusion	44
2.7	Appendix: Long Time Approximation for Exponential Current	44
 CHAPTER 3. EVALUATION OF TRANSIENT MAGNETIC FIELD		
	DUE TO A PLATE USING ASYMPTOTIC SOLUTIONS	48
3.1	Introduction	48
3.2	Theory	49
3.2.1	Short Time Approximation	51
3.2.2	Long Time Approximation	53
3.3	Results	56
3.4	Conclusion	58
 CHAPTER 4. TIME DOMAIN DYADIC GREEN'S FUNCTION FOR		
	AN ELECTRIC SOURCE IN A CONDUCTIVE PLATE	60
4.1	Introduction	60
4.2	Dyadic Kernels	63
4.2.1	Maxwell's Equations in Dyadic Form	63
4.2.2	Unbounded Domain Solution	64
4.2.3	Piecewise Homogeneous Layered Structure	65
4.3	Solutions for a Plate	68

4.3.1	Scalar Potentials	68
4.3.2	Explicit Dyadic Kernel for a Plate	71
4.3.3	Asymptotic Expansions	72
4.4	Magnetic field of an embedded electric source	74
4.4.1	Evaluation of $\mathcal{G}^{(\text{me})}(\mathbf{r}, \mathbf{r}', t, t')$	74
4.4.2	Integral Approximation	75
4.5	Numerical Results	79
4.6	Conclusion	79
 CHAPTER 5. TRANSIENT EDDY CURRENT RESPONSE DUE TO A SUBSURFACE CRACK IN A CONDUCTIVE PLATE . .		83
5.1	Introduction	83
5.2	Field Formula	86
5.2.1	Magnetic Vector Potential	86
5.2.2	Dyadic Kernel	87
5.2.3	Electric Field Integral Equation	89
5.3	Incident Field	90
5.4	Flaw Response	92
5.5	Regularization	94
5.6	Discrete Approximation of the Unbounded Domain Integral Operator . .	96
5.6.1	Integrated Kernel	96
5.6.2	Primary Kernel	98
5.7	Linear System	99
5.8	Numerical Results	103
5.9	Conclusion	110
5.10	Appendix	111
5.10.1	Plate Integrated Kernel	111

5.10.2 Plate Primary Kernel	115
CHAPTER 6. INDUCED VOLTAGE IN AN ENCIRCLING COIL DUE TO TRANSIENT EDDY CURRENTS IN A CYLINDRICAL ROD	117
6.1 Introduction	117
6.2 Induced Electromotive Force	118
6.3 Short Time Series	120
6.3.1 Large Argument Regime	120
6.3.2 Partial Fraction Series	121
6.4 Long Time Series	122
6.5 Results	122
6.6 Conclusion	126
CHAPTER 7. CONCLUSIONS AND FUTURE WORK	131
APPENDIX A. THE EFFECT OF OPENING ON EDDY CURRENT PROBE RESPONSE FOR AN IDEALIZED THROUGH CRACK	132
A.1 Abstract	132
A.2 Introduction	133
A.3 Formulation	134
A.3.1 Scalar Decomposition	134
A.3.2 Source Field	136
A.3.3 Impedance	138
A.4 Result	139
A.5 Conclusion	141
A.6 Acknowledgement	142
BIBLIOGRAPHY	143
ACKNOWLEDGEMENTS	153

LIST OF TABLES

Table 2.1	Plate parameters.	36
Table 2.2	Probe parameters.	36
Table 3.1	Probe parameters.	56
Table 5.1	Probe C29 parameters	102
Table 5.2	Aluminum alloy plate F5 and slot parameters	102
Table 5.3	Aluminum alloy plate F6 and slot parameters	103
Table 6.1	Probe parameters.	123
Table 6.2	Rod parameters.	124
Table 6.3	Partial list of the time constant τ_n for aluminum rod.	124
Table 6.4	Partial list of the time constant τ_n for steel rod.	124
Table A.1	Parameters for coil C5 and conductive plate	141

LIST OF FIGURES

Figure 1.1	Typical transient eddy current signal: (a) magnetic field (b) induced EMF	2
Figure 1.2	A filament (a) and a driver and pick-up coil (b) above a two layer conductive region with the domain in the radial direction truncated by a cylindrical boundary.	10
Figure 2.1	Driver coil (1), pick-up coil (2), and conductive plate.	24
Figure 2.2	Induced voltage in a pick up coil due to eddy currents in a conducting plate excited by a step current in a drive coil. Time range of data $20ns$ to $5\mu s$	36
Figure 2.3	Induced voltage in a pick up coil due to eddy currents in a conducting plate excited by a step current in a drive coil. Time range of data $5\mu s$ to $1200\mu s$	37
Figure 2.4	Error relative to the FFT prediction of the induced EMF in a pick-up coil due to eddy currents in a conducting plate excited by a step current in a driver coil.	38
Figure 2.5	Comparison of the relative error in the induced EMF calculated using the FFT compared with the results for the short time series solution. This shows the error in the FFT result decreasing as the number of sample points is increased from 2^{16} to 2^{17} and finally 2^{18} points.	39

Figure 2.6	Small time approximation for $m_{\max} = 0, 1, 2, 3, 4$	40
Figure 2.7	Large time approximation for $m_{\max} = 0, 1, 2$	41
Figure 2.8	Exponential driving current in the driver coil	41
Figure 2.9	Induced voltage in a pickup coil due to a conducting plate excited by an exponential current in the driver coil; drive current time constant $\tau_0 = 275\mu s$	42
Figure 2.10	Error relative to the FFT prediction of the induced EMF in a pick-up coil due to eddy currents in a conducting plate excited by an exponential current ($\tau_0 = 275\mu s$) in a driver coil.	42
Figure 2.11	Effect of plate thickness on the induced voltage in a pickup coil due to induced current in a conducting plate excited by an ex- ponential current in the driver coil; drive current time constant $\tau_0 = 275\mu s$	43
Figure 3.1	Driver coil and Hall sensor above a conductive plate.	50
Figure 3.2	Variation of ΔH_z at coil axis due to eddy currents in a conduct- ing plate excited by an exponential driver coil current with time constant, $\tau_0 = 250\mu s$	57
Figure 3.3	Error relative to the FFT prediction of ΔH_z at the coil axis due to eddy currents in a conducting plate excited by an exponential driver coil current with time constant, $\tau_0 = 250\mu s$	57
Figure 3.4	Variation of the peak amplitude of ΔH_z on the axis of coil with lift-off.	58
Figure 3.5	Variation of the peak amplitude of ΔH_z at the axis of coil with plate thickness.	59

Figure 4.1	Conductive plate of thickness d with non-conductive regions above and below it. The point P is a source point in the plate whose coordinates are represented as $\{\mathbf{r}', t'\}$	63
Figure 4.2	The evolution of the magnetic field component $H_z(\mathbf{r}, t)$, at $\mathbf{r} = [5, 5, 1]$ mm due to an impulsive embedded electric dipole at $\mathbf{r}' = [0, 0, -1]$ mm orientated in the x -direction. The circles denote the FFT results and the solid line denotes the asymptotic solution.	77
Figure 4.3	The evolution of the relative error in $H_z(\mathbf{r}, t)$ computed from the asymptotic solution and compared with FFT result at $\mathbf{r} = [5, 5, 1]$ mm due to an impulsive electric dipole at $\mathbf{r}' = [0, 0, -1]$ mm orientated in the x -direction.	78
Figure 4.4	Variation of $H_z(\mathbf{r}, t)$ in the plane $z = 1$ mm at $t = 1\mu\text{sec}$ due to to an impulsive electric dipole at $\mathbf{r}' = [0, 0, -1]$ mm orientated in the x -direction.	80
Figure 4.5	Variation of $H_z(\mathbf{r}, t)$ in the plane $z = 1$ mm at $t = 50\mu\text{sec}$ due to to an impulsive electric dipole at $\mathbf{r}' = [0, 0, -1]$ mm orientated in the x -direction.	80
Figure 4.6	Variation of $H_z(\mathbf{r}, t)$ in the plane $z = 1$ mm at $t = 200\mu\text{sec}$ due to an x oriented impulsive electric dipole whose coordinates are $\mathbf{r}' = [0, 0, -1]$ mm	81
Figure 4.7	Variation of $H_z(\mathbf{r}, t)$ in the plane $z = 1$ mm at $t = 500\mu\text{sec}$ due to an x oriented impulsive electric dipole whose coordinates are $\mathbf{r}' = [0, 0, -1]$ mm	81

Figure 5.1	Experimental setup for measuring the induced magnetic field due to a subsurface crack in a conductive plate. The driver coil has an exponentially rising current $I = I_0(1 - e^{-t/\tau_0})$ with the time constant $\tau_0 = 250\mu s$	102
Figure 5.2	Induced magnetic field due to unflawed region of plate F5 and F6.	104
Figure 5.3	Transient magnetic field due to a subsurface crack in plate F5 when the coil is centered above the crack. The amplitude of ΔH_z is normalized to an excitation current of 1A.	106
Figure 5.4	Transient magnetic field due to a subsurface crack in plate F6 when the coil is centered above the crack. The amplitude of ΔH_z is normalized to an excitation current of 1A.	106
Figure 5.5	Distribution of dipole density $p(\mathbf{r}, t)$ at $30\mu s$ in plate F5.	107
Figure 5.6	Distribution of dipole density $p(\mathbf{r}, t)$ at $750\mu s$ in plate F5.	107
Figure 5.7	Distribution of dipole density $p(\mathbf{r}, t)$ at $2250\mu s$ in plate F5.	108
Figure 5.8	Incident current density $J(\mathbf{r}, t)$ on crack surface at $30\mu s$ in plate F5.	108
Figure 5.9	Incident current density $J(\mathbf{r}, t)$ on crack surface at $750\mu s$ in plate F5.	109
Figure 5.10	Incident current density $J(\mathbf{r}, t)$ on crack surface at $2250\mu s$ in plate F5.	109
Figure 5.11	Variation of ΔH_z due to a long crack in plate F5 when the coil moves away from the crack.	110
Figure 6.1	Driver and pick up coil circling a conductive rod	119
Figure 6.2	Zeros of $\mathcal{R}_2(s)$ for aluminum rod	123
Figure 6.3	Zeros of $\mathcal{R}_2(s)$ for steel rod	125

Figure 6.4	Aluminum rod: induced EMF in a pick-up coil. The fields is excited by a step current in a driver coil.	127
Figure 6.5	Aluminum rod: relative error of the short and large time series solutions in Figure 6.4 compared with FFT results.	127
Figure 6.6	Steel rod: induced EMF in a pick-up coil. The fields is excited by a step current in a driver coil.	128
Figure 6.7	Steel rod: relative error of the short and large time series solutions in Figure 6.6 compared with FFT results	128
Figure 6.8	Aluminum rod: induced EMF in a pick-up coil. The fields is excited by an exponentially rising current $I(t) = I_0(1 - e^{-t/\tau_0})$ with $I_0 = 1$ and a time constant $\tau_0 = 275\mu s$ in a driver coil. . . .	129
Figure 6.9	Aluminum rod: relative error of the short and large time series solutions in Figure 6.8 compared with FFT results.	129
Figure 6.10	Steel rod: induced EMF in a pick-up coil. The fields is excited by an exponentially rising current $I(t) = I_0(1 - e^{-t/\tau_0})$ with $I_0 = 1$ and a time constant $\tau_0 = 275\mu s$ in a driver coil.	130
Figure 6.11	Steel rod: relative error of the short and large time series solutions in Figure 6.10 compared with FFT results.	130
Figure A.1	An eddy current coil over two coplanar conductive plates next to one another with their edges separated by a small distance. Note that in (c), (d), (e), (f), the arrow pointing to the left indicates the current in the coil flows in clockwise direction viewed from above and the arrow pointing to the right indicates current flows in counterclockwise direction.	135

- Figure A.2 Change in coil resistance and inductance as a function of coil center position for coil C5 scanned across two gaps between two coplanar Al 2024 plates at 480Hz with constant coil liftoff. The circles represent the experimental measurement and the lines are theory prediction. (a) Resistance change. (b) Inductance change. 140
- Figure A.3 Change in coil resistance and inductance as a function of coil center position for coil C5 scanned across two gaps between two coplanar Al 2024 plates at 9980Hz with constant coil liftoff. The circles represent the experimental measurement and the lines are theory prediction. (a) Resistance change. (b) Inductance change. 140
- Figure A.4 Normalized coil resistance change with coil C5 centering over the crack. (a) Resistance change. (b) Reactance change. 142

ABSTRACT

Eddy current nondestructive evaluation (NDE) is usually carried out by exciting a time harmonic field using an inductive probe. However, a viable alternative is to use transient eddy current NDE in which a current pulse in a driver coil produces a transient field in a conductor that decays at a rate dependent on the conductivity and the permeability of the material and the coil configuration. By using transient eddy current, it is possible to estimate the properties of the conductive medium and to locate and size potential flaws from the measured probe response. The fundamental study described in this dissertation seeks to establish a theoretical understanding of the transient eddy current NDE. Compared with the Fourier transform method, the derived analytical formulations are more convenient when the transient eddy current response within a narrow time range is evaluated. The theoretical analysis provides a valuable tool to study the effect of layer thickness, location of defect, crack opening as well as the optimization of probe design.

Analytical expressions have been developed to evaluate the transient response due to eddy currents in a conductive plate based on two asymptotic series. One series converges rapidly for a short time regime and the other for a long time regime and both of them agree with the results calculated by fast Fourier transform over all the times considered. The idea of asymptotic expansion is further applied to determine the induced electromotive force (EMF) in a pick-up coil due to eddy currents in a cylindrical rod.

Starting from frequency domain representation, a quasi-static time domain dyadic

Green's function for an electric source in a conductive plate has been derived. The resulting expression has three parts; a free space term, multiple image terms and partial reflection terms. The dyadic Green's function serves as the kernel of an electric field integral equation which defines the interaction of an ideal crack with the transient eddy currents in a conductive plate. The crack response is found using the reciprocity theorem. Good agreement is observed between the predictions of the magnetic field due to the crack and experimental measurements.

CHAPTER 1. INTRODUCTION

1.1 Transient Eddy Current Nondestructive Evaluation

Eddy current nondestructive evaluation is usually carried out by exciting a time harmonic field using an inductive probe. In transient eddy current nondestructive evaluation, by contrast, the source coil carries a pulse excitation current instead of a sinusoidal alternating current of a certain frequency. A current pulse in the driver coil produces a transient field in a conductor, which migrates according to a vector diffusion equation and decays at a rate that depends on the coil size, the conductivity and the permeability of the conductor. The signal is equivalent of a continuum of frequencies. It is possible to estimate layer thicknesses, locate and size the subsurface flaws by taking advantage of the full characteristic of the transient eddy current response.

1.2 Transient Eddy Current Signal

1.2.1 Data Category

The transient response can be measured either by a pick up coil or Hall device. The former measures the induced EMF and the latter generally monitors the normal component of the magnetic field. The Hall device has some advantages compared with induction coil. Firstly, it measures the magnetic field itself instead of the change rate of the magnetic field and possess constant sensitivity down to direct current (DC). Second, the size of Hall device is generally smaller than an induction coil, which contributes to

higher resolution. The Hall device signal, however, is band limited.

1.2.2 Signal Interpretation

The transient responses due to a flaw is found by subtracting a reference signal from the measured signal. A reference signal may be collected by scanning the probe over a flaw free specimen. Figure 1.1 gives the typical transient eddy current response due to a conductive plate. Any flaws, or conductivity and dimensional changes produce a change in the signal. Peak amplitude, peak arrival time and zero crossing time have been identified as important characteristics of the signal giving indications of the location and size of defects. Intuitively, the peak amplitude is dependent on the flaw size and depth. Variations in the depth of a flaw cause the signal to shift in time and the location of the peak amplitude of a transient Hall signal can be used to gain information about the depth of a feature of interest.

Figure 1.1 Typical transient eddy current signal: (a) magnetic field (b) induced EMF

The transient eddy current measurement has a number of advantages over time harmonic eddy current inspection as claimed by Smith and Hugo [1], [2]. Some of the important ones are quantitatively characterization of defect, fast data acquisition, conveniently response interpretation, low cost test equipment, and deep penetration depth.

In conventional eddy current measurement, all the information is contained in the real and imaginary parts of the impedance change at a number of discrete frequencies. Transient eddy current response contains a spectrum of frequency and potentially provides sufficient information to distinguish defect and structure changes and evaluate them quantitatively. A transient system gives the equivalent information as a multi-frequency measurement in just $10ms$. A multiple-frequency measurement usually takes several seconds and depends on the number of frequency. The transient eddy current response can be presented by C-scan image that allows us to inspect an intersection within the specimen at certain depth. Some structure effects, including lift-off and edge, may be compensated by post signal processing. The instrument cost of a transient system may be lower than a multi-frequency eddy current system. In latter case, the cost increases linearly with the number of frequency channels. The application of Hall device provides deeper penetration depth due to its high sensitivity to low frequency signal. Much higher total energy level can be used to excite the transient probe since the driven source is pulse instead of continuous.

In spite of the listed advantages, transient eddy current measurement has shortcomings as well. The absolute sensitivity may be lower than conventional eddy current because the signal to noise ratio in a wide band system is generally lower than that in a narrow-band system. The most challenge issue associated with transient-eddy current NDE is signal processing, which extracts flaw and structure features from the measured response containing abundant information.

1.2.3 Transient Eddy Current Theory

Numerous work have tried to establish a fast and accurate model that describes the interaction between the transient external source and any conducting medium with or without the presence of flaw. Most of the models are either integral equation based or differential equation based. In these models, the geometry of the source range from an

infinite wire or rod, current loop to pancake axisymmetric coil with rectangular cross section. The excitation function could be of any shape, but usually sinusoidal, ramp or step current is assumed. Complex conductor geometry, like flawed multi-layer plates, can be analyzed as the models become more and more sophisticated.

In finite element scheme, the differential equation defining the magnetic vector potential and electric scalar potential are transformed into algebraic matrix equations. These matrix equations are, then, solved using appropriate solvers, such as conjugate gradient method, at each time step. A large number of models [3] - [4] fall into this category. In 1990, Ludwig et al. [3] developed a weighted residual finite element model, implicitly employing finite difference time stepping, to calculate the two dimensional transient magnetic vector potential and eddy current distribution within a half space conductor. Their formulation was verified by comparing against two analytical models solved by inverse Laplace transform. However, the sources were relatively simple. Only an infinite wire and current loop excited by a step function were considered. No comparison was made with experimental data. Later, Xiao et al. [5] extended the analysis to more complex and realistic testing situations. For example, the axisymmetric pancake coil was excited by a pulsed Maxwell-distributed current. The variation of the induced voltage in a pickup coil was studied against the change of crack size and position. Allen et al. [6] compared two finite element based methods by computing the transmission of transient signal when it passed a conductive plate. The first model made use of the Fourier expansion of the source current, solved a steady-state problem for each significant frequency, then summed up all the harmonic component to compute the EMF in a pickup coil. It is reported that the frequency dependent discretization of the solution region may introduce errors. The transient time stepping method solved the magnetic vector potential matrix equation at each time step and eliminate the discretization problem. Patel et al. [7] computed the transient EMF in an absolute coil due to a conductive plate via three dimensional (3D) time stepping finite element method and suggested that it may

be beneficial to detect flaw by comparing the spectrum of the computed time domain EMF and that of experimental data. In their analysis, few sampling points were taken due to restrictions on the computational time. Under the finite element framework, edge element [8] - [9] was developed to calculate the transient magnetic field due to a half space conductor. The edge elements are vector finite elements whose tangential component is continuous at the interface of two adjacent elements, but jump is allowed for the normal component. Therefore, it is more suitable for describing the behavior of the magnetic field. The finite element methods are usually time consuming and needs a lot of computational resources. This situation may be allowed in solving forward problem, but is unacceptable for inversion when numerous forward predictions are required. Another point is that the finite element method, in nature, has difficulties in simulating the singular behavior of the field at the source. On the other hand, the integral equation based method is more suitable for simple geometry problem, which is the case in this thesis. For these reasons, the finite element method will not be discussed further in this report.

In boundary or volume element method, the unknown, usually a field quantity representing the scatter, is defined by an integral equation. Although these two methods are not as popular as the finite element method, they have their own advantages. Tsuboi et al [10] showed that the boundary element method could obtain the same computational accuracy as the finite element method with larger time step. Pávó [11] presented a computational method to evaluate the transient eddy current response due to planar flaw in a two plate system. The current pulse driving the pancake coil was decomposed into a series of time harmonic components. The flaw is assumed to be infinitesimally thin and represented by two imaginary sources, tangential magnetic and electric surface current. These imaginary sources are solved by boundary element method applying impedance type interface condition. Once the time harmonic response of the system was obtained the time domain induced voltage or magnetic field can be calculated by the inverse

Fourier transform. To investigate the application of transient eddy current technique in inspecting the corrosion of pipes used in petrochemical industry, S. M van den Berg [12] described a 3D forward model, which gives the induced EMF in a pick-up coil due to a volume defect in a planar layered configuration in frequency domain and the time domain signal is recovered using FFT. Both the conductivity and permeability contrast are considered in S. M van den Berg's work, and thus, his model can be applied to ferromagnetic materials. On the other hand, the curved surface of pipe is approximated by a planar surface, which leaves space for improvement. The computational cost of this model is relatively high, more than 100 hours for one forward calculation.

Bowler and Johnson [13], [14] have derived analytical expressions for the induced EMF and integrated EMF in an absolute coil excited by a step current due to a half space conductor using the inverse Laplace transform. Bowler [15] further developed a quasi-static dyadic Green's function for an electric source in a homogeneous conductive half space using transverse potential decomposition, which provides the kernel of an electric field integral equation defining the interaction between a volumetric flaw and the transient eddy currents induced in the half space. In practical application, it would be more desirable to consider the plate geometry instead of the half space case and the central goal of this thesis is to extend the analysis in [14], [15], [16] to a conductive plate.

There are some other methods that have been developed. Demens et al. [17] have applied the finite integration (FI) method to solve TEAM Workshop Problem 7. A set of matrix equation, called Maxwell-Grid Equation (MGE), were generated by mapping the Maxwell's equations in integral form on a dual grid-duplet $\{G, G'\}$. The field components were equivalently represented by facet fluxes and grid voltage with appropriate time-stepping technique. The matrix equation are solved efficiently by the preconditioned conjugate gradient method. Again, this method is time expensive to implement. Davey et al. [18] extended the surface impedance method to the transient regime. The time dependent response was built from the eigenvalues extracted from transient surface

impedance method. Recently, the Markov chain Monte Carlo method (MCMC) was introduced by Netter [19] in the transient eddy current calculations. This method is claimed to be more efficient than the finite element method since it can estimate the solution at any position without the knowledge of the whole field. To improve the accuracy, however, more random numbers need to be generated, which adds a significant amount of time cost. It was also reported that the MCMC method converges very slow.

It is deserved to mention that the TEAM Workshop Problem 27 has been proposed to facilitate the validation of various models that are able to predict the transient response due to a constant volume crack starting from screw holes at a certain depth in a conductive plate.

1.2.4 Transient Eddy Current Experiment

The first recorded practise of transient eddy current NDE was measuring the difference in the properties of the test and reference specimen. The transient signal was introduced by a clock and connected to two coils, one on a reference specimen and the other on a specimen to be tested. The measured differential signal was sent to a telephone receiver and adjustment was made such that no sound can be heard. The degree of adjustment reflected the relative difference in the properties of the test and reference specimen.

Recently, Smith and Hugo [1], [2] studied the capabilities and limitations of transient eddy current nondestructive evaluation through extensive experiments seeking methods to characterize defects and structural variations. Moulder et al. [20] tried using transient eddy current signal to identify the effect of metal loss and inter-layer separation. They found the quantity of metal loss can be estimated from the amplitude of the pulsed eddy current (PEC) signal and the gap effect can be read from the zero-crossing time. Bieber et al. [21], [22] used the peak amplitude data to evaluate the metal loss caused by corrosion. However, it is found that variations of lift-off and inter-layer gap can

cause noise which may overwhelm the material loss signature. Giguère et al. [23], [24], [25], [26] minimized the effect of lift-off by taking advantage of the amplitude of transient response at the lift-off point of intersection which occurs at a time following the initiation of the transient when the lift-off signal is at a null. Similarly, at later time, there is a narrow region where the effect of plate separation is at a null allowing the detection of material loss, which is insensitive to plate separation. Based on the asymptotic behavior of transient response in short and long time limit, Burke et al. [27] corrected the lift-off effect by multiplying the signal by a time varying function, called the lift-off scaling factor. To determine the conductive plate parameters, Johnson et al. [28], [29] developed a parameter based inversion scheme on the basis of a verified fast forward model. As a preliminary step, a database storing the information of plate conductivity, liftoff and magnetic field was constructed by the model. The plate parameters may be determined by minimizing a penalty function. A review of the early development of transient eddy current experimental technology is found in [30] by Beiber et al and [31] by Johnson.

1.3 Modified Dodd and Deeds's Formulae

Dodd and Deeds's model [32] has been widely referred to solve eddy current related problems in nondestructive evaluation, such as the analysis of coil impedance or magnetic field change due to a planar or cylindrical layered structure. Setting the boundary of interest to infinity, the electromagnetic fields are expressed in closed form in terms of Fourier-Bessel integral, which converges slow when evaluated numerically. One way of reducing the computational cost is to transform the integral into series form, where the accuracy and speed can be partly controlled by adjusting the number of turns. To achieve this goal, the solution region is truncated by imposing a magnetic boundary at appropriate distance from the source coil, for example, 10 times the outer radius of the excitation coil. It has been studied [33] that the replacement of the integral with a series

would not sacrifice the computational accuracy as long as sufficient terms are included. Usually, less than one hundred terms would give accurate results.

As a preparation for subsequent analysis, region truncation method is used to reformulate Dodd and Deeds's expressions for a two layered planar structure in this section. The magnetic field above the conductive region and the induced EMF in a pick-up coil due to the layered structure are expressed in series form. The results for the case of a conductive plate are obtained when the conductivity of the bottom layer reduces to zero.

1.3.1 Magnetic Vector Potential

Our analysis starts from the magnetic vector potential due to a filament above a two layered conductive region, which is shown in Figure. 1.2 (a). The axis of the filament is coincident with the z axis. The radius of the filament is r_0 and the distance between the filament and top surface of conductive region is z_0 . The top layer has a thickness of d and the bottom layer is infinite in the $-z$ direction. The region of interest is truncated by a cylindrical surface, with a diameter $2b$, where the magnetic vector potential is assumed to vanish. For the convenience of matching interface conditions, the region within the truncated domain is divided into four parts along the z axis.

Neglecting the displacement current, the magnetic vector potential $\mathbf{A}(r, z)$, in an isotropic, linear and homogeneous medium due to a filament coil at (r_0, z_0) satisfies

$$\frac{\partial^2 \mathbf{A}}{\partial r^2} + \frac{1}{r} \frac{\partial \mathbf{A}}{\partial r} + \frac{\partial^2 \mathbf{A}}{\partial z^2} - \frac{\mathbf{A}}{r^2} - j\omega\mu\sigma\mathbf{A} = -\mu I \delta(r - r_0)(z - z_0)\hat{\phi}, \quad (1.1)$$

where μ and σ are the permeability and conductivity of the medium. I is the amplitude of excitation current. The magnetic vector potential has azimuthal component $A_\phi(r, z)$ only because of symmetry, which satisfies

$$\frac{\partial^2 A_\phi}{\partial r^2} + \frac{1}{r} \frac{\partial A_\phi}{\partial r} + \frac{\partial^2 A_\phi}{\partial z^2} - \frac{A_\phi}{r^2} = 0 \quad (1.2)$$

Figure 1.2 A filament (a) and a driver and pick-up coil (b) above a two layer conductive region with the domain in the radial direction truncated by a cylindrical boundary.

in air and

$$\frac{\partial^2 A_\phi}{\partial r^2} + \frac{1}{r} \frac{\partial A_\phi}{\partial r} + \frac{\partial^2 A_\phi}{\partial z^2} - \frac{A_\phi}{r^2} - j\omega\mu\sigma A_\phi = 0 \quad (1.3)$$

in conductive region.

Separation of variable shows that $A_\phi(r, z)$ has the form [33]

$$A_\phi(r, z) = [A(\kappa_i)J_1(\kappa_i r) + B(\kappa_i)Y_1(\kappa_i r)][C(\kappa_i)e^{\gamma_i z} + D(\kappa_i)e^{-\gamma_i z}], \quad (1.4)$$

where $J_1(x)$ and $Y_1(x)$ are the the first order Bessel function of the first and second kind. $A(\kappa_i)$, $B(\kappa_i)$, $C(\kappa_i)$ and $D(\kappa_i)$ are the coefficients to be determined. Note that $\gamma_i = \sqrt{\kappa_i^2 + j\omega\mu_0\mu_{rj}\sigma_j}$, where μ_{rj} and σ_j are the relative permeability and conductivity of j^{th} layer. The discrete eigenvalue κ_i is introduced instead of the continuous κ in Dodd and Deeds' s model due to the imposed Dirichlet boundary for the magnetic vector potential at $r = b$. The eigenvalues are found by $J_1(\kappa_i b) = 0$. Due to the divergence of $Y_1(\kappa_i r)$ at the origin, $B(\kappa_i)$ vanishes in all regions. Since the fields converge at infinity either $C(\kappa_i)$ or $D(\kappa_i)$ goes to zero in region 1 or 4. Therefore, the solution of $A_\phi(r, z)$ in each region is simplified as

$$A_\phi^1(r, z) = \sum_{i=1}^{\infty} J_1(\kappa_i r) e^{-\kappa_i z} D_1, \quad (1.5)$$

$$A_\phi^2(r, z) = \sum_{i=1}^{\infty} [J_1(\kappa_i r) e^{\kappa_i z} C_2 + J_1(\kappa_i r) e^{-\kappa_i z} D_2], \quad (1.6)$$

$$A_\phi^3(r, z) = \sum_{i=1}^{\infty} [J_1(\kappa_i r) e^{\gamma_1 z} C_3 + J_1(\kappa_i r) e^{-\gamma_1 z} D_3], \quad (1.7)$$

and

$$A_\phi^4(r, z) = \sum_{i=1}^{\infty} J_1(\kappa_i r) e^{\gamma_2 z} C_4. \quad (1.8)$$

where

$$\gamma_1 = \sqrt{\kappa_i^2 + j\omega\mu_0\mu_{r1}\sigma_1} \quad \text{and} \quad \gamma_2 = \sqrt{\kappa_i^2 + j\omega\mu_0\mu_{r2}\sigma_2}. \quad (1.9)$$

Note that D_1, C_2, D_2, C_3, D_3 and D_4 are dependent on κ_i and they are solved by six linear equations generated from the continuity conditions of $A_\phi(r, z)$ at the interfaces $z = z_0$, $z = 0$ and $z = -d$, which are

$$A_\phi^1(r, z_0) = A_\phi^2(r, z_0), \quad (1.10)$$

$$\frac{\partial A_\phi^1(r, z_0)}{\partial z} = \frac{\partial A_\phi^2(r, z_0)}{\partial z} - \mu_0 I(j\omega) \delta(r - r_0), \quad (1.11)$$

$$A_\phi^2(r, 0) = A_\phi^3(r, 0), \quad (1.12)$$

$$\frac{\partial A_\phi^2(r, 0)}{\partial z} = \frac{1}{\mu_{r1}} \frac{\partial A_\phi^3(r, 0)}{\partial z}, \quad (1.13)$$

$$A_\phi^3(r, -d) = A_\phi^4(r, -d) \quad (1.14)$$

and

$$\frac{1}{\mu_{r1}} \frac{\partial A_\phi^3(r, -d)}{\partial z} = \frac{1}{\mu_{r2}} \frac{\partial A_\phi^4(r, -d)}{\partial z}. \quad (1.15)$$

The orthogonality of Bessel function can be used to obtain the equations defining the unknown coefficients in (1.5) – (1.8).

$$\int_0^b r J_1(\kappa_j r) J_1(\kappa_i r) dr = \begin{cases} \frac{b^2}{2} J_0^2(\kappa_i b) & i = j \\ 0 & i \neq j \end{cases} \quad (1.16)$$

For example, substitute (1.10) into (1.6), multiply both side of the resulted equation by $r J_1(q_k r)$ and integrate from 0 to b we get

$$e^{-\kappa_i z_0} D_1 = e^{\kappa_i z_0} C_2 + e^{-\kappa_i z_0} D_2 \quad (1.17)$$

Similarly,

$$-\kappa_i \frac{b^2}{2} J_0^2(\kappa_i b) e^{-\kappa_i z_0} D_1 = \kappa_i \frac{b^2}{2} J_0^2(\kappa_i b) (e^{\kappa_i z_0} C_2 - e^{-\kappa_i z_0} D_2) - \mu_0 I(j\omega) r_0 J_1^2(\kappa_i r_0), \quad (1.18)$$

$$C_2 + D_2 = C_3 + D_3, \quad (1.19)$$

$$\kappa_i (C_2 - D_2) = \frac{\gamma_1}{\mu_{r1}} (C_3 - D_3), \quad (1.20)$$

$$e^{-\gamma_1 d} C_3 + e^{\gamma_1 d} D_3 = e^{-\gamma_2 d} C_4 \quad (1.21)$$

and

$$\frac{\gamma_1}{\mu_{r1}} (e^{-\gamma_1 d} C_3 - e^{\gamma_1 d} D_3) = \frac{\gamma_2}{\mu_{r2}} e^{-\gamma_2 d} C_4 \quad (1.22)$$

Solving the linear equations (1.17) – (1.22) gives

$$\begin{aligned} D_1 = & \frac{\Psi(\kappa_i) e^{-\kappa_i z_0} J_1(\kappa_i r_0) r_0}{\kappa_i} \left[(e^{2d\gamma_1} - 1) (e^{2\kappa_i z_0} - 1) \mu_{r2} \gamma_1^2 \right. \\ & + (e^{2d\gamma_1} + 1) [(e^{2\kappa_i z_0} - 1) \gamma_2 + (e^{2\kappa_i z_0} + 1) \kappa_i \mu_{r2}] \mu_{r1} \gamma_1 \\ & \left. + (e^{2d\gamma_1} - 1) (e^{2\kappa_i z_0} + 1) \gamma_2 \kappa_i \mu_{r1}^2 \right], \end{aligned} \quad (1.23)$$

$$C_2 = \frac{e^{-\kappa_i z_0} J_1(\kappa_i r_0) I(j\omega) r_0 \mu_0}{b^2 J_0^2(\kappa_i b) \kappa_i}, \quad (1.24)$$

$$\begin{aligned} D_2 = & \frac{\Psi(\kappa_i) e^{-\kappa_i z_0} J_1(\kappa_i r_0) r_0}{\kappa_i} \left[- (e^{2d\gamma_1} - 1) \mu_{r2} \gamma_1^2 - (e^{2d\gamma_1} + 1) (\gamma_2 - \kappa_i \mu_{r2}) \mu_{r1} \gamma_1 \right. \\ & \left. + (e^{2d\gamma_1} - 1) \gamma_2 \kappa_i \mu_{r1}^2 \right], \end{aligned} \quad (1.25)$$

$$C_3 = 2\Psi(\kappa_i) e^{-\kappa_i z_0} J_1(\kappa_i r_0) r_0 \mu_{r1} e^{2d\gamma_1} (\gamma_2 \mu_{r1} + \gamma_1 \mu_{r2}), \quad (1.26)$$

$$D_3 = -2\Psi(\kappa_i) e^{-\kappa_i z_0} J_1(\kappa_i r_0) r_0 \mu_{r1} (\gamma_2 \mu_{r1} - \gamma_1 \mu_{r2}), \quad (1.27)$$

$$C_4 = 4\Psi(\kappa_i) e^{-\kappa_i z_0} J_1(\kappa_i r_0) r_0 \mu_{r1} \mu_{r2} \gamma_1 e^{d(\gamma_1 + \gamma_2)}, \quad (1.28)$$

where

$$\Psi(\kappa_i) = \frac{i(j\omega)\mu_0}{b^2 J_0^2(\kappa_i b) \left[(e^{2d\gamma_1} - 1)\mu_{r2}\gamma_1^2 + (e^{2d\gamma_1} + 1)(\gamma_2 + \kappa_i\mu_{r2})\mu_{r1}\gamma_1 + (e^{2d\gamma_1} - 1)\gamma_2\kappa_i\mu_{r1}^2 \right]} \quad (1.29)$$

and $i(j\omega)$ is defined as

$$i(j\omega) = \frac{N_1 I(j\omega)}{(r_2 - r_1)(z_2 - z_1)}. \quad (1.30)$$

Note that $I(j\omega)$ is the Fourier transform of the excitation current, N_1 is the number of turns of the driver coil and r_1, r_2, z_1, z_2 are defined in Figure 1.2 (b). Substitute (1.23) – (1.30) into (1.5) – (1.8), $A_\phi(r, z)$ due to a filament is determined. For a pancake coil with rectangular cross section, the total magnetic vector potential $A_\phi^{\text{total}}(r, z)$ is obtained by integrating the filament response over the coil cross section [32] ,

$$A_\phi^{\text{total}}(r, z) = \int_{r_1}^{r_2} \int_{z_1}^{z_2} A_\phi(r, z, r_0, z_0) dr_0 dz_0. \quad (1.31)$$

1.3.2 Electromagnetic Fields

With $A_\phi^{\text{total}}(r, z)$ in each region derived, the electromagnetic fields can be obtained by standard routines, such as $\mathbf{E} = j\omega\mathbf{A}$ and $\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A}$. In this thesis, we are interested in the magnetic field above the conductive region and the induced EMF in a pickup coil. It is straightforward that $H_\phi = 0$ because of the curl operator on $A_\phi^{\text{total}}(r, z)$. The radial component $H_r(r, z)$ is

$$\begin{aligned} H_r(r, z) = & - \sum_{i=1}^{\infty} \frac{J_1(\kappa_i r) \Psi(\kappa_i)}{\mu_0 \kappa_i^3} \chi(\kappa_i r_1, \kappa_i r_2) e^{-2\kappa_i z} \cdot \\ & \left\{ e^{-\kappa_i z_1} \left[(\gamma_1 \mu_{r2} - \gamma_2 \mu_{r1}) [e^{\kappa_i(z+2z_1)} (-\gamma_1 + \kappa_i \mu_{r1}) - (e^{-\kappa_i z} - 2e^{\kappa_i z_1}) (\gamma_1 + \kappa_i \mu_{r1})] \right. \right. \\ & + e^{2d\gamma_1} (\gamma_1 \mu_{r2} + \gamma_2 \mu_{r1}) [e^{\kappa_i(z+2z_1)} (\gamma_1 + \kappa_i \mu_{r1}) + (e^{\kappa_i z} - 2e^{\kappa_i z_1}) (\gamma_1 - \kappa_i \mu_{r1})] \\ & - e^{-\kappa_i z_2} \left[(\gamma_2 \mu_{r1} - \gamma_1 \mu_{r2}) [e^{3\kappa_i z} (\gamma_1 - \kappa_i \mu_{r1}) + (e^{\kappa_i z} - 2e^{\kappa_i z_2}) (\gamma_1 + \kappa_i \mu_{r1})] \right. \\ & \left. \left. + e^{2d\gamma_1} (\gamma_1 \mu_{r2} + \gamma_2 \mu_{r1}) [e^{3\kappa_i z} (\gamma_1 + \kappa_i \mu_{r1}) + (e^{\kappa_i z} - 2e^{\kappa_i z_2}) (\gamma_1 - \kappa_i \mu_{r1})] \right] \right\}, \quad (1.32) \end{aligned}$$

where $\chi(x_1, x_2)$ is defined as

$$\chi(x_1, x_2) = \int_{x_1}^{x_2} x J_1(x) dx. \quad (1.33)$$

Note that (1.33) can be evaluated through the Struve and Bessel function. The z component of the magnetic field $H_z(r, z)$ is

$$\begin{aligned} H_z(r, z) = & \sum_{i=1}^{\infty} \frac{2J_1(\kappa_i r) + r\kappa_i[J_0(\kappa_i r) - J_2(\kappa_i r)]}{2r\mu_0\kappa_i^4} \Psi(\kappa_i) \chi(\kappa_i r_1, \kappa_i r_2) e^{-2\kappa_i z} \cdot \\ & \left\{ (e^{\kappa_i(z-z_1)} - 1) [(e^{2d\gamma_1} - 1)(e^{\kappa_i(z+z_1)} - 1)\mu_{r2}\gamma_1^2 \right. \\ & + (e^{2d\gamma_1} + 1)[- \gamma_2 + \kappa_i\mu_{r2} + e^{\kappa_i(z+z_1)}(\gamma_2 + \kappa_i\mu_{r2})]\gamma_1\mu_{r1} \\ & + (e^{2d\gamma_1} - 1)(e^{\kappa_i(z+z_1)} + 1)\gamma_2\kappa_i\mu_{r1}^2] \\ & + (-e^{\kappa_i(z-z_2)} + 1) [(e^{2d\gamma_1} - 1)(e^{2\kappa_i z} - 1)\mu_{r2}\gamma_1^2 \\ & + (e^{2d\gamma_1} + 1)[(e^{2\kappa_i z} - 1)\gamma_2 + (e^{2\kappa_i z} + 1)\kappa_i\mu_{r2}]\gamma_1\mu_{r1} \\ & \left. + (e^{2d\gamma_1} - 1)(e^{2\kappa_i z} + 1)\gamma_2\kappa_i\mu_{r1}^2] \right\}. \quad (1.34) \end{aligned}$$

For a filament of radius r , the induced EMF is related to $A_\phi(r, z)$ by $V(r, z) = j\omega 2\pi r A_\phi(r, z)$. The total induced EMF, V , in a pick up coil with N_2 turns in region 1-2 can be found by superposition, which gives

$$V = \frac{j\omega 2\pi N_2}{(r_4 - r_3)(z_4 - z_3)} \sum_{i=1}^{\infty} \frac{\Psi(\kappa_i) \chi(\kappa_i r_1, \kappa_i r_2) \chi(\kappa_i r_3, \kappa_i r_4)}{\kappa_i^6} (\Theta_1 + \Theta_2), \quad (1.35)$$

where

$$\begin{aligned} \Theta_1 = & \left[z_4 - z_3 + \frac{e^{-2\kappa_i z_3} - e^{-2\kappa_i z_4} + 4(e^{-\kappa_i z_4} - e^{-\kappa_i z_3}) \cosh(\kappa_i z_1)}{2\kappa_i} \right] [(e^{2d\gamma_1} - 1)\mu_{r2}\gamma_1^2] \\ & + \left[z_4 - z_3 + \frac{e^{\kappa_i z_1}(e^{-\kappa_i z_4} - e^{-\kappa_i z_3})}{\kappa_i} \right] [(e^{2d\gamma_1} + 1)(\gamma_2 + \kappa_i\mu_{r2})\mu_{r1}\gamma_1] \\ & + \left[z_4 - z_3 + \frac{e^{-2\kappa_i z_4} - e^{-2\kappa_i z_3} + 4(e^{-\kappa_i z_4} - e^{-\kappa_i z_3}) \sinh(\kappa_i z_1)}{2\kappa_i} \right] [(e^{2d\gamma_1} - 1)\gamma_2\kappa_i\mu_{r1}^2] \\ & + \left[\frac{e^{-2\kappa_i z_4} - e^{-2\kappa_i z_3} - 2e^{-\kappa_i z_1}(e^{-\kappa_i z_4} - e^{-\kappa_i z_3})}{2\kappa_i} \right] [(e^{2d\gamma_1} + 1)(-\gamma_2 + \kappa_i\mu_{r2})\mu_{r1}\gamma_1] \quad (1.36) \end{aligned}$$

and

$$\begin{aligned} \Theta_2 = & \left[z_4 - z_3 + \frac{e^{-2\kappa_i z_4} - e^{-2\kappa_i z_3}}{2\kappa_i} + \frac{2e^{-\kappa_i z_2} [\cosh(\kappa_i z_3) - \cosh(\kappa_i z_4)]}{\kappa_i} \right] \\ & \left[(e^{2d\gamma_1} - 1)\mu_{r2}\gamma_1^2 + (e^{2d\gamma_1} + 1)\mu_{r1}\gamma_2\gamma_1 \right] \\ & + \left[z_4 - z_3 - \frac{e^{-2\kappa_i z_4} - e^{-2\kappa_i z_3}}{2\kappa_i} + \frac{2e^{-\kappa_i z_2} [\sinh(\kappa_i z_3) - \sinh(\kappa_i z_4)]}{\kappa_i} \right] \\ & \left[(e^{2d\gamma_1} - 1)\gamma_2\kappa_i\mu_{r1}^2 + (e^{2d\gamma_1} + 1)\kappa_i\gamma_1\mu_{r2}\mu_{r1} \right]. \end{aligned} \quad (1.37)$$

1.3.3 Fields Due to a Conductive Plate

Based on the formulation derived in previous section, we can write down the results when the conductive region is a conducting plate. In this case, the conductivity of the bottom layer is zero, which gives $\gamma_2 = \kappa_i$ and $\mu_{r2} = 1$. The electric field within the plate is of interest when an eddy current forward problem is considered. For reference, it is shown here,

$$\begin{aligned} E_\phi = & \sum_{i=1}^{\infty} \frac{2j\omega N_1 J_1(\kappa_i r) \mu_0 I(j\omega)}{(r_2 - r_1)(z_2 - z_1) b^2 \kappa_i^3 J_0^2(\kappa_i b)} \chi(\kappa_i r_1, \kappa_i r_2) (e^{-\kappa_i z_2} - e^{-\kappa_i z_1}) \cdot \\ & \frac{e^{-z\gamma_1} [e^{2(d+z)\gamma_1} (\gamma_1 + \kappa_i \mu_{r1}) + (\gamma_1 - \kappa_i \mu_{r1})]}{e^{2d\gamma_1} (\gamma_1 + \kappa_i \mu_{r1})^2 - (\gamma_1 - \kappa_i \mu_{r1})^2}. \end{aligned} \quad (1.38)$$

In later analysis, we focus on the z component of the magnetic field on the coil axis generated by the eddy currents in the plate ΔH_z , which is the difference of the magnetic field with and without plate in (1.34),

$$\begin{aligned} \Delta H_z = & - \sum_{i=1}^{\infty} \frac{N_1 \chi(\kappa_i r_1, \kappa_i r_2) I(j\omega)}{(r_2 - r_1)(z_2 - z_1) b^2 \kappa_i^3 J_0^2(\kappa_i b)} \cdot \\ & \frac{e^{-\kappa_i z} (e^{-\kappa_i z_2} - e^{-\kappa_i z_1}) (-1 + e^{2d\gamma_1}) (\gamma_1^2 - \kappa_i^2 \mu_{r1}^2)}{e^{2d\gamma_1} (\gamma_1 + \kappa_i \mu_{r1})^2 - (\gamma_1 - \kappa_i \mu_{r1})^2}. \end{aligned} \quad (1.39)$$

Similarly, the induced EMF due to eddy currents in the plate ΔV is

$$\begin{aligned} \Delta V = & \frac{j\omega 2\pi N_1 N_2 \mu_0 I(j\omega)}{(r_2 - r_1)(z_2 - z_1)(r_4 - r_3)(z_4 - z_3)} \cdot \sum_{i=1}^{\infty} \frac{\chi(\kappa_i r_1, \kappa_i r_2) \chi(\kappa_i r_3, \kappa_i r_4)}{b^2 \kappa_i^3 J_0^2(\kappa_i b)} \cdot \\ & \left\{ \frac{(e^{-\kappa_i z_4} - e^{-\kappa_i z_3})(e^{-\kappa_i z_2} - e^{-\kappa_i z_1}) (-1 + e^{2d\gamma_1}) (\gamma_1^2 - \kappa_i^2 \mu_{r1}^2)}{e^{2d\gamma_1} (\gamma_1 + \kappa_i \mu_{r1})^2 - (\gamma_1 - \kappa_i \mu_{r1})^2} \right\}. \end{aligned} \quad (1.40)$$

The $j\omega$ in (1.39) and (1.40) is replaced by the Laplace variable s in Appendix B and Chapter 2, which serve as the starting point of followed analysis in time domain.

1.4 Evaluation of Partial Fractions

In Chapter 2, the partial fraction expansion of the power of hyperbolic cotangent function, $\coth^n(z)$, is required to evaluate the terms in calculating the transient eddy current response using large time approximation formulae. However, this work is equivalent to expanding $\cot(z)$ into partial fraction series by taking advantage of $\coth^n(z) = i^n \cot^n(iz)$ [34] and we focus on the latter task in this section.

1.4.1 First Order Poles

Cauchy integral theorem shows that if a function f is to be analytic within and on a simple closed contour C , then the values of f interior to C are completely determined by the values of f on C ,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z} dw. \quad (1.41)$$

More generally, if $f(z)$ has n simple (first order) poles in C , Cauchy integral theorem can be written as

$$f(z) + S_C^{(0)} = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z} dw \quad (1.42)$$

using residue theorem. $S_C^{(0)}(z)$ is defined as

$$S_C^{(0)} = \sum_m \frac{b_m}{z_m - z}, \quad (1.43)$$

where z_m is the first order poles of $f(z)$ and b_m is the residue of $f(z)$ at z_m .

$$\text{Res}[f(z), z_m] = b_m. \quad (1.44)$$

As a example, let's consider $f(z) = \csc(z)$. we know that $\csc(z)$ has poles $a_m = \pi m$ where $m = -\infty \dots -2, -1, 0, 1, 2 \dots \infty$ with residues $(-1)^m$. Assuming that the contour

C encloses all poles and that the contour integration vanishes in the limit as it extends to infinity, (detailed consideration must be given to the vanishing of the integral as describes in text books [35], [36]. We shall omit these considerations here) applying (1.42) to $\csc(z)$ gives

$$\csc(z) = - \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{\pi m - z} \quad \text{or} \quad \csc(z) = \frac{1}{z} + 2z \sum_{m=1}^{\infty} \frac{(-1)^m}{(z^2 - \pi^2 m^2)} \quad (1.45)$$

Similarly, $\cot(z)$ has poles $a_m = \pi m$ where $m = -\infty \dots -2, -1, 0, 1, 2 \dots \infty$ with residues 1 at all poles. Assuming again that the contour C encloses all poles and that the contour integration vanishes in the limit as it extends to infinity, we get

$$\cot(z) = - \sum_{m=-\infty}^{\infty} \frac{1}{\pi m - z} \quad \text{or} \quad \cot(z) = \frac{1}{z} + 2z \sum_{m=1}^{\infty} \frac{1}{(z^2 - \pi^2 m^2)}. \quad (1.46)$$

1.4.2 Higher Order Poles

An extension of Cauchy's theorem is [36]

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w - z)^{n+1}} dw \quad (1.47)$$

where $f(w)$ is analytic in the region bounded by the contour C and the function $f^{(n)}(z)$ is the n th derivative of $f(z)$. Since $f(z)$ is an analytic function, the same is true of its derivatives, including $f^{(n)}(z)$.

The above relationship can be extended to the more general case in which $f(z)$ has simple (first order) poles in C , which gives

$$f^{(n)}(z) + S_C^{(n)} = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w - z)^{n+1}} dw, \quad (1.48)$$

where $S_C^{(n)}(z)$ is the sum of the residues of the poles of $f(z)$ in C and

$$S_C^{(n)}(z) = \sum_m \frac{n! b_m}{(z_m - z)^{n+1}}. \quad (1.49)$$

One notes that because $f(z)$ has first order poles, then $f^{(n)}(z)$ has poles of order $n + 1$ as can be seen immediately by expressing $f(z)$ in the form of a Laurent expansion and differentiate n times.

The task we set ourselves is to express the functions $\cot^n(z)$ in the form of partial fraction expansions and this is done on a case-by-case basis because the general power of the cotangent function seems too difficult to evaluate without first studying some examples. Section 1.4.1 gives the result for $n = 1$ hence we proceed to the case $n = 2$.

For $n = 2$, we consider $\cot^2(z) = \csc^2(z) - 1$. Clearly we need only to write $\csc^2(z)$ as a partial fraction expansion to achieve our aim of expressing $\cot^2(z)$ as an expansion in partial fractions. Note that the poles of $\csc^2(z)$ are second order, and that $\int \csc^2(z) dz = -\cot(z)$. Hence we choose $f(z) = -\cot(z)$, the poles of which are at $z_m = \pi m$ and as noted earlier the residues are $b_m = 1$. Assuming the integral on the right hand side of (1.48) vanishes, we immediately get

$$\csc^2(z) = \sum_{m=-\infty}^{\infty} \frac{1}{(\pi m - z)^2} \quad (1.50)$$

For $n = 3$, we consider $\cot^3(z) = \cos(z) \csc^3(z) - \cot(z)$. Since we already have an expansion for $\cot(z)$ and can easily see that the remaining term has third order poles, we shall let $f^{(2)}(z) = \cos(z) \csc^3(z)$. Integrating twice gives $\frac{1}{2}[z + \cot(z)]$, but the analytical term $z/2$ can be discarded. Thus we assign $f(w) = \frac{1}{2} \cot(z)$. Using the same poles and residues as before for this function to give

$$\cos(z) \csc^3(z) = - \sum_{m=-\infty}^{\infty} \frac{1}{(\pi m - z)^3} \quad (1.51)$$

and

$$\cot^3(z) = - \sum_{m=-\infty}^{\infty} \frac{1}{(\pi m - z)^3} + \sum_{m=-\infty}^{\infty} \frac{1}{(\pi m - z)}. \quad (1.52)$$

For $n = 4$, we consider $\cot^4(z) = \csc^4(z) - \csc^2(z) - 1$. Since the partial fraction expansion of $\csc^2(z)$ is known it would seem that $\csc^4(z)$ should be expanded. However, integrating $\csc^4(z)$ twice gives a $\log[\sin(z)]$ term, among others. The offending log term cannot be used in the Cauchy integral nor will the third integration help us recover a function whose singularities are just poles. The way round this difficulty is to note that $\csc^2(z)$ when integrated twice gives the same unwanted term, $\log[\sin(z)]$. By assigning

$f^{(3)}(z) = \csc^4(z) - \frac{2}{3} \csc^2(z)$, the offending term cancels and the three stages of integration yield $-\frac{1}{6}(\cot(z) + z)$. Discarding the analytical term, $-z/6$, we put $f(z) = -\frac{1}{6} \cot(z)$, and use the same poles and residues as before to give

$$\csc^4(z) - \frac{2}{3} \csc^2(z) = \sum_{m=-\infty}^{\infty} \frac{1}{(\pi m - z)^4} \quad (1.53)$$

and

$$\cot^4(z) = \sum_{m=-\infty}^{\infty} \frac{1}{(\pi m - z)^4} - \frac{1}{3} \sum_{m=-\infty}^{\infty} \frac{1}{(\pi m - z)^2} - 1. \quad (1.54)$$

For $n = 5$, we consider $\cot^5(z) = \cot(z) \csc^4(z) - 2 \cot(z) \csc^2(z) + \cot(z)$. Let $f(z) = \cot(z)/24$, then we get

$$\cot(z) \csc^4(z) - \frac{1}{3} \cot(z) \csc^2(z) = - \sum_{m=-\infty}^{\infty} \frac{1}{(\pi m - z)^5} \quad (1.55)$$

and

$$\cot^5(z) = - \sum_{m=-\infty}^{\infty} \frac{1}{(\pi m - z)^5} + \frac{5}{3} \sum_{m=-\infty}^{\infty} \frac{1}{(\pi m - z)^3} - \sum_{m=-\infty}^{\infty} \frac{1}{(\pi m - z)}. \quad (1.56)$$

For $n = 6$, we consider $\cot^6(z) = \csc^6(z) - 3 \csc^4(z) + 3 \csc^2(z) - 1$. Let $f(z) = -\cot(z)/120$, and then we get

$$\csc^6(z) - \csc^4(z) + \frac{2}{15} \csc^2(z) = \sum_{m=-\infty}^{\infty} \frac{1}{(\pi m - z)^6} \quad (1.57)$$

and

$$\cot^6(z) = \sum_{m=-\infty}^{\infty} \frac{1}{(\pi m - z)^6} - 2 \sum_{m=-\infty}^{\infty} \frac{1}{(\pi m - z)^4} + \frac{23}{15} \sum_{m=-\infty}^{\infty} \frac{1}{(\pi m - z)^2} - 1. \quad (1.58)$$

For $n = 7$, we consider $\cot^7(z) = \cot(z) \csc^6(z) - 3 \cot(z) \csc^4(z) + 3 \cot(z) \csc^2(z) - \cot(z)$. Let $f(z) = \cot(z)/720$, then we get

$$\cot(z) \csc^6(z) - \frac{2}{3} \cot(z) \csc^4(z) + \frac{2}{45} \cot(z) \csc^2(z) = - \sum_{m=-\infty}^{\infty} \frac{1}{(\pi m - z)^7} \quad (1.59)$$

and

$$\begin{aligned} \cot^7(z) = & - \sum_{m=-\infty}^{\infty} \frac{1}{(\pi m - z)^7} + \frac{7}{3} \sum_{m=-\infty}^{\infty} \frac{1}{(\pi m - z)^5} \\ & - \frac{98}{45} \sum_{m=-\infty}^{\infty} \frac{1}{(\pi m - z)^3} + \sum_{m=-\infty}^{\infty} \frac{1}{(\pi m - z)}. \end{aligned} \quad (1.60)$$

1.5 Scope of the Dissertation

This thesis begins with a general introduction of the transient eddy current NDE technique and is followed by four chapters which are essentially four manuscripts to be published or submitted to the journal of IEEE Transactions on Magnetics.

The four chapters discuss the following problems in sequence: the calculation of the transient EMF induced in a pick-up coil due to a conductive plate, the derivation of a time domain dyadic Green's function due to an electric source in a plate, the calculation of the magnetic field due to a subsurface open crack in a plate, and the induced EMF in a pick up coil encircling an infinite rod.

In Chapter 1, the concept of the transient eddy current nondestructive evaluation is introduced as well as its advantages and disadvantages. Given two dissertations that have studied the similar topic [12], [31], only the most recent research work has been reviewed and this process is organized in terms of theoretical models and experimental studies. An emphasis has been given to the theoretical aspect which relates more closely to the work to be described. Two mathematic formulations have been addressed as a preliminary derivation for those involved in Chapter 2. To save the computational cost, the infinite integrals given in Dodd and Deeds's work are transformed into series form by imposing a cylindrical magnetic boundary at a certain distance from the source. Next, the Cauchy integral theorem has been used to write the power of the cotangent function $\cot^n(z)$ as partial fraction expansions.

In Chapter 2, analytical formulas are derived for the temporal variation of the induced EMF in a normal coil due to eddy currents in a plate excited by a coaxial driver coil. Based on a frequency domain expression, the time dependent response is written in series form using the inverse Laplace transform where the transformation is performed term by term both in the short and long time limits.

In Chapter 4, a time domain plate dyadic Green's function is derived using trans-

verse potential decomposition. With a pre-defined direction, the electromagnetic fields are represented by TE and TM potential, which satisfy the Helmholtz equation in the conductive region and the Laplace equation in the air. The scalar potentials are solved in the Laplace and Fourier domain by applying appropriate interface conditions. The derived formulae contains three parts, the free space terms, the multiple image terms and the partial reflection terms.

In Chapter 5, we calculate the magnetic field due to a subsurface open crack in the plate. Based on an ideal crack assumption, the crack is represented as a surface layer of current dipole, which is defined by an electric field integral equation with the dyadic Green's function derived in Chapter 4 as the kernel. The integral equation is solved using the moment method and time stepping and the crack response found by using reciprocity theorem.

In Chapter 6, the transient EMF induced in a pick-up coil encircling an infinite rod with the fields excited by a step or exponential electric current in a driver coil are evaluated. Two asymptotic series in time domain are derived based on a suitable expansion of the Laplace domain formula or its poles. The work in this chapter made a step forward towards the evaluation of transient response due to a pipe or defects in a cylindrical rod.

In Chapter 7, the major accomplishments of this dissertation are summarized and future work suggested.

CHAPTER 2. TRANSIENT EDDY CURRENT DRIVER PICK-UP PROBE RESPONSE DUE TO A CONDUCTIVE PLATE

2.1 Introduction

In transient eddy current nondestructive evaluation, the source coil carries a pulse excitation current instead of a time harmonic current. A current pulse in the driver coil produces a transient field in a conductor which migrates according to a vector diffusion equation and decays at a rate that depends on the coil size, the conductivity of the material and the permeability. The signal sensed by the pick-up coil, equivalent to a continuum of frequencies, may be used to estimate layer thicknesses and locate subsurface flaws by relating time delays in the response with flaw depth.

Smith and Hugo [1], [2] studied the capabilities and limitations of transient eddy current nondestructive evaluation through extensive experiments seeking methods to characterize defects and structural variations. Theoretical efforts in this direction led Burke, Hugo and Harrison [37] to evaluate the asymptotic behavior of the magnetic field at the surface of a layered conductor. They showed that the long time tail of the magnetic field varies as $t^{-3/2}$ for a half-space conductor but decays as t^{-3} for a plate and for a finite thickness stack of plates. By measuring the long time tail, they were able to estimate the total metal thickness, the result being insensitive to air gaps between layers. This work has implication for assessing the loss of material due to corrosion in

layered structures such as aircraft skins.

In a markedly different approach Yioultsis, Charitou, Antonopoulos, and Tsiboukis [38] calculated the transient electromagnetic field inside a conductor using a hybrid method combining a finite difference scheme for the diffusion equation with a boundary element method for the open regions. Their work focuses on the search for effective methods of formulating the finite difference algorithm to achieve both stability and accuracy. Hybrid approaches can potentially deal with complex structures but the computational cost may be high even when the desired accuracy is modest. For this reason, we have preferred to proceed via an analytical treatment for elementary structures such as a defect free plate while acknowledging that flaw calculations will usually require a numerical scheme. Time domain flaw calculations can be done efficiently if analytical expressions are available for a representing the fundamental solution in the unflawed structure. The elementary solution is a point dipole field which can be regarded as the time domain Green's function for the plate. Such solutions, can be represented conveniently using the asymptotic techniques used in the present study.

Setting aside the fundamental solution and the calculation of flaw signals for the future, our immediate goal is to extend results from earlier work in which the transient eddy current probe response due to a half-space conductor was evaluated [14]. A simple, explicit time domain solution is available for a half-space field because the necessary inverse Laplace transform can be carried out analytically without forming a series expansion. From a practical point of view in nondestructive evaluation however, it is more useful to have temporal solutions for a conducting plate as described in the present article.

Our starting point is the integral expression for the frequency domain electromotive force (EMF) induced in a normal pick-up coil due to a electric current in conductive plate excited by a normal driver coil coaxial with the pick-up. A normal coil is one whose axis is normal to the surface of the conductor. Here the integral expression for

the EMF is approximated as a summation by limiting the domain of the field to one with a finite radius, Section 2.2. This approximation reduces the computational cost [33] and is therefore a useful option, although not essential. Our primary task however, is to express the frequency domain induced EMF due to a conductive plate in a form that allows an analytical transformation to a time domain. There is more than one way of doing this as has been shown by de Haan and de Jong [39]. Here we present alternatives to their short and long time solutions due to a step current excitation and, in addition, treat the more general case of an exponential drive current, Section 2.3 and 2.4.

2.2 Field Formula

A coaxial driver coil and pick-up coil above a conductive plate are shown in Figure 2.1.

For the coils shown, the induced EMF in the pick-up is given by [32]

$$\Delta\tilde{V}(s) = \pi\mu_0 I_0 \int_0^\infty \frac{s\tilde{R}(\kappa, s)}{\kappa^6} \mathcal{J}_1(\kappa) \mathcal{J}_2(\kappa) d\kappa, \quad (2.1)$$

where

$$s\tilde{R}(\kappa, s) = s\tilde{I}(s)\Gamma_p(\kappa, s)/I_0 \quad (2.2)$$

contains the frequency dependence expressed in terms of the Laplace variable s , including the normalized coil current $\tilde{I}(s)/I_0$ and the plate reflection coefficient $\Gamma_p(\kappa, s)$. The driver and pick-up coil functions are defined by

$$\mathcal{J}_1(\kappa) = \frac{N_1 \chi(\kappa r_1, \kappa r_2) (e^{-\kappa z_2} - e^{-\kappa z_1})}{(r_2 - r_1)(z_2 - z_1)} \quad (2.3)$$

and

$$\mathcal{J}_2(\kappa) = \frac{N_2 \chi(\kappa r_3, \kappa r_4) (e^{-\kappa z_4} - e^{-\kappa z_3})}{(r_4 - r_3)(z_4 - z_3)} \quad (2.4)$$

respectively where

$$\chi(x_1, x_2) = \int_{x_1}^{x_2} x J_1(x) dx, \quad (2.5)$$

which can be evaluated using standard functions. N_1 and N_2 are the number of turns in the driver and pick-up coils respectively. Spatial parameters used in equations (2.3) and (2.4) are shown in Figure 2.1. For a plate with thickness d , the reflection coefficient is

$$\Gamma_p = \frac{\Gamma(e^{-2\gamma\lambda} - 1)}{\Gamma^2 e^{-2\gamma\lambda} - 1}, \quad (2.6)$$

where $\lambda = \kappa d$ and Γ is the reflection coefficient of a half-space conductor;

$$\Gamma = \frac{1 - c\gamma}{1 + c\gamma}, \quad (2.7)$$

in which $c = 1/\mu_r$ and $\gamma = (1 + s\tau)^{1/2}$, with $\tau = \mu_0 \mu_r \sigma / \kappa^2$. A formula for the EMF in the pick-up of parallel axis coils offset from one another has been given by Burke and Ibrahim [40].

Equation (2.1) contains a spatial-frequency integral from zero to infinity that takes some time to evaluate. As indicated in the introduction, to speed up the computation

and better control the accuracy, the solution region is truncated at a cylindrical boundary at a suitable distance b from the axis of the probe [33]. A condition that the tangential electric field vanishes on the boundary is imposed (Perfect Electric Conductor) . This additional condition means that the magnetic vector potential representing the field of a circular current filament of radius r_0 in the plane $z = z_0$ is of the form

$$\tilde{A}(r, z, s) = \sum_{i=1}^{\infty} J_1(\kappa_i r) e^{-\kappa_i |z - z_0|} A_i^{(0)}(s), \quad (2.8)$$

where the κ_i are chosen such that $J_1(\kappa_i b) = 0$. The artificial boundary condition approximates the unbounded domain solutions but the error involved is easily controlled by making b as large as desired. The coefficients in the expansion are determined from the discontinuity in the magnetic field at the plane of the filament. Writing the filament current density as $\mathbf{J} = I\delta(r - r_0)\delta(z - z_0)\hat{\phi}$, from Ampère's law we get

$$\left. \frac{\partial \tilde{A}_\phi}{\partial z} \right|^{+} - \left. \frac{\partial \tilde{A}_\phi}{\partial z} \right|^{-} = -\mu_0 I(s) \delta(r - r_0). \quad (2.9)$$

where the $\pm$ superscripts indicates that the limiting value of the z -coordinate is approached from above or below the filament plane respectively. Substituting for the vector potential from (2.8), multiplying by $rJ_1(\kappa_j r)$ and integrating between zero and b gives

$$\tilde{A}_i^{(0)}(s) = \mu_0 I(s) \frac{J_1(\kappa_i r_0) r_0}{\kappa_i b^2 J_0(\kappa_i b)}, \quad (2.10)$$

where we have used the orthogonality condition (11.4.5 of reference [34])

$$\int_0^b J_1(\kappa_j r) J_1(\kappa_i r) r dr = \begin{cases} \frac{b^2}{2} J_0^2(\kappa_i b) & i = j \\ 0 & \text{otherwise.} \end{cases} \quad (2.11)$$

The field produced by a current in a coil of rectangular cross-section can be calculated by linear superposition of the filament field and effects of the conductor on the field at the pick-up coil taken into account by including a reflection term. In this way it is found that the induced EMF due to a conductive plate is

$$\Delta \tilde{V}(s) = 2\pi\mu_0 I_0 \cdot \sum_{i=1}^{\infty} \frac{s \tilde{R}(\kappa_i, s)}{b^2 \kappa_i^7 J_0^2(\kappa_i b)} \mathcal{J}_1(\kappa_i) \mathcal{J}_2(\kappa_i). \quad (2.12)$$

The inverse Laplace transform of (2.12) is written

$$\Delta V(t) = 2\pi\mu_0 I_0 \cdot \sum_{i=1}^{\infty} \frac{R'(\kappa_i, t)}{b^2 \kappa_i^7 J_0^2(\kappa_i b)} \mathcal{J}_1(\kappa_i) \mathcal{J}_2(\kappa_i) \quad (2.13)$$

where, using the prime denoting the time derivative, $R'(\kappa_i, t)$ is the inverse Laplace transform of $s\tilde{R}(\kappa_i, s)$:

$$R'(\kappa_i, t) = \frac{1}{2i\pi} \int_{\text{Br}} s\tilde{R}(\kappa_i, s) e^{st} ds. \quad (2.14)$$

The path of integration, denoted by Br, is the Bromwich contour following a path in the complex s plane parallel to the imaginary axis and passing to the right of any poles. In order to carry out the inverse Laplace transform for both short and long time regimes, $\tilde{R}(\kappa, s)$ is expanded as uniformly convergent series:

$$\tilde{R}(\kappa, s) = \sum_{m=0}^{\infty} \tilde{R}_m(\kappa, s). \quad (2.15)$$

In the next two sections, expansions are derived which allow a term-by-term transformation to the time domain giving explicit results in the form

$$R'(\kappa, t) = \sum_{m=0}^{\infty} R'_m(\kappa, t). \quad (2.16)$$

We investigate the accuracy of a large time series solution truncated at $m_{\text{max}} = 2$ and a short time series truncated at $m_{\text{max}} = 4$. The large time series is more complicated and difficult to extend beyond the 3 terms used here but is reasonably accurate.

2.3 Short Time Approximation

First, a step current excitation is considered and later in this section, an exponential current pulse is analyzed for the short time regime. It should be noted however that the expansions derived in this section can be used well beyond times that might ordinarily be considered “short”. The first term in the series is identical to the term one finds when dealing with a half-space. Physically intuition indicates that this is the only term require

until the field migration from the top to the bottom of the plate and back produces a significant effect on the pick-up coil signal. Then an addition term is needed for accurate results. Further terms in the series correspond to further migration of the field between upper and lower surface of the plate.

2.3.1 Step Current

For this subsection it is assumed that the coil carries a prescribed current varying in time as

$$I(t) = I_0 u(t), \quad (2.17)$$

where $u(t)$ is a unit step function. The Laplace transform of the current is simply

$$\tilde{I}(s) = I_0 \frac{1}{s}. \quad (2.18)$$

For the short time regime, the reflection coefficient of the plate, equation (2.6), is written as

$$\Gamma_p = \Gamma - \frac{1 - \Gamma^2}{\Gamma} \sum_{m=1}^{\infty} (\Gamma e^{-\gamma\lambda})^{2m}. \quad (2.19)$$

This expansion converges because $|\Gamma e^{-\gamma\lambda}| < 1$. Following further steps we get

$$\Gamma_p = \frac{1 - c\gamma}{1 + c\gamma} + \frac{4}{c\gamma} \sum_{m=1}^{\infty} \left[\left(1 - \frac{2}{1 + c\gamma}\right)^{2m} + \frac{1}{(1 + c\gamma)^2} \left(1 - \frac{2}{1 + c\gamma}\right)^{2m-1} \right] e^{-2m\gamma\lambda}. \quad (2.20)$$

In view of the definition (2.2) and the frequency dependence of the current (2.18), $s\tilde{R}(\kappa, s) = \Gamma_p(\kappa, s)$. Terms in the series whose general form is given by (2.15) are here assigned as follows:

$$s\tilde{R}_0(\kappa, s) = \frac{1 - c\gamma}{1 + c\gamma} \quad (2.21)$$

and, for $m > 0$,

$$s\tilde{R}_m(\kappa, s) = \frac{4}{c\gamma} \left[\sum_{j=0}^{2m} C_{2m}^j \left(\frac{-2}{1 + c\gamma} \right)^j + \frac{1}{(1 + c\gamma)^2} \sum_{j=0}^{2m-1} C_{2m-1}^j \left(\frac{-2}{1 + c\gamma} \right)^j \right] e^{-2m\gamma\lambda} \quad (2.22)$$

where the $C_n^j = n!/(n-j)!j!$ are the binomial coefficients. The corresponding time domain terms can be written in the form of (2.16). $R'_0(\kappa, t)$ is identical to the result that arises for a half-space conductor and $R'_m(\kappa, t)$ ($m > 0$) is due to multiple reflections in the plate.

By using the inverse Laplace transform 29.3.37 in [34] and a linear transformation of the variable commonly known as the shift theorem, 29.3.14 [34], it is found that the inverse Laplace transform of $s\tilde{R}_0(\kappa, s)$ is

$$R'_0(\kappa, t) = -\delta(t) + \frac{2}{c^2\tau} e^{-t/\tau} \cdot \left[\sqrt{\frac{c^2\tau}{\pi t}} - e^{t/c^2\tau} \operatorname{erfc} \left(\frac{1}{c} \sqrt{\frac{t}{\tau}} \right) \right], \quad (2.23)$$

where $\operatorname{erfc}(x)$ is the complimentary error function.

In order to evaluate $R'_m(\kappa, t)$ from (2.22), consider

$$\tilde{\Psi}_n(a, k, s) = \frac{1}{\sqrt{s}(a + \sqrt{s})^{n+1}} e^{-k\sqrt{s}}. \quad (2.24)$$

Note that

$$(-1)^n \frac{d^n}{dk^n} [e^{-ak} \tilde{\Psi}_n(a, k, s)] = \frac{1}{\sqrt{s}(a + \sqrt{s})} e^{-k(a+\sqrt{s})} \quad (2.25)$$

which can be inverted according to 29.3.90 of [34], and integrated n times to give

$$\Psi_n(a, k, t) = (4t)^{\frac{n}{2}} e^{ak+a^2t} i^n \operatorname{erfc} \left(a\sqrt{t} + \frac{k}{2\sqrt{t}} \right), \quad (2.26)$$

where $i^k \operatorname{erfc}(x)$ is the k^{th} integral of the complimentary error function. Thus transforming (2.22) gives

$$\begin{aligned} R'_m(\kappa, t) = & -\frac{2}{c^2\tau} e^{-t/\tau} \left[\sum_{j=0}^{2m} C_{2m}^j \frac{(-2)^{j+1}}{c^{j-1}} \cdot \Psi_{j-1} \left(\frac{1}{c}, 2m\lambda, \frac{t}{\tau} \right) \right. \\ & \left. + \sum_{j=0}^{2m-1} C_{2m-1}^j \cdot \frac{(-2)^{j+1}}{c^{j+1}} \Psi_{j+1} \left(\frac{1}{c}, 2m\lambda, \frac{t}{\tau} \right) \right]. \end{aligned} \quad (2.27)$$

Equation (2.23) and (2.27) are combined with (2.13) to give the induced EMF in the pick-up coil due to a plate excited by a step function current in a driver coil. Note that $R'(\kappa, t)$ is here expressed as a double summation whereas in reference [39] it is a triple summation.

2.3.2 Exponential Current

Consider a prescribed exponential coil current

$$I(t) = I_0(1 - e^{-t/\tau_0})u(t), \quad (2.28)$$

where τ_0 is the characteristic rise time of the current. The Laplace transform of equation (2.28) is

$$\tilde{I}(s) = I_0 \left(\frac{1}{s} - \frac{\tau_0}{s\tau_0 + 1} \right) \quad (2.29)$$

and in this case equation (2.2) becomes

$$s\tilde{R}(\kappa, s) = \frac{1}{s\tau_0 + 1} \Gamma_p(\kappa, s). \quad (2.30)$$

Clearly, the presence of the term $1/(s\tau_0 + 1)$ make the inverse Laplace transform more complicated. Terms in the series whose general form is given by (2.15) are here assigned as follows:

$$s\tilde{R}_0(\kappa, s) = \frac{1}{s\tau_0 + 1} \cdot \frac{1 - c\gamma}{1 + c\gamma} \quad (2.31)$$

and, for $m > 0$,

$$\begin{aligned} s\tilde{R}_m(\kappa, s) = & \frac{1}{s\tau_0 + 1} \cdot \frac{4}{c\gamma} \left[\sum_{j=0}^{2m} C_{2m}^j \left(\frac{-2}{1 + c\gamma} \right)^j \right. \\ & \left. + \frac{1}{(1 + c\gamma)^2} \sum_{j=0}^{2m-1} C_{2m-1}^j \left(\frac{-2}{1 + c\gamma} \right)^j \right] \cdot e^{-2m\gamma\lambda}. \end{aligned} \quad (2.32)$$

From (2.31) it is found that the inverse Laplace transform of $s\tilde{R}_0(\kappa, s)$ is [14]

$$\begin{aligned} \tau_0 R'_0(\kappa, t) = & \frac{2}{(1 - c^2\nu\tau)} \left\{ e^{-t/\tau_0} \left[1 - c\sqrt{\nu\tau} \operatorname{erf}\sqrt{\nu t} \right] \right. \\ & \left. - e^{(1-c^2)t/c^2\tau} \operatorname{erfc} \left(\frac{1}{c} \sqrt{\frac{t}{\tau}} \right) \right\} - e^{-t/\tau_0} \end{aligned} \quad (2.33)$$

where

$$\nu = \frac{1}{\tau} - \frac{1}{\tau_0}. \quad (2.34)$$

To evaluate $R'_m(\kappa, t)$ ($m > 0$) from the inverse Laplace transform of (2.32), we follow a similar procedure to that followed in the step current case. Consider

$$\tilde{\Phi}_n(a, b, k, s) = \frac{1}{\sqrt{s}} \frac{1}{(\sqrt{s} + a)^n (\sqrt{s} + b)} e^{-k\sqrt{s}} \quad (2.35)$$

and note that a standard form is obtain by differentiating n times with respect to k to give

$$(-1)^n \frac{d^n}{dk^n} [e^{-ak} \tilde{\Phi}_n(a, b, k, s)] = \frac{1}{\sqrt{s}(\sqrt{s} + b)} e^{-k(a+\sqrt{s})}. \quad (2.36)$$

Taking the inverse Laplace transform and integrating n times gives

$$\begin{aligned} \Phi_n(a, b, k, t) &= (-1)^n e^{ak} e^{b^2 t} \int \int \dots \int^n e^{(b-a)k} \cdot \\ &\quad \operatorname{erfc} \left(b\sqrt{t} + \frac{k}{2\sqrt{t}} \right) dk dk \dots dk. \end{aligned} \quad (2.37)$$

Using the standard integral 7.4.36 in [34]

$$\int e^{ax} \operatorname{erf}(bx) dx = \frac{1}{a} \left[e^{ax} \operatorname{erf}(bx) - e^{\frac{a^2}{4b^2}} \operatorname{erf} \left(bx - \frac{a}{2b} \right) \right], \quad (2.38)$$

we get

$$\int e^{(b-a)k} \operatorname{erfc} \left(b\sqrt{t} + \frac{k}{2\sqrt{t}} \right) dk = \frac{e^{-b^2 t - ka}}{(b-a)} \cdot [\Psi_0(b, k, t) - \Psi_0(a, k, t)]. \quad (2.39)$$

Integrating a further $n-1$ times and multiplying by $(-1)^n e^{ak} e^{b^2 t}$ gives the inverse Laplace transform of (2.35)

$$\Phi_n(a, b, k, t) = \frac{1}{(a-b)^n} \left[\Psi_0(b, k, t) - \sum_{j=0}^{n-1} (a-b)^j \Psi_j(a, k, t) \right]. \quad (2.40)$$

This results is then used to write the inverse Laplace transform of (2.32) as

$$\begin{aligned} \tau_0 R'_m(\kappa, t) &= \varphi \left(\sqrt{\tau\nu}, 2m\lambda, \frac{t}{\tau} \right) + \sum_{j=1}^{2m} C_{2m}^j (-2)^{j+2} \phi_j \left(\frac{1}{c}, \sqrt{\tau\nu}, 2m\lambda, \frac{t}{\tau} \right) \\ &\quad + \sum_{j=0}^{2m-1} C_{2m-1}^j (-2)^{j+2} \phi_{j+2} \left(\frac{1}{c}, \sqrt{\tau\nu}, 2m\lambda, \frac{t}{\tau} \right), \end{aligned} \quad (2.41)$$

where

$$\varphi(b, k, t) = \frac{2e^{-t}}{bc} [\Psi_0(-b, k, t) - \Psi_0(b, k, t)], \quad (2.42)$$

and

$$\phi_n(a, b, k, t) = \frac{e^{-t}}{c^{n+1}} \left[\frac{1}{a^2 - b^2} \Psi_{n-1}(a, k, t) + \frac{1}{2b(a+b)} \Phi_{n-1}(a, -b, k, t) - \frac{1}{2b(a-b)} \Phi_{n-1}(a, b, k, t) \right] \quad n > 0. \quad (2.43)$$

In equation (2.41), the argument $\sqrt{\tau\nu}$ is imaginary if $\tau_0 < \tau$, see (2.34). However, the imaginary parts cancel with each other because in both (2.42) and (2.43), they appear as the sum of a function and its complex conjugate. Equations (2.41-2.43) complete the definition of the terms in (2.16) for the short time series.

2.4 Long Time Approximation

2.4.1 Step Current

A series expansion in the large time regime, can be found following the approach used by de Haan and de Jong [39] by expressing the plate reflection coefficient from equations (2.6) and (2.7) as

$$\Gamma_p = \frac{1 - c^2\gamma^2}{1 + c^2\gamma^2 + 2c\gamma \coth(\gamma\lambda)}. \quad (2.44)$$

For a step function current excitation in the drive coil, this leads to the expansion

$$s\tilde{R}(\kappa, s) = \frac{3}{2u+3} \frac{u^2 - \gamma^2\lambda^2}{\gamma^2\lambda^2 + a} \cdot \sum_{m=0}^{\infty} \left[\left(\frac{2u}{2u+3} \right) \frac{H(\gamma\lambda)}{\gamma^2\lambda^2 + a} \right]^m, \quad (2.45)$$

where

$$a = \frac{3u(u+2)}{2u+3}, \quad u = \lambda/c, \quad (2.46)$$

and

$$H(z) = z^2 + 3 - 3z \coth(z), \quad (2.47)$$

The function $H(z)$ can be represented as a polynomial in even powers starting at the forth power of its argument, see 4.5.67 in [34]. It can therefore be approximated by a

few terms if its argument is small. Below we use a linear transformation to get a change of variable $\gamma\lambda \rightarrow \sqrt{s}$ which shows that the summation in (2.45) may be approximated by a few term provided only low temporal frequencies s are needed. This will be the case in the long time regime where the induced EMF varies slowly. Comparing (2.45) with the general form, equation (2.15), one finds that the terms of the expansion are given by

$$s\tilde{R}_m(\kappa, s) = \frac{3(2u)^m}{(2u+3)^{m+1}} \frac{u^2 - \gamma^2\lambda^2}{(\gamma^2\lambda^2 + a)^{m+1}} \sum_{k=0}^m C_m^k (-3\gamma\lambda)^k \cdot (\gamma^2\lambda^2 + 3)^{m-k} \coth^k(\gamma\lambda). \quad (2.48)$$

For $m = 0$, it is straightforward to get the inverse Laplace transform since there is only a first order pole to deal with associated with the factor $(\gamma^2\lambda^2 + a)$ in the denominator of (2.48):

$$R'_0(\kappa, t) = \frac{3}{(2u+3)} \left\{ \frac{e^{-t/\tau}}{\lambda^2\tau} (u^2 + a) e^{-\frac{a}{\lambda^2\tau}t} - \delta(t) \right\}. \quad (2.49)$$

For higher order terms, $\coth^j(z)$ can be expressed as partial fraction expansions using

$$\frac{d^n}{dz^n} \coth(z) = -n! \sum_{\nu=-\infty}^{\infty} \frac{1}{(i\pi\nu - z)^{n+1}}. \quad (2.50)$$

For example with $n = 0$,

$$\coth z = \frac{1}{z} + 2z \sum_{j=1}^{\infty} \frac{1}{z^2 + j^2\pi^2}, \quad (2.51)$$

see 4.5.12 and 4.3.91 of [34]. Equation (2.50) can be derived by taking the n th order derivative of $\coth(z)$ and the partial fraction expansion of $\coth(z)$ derived using a generalization of Cauchy's theorem [35]. Apart from the pole of order $m+1$ in (2.48), the $\coth$ function gives rise to a k th order poles ($k = 1, 2 \dots m$) written as $(\gamma^2\lambda^2 + j^2\pi^2)^{-k}$ which means $\tilde{R}_m(\kappa, s)$ takes the form

$$s\tilde{R}_m(\kappa, s) = \frac{3(2u)^m}{(2u+3)^{m+1}} \left\{ \frac{f_m(\gamma^2\lambda^2, u, a)}{(\gamma^2\lambda^2 + a)^{m+1}} - 1 + \sum_{k=1}^m \sum_{j=1}^{\infty} \frac{1}{(a - j^2\pi^2)^{m+k}} \frac{g_{m,k}(\gamma^2\lambda^2, u, a, j^2\pi^2)}{(\gamma^2\lambda^2 + a)^{m+1} (\gamma^2\lambda^2 + j^2\pi^2)^k} \right\} \quad (2.52)$$

where $f_m(z, u, a)$, and $g_{m,k}(z, u, a, b)$ are a polynomials in z of order m and $m + k$ respectively. Applying the Heaviside expansion theorem gives

$$\begin{aligned} R'_m(\kappa, t) = & \frac{3(2u)^m}{(2u+3)^{m+1}} \left\{ \frac{e^{-t/\tau}}{\lambda^2 \tau} F_m \left(\frac{t}{\lambda^2 \tau}, u, a \right) e^{-\frac{a}{\lambda^2 \tau} t} - \delta(t) \right. \\ & + \frac{e^{-t/\tau}}{\lambda^2 \tau} \sum_{k=1}^m \sum_{j=1}^{\infty} \left[\frac{P_{m,k} \left(\frac{t}{\lambda^2 \tau}, u, a, j^2 \pi^2 \right) e^{-\frac{a}{\lambda^2 \tau} t}}{(a - j^2 \pi^2)^{m+k}} \right. \\ & \left. \left. + \frac{Q_{m,k} \left(\frac{t}{\lambda^2 \tau}, u, a, j^2 \pi^2 \right) e^{-\frac{j^2 \pi^2}{\lambda^2 \tau} t}}{(a - j^2 \pi^2)^{m+k}} \right] \right\}. \end{aligned} \quad (2.53)$$

For $m = 0$ we get

$$F_0(t, u, a) = u^2 + a. \quad (2.54)$$

For $m = 1$, we get

$$F_1(t, u, a) = -at(a + u^2) + 2a + u^2 - \delta(t), \quad (2.55)$$

$$P_{1,1}(t, u, a, b) = -6[at(a - b)(u^2 + a) - a^2 + 2ab + bu^2], \quad (2.56)$$

and

$$Q_{1,1}(t, u, a, b) = 6b(u^2 + b). \quad (2.57)$$

For $m = 2$, we take advantage of 4.3.92 of [34] and expand $\coth^2(z)$ as

$$\coth^2(z) = 1 + \frac{1}{z^2} + \sum_{j=1}^{\infty} \frac{2(z^2 - j^2 \pi^2)}{(z^2 + j^2 \pi^2)^2}. \quad (2.58)$$

Also, we need

$$F_2(t, u, a) = \frac{t^2}{2} a(a - 9)(u^2 + a) - t(2u^2 a - 9u^2 + 3a^2 - 18a) + u^2 + 3a - 9, \quad (2.59)$$

$$\begin{aligned} P_{2,1}(t, u, a, b) = & -12 \left\{ \frac{t^2}{2} a(a - 3)(a - b)^2(u^2 + a) - t(a - b) \cdot \right. \\ & \left[u^2(a^2 - 2ab + 3b) + a[2a^2 - 3a(b + 1) + 6b] \right] \\ & \left. + a^3 - 3a^2 b + 3ab^2 + b[u^2(b - 3) - 3b] \right\}, \end{aligned} \quad (2.60)$$

$$Q_{2,1}(t, u, a, b) = 12b(b-3)(u^2 + b), \quad (2.61)$$

$$\begin{aligned} P_{2,2}(t, u, a, b) = 9 \bigg\{ & at^2(a+b)(a-b)^2(u^2 + a) \\ & - 2t(a-b)[a(a^2 - 3ab - 2b^2) - bu^2(3a + b)] \\ & + 2b[3u(a^2 + b) + b(5a + b)] \bigg\}, \end{aligned} \quad (2.62)$$

and

$$Q_{2,2}(t, u, a, b) = 18b[2bt(a-b)(u^2 + b) - 3u^2(a+b) - b(5a + b)]. \quad (2.63)$$

In this way, $R'_0(\kappa, t)$, $R'_1(\kappa, t)$ and $R'_2(\kappa, t)$ are obtained. Usually, three terms give results of good numerical accuracy. For an exponential driving current, a similar derivation can be given starting with $sR(\kappa, s) = 1/(s\tau_0 + 1)\Gamma_p(\kappa, s)$. The formulas for this case are given in the Appendix.

2.5 Numerical Results

The results given in this section are based on the plate parameters given in Table 2.1 and the probe parameters given in Table 2.2. The radius of the truncation boundary, b , was set at 40 times the outer radius of the coil and the infinite series approximating the spatial frequency integration, equation (2.13), truncated at sixty terms. Both step and exponential current excitations are considered in these calculations. Figures 2.2 and 2.3 show the EMF in a pick-up coil due to a step excitation calculated using both short time and long time asymptotic solutions. The small time and large time approximation agree with each other very well at intermediate times however, the short time algorithm starts to become unstable at large times.

Figure 2.4 shows the discrepancy of the results in comparison with calculations using a fast Fourier transform (FFT) with 2^{17} sample points. Note that in order to represent

Table 2.1 Plate parameters.

Thickness d (mm)	Relative Permeability μ_r	Conductivity σ (MS/m)
1	1	22.5

Table 2.2 Probe parameters.

Coil parameters	Pick-up coil	Drive coil
Inner radius (mm)	2.5	7.10
Outer radius (mm)	5.0	11.41
Length (mm)	2.5	5.0
Number of turns	100	2550
Liftoff (mm)	2.5	1.5
τ_0 (μ s)	-	250

Figure 2.2 Induced voltage in a pick up coil due to eddy currents in a conducting plate excited by a step current in a drive coil. Time range of data $20ns$ to $5\mu s$.

Figure 2.3 Induced voltage in a pick up coil due to eddy currents in a conducting plate excited by a step current in a drive coil. Time range of data $5\mu s$ to $1200\mu s$.

the discontinuity in the response at $t = 0$ accurately, the FFT requires a large bandwidth. The errors in the FFT can be reduced by increasing the number of sample points as shown in Figure 2.5, but a very large number of points is needed to achieve a accuracy similar to that of the short time series which is exact in the limit as $t \rightarrow 0$. The computational time for the FFT plus the assignment of data values to arrays is roughly the same as for the direct short time solution. For example 1024 values are calculated using the short time series in 155 sec. To achieve similar accuracy using the FFT, we need to evaluate the results at more than 2^{17} sample points but even this number of points requires 207 seconds. These results were obtained on a 2 GHz personal computer using Mathematica. Although the computational time for the FFT varies as $N \log N$, most of the time for both the series solution and the FFT is taken up with the evaluation of functions, the processing time being proportional to the number of sample points N .

It can be seen from Figure 2.4 that the short time solution gives better accuracy

Figure 2.4 Error relative to the FFT prediction of the induced EMF in a pick-up coil due to eddy currents in a conducting plate excited by a step current in a driver coil.

than the long time solution even in the range up to 1 msec. and beyond. This is partly because we are able to use more terms; 5 or more for the short time expansion compared with 3 for the long time expansion. A practical range for the truncated short time series is limited to a time t such that $t < m_0 c \lambda \tau$ where m_0 is order beyond which terms are negligible. Using the estimate $\kappa \approx 1/\bar{r}$ where $\bar{r}$ is the mean radius of the probe, we find that $c \lambda \tau \approx 200 \mu\text{sec}$ for the parameters in Table II. As Figure 2.6 shows, the solution represented by (2.23) and (2.27) with (2.16) is accurate at early times even when only one term of the expansion is used. It remains accurate at a longer time if a sufficient number of terms is used. The solutions found by terminating at different value of m_{max} give practically the same result at $200 \mu\text{sec}$. For later times, however, results generated with different values of m_{max} give rise to divergent results. After $400 \mu\text{sec}$ the solution with $m_{\text{max}} = 2$ starts to diverge from the higher order responses and after $600 \mu\text{sec}$ the transient estimated with $m_{\text{max}} = 3$ diverges from that with $m_{\text{max}} = 4$.

Figure 2.5 Comparison of the relative error in the induced EMF calculated using the FFT compared with the results for the short time series solution. This shows the error in the FFT result decreasing as the number of sample points is increased from 2^{16} to 2^{17} and finally 2^{18} points.

Figure 2.6 Small time approximation for $m_{\max} = 0, 1, 2, 3, 4$.

The solution defined by (2.49) and (2.53) with (2.16) is described as large time series because it is accurate at large times even when only one or two terms are used. It is usually accurate over a wide range of time if the plate thickness is relatively small compared with the coil radii. However it is difficult to quantify this range more precisely due to the complexity of the series. A conservative estimate would be to require $t \gg c\lambda\tau$. As Figure 2.7 shows, the solution is insensitive to the choice of $m_{\max}$ if this criterion is adopted (in the case considered $c\lambda\tau \approx 200\mu\text{sec}$). However it is also evident from Figure 2.7 that the series converges rapidly and can be used to estimate the response for $t < c\lambda\tau$.

An exponential current is shown in Figure 2.8 in which the time constant is $\tau_0 = 275\mu\text{s}$ and Figure 2.9 gives the corresponding response of the pick-up coil for a probe above a 1 mm thick plate. The result shows that the small time and large time approximation agree very well for all times considered. Figure 2.11, shows a response calculated using the large time approximation. The variation of EMF in a pick-up coil due to a exponential excitation in a drive coil is shown for various values of plate thickness. Note that for small plate thicknesses the response decays more rapidly.

Figure 2.7 Large time approximation for $m_{\text{max}} = 0, 1, 2$.

Figure 2.8 Exponential driving current in the driver coil

Figure 2.9 Induced voltage in a pickup coil due to a conducting plate excited by an exponential current in the driver coil; drive current time constant $\tau_0 = 275\mu\text{s}$.

Figure 2.10 Error relative to the FFT prediction of the induced EMF in a pick-up coil due to eddy currents in a conducting plate excited by an exponential current ($\tau_0 = 275\mu\text{s}$) in a driver coil.

Figure 2.11 Effect of plate thickness on the induced voltage in a pickup coil due to induced current in a conducting plate excited by an exponential current in the driver coil; drive current time constant $\tau_0 = 275\mu s$.

2.6 Conclusion

Analytical formulas have been derived for the temporal variation of induced EMF in a normal coil due to eddy currents in a plate excited by a coaxial driver coil. Asymptotic series expressions for short and long time regimes give consistent results at intermediate times.

The case of an exponential drive coil current has been considered here, the step current being a special case in which the rise time is zero. Other functional forms for the drive current can be accommodated by using the principle of superposition and approximations. For example, the present analysis can be extended to the case of a linear ramp current excitation of finite duration. An arbitrary current profile can then be approximated as piecewise linear, and an approximate solution for an arbitrary drive current variation expressed as a superposition of finite-duration ramp solutions.

Predictions of the coil signals due to a plate can also be made by the traditional method of transforming the frequency domain solutions numerically using the fast Fourier transform. However, the series representation offers new possibilities for signal and field analysis. Examples include the derivation of signals for more complex conductive layers and the representation of the Green's function for a plate in the form of a temporal asymptotic series.

2.7 Appendix: Long Time Approximation for Exponential Current

Here the long time approximation of Section 2.4 is extended to the case where the coil driver current is exponential in time. With an exponential rise time τ_0 , equation (2.52) becomes

$$s\tilde{R}_m(\kappa, s) = \frac{3(2u)^m}{(2u+3)^{m+1}} \frac{1}{s\tau_0+1} \left\{ \frac{f_m(\gamma^2\lambda^2, u, a)}{(\gamma^2\lambda^2+a)^{m+1}} + \sum_{k=1}^m \sum_{j=1}^{\infty} \frac{1}{(a-j^2\pi^2)^{m+k}} \cdot \frac{g_{m,k}(\gamma^2\lambda^2, u, a, j^2\pi^2)}{(\gamma^2\lambda^2+a)^{m+1}(\gamma^2\lambda^2+j^2\pi^2)^k} \right\}. \quad (2.64)$$

Following the approach used for step current case, $R'_m(\kappa, t)$ is written

$$R'_m(\kappa, t) = \frac{3(2u)^m}{(2u+3)^{m+1}} \cdot \left\{ \frac{E_m(u, a)e^{-t/\tau_0} + F_m(\frac{t}{\lambda^2\tau}, u, a)e^{-t/\tau}e^{-\frac{a}{\lambda^2\tau}t}}{[(\lambda^2\nu\tau+a)\tau_0]^{m+1}} + \sum_{k=1}^m \sum_{j=1}^{\infty} \left[\frac{O_{m,k}(u, a, j^2\pi^2)e^{-t/\tau_0}}{[(\lambda^2\nu\tau+a)\tau_0]^{m+1}[(\lambda^2\nu\tau+b)\tau_0]^k} + \frac{e^{-t/\tau}}{(a-j^2\pi^2)^{m+k}} \left[\frac{P_{m,k}(\frac{t}{\lambda^2\tau}, u, a, j^2\pi^2)e^{-\frac{a}{\lambda^2\tau}t}}{[(\lambda^2\nu\tau+a)\tau_0]^{m+1}} + \frac{Q_{m,k}(\frac{t}{\lambda^2\tau}, u, a, j^2\pi^2)e^{-\frac{j^2\pi^2}{\lambda^2\tau}t}}{[(\lambda^2\nu\tau+b)\tau_0]^k} \right] \right] \right\}. \quad (2.65)$$

For $m = 0$, we get

$$E_0(u, a) = u^2 - \lambda^2\nu\tau, \quad (2.66)$$

and

$$F_0(u, a) = -(u^2 + a), \quad (2.67)$$

where ν has been defined in (2.34). For $m = 1$, it is found that

$$E_1(u, a) = \lambda^2\nu\tau\tau_0(u^2 - \lambda^2\nu\tau), \quad (2.68)$$

$$F_1(t, u, a) = \frac{at}{\tau_0}(u^2 + a)(\lambda^2\nu\tau + a\tau_0^2) - \lambda^2\nu\tau\tau_0(u^2 + 2a) - a^2\tau_0, \quad (2.69)$$

$$O_{1,1}(u, a, b) = -6\lambda^2\nu\tau\tau_0(u^2 - \lambda^2\nu\tau)\tau_0, \quad (2.70)$$

$$P_{1,1}(t, u, a, b) = 6\frac{at}{\tau_0}(a-b)(u^2 + a)(\lambda^2\nu\tau + a\tau_0^2) + 6a^2\tau_0(u^2 + b) - 6\lambda^2\nu\tau\tau_0(a^2 - 2b - bu^2), \quad (2.71)$$

and

$$Q_{1,1}(t, u, a, b) = -6b(u^2 + b). \quad (2.72)$$

Finally, for $m = 2$,

$$E_2(u, a) = \lambda^2 \nu \tau \tau_0^2 (\lambda^2 \nu \tau - u^2) (\lambda^2 \nu \tau + 9), \quad (2.73)$$

$$\begin{aligned} F_2(t, u, a) = & -\frac{t^2}{2} a(a-9)(u^2+a)(\lambda^2 \nu \tau + a)^2 \tau_0^2 + t(\lambda^2 \nu \tau + a) \tau_0 [a^2 \tau_0 (u^2 + 2a - 9) \\ & - \lambda^2 \nu \tau \tau_0 [u^2(2a-9) + 3a(a-6)]] - \lambda^4 \nu^2 \tau^2 \tau_0^2 (u^2 + 3a - 9) \\ & - 3\lambda^2 \nu \tau \tau_0^2 (a^2 + 3u^2) - a^3 \tau_0^2, \end{aligned} \quad (2.74)$$

$$O_{2,1}(u, a, b) = 12\lambda^2 \nu \tau \tau_0^3 (\lambda^2 \nu \tau + 3)(\lambda^2 \nu \tau - u^2), \quad (2.75)$$

$$\begin{aligned} P_{2,1}(t, u, a, b) = & -6at^2(a-3)(u^2+a)(a-b)^2(\lambda^2 \nu \tau + a)^2 \tau_0^2 + 12t(a-b)^2(\lambda^2 \nu \tau + a) \tau_0 \cdot \\ & \left\{ \lambda^2 \nu \tau \tau_0 [u^2(a^2 - 2ab + 3b) + a[2a^2 - 3a(b+1) + 6b]] \right\} \\ & - 12 \left\{ \lambda^4 \nu^2 \tau^2 \tau_0^2 b [u^2(b-3) - 3b] + \lambda^4 (\tau^2 + \tau_0^2) (a^3 - 3a^2b + 3ab^2) \right. \\ & - \lambda^2 \tau \tau_0 [a^2(3b-a)(u^2 - 2\lambda^2 + b - 3) + 3ab(2b\lambda^2 - 3u^2) + 3b^2u^2] \\ & + \tau_0^2 \left[-\lambda^2 [a^2(a-3b)(b-3) + u^2[a^3 - 3ab(a-3) - 3b^2]] \right. \\ & \left. \left. + a^3(b-3)(u^2+b) \right] \right\}, \end{aligned} \quad (2.76)$$

$$Q_{2,1}(t, u, a, b) = 12b(u^2 + b)(b-3), \quad (2.77)$$

$$O_{2,2}(u, a, b) = -18\lambda^2 \nu \tau \tau_0^4 (\lambda^2 \nu \tau - b)(\lambda^2 \nu \tau - u^2), \quad (2.78)$$

$$\begin{aligned} P_{2,2}(t, u, a, b) = & -9at^2(a-b)^2(a+b)(u^2+a)(\lambda^2 \nu \tau + a)^2 \tau_0^2 \\ & + 18t(a-b) \left\{ \lambda^2 \tau_0^2 (\lambda^2 \nu \tau + a) \cdot [a(a^2 - 3ab - 2b^2) - bu^2(3a+b)] \right. \\ & - \lambda^2 \tau \tau_0 \cdot [a^2(a^2 - 6ab - 3b^2) - au^2(a^2 + 6ab + b^2)] \\ & \left. - \tau_0^2 (\lambda^2 + a) [a^2 u^2 (a+3b) + a^2 b(3a+b)] \right\} \end{aligned}$$

$$\begin{aligned}
& -18\tau_0^2 \left\{ b\lambda^4\nu^2\tau^2 \left[3u^2(a+b) + b(5a+b) \right] + \lambda^2\nu\tau \cdot \right. \\
& \left[a^4 - 4a^3b - 9a^2b(u^2+b) - 4ab^2u^2 + b^3u^2 \right] \\
& \left. - a^3 \left[u^2(a+5b) + 3b(a+b) \right] \right\}, \tag{2.79}
\end{aligned}$$

and

$$\begin{aligned}
Q_{2,2}(t, u, a, b) = 18b\tau_0 \left\{ -2bt(a-b)(u^2+b)(\lambda^2\nu\tau + b) + \lambda^2\nu\tau \left[3u^2(a+b) + b(5a+b) \right] \right. \\
\left. + b \left[u^2(a+5b) + 3b(a+b) \right] \right\}. \tag{2.80}
\end{aligned}$$

CHAPTER 3. EVALUATION OF TRANSIENT MAGNETIC FIELD DUE TO A PLATE USING ASYMPTOTIC SOLUTIONS

3.1 Introduction

Transient eddy current measurements provide a simple and convenient means of sensing flaws and estimating their depth below the surface of a conductor. The transient responses are usually measured in differential mode by subtracting a reference signal obtained from a defect free region. Peak amplitude, peak arrival time and zero crossing time have been identified as important characteristics of the signal giving indications of the location and size of defects. Bieber et al [21] used the peak amplitude data to evaluate the metal loss caused by corrosion. However, it is found that variations of lift-off and inter-layer gap can cause noise which may overwhelm the material loss signature. Giguère et al [23], [26] minimized the effect of lift-off by taking advantage of the amplitude of transient response at the lift-off point of intersection which occurs at a time following the initiation of the transient when the lift-off signal is at a null. Similarly, at a later time there is a point where the effect of plate separation is at a null allowing detection of material loss at a point in time that is insensitive to plate separation. Based on the asymptotic behavior of transient response in short and long time limit, Burke et al [27] corrected the lift-off effect by multiplying the signal by a time varying function, called the lift-off scaling factor. More extensive analysis, however, is

required to improve the reliability of quantitative transient eddy current nondestructive evaluation.

Due to the nature of electromagnetic field migration in a conductor, the reflected electromagnetic pulse is delayed in time and broadened spatially. As a result, the structural features nearer the material surface affect the initial part of eddy current response and deeper structure perturbs the long time signal behavior. According to recent studies [41], [39], the short and long time regimes lend themselves to analytical treatments using series expansions. Although transient fields can be computed from multi-frequency time-harmonic data, it is often more effective and accurate to calculate the field directly from time domain solutions. In this chapter, we extend the short and long time asymptotic series analysis in Chapter 2 to compute the transient magnetic field due to a conducting plate.

3.2 Theory

The probe and conductive plate used in this analysis are shown in Figure 3.1. The plate is assumed to have constant thickness and be infinite in the radial direction. The driver and pick-up coil share the same axis, which is defined as the z axis. The frequency domain coil response due to a multi-layer structure has been derived by first deducing the field of a circular filament and then superimposing filament solutions to find the field of a coil [32]. For the case of a coil above a half-space conductor, the frequency domain solution has been transformed analytically to the time domain [14] using the inverse Laplace transform. In the case of a conducting plate, the magnetic field on the axis of a driver coil has the form

$$\Delta \widetilde{H}_z(z, s) = \frac{I_0}{2} \int_0^\infty \frac{\widetilde{R}(\kappa, s)}{\kappa^2} \mathcal{J}_1(\kappa) e^{-\kappa z} d\kappa, \quad (3.1)$$

Figure 3.1 Driver coil and Hall sensor above a conductive plate.

where the frequency dependence is written in terms of the Laplace variable s . The function

$$\tilde{R}(\kappa, s) = I(s)\Gamma_p(\kappa, s)/I_0 \quad (3.2)$$

contains frequency dependent terms, including the normalized coil current $I(s)/I_0$ and the plate reflection coefficient, $\Gamma_p(\kappa, s)$. The coil functions are defined by (2.3). The reflection coefficient for a plate with thickness d is given in terms of the half-space reflection coefficient Γ by

$$\Gamma_p = \frac{\Gamma(1 - e^{2\gamma\lambda})}{\Gamma^2 - e^{2\gamma\lambda}} \quad \text{with} \quad \Gamma = \frac{1 - \gamma c}{1 + \gamma c}, \quad (3.3)$$

where $\gamma = (1 + s\tau)^{1/2}$, $\tau = \mu_0\mu_r\sigma/\kappa^2$, $\lambda = \kappa d$ and $c = 1/\mu_r$. Equations (3.1) involve spatial-frequency integrals from zero to infinity that are time consuming to evaluate. However, the computation cost can be reduced and better control of the accuracy achieved by truncating the solution region. This is done by imposing a cylindrical magnetic boundary at a chosen distance b from the axis of the probe [42], which means that the magnetic vector potential is set to zero at $r = b$. The presence of the boundary

introduces an error in the solution but the value of b can be made as large as necessary to reduce the domain truncation error. As a result, the coil solution is expressed in the form of series instead of an integral. Thus, the magnetic field on the coil axis is

$$\Delta \widetilde{H}_z(z, s) = I_0 \sum_{i=1}^{\infty} \frac{\widetilde{R}(\kappa_i, s)}{b^2 \kappa_i^3 J_0^2(\kappa_i b)} \cdot \mathcal{J}_1(\kappa_i) e^{-\kappa_i z}. \quad (3.4)$$

The discrete eigenvalue, κ_i , is introduced instead of the continuous κ due to an imposed homogeneous Dirichlet boundary condition on the magnetic vector potential: $A_\phi = 0$ at $r = b$ [42]. The eigenvalues are defined by the roots of the J_1 Bessel function:

$$\kappa_i = \frac{x_i}{b}, \quad J_1(x_i) = 0. \quad (3.5)$$

The inverse Laplace transform of (3.4) gives the time domain response as

$$\Delta H_z(z, t) = I_0 \sum_{i=1}^{\infty} \frac{R(\kappa_i, t)}{b^2 \kappa_i^3 J_0^2(\kappa_i b)} \cdot \mathcal{J}_1(\kappa_i) e^{-\kappa_i z}. \quad (3.6)$$

where $R(\kappa_i, t)$ is the inverse Laplace transform of $\widetilde{R}(\kappa_i, s)$,

$$R(\kappa_i, t) = \frac{1}{2i\pi} \int_{\text{Br}} \widetilde{R}(\kappa_i, s) e^{st} ds. \quad (3.7)$$

In order to carry out the inverse Laplace transform for both short and long time limits, $\widetilde{R}(\kappa, s)$ is expanded as asymptotic series assuming a prescribed coil current. Here the analysis is limited to brief account of the expansion method to determine $R(\kappa, t)$.

3.2.1 Short Time Approximation

For the short time range, we use the series representation of Γ_p derived in (2.20),

$$\Gamma_p = \frac{1 - c\gamma}{1 + c\gamma} + \frac{4}{c\gamma} \sum_{m=1}^{\infty} \left[\left(1 - \frac{2}{1 + c\gamma}\right)^{2m} + \frac{1}{(1 + c\gamma)^2} \left(1 - \frac{2}{1 + c\gamma}\right)^{2m-1} \right] e^{-2m\gamma\lambda}. \quad (3.8)$$

3.2.1.1 Step Current

Noting that $s\widetilde{R}(\kappa, s) = \Gamma_p$, for the case of a current step coil excitation, we get

$$\widetilde{R}(\kappa, s) = \sum_{m=0}^{\infty} \widetilde{R}_m(\kappa, s) \quad \text{where} \quad s\widetilde{R}_0(\kappa, s) = \frac{1 - \gamma c}{1 + \gamma c} \quad (3.9)$$

and for $m > 0$,

$$s\tilde{R}_m(\kappa, s) = \frac{4}{\gamma c} \left[\sum_{j=0}^{2m} C_{2m}^j \left(\frac{-2}{1 + \gamma c} \right)^j + \frac{1}{(1 + \gamma c)^2} \sum_{j=0}^{2m-1} C_{2m-1}^j \left(\frac{-2}{1 + \gamma c} \right)^j \right] e^{-2m\gamma\lambda}, \quad (3.10)$$

C_{2m}^j being the binomial coefficients [34]. Thus the corresponding time-domain reflection terms can be written as

$$R(\kappa, t) = \sum_{m=0}^{\infty} R_m(\kappa, t). \quad (3.11)$$

$R_0(\kappa, t)$ is the contribution of a half space conductor and $R_m(\kappa, t)$ is caused by multiple reflections in the plate. By using (29.3.37) in [34] and the shift theorem, we get

$$R_0(\kappa, t) = 2 \left[\sqrt{\frac{t}{\pi\tau}} e^{-t/\tau} + \frac{1}{2} \left(1 + \frac{2t}{\tau} \right) \operatorname{erf} \left(\sqrt{\frac{t}{\tau}} \right) - \frac{t}{\tau} \right] - u(t) \quad (3.12)$$

and

$$\begin{aligned} R_m(\kappa, t) = & \varphi \left(1, 2m\lambda, \frac{t}{\tau} \right) + \left[\sum_{j=1}^{2m} C_{2m}^j (-2)^{j+2} \phi_j \left(\frac{1}{c}, 2m\lambda, \frac{t}{\tau} \right) \right. \\ & \left. + \sum_{j=0}^{2m-1} C_{2m-1}^j (-2)^{j+2} \phi_{j+2} \left(\frac{1}{c}, 2m\lambda, \frac{t}{\tau} \right) \right] \end{aligned} \quad (3.13)$$

with

$$\phi_n(a, k, t) = \frac{a^n e^{-t/\tau}}{2} [\Phi_n(a, -a, k, t) - \Psi_n(a, k, t)] \quad (3.14)$$

where $\varphi(a, k, t)$, $\Psi_n(a, k, t)$ and $\Phi_n(a, b, k, t)$ have been defined in (2.42), (2.26), and (2.40).

3.2.1.2 Exponential Current

An exponential coil current can be described by

$$I(t) = I_0(1 - e^{-t/\tau_0})u(t) \quad (3.15)$$

with τ_0 being the characteristic rise time. The Laplace transform of equation (3.15) is

$$\tilde{I}(s) = I_0 \left(\frac{1}{s} - \frac{\tau_0}{s\tau_0 + 1} \right). \quad (3.16)$$

Therefore, we get

$$\tilde{R}(\kappa, s) = \left(\frac{1}{s} - \frac{\tau_0}{s\tau_0 + 1} \right) \Gamma_p(\kappa, s). \quad (3.17)$$

As we can see, the inverse Laplace transform of the first part on right hand side of (3.17) has been derived in Section 3.2.1.1 and time domain expression of the second part in Section 2.3.2.

3.2.2 Long Time Approximation

3.2.2.1 Step Current

For large time region, we adopted approach used by de Haan and de Jong and express Γ_p as [39]

$$\Gamma_p = \frac{1 - \gamma^2 c^2}{1 + \gamma^2 c^2 + 2\frac{c}{\lambda} + \frac{2\gamma^2 c \lambda}{3}} \cdot \sum_{m=0}^{\infty} \left[\frac{H(\gamma \lambda)}{1 + \frac{3c}{2\lambda} + \frac{3}{2\gamma^2 \lambda c} + \frac{3}{\gamma^2 \lambda^2}} \right]^m \quad (3.18)$$

where

$$H(z) = 1 + \frac{3}{z^2} - \frac{3 \coth(z)}{z}. \quad (3.19)$$

When $m = 0$, we get

$$\tilde{R}_0(\kappa, s) = \frac{1}{s} \cdot \frac{3}{2u + 3} \frac{-\gamma^2 \lambda^2 + u^2}{\gamma^2 \lambda^2 + a} \quad (3.20)$$

where

$$a = \frac{3u(u + 2)}{2u + 3}, \quad \text{and} \quad u = \lambda/c. \quad (3.21)$$

$\tilde{R}_0(\kappa, s)$ can be inverted to time domain easily since (3.20) contains partial fraction only and it is found

$$R_0(\kappa, t) = \frac{3}{2u + 3} \cdot \frac{1}{\lambda^2 + a} \left[u^2 - \lambda^2 - (u^2 + a)e^{-t/\tau} e^{-\frac{at}{\lambda^2 \tau}} \right]. \quad (3.22)$$

When $m = 1$, we take advantage of the expansion of $\coth(z)$ in (2.51) and write $\tilde{R}_1(\kappa, s)$ as

$$\tilde{R}_1(\kappa, s) = \frac{1}{s} \cdot \frac{36u}{(2u + 3)^2} \cdot \frac{\gamma^4 \lambda^4 (u^2 - \gamma^2 \lambda^2)}{(\gamma^2 \lambda^2 + a)^2} \cdot \sum_{j=1}^{\infty} \frac{1}{j^2 \pi^2 (\gamma^2 \lambda^2 + j^2 \pi^2)}. \quad (3.23)$$

As a result, we get

$$R_1(\kappa, t) = \frac{36u}{(2u+3)^2} \sum_{j=1}^{\infty} \left[C_1(j^2\pi^2, u, a) + \left(\frac{A_1\left(j^2\pi^2, u, a, \frac{t}{\lambda^2\tau}\right) e^{-\frac{a}{\lambda^2\tau}t} + B_1(j^2\pi^2, u, a) e^{-\frac{j^2\pi^2}{\lambda^2\tau}t}}{j^2\pi^2(a - j^2\pi^2)^2} \right) e^{-t/\tau} \right] \quad (3.24)$$

where

$$A_1(b, u, a, t) = \frac{a}{(a + \lambda^2)^2} \left[at(a - b^2)(a + u^2)(a + \lambda^2) - a(a^2 - 2ab^2 - b^2u^2) - \lambda^2 [a(2a - 3b^2) + u^2(a - 2b^2)] \right], \quad (3.25)$$

$$B_1(b, u, a) = -\frac{b^4(b^2 + u^2)}{(b^2 + \lambda^2)} \quad (3.26)$$

and

$$C_1(b, u, a) = \frac{\lambda^4(u^2 - \lambda^2)}{b^2(b^2 + \lambda^2)(a + \lambda^2)^2}. \quad (3.27)$$

When $m = 2$, we take advantage of the series representation of both $\coth(z)$ and $\coth^2(z)$ in (2.51) and (2.58) and get

$$\begin{aligned} \tilde{R}_2(\kappa, s) = & \frac{1}{s} \cdot \frac{12u^2}{(2u+3)^3} \cdot \frac{\gamma^4\lambda^4(u^2 - \gamma^2\lambda^2)}{(\gamma^2\lambda^2 + a)^3} \cdot \left[1 + \frac{9}{\gamma^2\lambda^2} \right. \\ & \left. - \frac{12(\gamma^2\lambda^2 + 3)}{\gamma^2\lambda^2} \sum_{j=1}^{\infty} \frac{1}{\gamma^2\lambda^2 + j^2\pi^2} - \frac{9}{\gamma^2\lambda^2} \sum_{j=1}^{\infty} \frac{2(\gamma^2\lambda^2 - j^2\pi^2)}{(\gamma^2\lambda^2 + j^2\pi^2)^2} \right]. \end{aligned} \quad (3.28)$$

The inverse Laplace transform of $\tilde{R}_2(\kappa, s)$ gives

$$\begin{aligned} R_2(\kappa, t) = & \frac{12u^2}{(2u+3)^3} \cdot \left\{ f_{21}(u, a) + f_{22}\left(u, a, \frac{t}{\lambda^2\tau}\right) e^{-\frac{a}{\lambda^2\tau}t} e^{-t/\tau} \right. \\ & + \sum_{j=1}^{\infty} \left[\frac{e^{-t/\tau}}{(a - j^2\pi^2)^4} \left[A_2\left(j^2\pi^2, u, a, \frac{t}{\lambda^2\tau}\right) e^{-\frac{a}{\lambda^2\tau}t} \right. \right. \\ & \left. \left. + B_2\left(j^2\pi^2, u, a, \frac{t}{\lambda^2\tau}\right) e^{-\frac{j^2\pi^2}{\lambda^2\tau}t} \right] + C_2(j^2\pi^2, u, a) \right] \left. \right\} \end{aligned} \quad (3.29)$$

where

$$f_{21}(u, a) = -\frac{\lambda^2(\lambda^2 + 9)(\lambda^2 - u^2)}{(\lambda^2 + a)^3}, \quad (3.30)$$

$$\begin{aligned}
f_{22}(u, a, t) = \frac{1}{(\lambda^2 + a)^3} \cdot \left\{ -\frac{at^2}{2}(a-9)(u^2+a)(\lambda^2+a)^2 \right. \\
+ t(\lambda^2+a) \left[a^2(u^2+2a-9) + \lambda^2 \left[u^2(2a-9) + 3a(a-6) \right] \right] \\
\left. - \left[\lambda^4(u^2+3a-9) + 3\lambda^2(a^2+3u^2) + a^3 \right] \right\}, \quad (3.31)
\end{aligned}$$

$$\begin{aligned}
A_2(b, u, a, t) = -\frac{1}{(\lambda^2 + a)^3} \left\{ \frac{at^2}{2}(a-b^2)^2 \left[3b^2 + a(-2b^2+2a-9) \right] (u^2+a)(\lambda^2+a)^2 \right. \\
- t(a-b^2)(\lambda^2+a) \left[a^2 \left[b^4(2u^2-3) - 6a^2b^2 + 3u^2b^2 + 2a^3 \right] \right. \\
+ a \left[4b^4 + b^2(15-2u^2) + 9u^2 \right] + \lambda^2 \left[u^2 \left[b^4(4a-3) - 3ab^2(2a-5) + 2a^3 \right] \right. \\
+ a \left[6b^4(a-1) - ab^2(10a-27) + a^2(4a-9) \right] \left. \right] \\
+ \lambda^4 \left[-6ab^6 + 3b^6 + 12a^2b^4 - 21ab^4 - 8a^3b^2 \right. \\
- u^2b^2(2b^4 - 2ab^2 + 3b^2 + 15a) + 2a^4 \left. \right] \\
- \lambda^2 \left[a^2 \left[6b^6 - b^4(8a-9) + 2ab^2(a+18) - 9a^2 \right] \right. \\
+ u^2 \left[3b^6 + 6ab^4(a-2) - a^2b^2(8a-45) + 2a^4 \right] \\
\left. + a^3 \left[-2b^6 + b^4(2a-3) - 15ab^2 + u^2 \left[-2b^4 + b^2(2a-9) - 9a \right] \right] \right\}, \quad (3.32)
\end{aligned}$$

$$\begin{aligned}
B_2(b, u, a, t) = -\frac{6b^2}{(b^2 + \lambda^2)^2} \left\{ 6b^2t(b^2-a)(b^2+u^2)(b^2+\lambda^2) + 2b^8 + b^6(2u^2+3) \right. \\
+ \lambda^2 \left[2b^6 + b^4(2u^2-3) + 3u^2b^2 + a \left[-2b^4 + b^2(21-2u^2) + 15u^2 \right] \right] \\
\left. + a \left[-2b^6 + b^4(15-2u^2) + 9u^2b^2 \right] + 9u^2b^4 \right\}, \quad (3.33)
\end{aligned}$$

and

$$C_2(b, u, a) = \frac{6\lambda^2(\lambda^2 - u^2) [2\lambda^4 + \lambda^2(2b^2 + 9) + 3b^2]}{(\lambda^2 + a)^3(b^2 + \lambda^2)^2}. \quad (3.34)$$

When the driving current is exponential, similar derivation can be obtained by simply replacing the $1/s$ in (3.20), (3.23), and (3.28) with $1/s - \tau_0/(s\tau_0 + 1)$.

3.3 Results

The transient response is calculated by series expansion formula, (3.6), directly in the time domain for the coil parameters shown in Table 3.1. The chosen conductivity, 22.5 MSm^{-1} is roughly that of an aluminum plate and the thickness is 1mm. The distance between the bottom of driver coil and upper plate surface (lift-off) is 1.5mm. The driver coil is excited by an exponential current with a time constant $250\mu\text{s}$.

Table 3.1 Probe parameters.

Coil parameters	Drive coil (mm)
Inner radius	7.10
Outer radius	11.41
Length	5.0
Number of turns	2550

To verify the series expansion results, data computed using the spectrum domain expressions, (3.4), were transformed to the time domain by fast Fourier transform (FFT). In performing the FFT we used 2^{16} points in time domain and the time step was $0.1\mu\text{s}$. Increasing the number of points made no significant difference to the results. Predictions for the normal component of magnetic field at the coil axis were calculated and shown in Figures 3.2. The corresponding relative errors with respect to the FFT results are given in Figures 3.3. The field point is 0.25mm below the bottom of the driver coil in all cases. As expected, the short and large time asymptotic approximations have a significant overlap at intermediate times. Both of them agree with the Fourier transform result very well.

Having verified the series expansion formulae for both short and long time limits, we studied how the key parameters of the transient response; peak amplitude, peak arrival time and zero-crossing time, relating them to the properties of plate thickness and coil lift-off. Figure 3.4 shows that the peak amplitude of the magnetic field decreases

Figure 3.2 Variation of ΔH_z at coil axis due to eddy currents in a conducting plate excited by an exponential driver coil current with time constant, $\tau_0 = 250\mu s$.

Figure 3.3 Error relative to the FFT prediction of ΔH_z at the coil axis due to eddy currents in a conducting plate excited by an exponential driver coil current with time constant, $\tau_0 = 250\mu s$.

Figure 3.4 Variation of the peak amplitude of ΔH_z on the axis of coil with lift-off.

exponentially with lift-off. Figure 3.5 shows the plate thickness effect on the magnetic field above the plate. The peak amplitude of the field tends to saturate as the thickness increases gradually and approaches the half space limit. From Figure 3.5 it can be seen that the metal below 10 mm deep plays negligible role in the transient response for the given current rise time, $250 \mu s$.

3.4 Conclusion

Time domain transient eddy current responses have been derived in series form both for the short and long time regime. At intermediate times, both solutions are accurate, agree with the fast Fourier transform results and therefore they combine to cover the entire time domain with a considerable region of overlap. Most of the key parameters, including peak amplitude, peak arrival time and zero-crossing time of transient response are found to be nonlinearly dependent on the plate thickness and lift-off. These nonlinear

Figure 3.5 Variation of the peak amplitude of ΔH_z at the axis of coil with plate thickness.

relationships can be used to evaluate the plate parameters.

CHAPTER 4. TIME DOMAIN DYADIC GREEN'S FUNCTION FOR AN ELECTRIC SOURCE IN A CONDUCTIVE PLATE

4.1 Introduction

Dyadic Green's functions are used to transform directly an electric or magnetic source into an electric or magnetic field through an integral over a source region or a boundary or both. Although the formalism is widely used in electromagnetic wave theory, it is less often applied to quasi-static problems yet for eddy current calculations in simple structures where the perturbation of the field due to a local inhomogeneity is sought, it can be an effective approach for formulating a numerical scheme. In eddy current nondestructive evaluation (NDE), the field is excited by an inductive probe in air which interacts with an embedded flaw in a conductor. Usually the field is time harmonic, but here we define a kernel that will be used for the computation of transient fields and probe signals.

For a unbounded domain, the time domain dyadic kernel is relatively simple, representing for example, the electric field due to an impulsive electric dipole of arbitrary orientation in an unbounded conductor. The differential equation for the electric field, contains a first order time derivative, which means that the dipole field can be expressed in terms of a three dimensional Gaussian function satisfying the diffusion equation. For a flaw in a half-space conductor it is advantageous to use a dyadic kernel which

satisfies the interface conditions at the surface. By using a dedicated kernel for the half-space [15], the corresponding electric field integral equation (EFIE) has a solution that automatically satisfies the interface conditions at the surface. Therefore numerical solutions can be found by rendering only an inhomogeneous region in a discrete form thereby limiting number of unknowns needed for a given accuracy. It is important to limit the number of unknowns because the discrete approximation of the problem leads to a dense matrix equation whose solution can have a high computational cost. The dedicated kernel approach has a computational cost benefit if the effort of computing a matrix approximation of the integral equation does not outweigh the advantage gained in reducing the number of unknowns. This is a reasonable expectation but there is a need to represent the kernel in a way that promotes efficient computation.

We seek here a kernel for a transient quasi-static field in a homogeneous conductive plate taking as a guide the literature on dyadic Green's functions for time-harmonic wave fields. In a piecewise constant layered structure, the field can be represented in terms of transverse electric (TE) and transverse magnetic (TM) scalar potentials defined with respect to a preferred direction [43] and the kernel expressed in terms of the corresponding scalar Green's functions. Equivalently, a dyadic kernel can be constructed from a set of solenoidal electric and magnetic vector potentials [29, 44] and recently, dyadic eigenfunctions have been used [45]. Although the dyadic Green's function formalism has been used mainly for time harmonic fields, transient fields have also been evaluated using this approach. In computing broadband signals, the calculations are simpler and more efficient if performed directly in the time domain than through frequency domain solutions that eventually recover the transient field using a Fourier transformation. The reason is that multi-frequency numerical solutions need a matrix for each frequency but in the equivalent time stepping calculation, only one matrix is required. The aim is to ensure that the matrix is small enough to be inverted or decomposed rapidly in order that multiple solutions can be generated.

Early work on the calculation of quasi-static fields in conductive media using the present formalism was done for application to geophysics [46]. Later the quasi-static dyadic Green's function for transient fields for a source in a homogenous conductive half-space was derived by transforming frequency domain expression directly to the time domain using the inverse Laplace transform [15]. In NDE, a plate kernel is more useful than a half space kernel but the frequency domain expressions are more complicated making it difficult to perform the inverse transform analytically. However, an approach using rapidly converging asymptotic expansions provides a way forward. These were used by de Jong and de Haan to determine the electromotive force in a pick-up coil due to a transient excitation in a driver coil over a conductive plate [39]. Series expansions were developed for both short and long time regimes the latter being defined with reference to a relaxation time determined by the plate thickness. Both series are accurate at intermediate times with a considerable region of overlap even when they are approximated by a small number of terms. Subsequently, a simpler short time expansion was developed by Fu and Bowler [41] for calculating the unflawed plate signal response. Here a similar short time series will be used for representing the plate dyadic kernel in a form suitable for computations.

In Section 4.2, the quasi-static dyadic Green's function for transforming an electric source in a plate into an electric field is expressed in terms of TE and TM scalar potentials. In Section 4.3, the scalar solutions for a singular source in a conductive plate are given and used to construct a dyadic integral kernel. In Section 4.4, expressions for the magnetic field above the plate due to point source are given. Ultimately these will be used to compute the external magnetic field due to a flaw in a plate. The formula for the electric field kernel contains a free space term and multiple image terms plus two partial reflection terms. The partial reflection terms are potentially costly to compute since they are expressed in terms of infinite integral but we show how such integrals can be approximated by series to improve the overall computational efficiency of evaluating

Figure 4.1 Conductive plate of thickness d with non-conductive regions above and below it. The point P is a source point in the plate whose coordinates are represented as $\{\mathbf{r}', t'\}$

both the electric and magnetic kernels. Numerical results are discussed in Section 4.5 followed by the conclusions.

4.2 Dyadic Kernels

4.2.1 Maxwell's Equations in Dyadic Form

Figure 4.1 shows an infinite homogenous plate with linear material properties having a conductivity σ_0 , permeability μ_0 , and thickness d . The conductivity in the regions above and below the plate is zero. In the quasi-static regime, the dyadic electric Green's function $\mathcal{G}^{(\text{ee})}(\mathbf{r}, \mathbf{r}', t, t')$ and its magnetic counterpart $\mathcal{G}^{(\text{me})}(\mathbf{r}, \mathbf{r}', t, t')$ due to a singular electric source at a point in the plate ($-d < z' < 0$) are related by Maxwell's equations written as

$$\nabla \times \mathcal{G}^{(\text{ee})}(\mathbf{r}, \mathbf{r}', t, t') = -\mu_0 \frac{\partial}{\partial t} \mathcal{G}^{(\text{me})}(\mathbf{r}, \mathbf{r}', t, t') \quad (4.1)$$

and

$$\nabla \times \mathcal{G}^{(\text{me})}(\mathbf{r}, \mathbf{r}', t, t') = \sigma_0 \mathcal{G}^{(\text{ee})}(\mathbf{r}, \mathbf{r}', t, t') + \mathcal{I} \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'), \quad (4.2)$$

where $\mathcal{I} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$ is the unit dyadic. Equation (4.1) implies that

$$\nabla \cdot \mathcal{G}^{(\text{me})}(\mathbf{r}, \mathbf{r}', t, t') = 0 \quad (4.3)$$

everywhere, whereas from (4.2) we find

$$\nabla \cdot \mathcal{G}^{(\text{ee})}(\mathbf{r}, \mathbf{r}', t, t') = -\frac{1}{\sigma_0} \nabla \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'). \quad (4.4)$$

First we consider for completeness the fundamental solution for an unbounded domain. Then the general form of the plate kernel is expressed in terms of scalar functions using a derivation based on source decomposition. Finally we determine the scalar functions explicitly and assemble the dyadic kernels for transforming an electric source in the conductor into an electric and magnetic field.

4.2.2 Unbounded Domain Solution

Combining (4.1) and (4.2) to eliminate $\mathcal{G}^{(\text{me})}(\mathbf{r}, \mathbf{r}', t, t')$, using the identity $\nabla \times \nabla \times \equiv \nabla \nabla \cdot - \nabla^2$ and equation (4.4) gives

$$\left(\nabla^2 - \mu_0 \sigma_0 \frac{\partial}{\partial t} \right) \mathcal{G}^{(\text{ee})}(\mathbf{r}, \mathbf{r}', t, t') = \left(\mu_0 \frac{\partial}{\partial t} \mathcal{I} - \frac{1}{\sigma_0} \nabla \nabla \right) \mathcal{I} \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'). \quad (4.5)$$

By direct substitution one shows that

$$\mathcal{G}_0(\mathbf{r}, \mathbf{r}', t, t') = - \left(\mu_0 \frac{\partial}{\partial t} \mathcal{I} - \frac{1}{\sigma_0} \nabla \nabla \right) G_0(\mathbf{r}, \mathbf{r}', t, t'), \quad (4.6)$$

is a solution where $G_0(\mathbf{r}, \mathbf{r}', t, t')$ vanishes for $t < t'$, and satisfies the scalar diffusion equation

$$\left(\nabla^2 - \mu_0 \sigma_0 \frac{\partial}{\partial t} \right) G_0(\mathbf{r}, \mathbf{r}', t, t') = -\delta(\mathbf{r} - \mathbf{r}') \delta(t - t'). \quad (4.7)$$

Equation (4.7) can be solved using the Fourier-Laplace transform to give [15]

$$G_0(\mathbf{r}, \mathbf{r}', t, t') = \frac{1}{8} \sqrt{\frac{\mu_0 \sigma_0}{\pi^3 (t - t')^3}} e^{-\mu_0 \sigma_0 |\mathbf{r} - \mathbf{r}'|^2 / 4(t - t')} H(t - t'), \quad (4.8)$$

vanishing as $|\mathbf{r} - \mathbf{r}'| \rightarrow \infty$, where $H(t)$ is the Heaviside step function; $H(t) = 1.0$ for a nonnegative argument and is otherwise zero.

4.2.3 Piecewise Homogeneous Layered Structure

The decomposition of the solution will be accomplished by establishing a link between TE and TM fields and their corresponding sources. According to Lindell [47], a source can be characterized as TE, TM or non-radiating or a combination of all three. By definition, the non-radiating source does not contribute to the field outside a surface surrounding the source region. Because it is inaccurate to use the term ‘radiation’ to describe fields that satisfy a vector diffusion equation, the evolution of the field in space and time is called ‘migration’, a term used in geophysics. Hence the quasi-static sources are here characterized as TE, TM and non-migrating. As with radiating fields, it is convenient to add a non-migrating term to either TE or TM modes. In fact we have found it convenient to group a non-migrating and a TM source together.

The singular electric source in the system of equations (4.1) and (4.2) is $\mathcal{I}\delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$. Choosing a preferred direction as the z -direction, we decompose the unity dyadic using the identity, [43] p. 18,

$$\mathcal{I} = \hat{z}\hat{z} + \frac{\nabla_t \nabla_t}{\nabla_t^2} + \frac{\nabla \times \hat{z} \nabla \times \hat{z}}{\nabla_t^2} \quad (4.9)$$

where ∇_t represents the transverse gradient,

$$\nabla_t = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} . \quad (4.10)$$

Operating with the scalar product of the unity dyadic on an electric source vector $\mathbf{P}$ and expanding according to right side of (4.9), decomposes the source. The z -component represents a z -directed electric dipole giving rise to a TM field. The components transverse to $\hat{z}$ give rise to the equivalent of a magnetic dipole in the z -direction generating a TE field, unless they are expressible as the transverse gradient of a scalar in which

case we have a non-migrating source [47]. Similarly, the singular source on the right side of (4.2) can be viewed as the sum of terms that are TM, non-migrating and TE.

In general, the solution of (4.5) is a superposition of TE and TM modes expressible in terms of scalar functions that satisfy the same inhomogeneous diffusion equation but different boundary conditions. By defining modes with respect to a preferred direction normal to the interfaces of a piecewise uniform isotropic stratified structure, we ensure that they are not coupled. Therefore a TE source will produce a TE field everywhere and likewise a TM source will produce a TM field everywhere. Because they are uncoupled, we can develop separately dyadic solutions corresponding to the distinct modes and then added them together to form the complete solution.

4.2.3.1 TE mode

The equation for TE solution is found by replacing $\mathcal{I}\delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$ in (4.5) with the TE source to give

$$\left(\nabla^2 - \mu_0\sigma_0\frac{\partial}{\partial t}\right)\mathcal{G}'(\mathbf{r}, \mathbf{r}', t, t') = \mu_0\frac{\partial}{\partial t}\frac{\nabla \times \hat{z}\nabla \times \hat{z}}{\nabla_t^2}\delta(\mathbf{r} - \mathbf{r}')\delta(t - t'). \quad (4.11)$$

By direct substitution one shows that the solution is of the form

$$\mathcal{G}'(\mathbf{r}, \mathbf{r}', t, t') = -\mu_0\frac{\partial}{\partial t}\nabla \times \hat{z}[\nabla' \times \hat{z}U'(\mathbf{r}, \mathbf{r}', t, t')], \quad (4.12)$$

where $U'(\mathbf{r}, \mathbf{r}', t, t')$ satisfies

$$\left(\nabla^2 - \mu\sigma_0\frac{\partial}{\partial t}\right)\nabla_t^2 U(\mathbf{r}, \mathbf{r}', t, t') = \delta(\mathbf{r} - \mathbf{r}')\delta(t - t'). \quad (4.13)$$

4.2.3.2 TM mode

The equation for the TM kernel is found in a similar way by replacing the source term $\mathcal{I}\delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$ in (4.5), this time with the TM and non-migrating term

$$\left(\hat{z}\hat{z} + \frac{\nabla_t\nabla_t}{\nabla_t^2}\right)\delta(\mathbf{r} - \mathbf{r}')\delta(t - t').$$

In addition the identity [43]

$$\nabla \nabla = \left(\hat{a} \hat{a} + \frac{\nabla_t \nabla_t}{\nabla_t^2} \right) \nabla^2 - \frac{[\nabla \times (\nabla \times \hat{a})][\nabla \times (\nabla \times \hat{a})]}{\nabla_t^2} \quad (4.14)$$

is used to transform the right side of (4.5) to give

$$\begin{aligned} \left(\nabla^2 - \mu_0 \sigma_0 \frac{\partial}{\partial t} \right) \mathcal{G}''(\mathbf{r}, \mathbf{r}', t, t') &= -\frac{1}{\sigma_0} \left(\hat{z} \hat{z} + \frac{\nabla_t \nabla_t}{\nabla_t^2} \right) \\ &\quad \left(\nabla^2 - \mu_0 \sigma_0 \frac{\partial}{\partial t} \right) \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') + \\ &\quad \frac{1}{\sigma_0} \frac{[\nabla \times (\nabla \times \hat{z})][\nabla \times (\nabla \times \hat{z})]}{\nabla_t^2} \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'). \end{aligned} \quad (4.15)$$

By direct substitution, one verifies that the solution has the form

$$\begin{aligned} \mathcal{G}''(\mathbf{r}, \mathbf{r}', t, t') &= -\frac{1}{\sigma_0} \left(\hat{z} \hat{z} + \frac{\nabla_t \nabla_t}{\nabla_t^2} \right) \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') \\ &\quad + \frac{1}{\sigma_0} [\nabla \times \nabla \times \hat{z}] [\nabla' \times \nabla' \times \hat{z} U''(\mathbf{r}, \mathbf{r}', t, t')], \end{aligned} \quad (4.16)$$

where $U''(\mathbf{r}, \mathbf{r}', t, t')$ satisfies (4.13).

4.2.3.3 Complete formal solution

The electric-electric dyadic Green's function is the sum of TE and TM terms, equations (4.12) and (4.16), thus having the form

$$\begin{aligned} \mathcal{G}^{(ee)}(\mathbf{r}, \mathbf{r}', t, t') &= -\frac{1}{\sigma_0} \left(\hat{z} \hat{z} + \frac{\nabla_t \nabla_t}{\nabla_t^2} \right) \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') \\ &\quad - \mu_0 \frac{\partial}{\partial t} [\nabla \times \hat{z}] [\nabla' \times \hat{z} U'(\mathbf{r}, \mathbf{r}', t, t')] \\ &\quad + \frac{1}{\sigma_0} [\nabla \times \nabla \times \hat{z}] [\nabla' \times \nabla' \times \hat{z} U''(\mathbf{r}, \mathbf{r}', t, t')]. \end{aligned} \quad (4.17)$$

Note that the divergence of this result agrees with equation (4.4). For the nonconductive region, we write the solution of

$$\nabla \times \nabla \times \mathcal{G}^{(ee)}(\mathbf{r}, \mathbf{r}', t, t') = 0 \quad (4.18)$$

as

$$\begin{aligned}\mathcal{G}^{(ee)}(\mathbf{r}, \mathbf{r}', t, t') &= -\mu_0 \frac{\partial}{\partial t} [\nabla \times \hat{z}] [\nabla' \times \hat{z} U'(\mathbf{r}, \mathbf{r}', t, t')] \\ &\quad + \frac{1}{\sigma_0} [\nabla \times \nabla \times \hat{z}] [\nabla' \times \nabla' \times \hat{z} U''(\mathbf{r}, \mathbf{r}', t, t')],\end{aligned}\quad (4.19)$$

where $U'(\mathbf{r}, \mathbf{r}', t, t')$ and $U''(\mathbf{r}, \mathbf{r}', t, t')$ satisfy

$$\nabla^2 \nabla_t^2 U(\mathbf{r}, \mathbf{r}', t, t') = 0. \quad (4.20)$$

Here in (4.20) and elsewhere, the primes are suppressed when we refer to both TE and TM modes. In general, the possibility exists of adding no-trivial solutions of $\nabla_t^2 U(\mathbf{r}, \mathbf{r}', t, t') = 0$ to a $U(\mathbf{r}, \mathbf{r}', t, t')$ that satisfies the diffusion equation but on an infinite plane where z is constant, where the potentials vanish as $|\mathbf{r}| \rightarrow \infty$ and there are no sources at the interface, for example injected current, the two dimensional Laplace problem has only a trivial zero solution.

4.3 Solutions for a Plate

4.3.1 Scalar Potentials

To find $U'(\mathbf{r}, \mathbf{r}', t, t')$ due to the TE source or $U''(\mathbf{r}, \mathbf{r}', t, t')$ due to the TM source, it is convenient to define

$$G(\mathbf{r}, \mathbf{r}', t, t') = -\nabla_t^2 U(\mathbf{r}, \mathbf{r}', t, t'). \quad (4.21)$$

Then from (4.20) and (4.13), the scalar Green's function satisfies

$$\nabla^2 G(\mathbf{r}, \mathbf{r}', t, t') = 0, \quad (4.22)$$

for the nonconductive regions ($z > 0$, and $z < -d$) and

$$\left[\nabla^2 - \mu \sigma_0 \frac{\partial}{\partial t} \right] G(\mathbf{r}, \mathbf{r}', t, t') = -\delta(\mathbf{r} - \mathbf{r}') \delta(t - t'), \quad (4.23)$$

for the conductive region ($-d < z < 0$). The scalar Green's functions are found by taking a Laplace transform with respect to time and Hankel transforms of (4.22) and (4.23) with respect to the cylindrical radial coordinate to give

$$\left(\frac{\partial^2}{\partial z^2} - \kappa^2\right) \tilde{G}(\kappa, z, z', s) = 0, \quad z > 0 \quad \& \quad z < -d \quad (4.24)$$

$$\left[\frac{\partial^2}{\partial z^2} - (\kappa^2 + k^2)\right] \tilde{G}(\kappa, z, z', s) = -\delta(z - z'), \quad -d < z < 0 \quad (4.25)$$

where $k^2 = s\mu_0\sigma_0$,

$$\tilde{G}(\kappa, z, z', s) = \frac{1}{2\pi} \int_0^\infty \int_0^\infty G(\mathbf{r}, \mathbf{r}', t, t') e^{-s(t-t')} J_0(\kappa\rho) \rho d\rho d(t-t') \quad (4.26)$$

with $\rho = [(x - x')^2 + (y - y')^2]^{1/2}$ the cylindrical polar radial coordinate. Using Br to denote the Bromwich integral contour,

$$G(\mathbf{r}, \mathbf{r}', t, t') = \frac{1}{(2\pi)^2 i} \int_0^\infty \int_{\text{Br}} \tilde{G}(\kappa, z, z', s) e^{s(t-t')} J_0(\kappa\rho) \kappa d\kappa ds. \quad (4.27)$$

The solution of (4.24) and (4.25) vanishing as $|z| \rightarrow \infty$ has the form

$$\tilde{G}(\kappa, z, z', s) = \frac{1}{2\gamma} A(\kappa, z') e^{-\kappa z} \quad (4.28)$$

$$\tilde{G}(\kappa, z, z', s) = \frac{1}{2\gamma} [e^{-\gamma|z-z'|} + B(\kappa, z') e^{-\gamma z} + C(\kappa, z') e^{\gamma z}] \quad (4.29)$$

$$\tilde{G}(\kappa, z, z', s) = \frac{1}{2\gamma} D(\kappa, z') e^{\kappa(z+d)} \quad (4.30)$$

for $z > 0$, $-d < z < 0$, and $z < -d$ respectively, where $\gamma = \sqrt{\kappa^2 + s\mu_0\sigma_0}$, taking the root with a positive real part. The coefficients in (4.30) are found by imposing interface conditions deduced from the continuity of the tangential electric and magnetic fields at these two interfaces. It is found that

$$\mu G', \quad \frac{\partial G'}{\partial z}, \quad \sigma G'', \quad \text{and} \quad \frac{\partial G''}{\partial z} \quad (4.31)$$

are continuous. By applying the continuity conditions at $z = 0$ and $z = -d$, we get

$$A(\kappa, z') = \frac{\gamma (1 - \Gamma) [e^{\gamma z'} + \Gamma e^{-\gamma(z'+2d)}]}{\kappa (1 - \Gamma^2 e^{-2\gamma d})}, \quad (4.32)$$

$$B(\kappa, z') = \frac{\Gamma e^{-2\gamma d}(e^{-\gamma z'} + \Gamma e^{\gamma z'})}{1 - \Gamma^2 e^{-2\gamma d}}, \quad (4.33)$$

$$C(\kappa, z') = \frac{\Gamma [e^{\gamma z'} + \Gamma e^{-\gamma(z'+2d)}]}{1 - \Gamma^2 e^{-2\gamma d}}, \quad (4.34)$$

and

$$D(\kappa, z') = \frac{\gamma (1 - \Gamma) e^{-\gamma d}(e^{-\gamma z'} + \Gamma e^{\gamma z'})}{\kappa (1 - \Gamma^2 e^{-2\gamma d})}, \quad (4.35)$$

where Γ , the reflection coefficient for an electromagnetic disturbance originating in the conductor, is given by

$$\Gamma' = \frac{\gamma/\mu_r - \kappa}{\gamma/\mu_r + \kappa} \quad \text{and} \quad \Gamma'' = -1, \quad (4.36)$$

for the TE and TM modes respectively.

From (4.29), (4.33), (4.34), and (4.36), the TM mode for the conductive region ($-d < z < 0$) is

$$\tilde{G}''(\kappa, z, z', s) = \tilde{G}_0(\kappa, z, z', s) + \tilde{G}_-(\kappa, z, z', s) - \tilde{G}_+(\kappa, z, z', s) \quad (4.37)$$

where the unbounded domain term is represented as

$$\tilde{G}_0(\kappa, z, z', s) = \frac{1}{2\gamma} e^{-\gamma|z-z'|} \quad (4.38)$$

and can be transformed using (4.27) to give (4.8) [15]. In addition we have

$$\tilde{G}_-(\kappa, z, z', s) = \frac{1}{2\gamma} \frac{e^{-2\gamma d}}{1 - e^{-2\gamma d}} [e^{\gamma(z-z')} + e^{-\gamma(z-z')}] \quad (4.39)$$

and

$$\tilde{G}_+(\kappa, z, z', s) = \frac{1}{2\gamma} \frac{1}{1 - e^{-2\gamma d}} [e^{-\gamma(z+z'+2d)} + e^{\gamma(z+z')}] \quad (4.40)$$

On expanding $(1 - e^{-2\gamma d})^{-1}$ these can be seen as a series of fundamental solutions, expressible in terms of $G_0(\mathbf{r}, \mathbf{r}', t, t')$, equation (4.8), emanating from multiple image sources at regular intervals in the z -direction.

The TE Green's functions for the conductive region ($-d < z < 0$) is written as

$$\begin{aligned} \tilde{G}'(\kappa, z, z', s) &= \tilde{G}_0(\kappa, z, z', s) + \tilde{G}_-(\kappa, z, z', s) + \tilde{G}_+(\kappa, z, z', s) \\ &\quad + \kappa^2 \tilde{V}_-(\kappa, z, z', s) + \kappa^2 \tilde{V}_+(\kappa, z, z', s) \end{aligned} \quad (4.41)$$

where

$$\tilde{V}_-(\kappa, z, z', s) = \frac{1}{2\gamma\kappa^2} \left(\frac{\Gamma'^2}{1 - \Gamma'^2 e^{-2\gamma d}} - \frac{1}{1 - e^{-2\gamma d}} \right) \left[e^{\gamma(z-z')} + e^{-\gamma(z-z')} \right] e^{-2\gamma d} \quad (4.42)$$

and

$$\tilde{V}_+(\kappa, z, z', s) = \frac{1}{2\gamma\kappa^2} \left(\frac{\Gamma'}{1 - \Gamma'^2 e^{-2\gamma d}} - \frac{1}{1 - e^{-2\gamma d}} \right) \left[e^{\gamma(z+z')} + e^{-\gamma(z+z'+2d)} \right]. \quad (4.43)$$

4.3.2 Explicit Dyadic Kernel for a Plate

From (4.19) and (4.21) with (4.37) and (4.41), $\mathcal{G}^{(ee)}(\mathbf{r}, \mathbf{r}', t, t')$ can be rewritten as

$$\begin{aligned} \mathcal{G}^{(ee)}(\mathbf{r}, \mathbf{r}', t, t') &= \mathcal{G}_0(\mathbf{r}, \mathbf{r}', t, t') + \mathcal{G}_-(\mathbf{r}, \mathbf{r}', t, t') + \mathcal{G}_+(\mathbf{r}, \mathbf{r}', t, t') \\ &\quad + \mu_0 \frac{\partial}{\partial t} [\nabla \times \hat{z}] [\nabla' \times \hat{z} V_-(\mathbf{r}, \mathbf{r}', t, t')] \\ &\quad + \mu_0 \frac{\partial}{\partial t} [\nabla \times \hat{z}] [\nabla' \times \hat{z} V_+(\mathbf{r}, \mathbf{r}', t, t')] \end{aligned} \quad (4.44)$$

Equation (4.44) is found by operating on terms that are common to both (4.37) and (4.41) using the identity

$$\begin{aligned} - \left[\mu_0 \mathcal{I} \frac{\partial}{\partial t} - \frac{1}{\sigma_0} \nabla \nabla \right] \nabla_t^2 &= \frac{1}{\sigma_0} (\hat{z} \hat{z} \nabla_t^2 + \nabla_t \nabla_t) \left(\nabla^2 - \mu_0 \sigma_0 \frac{\partial}{\partial t} \right) - \mu_0 \frac{\partial}{\partial t} [\nabla \times \hat{z}] [\nabla \times \hat{z}] \\ &\quad - \frac{1}{\sigma_0} [\nabla \times \nabla \times \hat{z}] [\nabla \times \nabla \times \hat{z}], \end{aligned} \quad (4.45)$$

derived from (4.9) and (4.14). Operating first on $U_0 = -\nabla_t^{-2} G_0$ and using the definition (4.6) gives the first term in (4.44). Operating on $U_- = -\nabla_t^{-2} G_-$ and defining

$$\mathcal{G}_-(\mathbf{r}, \mathbf{r}', t, t') = - \left[\mu_0 \mathcal{I} \frac{\partial}{\partial t} - \frac{1}{\sigma_0} \nabla \nabla \right] G_-(\mathbf{r}, \mathbf{r}', t, t') \quad (4.46)$$

gives the second term. The contribution from $\tilde{G}_+$, which is negative in the TM potential, equation (4.37), is dealt with by operating on $\mathcal{I}' U_+ = \mathcal{I}' \nabla_t^{-2} G_+$, where $\mathcal{I}' = \hat{x}\hat{x} + \hat{y}\hat{y} - \hat{z}\hat{z}$ and defining

$$\mathcal{G}_+(\mathbf{r}, \mathbf{r}', t, t') = - \left[\mu_0 \mathcal{I}' \frac{\partial}{\partial t} + \frac{1}{\sigma_0} \nabla \nabla' \right] G_+(\mathbf{r}, \mathbf{r}', t, t'). \quad (4.47)$$

Next consider terms representing multiple images. To evaluate $G_-(\mathbf{r}, \mathbf{r}', t, t')$, we shall use

$$\frac{1}{1 - e^{-2\gamma d}} = \sum_{n=0}^{\infty} e^{-2n\gamma d} \quad (4.48)$$

to show that $\tilde{G}_-(\kappa, z, z', s)$ may be expanded to give

$$\tilde{G}_-(\kappa, z, z', s) = \frac{1}{2\gamma} \sum_{n=1}^{\infty} \left\{ e^{-\gamma[2nd - (z - z')]} + e^{-\gamma[2nd + (z - z')]} \right\}. \quad (4.49)$$

The inverse Laplace transform and Hankel transform of (4.49) gives

$$G_-(\mathbf{r}, \mathbf{r}', t, t') = \sum_{n=1}^{\infty} [G_0(\mathbf{r}, \mathbf{r}_{1n}, t, t') + G_0(\mathbf{r}, \mathbf{r}_{2n}, t, t')], \quad (4.50)$$

where $\mathbf{r}_{1n}$ denotes a point at $(x', y', z' + 2nd)$ and $\mathbf{r}_{2n}$ denotes a point at $(x', y', z' - 2nd)$.

Following a similar procedure, we get

$$G_+(\mathbf{r}, \mathbf{r}', t, t') = \sum_{n=1}^{\infty} [G_0(\mathbf{r}, \mathbf{r}_{3n}, t, t') + G_0(\mathbf{r}, \mathbf{r}_{4n}, t, t')], \quad (4.51)$$

where $\mathbf{r}_{3n}$ denotes a point at $[x', y', -z' - 2nd]$ and $\mathbf{r}_{4n}$ denotes a point at $[x', y', -z' + 2(n-1)d]$. As we can see, the contribution of multiple images is simply the sum of infinite free space terms centered at the image positions. The structure of the plate kernel in this section is somewhat simpler than the comparable results for the time harmonic case in [48].

4.3.3 Asymptotic Expansions

The evaluation of $V_-(\mathbf{r}, \mathbf{r}', t - t')$ and $V_+(\mathbf{r}, \mathbf{r}', t - t')$ needs more attention due to the complexity of the Laplace domain expression. In eddy current nondestructive evaluation, the electric field scattered by a flaw can be written as an integral over the flaw region with the dyadic Green's function $\mathcal{G}^{(ee)}(\mathbf{r}, \mathbf{r}', t, t')$ as the kernel. In anticipation of the z integral $\tilde{V}_-(\kappa, z, z', s)$ is expressed as a derivative with respect to z . Thus we write

$$\tilde{V}_-(\kappa, z, z', s) = \frac{1}{2\kappa^2} \frac{\partial}{\partial z} \left[\tilde{F}_2(\kappa, 2d - (z - z'), s) - \tilde{F}_2(\kappa, 2d + (z - z'), s) \right], \quad (4.52)$$

where

$$\tilde{F}_m(\kappa, \zeta, s) = \frac{e^{-\gamma\zeta}}{\gamma^2} \left[\frac{\Gamma^m}{1 - \Gamma'^2 e^{-2\gamma d}} - \frac{1}{1 - e^{-2\gamma d}} \right]. \quad (4.53)$$

Similarly,

$$\tilde{V}_+(\kappa, z, z', s) = \frac{1}{2\kappa^2} \frac{\partial}{\partial z} \left[\tilde{F}_1(\kappa, -(z + z'), s) - \tilde{F}_1(\kappa, 2d + (z + z'), s) \right]. \quad (4.54)$$

To facilitate the inverse Laplace transform, we expand (4.53) to give

$$\tilde{F}_m(\kappa, \zeta, s) = \frac{e^{-\gamma\zeta}}{\gamma^2} \sum_{n=0}^{\infty} e^{-2n\gamma d} \sum_{j=1}^{2n+m} C_{2n+m}^j (-2)^j \left(\frac{\kappa}{\gamma + \kappa} \right)^j \quad (4.55)$$

where $C_n^j = n!/(n-j)!j!$ is the binomial coefficient. A term-by-term inversion of (4.55) gives

$$F_m(\kappa, \zeta, t) = \frac{e^{-t/\tau}}{\mu_0 \sigma_0} \sum_{n=0}^{\infty} \sum_{j=1}^{2n+m} C_{2n+m}^j (-2)^j \Phi_j(1, 0, \kappa(2nd + \zeta), t/\tau), \quad (4.56)$$

where $\tau = \mu_0 \sigma_0 / \kappa^2$ and $\Phi_n(a, b, k, t)$ is the inverse Laplace transform of

$$\tilde{\Phi}_n(a, b, k, s) = \frac{1}{\sqrt{s}(\sqrt{s} + a)^n(\sqrt{s} + b)} e^{-k\sqrt{s}}. \quad (4.57)$$

By writing

$$\frac{1}{(\sqrt{s} + a)(\sqrt{s} + b)} = \frac{1}{a - b} \left[\frac{1}{(\sqrt{s} + b)} - \frac{1}{(\sqrt{s} + a)} \right]$$

it can be seen that $\tilde{\Phi}_n(a, b, k, s)$ satisfies the recursive relationship

$$\tilde{\Phi}_n(a, b, k, s) = \frac{1}{a - b} \left[\tilde{\Phi}_{n-1} - \tilde{\Psi}_{n-1} \right], \quad (4.58)$$

where

$$\tilde{\Psi}_n(a, k, s) = \frac{1}{\sqrt{s}(a + \sqrt{s})^{n+1}} e^{-k\sqrt{s}}. \quad (4.59)$$

By repeated use of (4.58) we find that [41]

$$\Phi_n(a, b, k, t) = \frac{1}{(a - b)^n} \left[\Psi_0(b, k, t) - \sum_{j=0}^{n-1} (a - b)^j \Psi_j(a, k, t) \right] \quad (4.60)$$

where

$$\Psi_n(a, k, t) = (4t)^{\frac{n}{2}} e^{ak + a^2 t} i^n \operatorname{erfc} \left(a\sqrt{t} + \frac{k}{2\sqrt{t}} \right). \quad (4.61)$$

For numerical evaluation, $i^k \text{erfc}(z)$ can be calculated by lower order integral of the complimentary error function using the recursive relationship, 7.2.5 [34].

The combination of (4.52), (4.54), (4.56) and (4.27) gives

$$\begin{aligned} V_-(\mathbf{r}, \mathbf{r}', t, t') = & \frac{1}{4\pi} \frac{\partial}{\partial z} \int_0^\infty \frac{1}{\kappa} [F_2(\kappa, 2d - (z - z'), t) \\ & - F_2(\kappa, 2d + z - z', t)] J_0(\kappa \rho) d\kappa \end{aligned} \quad (4.62)$$

and

$$\begin{aligned} V_+(\mathbf{r}, \mathbf{r}', t, t') = & \frac{1}{4\pi} \frac{\partial}{\partial z} \int_0^\infty \frac{1}{\kappa} [F_1(\kappa, -(z + z'), t) \\ & - F_1(\kappa, 2d + (z + z'), t)] J_0(\kappa \rho) d\kappa. \end{aligned} \quad (4.63)$$

This completes the definition of the terms in the expression for the dyadic kernel, equation (4.44), that transforms an electric source in a plate into the quasi-static electric field.

4.4 Magnetic field of an embedded electric source

4.4.1 Evaluation of $\mathcal{G}^{(\text{me})}(\mathbf{r}, \mathbf{r}', t, t')$

In order to evaluate the transient magnetic field above plate due to an embedded electric dipole, we use an expression for magnetic-electric dyadic Green's function $\mathcal{G}^{(\text{me})}(\mathbf{r}, \mathbf{r}', t, t')$ with $z > 0$ and the source point is in the plate; $-d < z' < 0$. The main role of the function is to compute the magnetic field in air due to a flaw in the conductor and thus predict field and impedance measurements. Substituting (4.19) into the left side of (4.1), and using (4.20) gives

$$\mathcal{G}^{(\text{me})}(\mathbf{r}, \mathbf{r}', t, t') = [\nabla \times \nabla \times \hat{z}][\nabla' \times \hat{z} U'(\mathbf{r}, \mathbf{r}', t, t')] \quad (4.64)$$

which contains the TE mode only and when used as a scalar product to operate on a vector source, annihilates the z component of the source.

The Laplace and Hankel transform of $U'(\mathbf{r}, \mathbf{r}', t, t')$ for $z > 0$ and $-d < z' < 0$ can be derived from (4.21), (4.28) and (4.32), giving

$$\tilde{U}'(\kappa, z, z', s) = \frac{1}{\kappa^2} \tilde{G}'(\kappa, z, z', s) = \frac{e^{-\gamma z'} (\Gamma' e^{-2\gamma d} + e^{2\gamma z'})}{\kappa^2 (\gamma + \kappa) (1 - \Gamma'^2 e^{-2\gamma d})} e^{-\kappa z}. \quad (4.65)$$

To perform the inverse Laplace transform of $\tilde{U}'(\kappa, z, z', s)$, define $\tilde{L}_m(\kappa, \zeta, s)$ and expand it into series:

$$\tilde{L}_m(\kappa, \zeta, s) = \frac{1}{\gamma + \kappa} \frac{\Gamma'^m e^{-\gamma \zeta}}{(1 - \Gamma'^2 e^{-2\gamma d})} = \sum_{n=0}^{\infty} \sum_{j=0}^{2n+m} C_{2n+m}^j (-2\kappa)^j \frac{e^{-(2nd+\zeta)\gamma}}{(\gamma + \kappa)^{j+1}}. \quad (4.66)$$

Then by using (4.61) to transform (4.66), it can be shown that

$$\begin{aligned} L_m(\kappa, \zeta, t) &= \frac{e^{-t/\tau}}{\kappa \tau} \sum_{n=0}^{\infty} \sum_{j=0}^{2n+m} C_{2n+m}^j (-2)^j [\Psi_{j-1}(1, \kappa(2nd + \zeta), t/\tau) \\ &\quad - \Psi_j(1, \kappa(2nd + \zeta), t/\tau)]. \end{aligned} \quad (4.67)$$

Using (4.67), the inverse Laplace transform of (4.65) may be written as

$$\tilde{U}'(\kappa, z, z', t) = \frac{e^{-\kappa z}}{\kappa^2} [L_0(\kappa, -z', t) + L_1(\kappa, 2d + z', t)] \quad (4.68)$$

Hence the required function expressed as a Hankel transform of a time dependent function is

$$U'(\mathbf{r}, \mathbf{r}', t, t') = \frac{1}{2\pi} \int_0^{\infty} \frac{e^{-\kappa z}}{\kappa} [L_0(\kappa, -z', t) + L_1(\kappa, 2d + z', t)] J_0(\kappa \rho) d\kappa. \quad (4.69)$$

Equation (4.69) combined with (4.64) can be used to determine the magnetic field due to a point source above a plate. In order to perform numerical calculations efficiently, (4.69) is approximated as described in the following section.

4.4.2 Integral Approximation

The evaluation of the Hankel transforms, (4.62), (4.63) and (4.69), is potentially time consuming but the computational cost can be reduced using an approximation which limits the range of allowed values for the radial coordinate ρ . This leads to

a series approximation of the corresponding integral that is accurate for a truncated range of the argument. The truncation of a region or the range of a variable has been used to approximate solutions of the electromagnetic field equations in integral form by eigenfunction expansions [42]. Here, however, the range truncation is simply used to approximate Hankel integral transforms.

In general, the numerical value of a function $F(r)$ given by

$$F(r) = \int_0^\infty f(\kappa) J_m(\kappa r) \kappa d\kappa, \quad (4.70)$$

is sought using the approximation

$$F(r) \simeq \sum_{i=1}^{\infty} A(\kappa_i) f(\kappa_i) J_m(\kappa_i r), \quad (4.71)$$

where the coefficients $A(\kappa_i)$ are to be determined. From (4.70), we know that $f(\kappa)$ is the inverse Hankel transform of $F(r)$ having the form

$$f(\kappa) = \int_0^\infty F(r) J_m(\kappa r) r dr. \quad (4.72)$$

Assuming that $F(r)$ decays sufficiently fast such that $F(r)$ is negligible for $r > b$ means that the integral (4.72) can be approximated by setting the upper limit to b . Substituting for $F(r)$ using (4.71) and using the orthogonality properties of Bessel function, see for example [34] equation 11.4.5, shows that

$$A(\kappa_i) = \frac{2}{b^2 J_{m+1}^2(\kappa_i b)}. \quad (4.73)$$

Therefore,

$$F(r) \simeq \sum_{i=1}^{\infty} \frac{2 J_m(\kappa_i r)}{b^2 J_{m+1}^2(\kappa_i b)} f(\kappa_i), \quad (4.74)$$

where κ_i are the zeros of the m th order Bessel function, satisfying $J_m(\kappa_i b) = 0$. Thus (4.69) can be approximated a truncation of the series

$$U'(\mathbf{r}, \mathbf{r}', t, t') = \frac{1}{\pi b^2} \sum_{i=1}^{\infty} \frac{e^{-\kappa_i z}}{\kappa_i^2 J_1^2(\kappa_i b)} [L_0(\kappa_i, -z', t) + L_1(\kappa_i, 2d + z', t)] J_0(\kappa_i \rho). \quad (4.75)$$

Figure 4.2 The evolution of the magnetic field component $H_z(\mathbf{r}, t)$, at $\mathbf{r} = [5, 5, 1]$ mm due to an impulsive embedded electric dipole at $\mathbf{r}' = [0, 0, -1]$ mm orientated in the x -direction. The circles denote the FFT results and the solid line denotes the asymptotic solution.

Equations (4.62), (4.63) are evaluated using similar approximations.

To complete the discussion of the magnetic field kernel, note that the z -component of the magnetic field above the plate due to an x -orientated electric dipole, can be expressed from (4.64) as

$$\hat{z} \cdot \mathcal{G}^{(\text{me})}(\mathbf{r}, \mathbf{r}', t, t') \cdot \hat{x} = G_{zx}^{(\text{me})}(\mathbf{r}, \mathbf{r}', t, t') = -\hat{z} \nabla_t^2 \frac{\partial}{\partial y'} U'(\mathbf{r}, \mathbf{r}', t, t'). \quad (4.76)$$

Evaluating the partial derivatives explicitly and using (4.75), it is found that

$$G_{zx}^{(\text{me})}(\mathbf{r}, \mathbf{r}', t, t') = \frac{y - y'}{\pi b^2} \sum_{i=1}^{\infty} \frac{\kappa_i e^{-\kappa_i z} J_1(\kappa_i \rho)}{J_0^2(\kappa_i b)} [L_0(\kappa, -z', t) + L_1(\kappa, 2d + z', t)] \quad (4.77)$$

which can be used to determine the magnetic field above the plate due to a buried electric point source or an extended source due to a flaw [16].

Figure 4.3 The evolution of the relative error in $H_z(\mathbf{r}, t)$ computed from the asymptotic solution and compared with FFT result at $\mathbf{r} = [5, 5, 1]$ mm due to an impulsive electric dipole at $\mathbf{r}' = [0, 0, -1]$ mm orientated in the x -direction.

4.5 Numerical Results

The numerical results summarized here are for a 2 mm thick aluminum plate of conductivity 22.5 MS/m and relative permeability 1. The source point is located at $\mathbf{r}' = [0, 0, -1]$ mm at the mid-plane of the plate and the field point is at $\mathbf{r} = [5, 5, 1]$ mm. The series in (4.75) is formed on the assumption that the electric potential is negligible beyond $|\mathbf{r} - \mathbf{r}'| = 200$ mm. In the calculations, this series is truncated at 250 values of κ_i and the series in equation (4.67) truncated at $n = 3$. Figure 4.2 shows the z -component of the magnetic field calculated from (4.77) and the corresponding time dependence computed using a FFT. Figure 4.3 displays the discrepancy in (4.77) relative to FFT results computed using 200k data points, showing good numerical agreement.

Figure 4.4 to Figure 4.7 show the z -component of the magnetic field evolving in time at the plane $z = 1$ mm due to a subsurface impulsive dipole in the mid-plane of the plate. The field, which exhibits the characteristics of a dipole, is initially localized and rising to a maximum while spreading laterally and eventually relaxing towards zero.

4.6 Conclusion

Time domain electric and magnetic dyadic Green's functions for an electric source in a plate has been derived by using a decomposition into TE and TM modes. Asymptotic series expansions are used to facilitate the transformation of frequency domain expressions to the time domain. The series converge faster at shorter times but are accurate also for intermediate times which means that predictions of a typical transient flaw response can be made for all epochs except in the long time regime where the signals are likely to be very small. The kernel is needed for numerical calculations based on a transient electric field integral equation.

The electric field dyadic kernel has been presented in a form that is convenient for inclusion in a time stepping algorithm for computing numerical approximations of a

Figure 4.4 Variation of $H_z(\mathbf{r}, t)$ in the plane $z = 1$ mm at $t = 1 \mu\text{sec}$ due to an impulsive electric dipole at $\mathbf{r}' = [0, 0, -1]$ mm orientated in the x -direction.

Figure 4.5 Variation of $H_z(\mathbf{r}, t)$ in the plane $z = 1$ mm at $t = 50 \mu\text{sec}$ due to an impulsive electric dipole at $\mathbf{r}' = [0, 0, -1]$ mm orientated in the x -direction.

Figure 4.6 Variation of $H_z(\mathbf{r}, t)$ in the plane $z = 1$ mm at $t = 200 \mu\text{sec}$ due to an x oriented impulsive electric dipole whose coordinates are $\mathbf{r}' = [0, 0, -1]$ mm .

Figure 4.7 Variation of $H_z(\mathbf{r}, t)$ in the plane $z = 1$ mm at $t = 500 \mu\text{sec}$ due to an x oriented impulsive electric dipole whose coordinates are $\mathbf{r}' = [0, 0, -1]$ mm .

transient electric field using a boundary or volume integral method. In such a scheme, a discrete representation of the transient field is updated at each step using the solution of a matrix equation. With a fixed time interval for each step, the coefficient matrix is also fixed. The costly process of performing an LU decomposition [49] on the matrix may then be justified because the decomposed matrix can produce multiple solutions efficiently. The matrix elements can be computed by integrating in space over discrete cells with the integrand formed by a product of the explicit time domain kernel and a basis function.

The fast Fourier transform gives results that agree with the asymptotic expansions but one is often forced to use a large number of data points to achieve good accuracy with an FFT even when only a few points are needed. When a sequence of numerical results is the required outcome, the times taken for the FFT and the asymptotic series expressions is of a similar magnitude. The main advantage of the asymptotic series is the flexibility of being able to compute the field at an arbitrary point in space using an expression that involves the time dependence explicitly and to be able to integrate the expression spatially at a fixed time.

CHAPTER 5. TRANSIENT EDDY CURRENT RESPONSE DUE TO A SUBSURFACE CRACK IN A CONDUCTIVE PLATE

5.1 Introduction

In eddy current nondestructive evaluation, flaws can be detected by measuring the changes in the magnetic field that occur when an induced current is perturbed. Usually alternating current is used for eddy current testing but in the problem considered in this article, the field is excited by an induction coil carrying current pulses. The excitation coil generates a transient field which interacts with a crack in a conductive plate giving rise to a perturbed field. The disturbance of the field can be detected by using magnetic field sensors. In order to predict the observed signals for comparison with experiment, it is necessary to calculate the time dependent scattered field.

Closely related time domain problems are to be found in the literature on geophysical surveying where volume element [46] and finite difference time domain [50] methods have been used to compute the evolution of the electromagnetic field. Many researchers choose to compute the time dependent response by using the inverse Fourier transform and therefore formulate the problem in the frequency domain. Tsuboi [9] calculated the frequency response of a crack in a plate based on an edge-element finite element method (FEM) using the Fourier transform and time stepping. Within this category, various formulations have been developed to address different issues such as the do-

main discretization, computation efficiency, and accuracy [51], [52], [53]. The finite element method is advantageous when dealing with nonlinear or inhomogeneous media and problems with complex geometries. However, these calculations typically require large computer resources, and therefore it is important to consider strategies for restraining the computational cost where integral equation based method may be utilized [52], [54], [11]. Recently, a boundary meshless method has been introduced to calculate two dimensional transient eddy current distributions within a rectangular bar [55]. The boundary is represented by a number of scattered points instead of discretized elements. The formulation is based on an electric field integral equation with an unbounded domain kernel.

There are few studies that have addressed the situations encountered in transient eddy current nondestructive evaluation either using differential equation based or integral equation based methods. The interaction between transient eddy currents with the crack has been evaluated using an electric field integral equation [16] [56] with half space dyadic Green's function kernel [15]. Given a plate Green's function, our goal in this paper is extending the analysis in to more practical case.

An ideal crack, by acting as a barrier to current, introduces a discontinuity in the electric field. The continuity of the normal component of the magnetic field allows the jump in the electric field to be represented as the gradient of a surface scalar function [57]. This scalar function represents a volume of current dipoles whose orientation is normal to the crack surface and whose density changes with time as the incident field changes. Note that a similar relationship between the jump in the tangential electric field at a surface and a surface dipole density arises in electrostatics as discussed in Stratton's textbook [58]. By representing the effect of the crack in terms of an equivalent source, the scattered field can be written as an integral of the source density over crack volume and over time, using a dyadic Green's kernel [59].

For the case of a narrow crack, the computational burden is reduced by treating

flaw as a surface scatterer and by representing the field on the crack by scalar function. In this way, a three dimensional vector field problem is reduced to the evaluation of a surface scalar defined on an open surface, S_0 , referred to as the crack surface. The justification for this scalar reduction is based on the properties of the field at an ideal crack. By using a Green's kernel that is tailored to the conductor geometry, such as that for an unbounded domain, a half-space or an infinite plate [60], the problem reduces to one of finding the current dipole density at the surface S_0 .

The fact that the crack is a perfect barrier implies that it is impenetrable to eddy currents. Consequently, the normal component of the current density at a point on the crack surface is zero. This condition means that the normal component of the incident and scattered electric fields are equal and opposite at such a point. Because the scattered field is expressed in integral form, the boundary condition on the electric field leads to a Fredholm integral equation of the first kind for the dipole density on the crack. This integral equation, Section 5.2, is expressed in a form suitable for numerical approximation and a solution sought using the moment method [61].

An equation expressing the external magnetic field in terms of the dipole density has been derived [60]. This equation allows field predictions to be calculated for comparison with experimental measurements once the dipole density has been found. However, before any predictions can be made, the unperturbed electric field must be calculated. These calculations are carried out for a conductive plate, Section 5.3. It is assumed that the unperturbed field is excited by a normal coil carrying a current pulse that rises exponentially to a constant value. A special case is one where the coil current rise time is zero, in which case the current excitation is a step function in time.

In order to approximate the integral equation by a discrete form, it is assumed that the dipole density can be represented, with reasonable accuracy, as piecewise constant in space and piecewise linear in time, the regions of constant density in space being defined on a rectangular lattice of boundary elements covering the crack surface. This

piecewise constant approximation, together with the process of point matching, dictates the definition of a matrix derived from the integral kernel. The evaluation of this matrix is carried out for unbounded domain in Section 5.6, and plate domain in Appendix. Thus a linear system of equations, Section 5.7, is defined and solved to give a numerical approximation of the dipole density. Because the number of unknowns is relatively small, a few thousand at most, and the same matrix is used for many different excitations, the solutions are obtained by matrix inversion via a LU decomposition [49]. The relatively high computational cost of the matrix inversion is justified because the subsequent cost of calculating numerous solutions is low.

Calculations have been carried out to determine the time variation of the magnetic field at a point on the axis of an excitation coil due to interaction with a subsurface rectangular slot in a conductive plate. These results are presented in Section 5.8 and agree with the experimental measurements.

5.2 Field Formula

5.2.1 Magnetic Vector Potential

Neglecting the displacement current, the magnetic field in a linear nonmagnetic conductor containing a local conductivity variation satisfies Faraday's induction law

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\mu_0 \frac{\partial \mathbf{H}(\mathbf{r}, t)}{\partial t} \quad (5.1)$$

and Ampere's law in differential form,

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \sigma_0 \mathbf{E}(\mathbf{r}, t) + \mathbf{P}(\mathbf{r}, t), \quad (5.2)$$

where σ_0 is the host conductivity and $\mathbf{P}(\mathbf{r}, t) = [\sigma(\mathbf{r}) - \sigma_0]\mathbf{E}(\mathbf{r}, t)$ is a current dipole distribution representing the effect of the flaw as an equivalent source.

A solution of Maxwell's equations for a conductor with linear material properties will be sought using the magnetic vector potential by following standard electromagnetic

theory. The vector potential is defined such that $\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$, hence, for a medium with the permeability of free space

$$\mathbf{H}(\mathbf{r}, t) = \frac{1}{\mu_0} \nabla \times \mathbf{A}(\mathbf{r}, t). \quad (5.3)$$

By substituting for $\mu_0 \mathbf{H}(\mathbf{r}, t)$ in the induction law, one concludes that the electric field has the form

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} - \nabla \psi(\mathbf{r}, t), \quad (5.4)$$

where $\psi(\mathbf{r}, t)$ is a scalar potential. Substituting (5.3) and (5.4) into (5.2) and choosing the gauge condition $\psi(\mathbf{r}, t) = -\nabla \cdot \mathbf{A}(\mathbf{r}, t)/\mu_0 \sigma_0$ it is found that the vector potential satisfies a three dimensional vector diffusion equation

$$\nabla^2 \mathbf{A}(\mathbf{r}, t) + \mu_0 \sigma_0 \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} = -\mu_0 \mathbf{P}(\mathbf{r}, t). \quad (5.5)$$

Combining the gauge condition with (5.4) shows that the electric field is given by

$$\mathbf{E}(\mathbf{r}, t) = -\left(\mathcal{I} \frac{\partial}{\partial t} - \frac{1}{\mu_0 \sigma_0} \nabla \nabla\right) \cdot \mathbf{A}(\mathbf{r}, t) \quad (5.6)$$

where $\mathcal{I} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$ is the unit dyad. The solution of (5.5) can be written in the form

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) &= \mathbf{A}^{(0)}(\mathbf{r}, t) \\ &+ \mu_0 \int_0^t \int_{\Omega} \mathcal{G}_A(\mathbf{r}, \mathbf{r}', t, t') \cdot \mathbf{P}(\mathbf{r}', t') d\mathbf{r}' dt', \end{aligned} \quad (5.7)$$

where $\mathbf{A}^{(0)}(\mathbf{r}, t)$ is the unperturbed field and the dyadic Green's function $\mathcal{G}_A(\mathbf{r}, \mathbf{r}', t, t')$ is the vector potential due to a point electric current source.

5.2.2 Dyadic Kernal

In general, $\mathcal{G}_A(\mathbf{r}, \mathbf{r}', t, t')$ is related to the electric field dyadic Green's function $\mathcal{G}^{(ee)}(\mathbf{r}, \mathbf{r}', t, t')$ by

$$\mathcal{G}^{(ee)}(\mathbf{r}, \mathbf{r}', t, t') = -\left(\mu_0 \frac{\partial}{\partial t} \mathcal{I} - \frac{1}{\sigma_0} \nabla \nabla\right) \cdot \mathcal{G}_A(\mathbf{r}, \mathbf{r}', t, t'), \quad (5.8)$$

reflecting (5.6). The expression for $\mathcal{G}^{(ee)}(\mathbf{r}, \mathbf{r}', t - t')$ for a conducting plate has been given in [60], from which it can be deduced that

$$\begin{aligned} \mathcal{G}_A(\mathbf{r}, \mathbf{r}', t, t') = & \mathcal{I}G_-(\mathbf{r}, \mathbf{r}', t, t') + \mathcal{I}'G_+(\mathbf{r}, \mathbf{r}', t, t') - \nabla \times \hat{z}[\nabla' \times \hat{z}]V_-(\mathbf{r}, \mathbf{r}', t, t') \\ & - \nabla \times \hat{z}[\nabla' \times \hat{z}]V_+(\mathbf{r}, \mathbf{r}', t, t') \end{aligned} \quad (5.9)$$

where $\mathcal{I}' = \hat{x}\hat{x} + \hat{y}\hat{y} - \hat{z}\hat{z}$. The first and second term in (5.9) are defined as [60],

$$G_-(\mathbf{r}, \mathbf{r}', t, t') = \sum_{n=-\infty}^{\infty} G_0(\mathbf{r}, \mathbf{r}_{1n}, t, t') \quad (5.10)$$

and

$$G_+(\mathbf{r}, \mathbf{r}', t, t') = \sum_{n=-\infty}^{\infty} G_0(\mathbf{r}, \mathbf{r}_{2n}, t, t') \quad (5.11)$$

where $\mathbf{r}_{1n}$ and $\mathbf{r}_{2n}$ denote image points at $(x', y', z' + 2nd)$ and $(x', y', -z' + 2nd)$ with d being the plate thickness. The scalar Green's function $G_0(\mathbf{r}, \mathbf{r}', t - t')$ has a similar form to that used in heat conduction problems and is given by [15]

$$G_0(\mathbf{r}, \mathbf{r}', t - t') = \frac{1}{\pi^{3/2}\mu_0\sigma_0\alpha^3} e^{-R^2/\alpha^2} H(t - t'), \quad (5.12)$$

with $\alpha = 2\sqrt{(t - t')/\mu_0\sigma_0}$ being the migration length, $R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$, $H(t)$ is the Heavyside step function. The third and fourth terms in (5.9) are defined as [60]

$$\begin{aligned} V_-(\mathbf{r}, \mathbf{r}', t, t') = & \frac{1}{4\pi} \frac{\partial}{\partial z} \int_0^\infty \frac{1}{\kappa} [F_2(\kappa, 2d - (z - z'), t) \\ & - F_2(\kappa, 2d + z - z', t)] J_0(\kappa\rho) d\kappa \end{aligned} \quad (5.13)$$

and

$$\begin{aligned} V_+(\mathbf{r}, \mathbf{r}', t, t') = & \frac{1}{4\pi} \frac{\partial}{\partial z} \int_0^\infty \frac{1}{\kappa} [F_1(\kappa, -(z + z'), t) \\ & - F_1(\kappa, 2d + (z + z'), t)] J_0(\kappa\rho) d\kappa. \end{aligned} \quad (5.14)$$

where

$$F_m(\kappa, \zeta, t) = \frac{e^{-t/\tau}}{\mu_0\sigma_0} \sum_{n=0}^{\infty} \sum_{j=1}^{2n+m} C_{2n+m}^j (-2)^j \Phi_j(1, 0, \kappa(2nd + \zeta), t/\tau) \quad (5.15)$$

with $\tau = \mu_0\sigma_0/\kappa^2$, $C_n^j = n!/(n-j)!/j!$ being the binomial coefficient. $\Phi_n(a, b, k, t)$ is defined as [41]

$$\Phi_n(a, b, k, t) = \frac{1}{(a-b)^n} \left[\Psi_0(b, k, t) - \sum_{j=0}^{n-1} (a-b)^j \Psi_j(a, k, t) \right], \quad (5.16)$$

which is derived from its Laplace transform,

$$\tilde{\Phi}_n(a, b, k, s) = \frac{1}{\sqrt{s}(\sqrt{s}+a)^n(\sqrt{s}+b)} e^{-k\sqrt{s}}, \quad (5.17)$$

and

$$\Psi_n(a, k, t) = (4t)^{\frac{n}{2}} e^{ak+a^2t} i^n \operatorname{erfc} \left(a\sqrt{t} + \frac{k}{2\sqrt{t}} \right). \quad (5.18)$$

5.2.3 Electric Field Integral Equation

From (5.6) and (5.7), the electric field in the presence of a volumetric flaw is written as

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) = \mathbf{E}^{(0)}(\mathbf{r}, t) - \left(\mu_0 \mathcal{I} \frac{\partial}{\partial t} - \frac{1}{\sigma_0} \nabla \nabla \right) \cdot \\ \int_0^t \int_{\Omega} \mathcal{G}_A(\mathbf{r}, \mathbf{r}', t, t') \cdot \mathbf{P}(\mathbf{r}', t') d\mathbf{r}' dt', \end{aligned} \quad (5.19)$$

The variation of dipole density $\mathbf{P}(\mathbf{r}', t')$ across the crack is negligible when the crack opening is much smaller than crack length. In this case, the volume integral in the right hand side of (5.19) can be approximated by a surface integral. Multiplying equation (5.19) by σ_0 and using the fact that the normal component of the current at the surface of an ideal crack is zero gives

$$J_x^{(0)}(\mathbf{r}_{\pm}, t) = \lim_{\mathbf{r} \rightarrow \mathbf{r}_{\pm}} \left(\mu_0 \sigma_0 \hat{x} \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \nabla \right) \cdot \int_0^t \int_{S_0} \mathbf{L}_A(\mathbf{r}, \mathbf{r}'_0, t, t') p(\mathbf{r}'_0, t') dS'_0 dt' \quad (5.20)$$

where $\hat{x}$ denotes the normal direction to the crack surface, see Fig. 5.1, and $J_x^{(0)}(\mathbf{r}, t) = \sigma_0 \hat{x} \cdot \mathbf{E}^{(0)}(\mathbf{r}, t)$ is the normal component of the incident field on crack surfaces. Note that $\mathbf{P}(\mathbf{r}, t)$ has normal component only and $\mathbf{P}(\mathbf{r}'_0, t) \simeq \hat{x} p(\mathbf{r}'_0, t)(x_+ - x_-)$. In (5.20), $\mathbf{r}'_0$ is a coordinate in the direction normal to $\hat{x}$; $\mathbf{r}'_0 = \mathbf{r}' - \hat{x}x$. $\mathbf{r}_{\pm}$ is a limiting point on one or

other of the crack surfaces S_0 approached from within the conductor. $\mathbf{L}_A(\mathbf{r}, \mathbf{r}'_0, t, t')$ is defined as

$$\mathbf{L}_A(\mathbf{r}, \mathbf{r}'_0, t, t') = \int_{x_-}^{x_+} \mathcal{G}_A(\mathbf{r}, \mathbf{r}', t, t') dx' \cdot \hat{x} \quad (5.21)$$

where x_- and x_+ denote the lower and upper limit of crack in the normal direction.

5.3 Incident Field

Integral expressions for the time-harmonic field due to an axially symmetric coil whose axis is perpendicular to the surface of the conductor are well known and used extensively in eddy current calculations [32]. These solutions can be transformed to the time domain giving the field excited by a normal coil carrying a transient current [14]. Thus transforming the time-harmonic azimuthal electric field in a homogeneous planar conductor, gives the form

$$E^{(0)}(\rho, z, t) = \mu_0 I_0 \int_0^\infty \frac{1}{\kappa^2} \chi(\tau_0, \kappa, -z, t) \mathcal{J}(\kappa) J_1(\kappa \rho) d\kappa \quad (5.22)$$

where $\mathcal{J}(\kappa)$ is called the coil function and defined by

$$\mathcal{J}(\kappa) = \frac{N \chi(\kappa r_1, \kappa r_2) (e^{-\kappa z_2} - e^{-\kappa z_1})}{(r_2 - r_1)(z_2 - z_1)} \quad (5.23)$$

for a normal coil above a conductive plate [41], see Figure 5.1. I_0 is a reference current such that the Laplace transform of the current in the coil is $I_0 i(s)$. The waveform of the excitation current and the structure of the conductive medium are reflected in function $\chi(\tau_0, \kappa, -z, t)$. For the electric field within a conductive plate, thickness d , excited by an exponentially rising current with a time constant τ_0 , $\chi(\tau_0, \kappa, -z, t)$ can be sought from its Laplace transform [32]

$$\tilde{\chi}(\tau_0, \kappa, \zeta, s) = \frac{1}{1 + s\tau_0} \frac{1}{\kappa + \gamma} \frac{1 + \Gamma e^{-2\gamma(d-\zeta)}}{1 - \Gamma^2 e^{-2\gamma d}} e^{-\gamma \zeta} \quad (5.24)$$

where Γ is the reflection coefficient for an electromagnetic disturbance originating in the conductor and defined by

$$\Gamma = \frac{\gamma - \kappa}{\gamma + \kappa} \quad (5.25)$$

where $\gamma = \sqrt{\kappa^2 + s\mu_0\sigma_0}$ with positive real part, and $\zeta = -z$.

It is convenient to define

$$\tilde{\eta}_m(\tau_0, \kappa, \zeta, s) = \frac{1}{1 + s\tau_0} \frac{1}{\kappa + \gamma} \frac{\Gamma^m e^{-\gamma\zeta}}{1 - \Gamma^2 e^{-2\gamma d}} \quad (5.26)$$

with $m = 0$ or $m = 1$. Then, (5.24) is written as

$$\chi(\tau_0, \kappa, \zeta, t) = \eta_0(\tau_0, \kappa, \zeta, t) + \eta_1(\tau_0, 2d - \zeta, t). \quad (5.27)$$

Expand $\tilde{\eta}_m(\tau_0, \kappa, \zeta, s)$, $m = 0, 1$ into a series such that the inverse Laplace transform can be performed analytically in a term by term manner,

$$\tilde{\eta}_m(\tau_0, \kappa, \zeta, s) = \frac{1}{1 + s\tau_0} \sum_{n=0}^{\infty} \sum_{j=0}^{2n+m} C_{2n+m}^j (-2)^j \cdot \frac{\kappa^j}{(\kappa + \gamma)^{j+1}} e^{-(2nd+\zeta)\gamma}. \quad (5.28)$$

To find the inverse Laplace transform of each term in (5.28) we define

$$\tilde{\phi}_n(a, b, k, s) = \frac{\partial}{\partial k} \tilde{\Phi}_n(a, b, k, s) \quad (5.29)$$

where $\tilde{\Phi}_n(a, b, k, s)$ is defined in (5.17). The time domain version for $\tilde{\phi}_n(a, b, k, s)$ is given by taking the k derivative of (5.16) to give

$$\phi_n(a, b, k, t) = \frac{1}{(a - b)^n} \cdot \left[\psi_0(b, k, t) - \sum_{j=0}^{n-1} (a - b)^j \psi_j(a, k, t) \right] \quad (5.30)$$

where

$$\psi_n(a, k, t) = a\Psi_n(a, k, t) - \Psi_{n-1}(a, k, t) \quad (5.31)$$

and $\Psi_n(a, k, t)$ is defined in (5.18). To apply (5.30), equation (5.28) is split into two parts by partial fractions and its inverse Laplace transform is found to be

$$\begin{aligned} \eta_m(\tau_0, \kappa, \zeta, t) = & \frac{e^{-t/\tau}}{2\kappa\tau_0\sqrt{\nu\tau}} \sum_{n=0}^{\infty} \sum_{j=0}^{2n+m} C_{2n+m}^j (-2)^j \cdot \left[\phi_{j+1} \left(1, \sqrt{\nu\tau}, \kappa(2nd + \zeta), t/\tau \right) \right. \\ & \left. - \phi_{j+1} \left(1, -\sqrt{\nu\tau}, \kappa(2nd + \zeta), t/\tau \right) \right] \end{aligned} \quad (5.32)$$

where

$$\nu = \frac{1}{\tau} - \frac{1}{\tau_0}. \quad (5.33)$$

Substitute (5.27) into (5.22) the incident electric field in a plate can be evaluated by a radial integral with respect to κ .

5.4 Flaw Response

A flaw is detectable through measurements of the magnetic field above the surface of the conductor. For the case of an ideal crack, where the source of the perturbed field is a surface layer of current dipoles, the perturbed magnetic field can be written in the form

$$\mathbf{H}(\mathbf{r}, t) = \int_0^t \int_{S_0} \mathcal{G}^{(\text{me})}(\mathbf{r}, \mathbf{r}', t - t') \cdot \mathbf{p}(\mathbf{r}', t') dS' dt', \quad (5.34)$$

where $\mathcal{G}^{(\text{me})}(\mathbf{r}, \mathbf{r}', t - t')$ is the magnetic field due to a point electric source whose coordinate is $\mathbf{r}'$. With the field point in air, the magnetic-electric dyad is given by [60]

$$\mathcal{G}^{(\text{me})}(\mathbf{r}, \mathbf{r}', t - t') = [\nabla \times \nabla \times \hat{z}][\nabla' \times \hat{z}U'(\mathbf{r}, \mathbf{r}', t - t')], \quad (5.35)$$

where $U'(\mathbf{r}, \mathbf{r}', t - t')$ is the TE scalar potential above the plate due to a point electric source in the plate satisfying

$$\nabla_t^2 U'(\mathbf{r}, \mathbf{r}', t - t') = \frac{1}{2\pi} \int_0^\infty \chi(0, \kappa, -z', t - t') \cdot e^{-\kappa z} J_0(\kappa \rho) \kappa d\kappa \quad (5.36)$$

with $\rho^2 = (x - x')^2 + (y - y')^2$. The function $\chi(0, \kappa, \zeta, t)$ is defined by [60]

$$\tilde{\chi}(0, \kappa, \zeta, s) = \frac{1}{\kappa + \gamma} \frac{1 + \Gamma e^{-2\gamma(d-\zeta)}}{1 - \Gamma^2 e^{-2\gamma d}} e^{-\gamma \zeta}. \quad (5.37)$$

In due course, predictions will be compared with measurements of the magnetic field component normal to the surface of the conductor as measured using a Hall device. Taking the z -component (5.34), it is found that

$$H_z(\mathbf{r}, t) = \int_0^t \int_{S_0} \mathbf{E}^{(\text{m})}(\mathbf{r}, \mathbf{r}', t - t') \cdot \mathbf{p}(\mathbf{r}', t') dS' dt', \quad (5.38)$$

where $\mathbf{E}^{(m)}(\mathbf{r}, \mathbf{r}', t - t') = \hat{\phi} E^{(m)}(\rho, z, z', t - t')$, $\hat{\phi}$ being the azimuthal unit vector and

$$\hat{\phi} E^{(m)}(\rho, z, z', t - t') = \hat{z} \cdot \mathcal{G}^{(me)}(\mathbf{r}, \mathbf{r}', t - t') = \nabla_t^2 \nabla' \times [\hat{z} U'(\mathbf{r}, \mathbf{r}', t - t')]. \quad (5.39)$$

From (5.35), (5.36) and (5.39), it is found that

$$\begin{aligned} E^{(m)}(\rho, z, z', t - t') &= -\nabla_t^2 \frac{\partial}{\partial \rho} U'(\mathbf{r}, \mathbf{r}', t - t') \\ &= \frac{1}{2\pi} \int_0^\infty \chi(0, \kappa, -z', t - t') \cdot e^{-\kappa z} J_1(\kappa \rho) \kappa^2 d\kappa. \end{aligned} \quad (5.40)$$

It is convenient to define

$$\tilde{\xi}_m(\kappa, \zeta, s) = \frac{1}{\kappa + \gamma} \frac{\Gamma^m e^{-\gamma \zeta}}{1 - \Gamma^2 e^{-2\gamma d}} \quad (5.41)$$

such that

$$\chi(0, \kappa, \zeta, t) = \xi_0(\kappa, \zeta, t) + \xi_1(\kappa, 2d - \zeta, t). \quad (5.42)$$

Expanding (5.41) gives

$$\tilde{\xi}_m(\kappa, \zeta, s) = \sum_{n=0}^{\infty} \sum_{j=0}^{2n+m} C_{2n+m}^j (-2)^j \cdot \frac{\kappa^j}{(\kappa + \gamma)^{j+1}} e^{-(2nd+\zeta)\gamma}. \quad (5.43)$$

The time domain expression of the terms in (5.43) requires a consideration of the inverse Laplace transform of

$$\tilde{\psi}_n(a, k, s) = \frac{1}{(\sqrt{s} + a)^n} e^{-k\sqrt{s}}, \quad (5.44)$$

which is a special case of $\tilde{\phi}_n(a, b, k, s)$ with $b = 0$ as can be seen from the k derivative of (5.17). It is found that

$$\xi_m(\kappa, \zeta, t) = -\frac{e^{-t/\tau}}{\kappa \tau} \sum_{n=0}^{\infty} \sum_{j=0}^{2n+m} C_{2n+m}^j (-2)^j \cdot \psi_j(1, \kappa(2nd + \zeta), t/\tau). \quad (5.45)$$

With (5.38), (5.40), (5.42), and (5.45), the normal component of the induced magnetic field due to an buried flaw in a plate can be evaluated.

5.5 Regularization

In order to avoid using difference formulae for the derivatives in a numerical approximation of (5.20), the differential operators are brought inside the integrals to act on the kernel directly. The integral operator with the modified kernel can then be approximated numerically by a matrix. However, there is a potential problem with this procedure as illustrated by the following general relationship:

$$H(t, t_0) = \frac{\partial}{\partial t} \int_{t_0}^t F(t, t') dt' = \int_{t_0}^t \frac{\partial F(t, t')}{\partial t} dt' + F(t, t) \quad (5.46)$$

If one is to evaluate $H(t, t_0)$ from the second form, then it is necessary to evaluate $F(t, t')$ at $t' = t$ and this is not always possible. For example, in equation (5.20), the integral over $S_{\pm}$ may be identified with $F(t, t')$:

$$F(t, t') = \int_{S_0} \mathbf{L}_A(\mathbf{r}, \mathbf{r}', t - t') p(\mathbf{r}', t') dS'_0, \quad (5.47)$$

but the transformation in (5.46) cannot be carried out. To see that this is the case, note that when $t - t'$ is small, the Gaussian kernel is highly localized. Therefore, for an arbitrary point $\mathbf{r}_0$ away from the crack edge where the dipole density varies smoothly, the following approximation can be made:

$$\begin{aligned} F(t, t') &\simeq \int_{S_0} \mathbf{L}_A(\mathbf{r}_0, \mathbf{r}', t - t') dS'_0 p(\mathbf{r}_0, t') \\ &= (2\pi\mu_0\sigma_0)^{-1/2} (t - t')^{-1/2} p(\mathbf{r}_0, t). \end{aligned} \quad (5.48)$$

where, in view of the localization of the kernel, the limits integration about the point whose coordinate is $\mathbf{r}_0$ has been extended to infinity. Clearly, the time dependence $(t - t')^{-1/2}$ means that $F(t, t')$ cannot be evaluated at $t' = t$.

Before the differential operator can be moved inside the integrals, equation (5.20) must be recast. Subtracting an identical expression from the right and left hand sides gives

$$J_x^{(0)}(\mathbf{r}_{\pm}, t) - \lim_{\mathbf{r} \rightarrow \mathbf{r}_{\pm}} \left(\mu_0 \sigma_0 \hat{x} \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \nabla \right) \cdot \int_{t_n}^t \int_{S_{\pm}} \mathbf{L}_A(\mathbf{r}, \mathbf{r}', t - t') p(\mathbf{r}', t) dS' dt' =$$

$$\begin{aligned}
& \lim_{\mathbf{r} \rightarrow \mathbf{r}_{\pm}} \left(\mu_0 \sigma_0 \hat{x} \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \nabla \right) \cdot \int_0^{t_n} \int_{S_{\pm}} \mathbf{L}_A(\mathbf{r}, \mathbf{r}'_0, t - t') p(\mathbf{r}'_0, t') dS' dt' + \\
& \lim_{\mathbf{r} \rightarrow \mathbf{r}_{\pm}} \left(\mu_0 \sigma_0 \hat{x} \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \nabla \right) \cdot \int_{t_n}^t \int_{S_{\pm}} \mathbf{L}_A(\mathbf{r}, \mathbf{r}'_0, t - t') \cdot [p(\mathbf{r}'_0, t') - p(\mathbf{r}'_0, t)] dS' dt'.
\end{aligned} \tag{5.49}$$

where an arbitrary breakpoint in time, t_n , has been introduced. It is convenient to specify t_n such that the interval $t - t_n$ is constant. In so doing, the integral operator acting on $\mathbf{p}(\mathbf{r}'_0, t)$ on the left hand side of equation (5.49) is time independent. The significance of the time invariance of the operator is that its matrix approximation also does not change with time. As a result, the linear system to be solved at each time step involves the same matrix.

Equation (5.49) is rewritten in terms of a function referred to as the integrated kernel and defined by

$$K(\mathbf{r}, \mathbf{r}'_0, t - t_n) = -\mu_0 \sigma_0 \hat{x} \cdot \mathbf{L}_A(\mathbf{r}, \mathbf{r}'_0, t - t_n) + g(\mathbf{r}, \mathbf{r}'_0, t - t_n), \tag{5.50}$$

where

$$g(\mathbf{r}, \mathbf{r}'_0, t - t') = \frac{\partial}{\partial x} \nabla \cdot \int \mathbf{L}_A(\mathbf{r}, \mathbf{r}'_0, t - t') d(t - t'). \tag{5.51}$$

The transformation of (5.49) is brought about by changing the order of the integration on the left hand side and integrating with respect to time. On the right hand side the differential operators are moved inside the temporal and spatial integrals, allowing them to act on the vector potential kernel. As a results, it is found that

$$\begin{aligned}
J_x^{(0)}(\mathbf{r}_{\pm}, t) + \lim_{\mathbf{r} \rightarrow \mathbf{r}_{\pm}} \int_{S_{\pm}} K(\mathbf{r}, \mathbf{r}'_0, t - t_n) p(\mathbf{r}'_0, t) dS' = \\
- \lim_{\mathbf{r} \rightarrow \mathbf{r}_{\pm}} \int_0^{t_n} \int_{S_{\pm}} G(\mathbf{r}, \mathbf{r}'_0, t - t') p(\mathbf{r}'_0, t') dS' dt' \\
- \lim_{\mathbf{r} \rightarrow \mathbf{r}_{\pm}} \int_{t_n}^t \int_{S_{\pm}} G(\mathbf{r}, \mathbf{r}'_0, t - t') \cdot [p(\mathbf{r}'_0, t') - p(\mathbf{r}'_0, t)] dS' dt'
\end{aligned} \tag{5.52}$$

where the function

$$G(\mathbf{r}, \mathbf{r}'_0, t - t') = -\left(\mu_0 \sigma_0 \hat{x} \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \nabla \right) \cdot \mathbf{L}_A(\mathbf{r}, \mathbf{r}'_0, t - t') \tag{5.53}$$

has been introduced which will be referred to as the primary kernel to distinguish it from $K(\mathbf{r}, \mathbf{r}'_0, t - t_n)$. Note that

$$K(\mathbf{r}, \mathbf{r}'_0, t) = \int_0^t G(\mathbf{r}, \mathbf{r}'_0, t') dt' \quad (5.54)$$

hence the designation of $K(\mathbf{r}, \mathbf{r}', t)$ as the integrated kernel. Confirmation of (5.54) is found by reference to equations (5.50) and (5.51).

5.6 Discrete Approximation of the Unbounded Domain

Integral Operator

5.6.1 Integrated Kernel

For a current dipole source in an unbounded conductor, equation (5.21) gives the vector potential kernel used in (5.50) to define the integrated kernel $K(\mathbf{r}, \mathbf{r}'_0, t)$. Thus, for a volume crack between two planes $x = x_-$ and $x = x_+$, it is found that

$$\begin{aligned} K_0(\mathbf{r}, \mathbf{r}'_0, t) &= \int_{x_-}^{x_+} -\mu_0 \sigma_0 G_0(\mathbf{r}, \mathbf{r}', t, t') \\ &\quad + \frac{\partial^2}{\partial x^2} \int G_0(\mathbf{r}, \mathbf{r}', t - t') d(t - t') dx'. \end{aligned} \quad (5.55)$$

With $U(s) = \pi^{-1/2} e^{-s^2}$, equation (5.55) may be expanded to give

$$\begin{aligned} 4\pi K_0(\mathbf{r}, \mathbf{r}'_0, t) &= \frac{2}{\alpha^2} \exp \left[-\frac{(y - y')^2 + (z - z')^2}{\alpha^2} \right] \cdot \left[\operatorname{erf} \left(\frac{x - x_+}{\alpha} \right) - \operatorname{erf} \left(\frac{x - x_-}{\alpha} \right) \right] \\ &\quad + \frac{x - x_+}{R_+^3} \left[\operatorname{erfc} \left(\frac{R_+}{\alpha} \right) + 2 \frac{R_+}{\alpha} U \left(\frac{R_+}{\alpha} \right) \right] \\ &\quad - \frac{x - x_-}{R_-^3} \left[\operatorname{erfc} \left(\frac{R_-}{\alpha} \right) + 2 \frac{R_-}{\alpha} U \left(\frac{R_-}{\alpha} \right) \right] \end{aligned} \quad (5.56)$$

where $R_+^2 = (x - x_+)^2 + (y - y')^2 + (z - z')^2$ and $R_-^2 = (x - x_-)^2 + (y - y')^2 + (z - z')^2$.

With source and field point in the plane $x = x_{\pm}$, (5.56) becomes

$$\begin{aligned} 4\pi K_0(\mathbf{r}, \mathbf{r}'_0, t) &= -\frac{2}{\alpha^2} \exp \left[-\frac{(y - y')^2 + (z - z')^2}{\alpha^2} \right] \cdot \operatorname{erf} \left(\frac{a}{\alpha} \right) \\ &\quad - \frac{a}{R_a^3} \left[\operatorname{erfc} \left(\frac{R_a}{\alpha} \right) + 2 \frac{R_a}{\alpha} U \left(\frac{R_a}{\alpha} \right) \right] \end{aligned} \quad (5.57)$$

where $a = x_+ - x_-$ and $R_a^2 = a^2 + (y - y')^2 + (z - z')^2$.

Using the moment method, the integral equation for the dipole density is approximated by a matrix equation. This process has been carried out by expressing the dipole density as an expansion in pulse functions defined with respect to a set of rectangular boundary elements. The equation for the expansion coefficients is found by point matching. In this scheme, the matrix elements of the integrated kernel are given by

$$K_{\alpha\beta}^{(0)}(t) = \int_{S_\beta} K_0(\mathbf{r}_\alpha, \mathbf{r}'_0, t) dS'_0 \quad (5.58)$$

where $\mathbf{r}_\alpha$ is the matching point and S_β is the area of surface element β . In the case of the self element, a variation on the definition, equation (5.58) is adopted, as shown below. This is necessary because the singularity in the integrand means that the limit in equation (5.20) must be taken following the integration.

$$K_{00}^{(0)}(t) = \int_{S_{00}-S_x} K_0(\mathbf{0}, \mathbf{r}'_0, t) dS'_0 + I_{00} \quad (5.59)$$

where S_x is the exclusion region, centered at the origin of a local coordinate system and I_{00} is the integral over the exclusion zone given by

$$I_{00} = \lim_{\mathbf{r} \rightarrow \mathbf{r}_\pm} \int_{S_x} K_0(0, \mathbf{r}'_0, t) dS'_0. \quad (5.60)$$

The integral over the exclusion zone is evaluated assuming that the dimensions of S_x are small compared with the diffusion length α .

Initially (5.57) is expanded as a power series in R_0/α and integrated term by term. The result is expressed as a power series in a small dimensionless parameter $\epsilon = b/\alpha$ where b is a linear dimension of the exclusion zone. For a square exclusion zone of dimension $2b \times 2b$ it is found that [16]

$$I_{00} = c - \epsilon^3 \sum_{\nu=0}^{\infty} c_\nu \epsilon^{2\nu} + \operatorname{erf}\left(\frac{a}{\alpha}\right) \epsilon^2 \sum_{\nu=0}^{\infty} d_\nu \epsilon^{2\nu}. \quad (5.61)$$

Provided ϵ is small, only a few terms in the summation are needed to get a reasonable approximation of I_{00} . The coefficients in the series are given by

$$c = \lim_{x_0 \rightarrow x_{\pm}} \frac{1}{\pi} \int_0^b \int_0^b \frac{x_0 - x_+}{[(x_0 - x_+)^2 + y^2 + z^2]^{3/2}} - \frac{x_0 - x_-}{[(x_0 - x_-)^2 + y^2 + z^2]^{3/2}} dy dz \quad (5.62)$$

In addition,

$$c_\nu = \frac{a}{b\pi} \frac{4}{\sqrt{\pi}} \frac{(-1)^{\nu+1}}{\nu!(2\nu+3)} \int_0^1 \int_0^1 \left[\left(\frac{a}{b} \right)^2 + y^2 + z^2 \right]^\nu dy dz \quad (5.63)$$

and

$$d_\nu = -\frac{2}{\pi} \frac{(-1)^\nu}{\nu!} \int_0^1 \int_0^1 [y^2 + z^2]^\nu dy dz. \quad (5.64)$$

Evaluating c gives

$$c = \frac{1}{2} - \frac{1}{\pi} \arctan \left[\frac{b^2}{a\sqrt{2b^2 + a^2}} \right]. \quad (5.65)$$

Evaluating the other coefficients gives

$$c_0 = -\frac{4a}{3b\pi^{3/2}}, \quad \text{and} \quad c_1 = \frac{8a}{15b\pi^{3/2}} + \frac{4a^3}{5b^3\pi^{3/2}}, \quad (5.66)$$

$$d_0 = -\frac{2}{\pi}, \quad \text{and} \quad d_1 = \frac{4}{3\pi}, \quad (5.67)$$

Note that it is necessary to ensure that the condition $b \ll \alpha$ is valid even at the first time increment where α is smallest. For the numerical calculations, the series (5.61) is terminated at $\nu = 2$. The arbitrary parameter b may be varied, subject to the requirement that it is much smaller than the diffusion length, yet the numerical value of $K_{00}^{(0)}(t)$ should remain largely unchanged. Numerical experiments confirm that this consistency check is satisfied.

5.6.2 Primary Kernel

From (5.53), it is found that for the unbounded domain term

$$\begin{aligned} G^{(0)}(\mathbf{r}, \mathbf{r}', t - t') &= \lim_{x \rightarrow x_{\pm}} \int_{x_-}^{x_+} -\left(\mu_0 \sigma_0 \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) \cdot G_0(\mathbf{r}, \mathbf{r}', t, t') dx' \\ &= -\frac{\text{erf}\left(\frac{a}{\alpha}\right)}{2(t-t')\alpha^2} U\left(\frac{y}{\alpha}\right) U\left(\frac{z}{\alpha}\right) \cdot \left[\frac{(y-y')^2 + (z-z')^2}{\alpha^2} - 1 \right]. \end{aligned} \quad (5.68)$$

In general, the primary matrix elements are defined by

$$G_{\alpha\beta}(t) = \int_{S_\beta} G^{(0)}(\mathbf{r}_\alpha, \mathbf{r}'_0, t) d\mathbf{r}'_0. \quad (5.69)$$

For an unbounded domain, a shift in the coordinates may be carried out to give

$$G_{\alpha\beta}(t) = G_{\alpha-\beta}^{(0)}(t) = \int_{S_\alpha} G^{(0)}(\mathbf{r}_\beta, \mathbf{r}'_0, t) d\mathbf{r}'_0 = \int_{S_{\beta-\alpha}} G^{(0)}(\mathbf{0}, \mathbf{r}'_0, t) d\mathbf{r}'_0, \quad (5.70)$$

where $G(\mathbf{r}, \mathbf{r}'_0, t)$ is given by (5.68). For the self element

$$\begin{aligned} G_{00}^{(0)}(t - t') &= -\frac{2}{(t - t')\alpha^2} \operatorname{erf}\left(\frac{a}{\alpha}\right) \int_0^{\Delta y/2} \int_0^{\Delta z/2} \left[\frac{y^2 + z^2}{\alpha^2} - 1 \right] U\left(\frac{y}{\alpha}\right) U\left(\frac{z}{\alpha}\right) dy dz \\ &= -\frac{\operatorname{erf}\left(\frac{a}{\alpha}\right)}{(t - t')} \int_0^{\Delta y/2\alpha} \int_0^{\Delta z/2\alpha} (-1 + y^2 + z^2) U(y) U(z) dy dz \\ &= \frac{\operatorname{erf}\left(\frac{a}{\alpha}\right)}{2(t - t')} \left[\left(\frac{\Delta y}{2\alpha} \right) U\left(\frac{\Delta y}{2\alpha}\right) \operatorname{erf}\left(\frac{\Delta z}{2\alpha}\right) + \left(\frac{\Delta z}{2\alpha} \right) U\left(\frac{\Delta z}{2\alpha}\right) \operatorname{erf}\left(\frac{\Delta y}{2\alpha}\right) \right] \end{aligned} \quad (5.71)$$

For a general matrix element

$$\begin{aligned} G_{jk}^{(0)}(t - t') &= \frac{-\operatorname{erf}\left(\frac{a}{\alpha}\right)}{2(t - t')} \int_{(j-1/2)\Delta y/\alpha}^{(j+1/2)\Delta y/\alpha} \int_{(k-1/2)\Delta z/\alpha}^{(k+1/2)\Delta z/\alpha} (-1 + y^2 + z^2) U(y) U(z) dy dz \\ &= \frac{\operatorname{erf}\left(\frac{a}{\alpha}\right)}{8(t - t')} \{ [y_j U(y_j) - y_{j-1} U(y_{j-1})] [\operatorname{erf}(z_k) - \operatorname{erf}(z_{k-1})] \\ &\quad + [\operatorname{erf}(y_j) - \operatorname{erf}(y_{j-1})] [z_k U(z_k) - z_{k-1} U(z_{k-1})] \} \end{aligned} \quad (5.72)$$

where $y_j = (j + 1/2)\Delta y/\alpha$ and $z_k = (k + 1/2)\Delta z/\alpha$.

The image and partial reflection terms of the integrated and primary kernel are given in Section 5.10.

5.7 Linear System

In the process of approximating equation (5.52) for the surface dipole density by a discrete form, it is assumed that the field can be represented with reasonable accuracy as piecewise constant in space and piecewise linear in time. The discrete solution is imposed by first expanding the solution in terms of a superposition of pulse functions

in space and forming a time dependent matrix equation by point matching. This gives the form

$$\overline{J}(t) + \overline{K}(t - t_n)\overline{p}(t) = - \int_0^{t_n} \overline{G}(t - t')\overline{p}(t') dt' - \int_{t_n}^t \overline{G}(t - t')[\overline{p}(t') - \overline{p}(t)] dt' \quad (5.73)$$

where the matrix elements of $\overline{G}(t)$ and $\overline{K}$ are defined by (5.69) and (5.58) respectively.

In parallel with equation (5.54),

$$\overline{K}(t) = \int_0^t \overline{G}(t') dt'. \quad (5.74)$$

The discrete equation with a piecewise linear approximation of the time variation requires a further matrix defined by

$$\overline{M}(t) = \int_0^t \overline{G}(t') t' dt'. \quad (5.75)$$

The dipole density is approximated as piecewise linear, in equal time intervals, Δ . As such it can be expanded as

$$\overline{p}(t) = \sum_{j=1}^k \overline{p}(t_j) \psi_j(t), \quad (5.76)$$

where $t_j = j\Delta$, $t \leq t_k$ and

$$\psi_j(t) = \begin{cases} 1 - |t_j - t|/\Delta, & t_{j-1} \leq t < t_{j+1} \\ 0, & \text{otherwise.} \end{cases} \quad (5.77)$$

Substitute (5.76) into (5.73) and let $t_k - t_n = \Delta_t$ to give

$$\begin{aligned} \overline{J}(t_k) + \overline{K}(\Delta_t)\overline{p}(t_k) &= - \sum_{j=1}^{k-1} \overline{p}(t_j) \int_{t_{j-1}}^{t_{j+1}} \overline{G}(t_k - t') dt' \\ &\quad + \frac{1}{\Delta_t} \sum_{j=1}^{k-1} \overline{p}(t_j) \int_{t_{j-1}}^{t_{j+1}} \overline{G}(t_k - t') |t_j - t'| dt' \\ &\quad + \frac{1}{\Delta_t} \int_{t_{k-1}}^{t_k} \overline{G}(t_k - t') \overline{p}(t_k) |t_k - t'| dt'. \end{aligned} \quad (5.78)$$

Next, we evaluate the three terms in right hand side of (5.78) explicitly. The first term is evaluated using equation (5.74), which gives

$$- \sum_{j=1}^{k-1} \overline{p}(t_j) \int_{t_{j-1}}^{t_{j+1}} \overline{G}(t_k - t') dt' = \sum_{j=1}^{k-1} \overline{p}(t_j) [\overline{K}(t_k - t_{j+1}) - \overline{K}(t_k - t_{j-1})], \quad (5.79)$$

The second term is determined by taking advantage both (5.74) and (5.75),

$$\begin{aligned}
& \frac{1}{\Delta_t} \sum_{j=1}^{k-1} \bar{p}(t_j) \int_{t_{j-1}}^{t_{j+1}} \bar{G}(t_k - t') |t_j - t'| dt' \\
&= \frac{1}{\Delta_t} \sum_{j=1}^{k-1} \bar{p}(t_j) \left\{ (t_k - t_j) [\bar{K}(t_k - t_j) - \bar{K}(t_k - t_{j-1})] - [\bar{M}(t_k - t_j) - \bar{M}(t_k - t_{j-1})] \right. \\
&\quad \left. - (t_k - t_j) [\bar{K}(t_k - t_{j+1}) - \bar{K}(t_k - t_j)] + [\bar{M}(t_k - t_{j+1}) - \bar{M}(t_k - t_j)] \right\}, \quad (5.80)
\end{aligned}$$

and the third term is found to be

$$\frac{1}{\Delta_t} \int_{t_{k-1}}^{t_k} \bar{G}(t_k - t') \bar{p}(t_k) |t_k - t'| dt' = \frac{\bar{p}(t_k)}{\Delta_t} \bar{M}(\Delta_t). \quad (5.81)$$

Combing (5.78), (5.79), (5.80) and (5.81) gives

$$\bar{L}(\Delta_t) \bar{p}(t_k) = \bar{Y}(t_k), \quad (5.82)$$

where

$$\bar{L}(\Delta_t) = \bar{K}(\Delta_t) - \frac{1}{\Delta_t} \bar{M}(\Delta_t), \quad (5.83)$$

and

$$\bar{Y}(t_k) = -\bar{J}(t_k) + \sum_{i=1}^{k-1} \bar{L}(t_k - t_i) \bar{p}(t_i) \quad (5.84)$$

with

$$\begin{aligned}
\Delta_t \bar{L}(t_k - t_i) &= (t_k - t_{i-1}) [\bar{K}(t_k - t_i) - \bar{K}(t_k - t_{i-1})] \\
&\quad - (t_k - t_{i+1}) [\bar{K}(t_k - t_{i+1}) - \bar{K}(t_k - t_i)] \\
&\quad + \bar{M}(t_k - t_{i+1}) + \bar{M}(t_k - t_{i-1}) - 2\bar{M}(t_k - t_i) \quad (5.85)
\end{aligned}$$

Initially $\bar{p}(t_1)$ is calculated by solving (5.82) with $\bar{Y}(t_1) = -\bar{J}(t_1)$. At the next time interval $\bar{p}(t_1)$ is used to in calculating $\bar{Y}(t_2)$ but the same matrix, $\bar{L}(\Delta_t)$ is used to find the new solution $\bar{p}(t_2)$ and so on. Because the same matrix is used at each stage, it is worthwhile computing the inverse of $\bar{L}(\Delta_t)$ via an LU decomposition [49] allowing $\bar{p}(t_k)$ to be determined very quickly for all k .

Figure 5.1 Experimental setup for measuring the induced magnetic field due to a subsurface crack in a conductive plate. The driver coil has an exponentially rising current $I = I_0(1 - e^{-t/\tau_0})$ with the time constant $\tau_0 = 250\mu s$.

Table 5.1 Probe C29 parameters

Inner radius r_1 (mm)	8.72
Outer radius r_2 (mm)	15.50
Length of windings $z_2 - z_1$ (mm)	6.88
Number of turns N	1008
Coil lift-off z_1 (mm)	2.62
Hall lift-off z_H (mm)	0.82
Current rise time τ_0 (μs)	250
DC coil resistance R_0 (Ω)	123
Isolated coil inductance L_0 (mH)	23.13

Table 5.2 Aluminum alloy plate F5 and slot parameters

Plate thickness d (mm)	6.45
Conductivity σ_0 (S/m)	2.28×10^7
Permeability μ_0 (H/m)	$4\pi \times 10^{-7}$
Ligament thickness g (mm)	1.55 ± 0.05
Slot width a (mm)	0.35

Table 5.3 Aluminum alloy plate F6 and slot parameters

Plate thickness d (mm)	6.45
Conductivity σ_0 (S/m)	2.28×10^7
Permeability μ_0 (H/m)	$4\pi \times 10^{-7}$
Ligament thickness g (mm)	1.09 ± 0.05
Slot width a (mm)	0.35

5.8 Numerical Results

A calibrated Hall sensor, mounted on the axis of an excitation coil was used to determine the variation of the axial magnetic field with time. The probe was placed on a conducting plate containing a subsurface slot used to simulate a crack and eddy currents excited by passing regular, well defined current pulses through the coil via a transconductance amplifier [14]. The current pulses rise exponentially to 70mA with a time constant, $250\mu s.$, according to component value in an RC filter. By subtracting the background plate signal, Figure 5.2, the perturbed magnetic field due to eddy current interaction with the subsurface slot is found.

Simulations have been performed to study the normal component of the magnetic field along the coil axis perturbed by crack. The integral equation defined in (5.20) is approximated as a matrix equation and solved using the method of moment where the crack region is divided into 40 cells in y axis and 30 cells in z axis. Note that the crack is bisected by the xz plane. The matching points are located at the center of each element. Assuming the dipole density within each cell is piecewise linear in the time and piecewise constant in space, equation (5.38) can be approximated as

$$\begin{aligned}
 H_z(\mathbf{r}, t_k) = \frac{\Delta t \Delta y \Delta z}{6} \sum_{i=1}^k \sum_{j=1}^N \left\{ \right. \\
 \left. \begin{aligned}
 & \left[2\mathbf{E}^{(m)}(\mathbf{r}, \mathbf{r}'_j, t_{k-i+1}) + \mathbf{E}^{(m)}(\mathbf{r}, \mathbf{r}'_j, t_{k-i}) \right] \cdot \mathbf{p}(\mathbf{r}'_j, t_{i-1}) \\
 & + \left[\mathbf{E}^{(m)}(\mathbf{r}, \mathbf{r}'_j, t_{k-i+1}) + 2\mathbf{E}^{(m)}(\mathbf{r}, \mathbf{r}'_j, t_{k-i}) \right] \cdot \mathbf{p}(\mathbf{r}'_j, t_i) \end{aligned} \right\} \quad (5.86)
 \end{aligned}$$

where $N = N_y \times N_z$, $\mathbf{r}'_j$ is the center of each cell and $\mathbf{E}^{(m)}(\mathbf{r}, \mathbf{r}'_j, 0) = \mathbf{p}(\mathbf{r}, 0) = 0$. Four

Figure 5.2 Induced magnetic field due to unflawed region of plate F5 and F6.

even images and five odd images are considered in the evaluation of plate integrated and primary kernel. The partial reflection terms are represented as asymptotic series where the first 3 terms are taken for numerical calculation. It has been tested the results would not change much with more terms added. The radial spatial integral defined in (5.22), (5.40), (5.102), and (5.108) are approximated as series to save the computational cost using the method described in [60] where the boundary is set to be 0.2m and the series is truncated at 250 terms. Figure 5.3 and 5.4 plot the estimated magnetic field due the slots in plate F5 and F6 together with associated experimental measurements. They show that numerical calculations successfully predict the induced magnetic field due to crack. The predications are only 2% and 6% higher than the experimental data at the peak for the ideal crack case and the crack with 0.05mm opening. Instability issue occurs when the crack opening continue to increase, as it is shown in Figure 5.4. It is observed that the induced magnetic field reaches its peak at about three times

the time constant of the excitation current. The theoretical predictions are expected to increase 10% when the crack opening is considered. The errors may be introduced in a number of ways. The size of the cell, especially in y axis, may be too large to represent the field within the cell accurately. The experiment was performed assuming that the slots in plate F5 and F6 are infinite long along y axis. However, they are truncated to be 120mm in the numerical modelling. Due to limited computational resources, only 40 cells are used in the y axis. On the other hand, Figure 5.5, 5.6, and 5.7 show the distribution of dipole density evolving with time. It can be seen that the dipole density varies significantly along ρ direction at early times, whereas the dipole density is assumed constant within each cell. Another source of numerical error may arise from the non-vanishing behavior of the dipole density along the buried top edge due to the piecewise constant discretization of the crack. A special element forcing the zero dipole density at crack edge is being developed to overcome this problem. A critical parameter in measurements on subsurface cracks is the distance from the crack edge to the surface of the conductor, known as the ligament size. The results are sensitive to the value of this parameter therefore a small error in its value may lead to significant differences between theory and experiment.

Figure 5.8, 5.9, and 5.10 show the induced electric current density on the crack surface excited by the driver coil at three time moments. At early times, the field is localized near the plate surface. As time goes by, the field penetrates into the plate deeper and spreads out. Figure 5.11 shows the effect of coil displacement. The induced magnetic field due to the crack in plate F5 decays very rapidly when the coil is moved away from the coil.

Figure 5.3 Transient magnetic field due to a subsurface crack in plate F5 when the coil is centered above the crack. The amplitude of ΔH_z is normalized to an excitation current of 1A.

Figure 5.4 Transient magnetic field due to a subsurface crack in plate F6 when the coil is centered above the crack. The amplitude of ΔH_z is normalized to an excitation current of 1A.

Figure 5.5 Distribution of dipole density $p(\mathbf{r}, t)$ at $30 \mu s$ in plate F5.

Figure 5.6 Distribution of dipole density $p(\mathbf{r}, t)$ at $750 \mu s$ in plate F5.

Figure 5.7 Distribution of dipole density $p(\mathbf{r}, t)$ at $2250 \mu s$ in plate F5.

Figure 5.8 Incident current density $J(\mathbf{r}, t)$ on crack surface at $30 \mu s$ in plate F5.

Figure 5.9 Incident current density $J(\mathbf{r}, t)$ on crack surface at $750 \mu s$ in plate F5.

Figure 5.10 Incident current density $J(\mathbf{r}, t)$ on crack surface at $2250 \mu s$ in plate F5.

Figure 5.11 Variation of ΔH_z due to a long crack in plate F5 when the coil moves away from the crack.

5.9 Conclusion

An integral formulation has been developed to calculate the interaction of an ideal crack with transient eddy currents in a conductor. By using the moment method, the integral equation for the field at the crack is approximated by a matrix equation and a numerical solution computed. The discontinuity in the field at the crack is expressed in terms of a current dipole density which is found from the numerical approximation and used to predict the external magnetic field. These predictions have been compared with measurements for the case of a subsurface rectangular slot which is used to simulate the effects of a crack on the eddy current distribution. The prediction and the experiment show reasonable agreement.

5.10 Appendix

5.10.1 Plate Integrated Kernel

In Section 5.6, matrix elements are evaluated from the unbounded domain integral kernels. In this Appendix, the matrix elements for plate kernels are evaluated assuming, as in the main text, that the crack lies between the plane $x = x_{\pm}$. Initially, the integrated kernel, defined by equation (5.50), is written in an expanded form. Combining (5.9), (5.21), (5.50) and (5.51) gives

$$\begin{aligned} K(\mathbf{r}, \mathbf{r}'_0, t, t') &= K_-(\mathbf{r}, \mathbf{r}'_0, t, t') + K_+(\mathbf{r}, \mathbf{r}'_0, t, t') \\ &\quad - V_{yy-}(\mathbf{r}, \mathbf{r}'_0, t, t') - V_{yy+}(\mathbf{r}, \mathbf{r}'_0, t, t') \end{aligned} \quad (5.87)$$

The terms in (5.87) are defined by

$$K_-(\mathbf{r}, \mathbf{r}'_0, t, t') = \sum_{n=-\infty}^{\infty} K_0(\mathbf{r}, \mathbf{r}_{1n}, t, t'), \quad (5.88)$$

$$K_+(\mathbf{r}, \mathbf{r}'_0, t, t') = \sum_{n=-\infty}^{\infty} K_0(\mathbf{r}, \mathbf{r}_{2n}, t, t'), \quad (5.89)$$

with the free space kernel $K_0(\mathbf{r}, \mathbf{r}'_0, t, t')$ being given by (5.55),

$$V_{yy-}(\mathbf{r}, \mathbf{r}'_0, t, t') = \mu_0 \sigma_0 \int_{x_-}^{x_+} \frac{\partial^2}{\partial y^2} V_-(\mathbf{r}, \mathbf{r}', t, t') dx', \quad (5.90)$$

and

$$V_{yy+}(\mathbf{r}, \mathbf{r}'_0, t, t') = \mu_0 \sigma_0 \int_{x_-}^{x_+} \frac{\partial^2}{\partial y^2} V_+(\mathbf{r}, \mathbf{r}', t, t') dx', \quad (5.91)$$

Note that $K_-(\mathbf{r}, \mathbf{r}', t - t')$ contains a term relating to the fundamental solution and reflection contributions from the even images which means they are produced by an even number of reflections. $K_+(\mathbf{r}, \mathbf{r}', t - t')$ is due to odd images resulting from $2n - 1$ reflections. V_{yy-} and V_{yy+} are the second order derivative of $V_-(\mathbf{r}, \mathbf{r}', t, t')$ and $V_+(\mathbf{r}, \mathbf{r}', t, t')$,

see (5.13) and (5.14). The matrix elements sought are expressed in terms of a function $\Pi(x, y, z, t)$ defined such that

$$\left. \frac{\partial \Pi_{\pm}(x - x', y - y', z \pm z', t - t')}{\partial z} \right|_{x=x_{\pm}} = \left[\frac{\partial V_{\pm}(\mathbf{r}, \mathbf{r}', t - t')}{\partial y} \right]_{x=x_{\pm}}. \quad (5.92)$$

Using $J'_0(x) = -J_1(x)$, it is found that

$$\Pi_{-}(x, y, z, t) = -\frac{y}{4\pi\rho} \int_0^{\infty} [F_2(\kappa, 2d - z, t) - F_2(\kappa, 2d + z, t)] J_1(\kappa\rho) d\kappa \quad (5.93)$$

and

$$\Pi_{+}(x, y, z, t) = -\frac{y}{4\pi\rho} \int_0^{\infty} [F_1(\kappa, -z, t) - F_1(\kappa, 2d + z, t)] J_1(\kappa\rho) d\kappa \quad (5.94)$$

where $F_m(\kappa, \zeta, t)$ is defined in (5.15) and $\rho = \sqrt{x^2 + y^2}$.

The corresponding matrix form of (5.87) is written as

$$K_{lm,LM}(t) = K_{l-L,m-M}^{(-)}(t) + V_{l-L,m-M}^{(-)}(t) + K_{l-L,m+M+1}^{(+)}(t) + V_{l-L,m+M+1}^{(+)}(t) \quad (5.95)$$

where $l, L = 0, 1, \dots, N_y - 1$, $m, M = 0, 1, \dots, N_z - 1$.

5.10.1.1 Even Image Terms

The first term in (5.95) is evaluated by

$$\begin{aligned} K_{l-L,m-M}^{(-)}(t) &= \sum_{n=-\infty}^{\infty} \int_{S_{LM}} K_0(\mathbf{r}_{lm}, \mathbf{r}_{1n}, t) dS' \\ &= \sum_{n=-\infty}^{\infty} \int_{(L-1/2)\Delta y}^{(L+1/2)\Delta y} \int_{(M-1/2)\Delta z}^{(M+1/2)\Delta z} K_0(y_l - y', z_m - z' - 2nd, t) dy' dz'. \end{aligned} \quad (5.96)$$

where $\mathbf{r}_{lm}$ denotes a field point on the crack surface $(x_{\pm}, y, z)$. Let $y_l - y' = p$, $z_m - z' - 2nd = q$. Then $dy' = -dp$, $dz' = -dq$. Therefore,

$$K_{l-L,m-M}^{(-)}(t) = \sum_{n=-\infty}^{\infty} \int_{(l-L+1/2)\Delta y}^{(l-L-1/2)\Delta y} \int_{(m-M+1/2)\Delta z-2nd}^{(m-M-1/2)\Delta z-2nd} K_0(p, q, t) dp dq \quad (5.97)$$

By collapsing $l - L$ to l and $m - M$ to m we further write

$$K_{l,m}^{(-)}(t) = \sum_{n=-\infty}^{\infty} \int_{(l-1/2)\Delta y}^{(l+1/2)\Delta y} \int_{(m-1/2)\Delta z+2nd}^{(m+1/2)\Delta z+2nd} K_0(y, z, t) dy dz \quad (5.98)$$

where $l = 0, \dots, N_y - 1$ and $m = -N_z + 1, \dots, N_z - 1$. $K_0(y, z, t)$ is derived from (5.57),

$$4\pi K_0(y, z, t) = -\frac{2}{\alpha^2} \exp\left[-\frac{y^2 + z^2}{\alpha^2}\right] \operatorname{erf}\left(\frac{a}{\alpha}\right) - \frac{a}{R_a^3} \left[\operatorname{erfc}\left(\frac{R_a}{\alpha}\right) + 2\frac{R_a}{\alpha} U\left(\frac{R_0}{\alpha}\right) \right]. \quad (5.99)$$

Note that $R_a^2 = a^2 + y^2 + z^2$.

5.10.1.2 Even Partial Reflection Terms

The second term in (5.95) is evaluated by

$$\begin{aligned} V_{l-L, m-M}^{(-)}(t) &= -\mu_0 \sigma_0 \int_{S_{LM}} \int_{x_-}^{x_+} \frac{\partial^2}{\partial y'^2} V_-(\mathbf{r}_{lm}, \mathbf{r}'_0, t, t') dx' dS' \\ &= -\mu_0 \sigma_0 \int_{x_-}^{x_+} \int_{(L-1/2)\Delta y}^{(L+1/2)\Delta y} \int_{(M-1/2)\Delta z}^{(M+1/2)\Delta z} \\ &\quad \frac{\partial^2 \Pi_-(x_{\pm} - x', y_l - y, z_m - z, t - t')}{\partial y \partial z} dy dz dx' \end{aligned} \quad (5.100)$$

Following a transformation of integral variable illustrated in deriving (5.97) we get

$$\begin{aligned} V_{l-L, m-M}^{(-)}(t) &= -\mu_0 \sigma_0 \int_{x_-}^{x_+} \int_{(l-L-1/2)\Delta y}^{(l-L+1/2)\Delta y} \int_{(m-M-1/2)\Delta z}^{(m-M+1/2)\Delta z} \\ &\quad \frac{\partial^2 \Pi_-(x_{\pm} - x', p, q, t - t')}{\partial p \partial q} dp dq dx' \end{aligned} \quad (5.101)$$

and

$$\begin{aligned} V_{l,m}^{(-)}(t) &= \frac{\mu_0 \sigma_0}{4\pi} \int_0^{\infty} \int_{x_-}^{x_+} \left[\frac{y_l J_1(\kappa \rho_l)}{\rho_l} - \frac{y_{l-1} J_1(\kappa \rho_{l-1})}{\rho_{l-1}} \right] dx' \\ &\quad \left[F_2(\kappa, 2d - z_m, t) - F_2(\kappa, 2d + z_m, t) \right. \\ &\quad \left. - F_2(\kappa, 2d - z_{m-1}, t) + F_2(\kappa, 2d + z_{m-1}, t) \right] d\kappa \end{aligned} \quad (5.102)$$

where $y_l = (l + 1/2)\Delta y$, $z_m = (m + 1/2)\Delta z$ and $\rho_l = \sqrt{(x_{\pm} - x')^2 + y_l^2}$.

5.10.1.3 Odd Image Terms

The third term in (5.95) is evaluated by

$$\begin{aligned}
 K_{l-L, m+M+1}^{(+)}(t) &= \sum_{n=-\infty}^{\infty} \int_{S_{LM}} K_0(\mathbf{r}_{lm}, \mathbf{r}_{2n}, t) dS' \\
 &= \sum_{n=1}^{\infty} \int_{(L-1/2)\Delta y}^{(L+1/2)\Delta y} \int_{(M-1/2)\Delta z}^{(M+1/2)\Delta z} \\
 &\quad K_0(y_l - y', z_m + z' + \Delta z + 2g + 2nd, t) dy' dz' \quad (5.103)
 \end{aligned}$$

where g is the ligament. Let $y_l - y' = p$, $z_m + z' + \Delta z + 2g + 2nd = q$. Then $dy' = -dp$, $dz = dq$ and (5.103) becomes

$$\begin{aligned}
 K_{l-L, m+M+1}^{(+)}(t) &= - \sum_{n=-\infty}^{\infty} \int_{(l-L+1/2)\Delta y}^{(l-L-1/2)\Delta y} \\
 &\quad \int_{(m+M+1/2)\Delta z + 2g + 2nd}^{(m+M+3/2)\Delta z + 2g + 2nd} K_0(p, q, t) dp dq. \quad (5.104)
 \end{aligned}$$

Again, we write

$$K_{l,m}^{(+)}(t) = \sum_{n=-\infty}^{\infty} \int_{(l-1/2)\Delta y}^{(l+1/2)\Delta y} \int_{(m-1/2)\Delta z + 2g + 2nd}^{(m+1/2)\Delta z + 2g + 2nd} K_0(y, z, t) dy dz \quad (5.105)$$

where $l = 0, \dots, N_y - 1$ and $m = 1, \dots, 2N_z - 1$.

5.10.1.4 Odd Partial Reflection Terms

The fourth term in (5.95) is evaluated by

$$\begin{aligned}
 V_{l-L, m+M+1}^{(+)}(t) &= -\mu_0 \sigma_0 \int_{S_{LM}} \int_{x_-}^{x_+} \frac{\partial^2}{\partial y'^2} V_+(\mathbf{r}_{lm}, \mathbf{r}'_0, t, t') dx' dS' \\
 &= \mu_0 \sigma_0 \int_{x_-}^{x_+} \int_{(L-1/2)\Delta y}^{(L+1/2)\Delta y} \int_{(M-1/2)\Delta z}^{(M+1/2)\Delta z} \\
 &\quad \frac{\partial^2 \Pi_+(x_{\pm} - x', y_l - y, z_m + z + \Delta z + 2g, t - t')}{\partial y \partial z} dy dz dx' \quad (5.106)
 \end{aligned}$$

Let $y_l - y = p$, $z_m + z + \Delta z + 2g = q$. Then $dy = -dp$, $dz = dq$ and (5.106) becomes

$$\begin{aligned}
 V_{l-L, m+M+1}^{(+)}(t) &= -\mu_0 \sigma_0 \int_{x_-}^{x_+} \int_{(l-L+1/2)\Delta y}^{(l-L-1/2)\Delta y} \int_{(m+M+1/2)\Delta z + 2g}^{(m+M+3/2)\Delta z + 2g} \\
 &\quad \frac{\partial^2 \Pi_+(x_{\pm} - x', p, q, t - t')}{\partial p \partial q} dp dq dx'. \quad (5.107)
 \end{aligned}$$

Further, we get

$$\begin{aligned}
V_{l,m}^{(+)}(t) = & \frac{\mu_0 \sigma_0}{4\pi} \int_0^\infty \int_{x_-}^{x_+} \left[\frac{y_l J_1(\kappa \rho_l)}{\rho_l} - \frac{y_{l-1} J_1(\kappa \rho_{l-1})}{\rho_{l-1}} \right] dx' \\
& \left[F_1(\kappa, -z_m, t) - F_1(\kappa, 2d + z_m, t) \right. \\
& \left. - F_1(\kappa, -z_{m-1}, t) + F_1(\kappa, 2d + z_{m-1}, t) \right] d\kappa
\end{aligned} \tag{5.108}$$

where $y_l = (l + 1/2)\Delta y$ and $z_m = (m + 1/2)\Delta z + 2g$.

5.10.2 Plate Primary Kernel

The primary kernel, combining (5.9), (5.21) and (5.53), is given by

$$\begin{aligned}
G(\mathbf{r}, \mathbf{r}'_0, t, t') = & - \int_{x_-}^{x_+} \left(\sigma_0 \mu_0 \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) \cdot \\
& \sum_{n=-\infty}^{\infty} \left[G_0(\mathbf{r}, \mathbf{r}_{1n}, t, t') + G_0(\mathbf{r}, \mathbf{r}_{2n}, t, t') \right] \\
& - \mu_0 \sigma_0 \frac{\partial^2}{\partial y^2} \frac{\partial}{\partial t} V_-(\mathbf{r}, \mathbf{r}'_0, t, t') - \mu_0 \sigma_0 \frac{\partial^2}{\partial y^2} \frac{\partial}{\partial t} V_+(\mathbf{r}, \mathbf{r}'_0, t, t') dx'.
\end{aligned} \tag{5.109}$$

From the primary matrix definition (5.69) we write

$$\begin{aligned}
G_{lm,LM}(t) = & G_{l-L,m-M}^{(-)}(t) + G_{l-L,m-M}^{(V-)}(t) \\
& + G_{l-L,m+M+1}^{(+)}(t) + G_{l-L,m+M+1}^{(V+)}(t).
\end{aligned} \tag{5.110}$$

The explicit expression of each terms in (5.110) can be derived by following the exact methodologies used in deriving the integrated kernel.

The even image term $G_{l-L,m-M}^{(-)}(t)$ is given by

$$G_{l,m}^{(-)}(t) = \sum_{n=-\infty}^{\infty} \int_{(l-1/2)\Delta y}^{(l+1/2)\Delta y} \int_{(m-1/2)\Delta z+2nd}^{(m+1/2)\Delta z+2nd} G_0(y, z, t) dy dz \tag{5.111}$$

where $l = 0, \dots, N_y - 1$ and $m = -N_z + 1, \dots, N_z - 1$. $G_0(y, z, t)$ is derived from (5.68),

$$G_0(y, z, t) = - \frac{\text{erf}\left(\frac{a}{\alpha}\right)}{2(t-t')\alpha^2} \left[\frac{y^2 + z^2}{\alpha^2} - 1 \right] U\left(\frac{y}{\alpha}\right) U\left(\frac{z}{\alpha}\right). \tag{5.112}$$

The even partial reflection kernel term $G_{l-L, m-M}^{(V-)}(t)$ is given by

$$G_{l-L, m-M}^{(V-)}(t) = \frac{\mu_0 \sigma_0}{4\pi} \int_0^\infty \int_{x_-}^{x_+} \left[\frac{y_l J_1(\kappa \rho_l)}{\rho_l} - \frac{y_{l-1} J_1(\kappa \rho_{l-1})}{\rho_{l-1}} \right] dx' \\ \left[F_2'(\kappa, 2d - z_m, t) - F_2'(\kappa, 2d + z_m, t) \right. \\ \left. - F_2'(\kappa, 2d - z_{m-1}, t) + F_2'(\kappa, 2d + z_{m-1}, t) \right] d\kappa \quad (5.113)$$

where $F_m'(\kappa, \zeta, t)$ is the derivative of $F_m(\kappa, \zeta, t)$ with respect to t ,

$$F_m'(\kappa, \zeta, t) = \frac{\kappa^2 e^{-t/\tau}}{(\mu_0 \sigma_0)^2} \sum_{n=0}^\infty \sum_{j=1}^{2n+m} C_{2n+m}^j (-2)^j [\Phi_j'(1, 0, \kappa(2nd + \zeta), t/\tau) \\ - \Phi_j(1, 0, \kappa(2nd + \zeta), t/\tau)] \quad (5.114)$$

with

$$\Phi_n'(a, b, k, t) = \frac{1}{(a-b)^n} \left[\Psi_0'(b, k, t) - \sum_{j=0}^{n-1} (a-b)^j \Psi_j'(a, k, t) \right] \quad (5.115)$$

and

$$\Psi_n'(a, k, t) = \left(\frac{n}{2t} + a^2 \right) \Psi_n(a, k, t) + \left(a - \frac{k}{2t} \right) \Psi_{n-1}(a, k, t). \quad (5.116)$$

Note that $y_l = (l + 1/2)\Delta y$, $z_m = (m + 1/2)\Delta z$, and $\rho_l = \sqrt{(x_\pm - x')^2 + y_l^2}$.

The odd image term $G_{l-L, m+M+1}^{(+)}(t)$ is given by

$$G_{l,m}^{(+)}(t) = \sum_{n=-\infty}^\infty \int_{(l-1/2)\Delta y}^{(l+1/2)\Delta y} \int_{(m-1/2)\Delta z + 2g + 2nd}^{(m+1/2)\Delta z + 2g + 2nd} G_0(y, z, t) dy dz \quad (5.117)$$

where $l = 0, \dots, N_y - 1$, and $m = 1, \dots, 2N_z - 1$.

The odd partial reflection term $G_{l-L, m+M+1}^{(V+)}(t)$ is given by

$$G_{l,m}^{(V+)}(t) = \frac{\mu_0 \sigma_0}{4\pi} \int_0^\infty \int_{x_-}^{x_+} \left[\frac{y_l J_1(\kappa \rho_l)}{\rho_l} - \frac{y_{l-1} J_1(\kappa \rho_{l-1})}{\rho_{l-1}} \right] dx' \\ \left[F_1'(\kappa, -z_m, t) - F_1'(\kappa, 2d + z_m, t) \right. \\ \left. - F_1'(\kappa, -z_{m-1}, t) + F_1'(\kappa, 2d + z_{m-1}, t) \right] d\kappa. \quad (5.118)$$

where $y_l = (l + 1/2)\Delta y$, and $z_m = (m + 1/2)\Delta z + 2g$. $\square$

CHAPTER 6. INDUCED VOLTAGE IN AN ENCIRCLING COIL DUE TO TRANSIENT EDDY CURRENTS IN A CYLINDRICAL ROD

6.1 Introduction

A fundamental problems in eddy current nondestructive evaluation is to determine the response measured by an induction coil due to its electromagnetic interaction with a homogeneous conductive workpiece. For simple conductor geometries, closed form analytical solutions can be used to compute probe signals and these solutions may form a component of a numerical scheme to determine the probe response due to flaws [32]. Using analytical methods and integrating such solutions into numerical code often enables the accuracy and reliability of at least part calculation to be maintained while keeping the computational cost low. This is especially true of transient field calculations since these are likely to be an order of magnitude more computer intensive than the corresponding single frequency calculations.

An examples of analytical results for modelling transient eddy currents includes the calculation of the normal coil voltage above a half-space conductor [14]. Similarly the electromagnetic field in a half-space excited by a normal coil can be determined directly from time domain expressions. An efficient calculation of the signal due to a flaw in a half-space can be achieved using integral methods and a dedicated half-space integral kernel [15] but it is more general and more useful in a planar geometry to work with

a plate as the host. In the case of a plate, the frequency domain expressions are more complicated making it more difficult to transform to the time domain directly using the inverse Laplace transform. However, by expanding the expression for the pick-up voltage as an asymptotic series, it is possible to transform to the time domain term-by-term [39]. For numerical computations it is necessary to terminate such series and because they converge rapidly, only a few terms are needed [41], [62]. An alternative way of examining the asymptotic series for large time regime is to find the poles of the frequency expression and represent the time domain solution as a sum of exponential functions [63]. This method has been used to determine the conductivity of a tube using the long time tail of the transient magnetic field within the tube where a single pole was considered [64]. Two basic types of asymptotic series are available, one based on the short time limit and one based on the long time limit. Thus a short or long time asymptotic series $\sum_{n=1}^N f_n(t)$ is one that is asymptotic to a function $F(t)$ in sense that the difference vanishes in the limit as the time t goes to zero or infinity, respectively. There is a considerable region of overlap in their useful time range allowing tests consistency to be made.

Here we examine short and long time series for describing the quasi-static electromagnetic field interaction with a homogeneous conducting rod of infinite length. This is the most basic cylindrical conductor geometry providing a test case for examining the feasibility representing solutions for more complex situations that may arise for in the modelling to transient eddy current inspection of pipelines, bore holes and tubing.

6.2 Induced Electromotive Force

We consider the induced voltage in a pick-up coil due to eddy currents in an infinite rod of radius r . A driver coil carries current pulses and a pick-up coil senses changes in the field by electromagnetic induction. Both driver and pick-up are concentric with the rod,

Figure 6.1 Driver and pick up coil circling a conductive rod

Figure 6.1. The field in air satisfies the Laplace equation and can be found by separation of variables. Expressing the solution for a coil in terms of a Fourier transform in the axial direction, means that the radial dependence is represented by associate Bessel functions. In the standard theory, one represents the field in air due to eddy currents in the rod as an integral superposition of terms having the form $A(\kappa)K_0(\kappa\rho)\cos(\kappa z)$. Similarly, the field in the conductor, which satisfies the Helmholtz equation, is written as an integral with respect to the spatial frequency κ of an expression having the form $B(\kappa)I_0(\gamma\rho)\cos(\kappa z)$, where $\gamma = \sqrt{\kappa^2 + s\mu_r\mu_0\sigma}$, s being the Laplace variable. An elementary field due to a current filament can be found in this way. The field due to a coil of finite axial and radial extent is then determined by integrating over the coil cross section. The pick-up coil voltage can then be found by integrating the electric field over the cross section of the pick-up coil. Using the Laplace variable s to represent frequency, the result is given by [32]

$$v(s) = sI(s)\mu_0 \int_0^\infty \mathcal{J}_1(\kappa)\mathcal{J}_2(\kappa) \frac{\kappa\mu_r I_0(\kappa r)I_1(\gamma r) - \gamma I_1(\kappa r)I_0(\gamma r)}{\kappa\mu_r K_0(\kappa r)I_1(\gamma r) + \gamma K_1(\kappa r)I_0(\gamma r)} d\kappa \quad (6.1)$$

where $I(s)$ is the driver current. The coil geometry is represented by the coil functions $\mathcal{J}_1(\kappa)$ and $\mathcal{J}_2(\kappa)$ for driver and pick-up respectively and given by

$$\mathcal{J}_1(\kappa) = \frac{4N_1}{\kappa^3 l_1 (r_2 - r_1)} \mathcal{K}(\kappa r_1, \kappa r_2) \sin\left(\frac{\kappa l_1}{2}\right), \quad (6.2)$$

$$\mathcal{J}_2(\kappa) = \frac{4N_2}{\kappa^3 l_2(r_4 - r_3)} \mathcal{K}(\kappa r_3, \kappa r_4) \sin\left(\frac{\kappa l_2}{2}\right) \quad (6.3)$$

and

$$\mathcal{K}(x_1, x_2) = \int_{x_1}^{x_2} x K_1(x) dx. \quad (6.4)$$

For a unit current step function excitation $I(s) = 1/s$, the inverse Laplace transformation of (6.1) gives a result of the form

$$V(t) = \frac{1}{\mu_r \sigma} \int_0^\infty \mathcal{J}_1(\kappa) \mathcal{J}_2(\kappa) R(\kappa r, t\tau/\tau) e^{-t/\tau} \kappa^2 d\kappa \quad (6.5)$$

where $\tau = \mu_0 \mu_r \sigma / \kappa^2$,

$$R(r, t) = \mathcal{L}^{-1}\{\mathcal{R}(r, s), t\} = \frac{1}{2\pi i} \int_{\text{Br}} \mathcal{R}(r, s) e^{st} ds, \quad (6.6)$$

$$\mathcal{R}(r, s) = \frac{I_0(r)I_1(\sqrt{s}r) - c\sqrt{s}I_1(r)I_0(\sqrt{s}r)}{K_0(r)I_1(\sqrt{s}r) + c\sqrt{s}K_1(r)I_0(\sqrt{s}r)}, \quad (6.7)$$

and $c = 1/\mu_r$.

6.3 Short Time Series

6.3.1 Large Argument Regime

To determine the inverse Laplace transform of the function (6.7), it is convenient to write it in the form

$$\mathcal{R}(r, s) = \frac{I_1(r)}{K_1(r)} \left\{ \frac{p(r) + q(r)}{q(r) + c\sqrt{s}p(\sqrt{s}r)} - 1 \right\} \quad (6.8)$$

where

$$p(z) = I_0(z)/I_1(z) \quad \text{and} \quad q(z) = K_0(z)/K_1(z). \quad (6.9)$$

Now, the frequency dependent part is contained in the denominator of the first term in (6.8) and we focus on obtaining the time domain expression of $F(s) = 1/[q(r) + c\sqrt{s}p(\sqrt{s}r)]$. For the large argument regime, we write

$$p(z) = 1 + \frac{1}{2z} + \frac{3}{8} \frac{1}{z^2} + \frac{3}{8} \frac{1}{z^3} + \frac{63}{128} \frac{1}{z^4} + \dots = 1 + \frac{1}{2z} + \frac{1}{z} \Delta(z) \quad (6.10)$$

where

$$\Delta(z) = \frac{3}{8} \frac{1}{z} + \frac{3}{8} \frac{1}{z^2} + \frac{63}{128} \frac{1}{z^3} + \dots \quad (6.11)$$

Then, substituting (6.10) into $F(s)$ gives

$$\begin{aligned} & \frac{1}{q(r) + c\sqrt{s}p(\sqrt{s}r)} \\ &= \frac{1}{q(r) + c\sqrt{s} \left[1 + \frac{1}{2\sqrt{s}r} + \frac{1}{\sqrt{s}r} \Delta(\sqrt{s}r) \right]} \\ &= \frac{1}{q(r) + c/2r + c\sqrt{s}} \left[1 + \frac{c\Delta(\sqrt{s}r)/r}{q(r) + c/2r + c\sqrt{s}} \right]^{-1} \\ &= \frac{1}{c} \frac{1}{\sqrt{s} + a} \left[1 + \frac{\Delta(\sqrt{s}r)}{r(\sqrt{s} + a)} \right]^{-1} \\ &= \frac{1}{c} \left[\frac{1}{\sqrt{s} + a} - \frac{\Delta(\sqrt{s}r)}{r(\sqrt{s} + a)^2} + \frac{\Delta^2(\sqrt{s}r)}{r^2(\sqrt{s} + a)^3} - \frac{\Delta^3(\sqrt{s}r)}{r^3(\sqrt{s} + a)^4} + \dots \right] \end{aligned} \quad (6.12)$$

where

$$a = \frac{q(r)}{c} + \frac{1}{2r}. \quad (6.13)$$

Since $\Delta(\sqrt{s}r)$ decreases with s , equation (6.12) is valid when s is sufficiently large. The inverse Laplace transform of (6.12) represents an asymptotic solution of $R(r, t)$ for short time regime.

6.3.2 Partial Fraction Series

Equation (6.12) can be inverted to the time domain term by term and this process requires considering the inverse Laplace transform of

$$\tilde{\psi}_{mn} = \frac{1}{(\sqrt{s})^m (\sqrt{s} + a)^n}, \quad (6.14)$$

which can be expressed as a partial fraction series

$$\tilde{\psi}_{mn} = \frac{1}{a^n} \sum_{i=1}^m \frac{(-1)^{i-1}}{a^{i-1}} \frac{C_{n,i}}{(\sqrt{s})^{m-i+1}} + \frac{(-1)^m}{a^m} \sum_{i=1}^n \frac{1}{a^{i-1}} \frac{C_{m,i}}{(\sqrt{s} + a)^{n-i+1}} \quad (6.15)$$

where $C_{i,1} = C_{1,i} = 1$ and $C_{i,j} = C_{i-1,j} + C_{i,j-1}$, or $C_{i,j} = (i+j-2)!/(i-1)!(j-1)!$.

The first summation is a linear combination of the term $1/(\sqrt{s})^n$ whose time domain

expressions can be looked up from the Laplace transform table [34]. For the second summation, it has been derived that [41]

$$\begin{aligned} & \mathcal{L}^{-1} \left[\frac{1}{(\sqrt{s} + a)^{n+1}} \right] \\ &= (4t)^{\frac{n-1}{2}} e^{a^2 t} t^{n-1} \operatorname{erfc}(a\sqrt{t}) - a(4t)^{\frac{n}{2}} e^{a^2 t} t^n \operatorname{erfc}(a\sqrt{t}) \quad (n = 0, 1, 2, \dots). \end{aligned} \quad (6.16)$$

6.4 Long Time Series

For large time regime, we'd like to rewrite $\mathcal{R}(r, s)$ in (6.7) as

$$\mathcal{R}(r, s) = \frac{\mathcal{R}_1(s)}{\mathcal{R}_2(s)} = \frac{I_0(r)I_1(\sqrt{sr})/\sqrt{s} - cI_1(r)I_0(\sqrt{sr})}{K_0(r)I_1(\sqrt{sr})/\sqrt{s} + cK_1(r)I_0(\sqrt{sr})}, \quad (6.17)$$

The nature of $I_1(\sqrt{sr})$ and $I_0(\sqrt{sr})$ reveals that both $\mathcal{R}_1(s)$ and $\mathcal{R}_2(s)$ are polynomials of s . Inspired by the Heaviside expansion theorem, we write

$$R(r, t) \simeq \sum_{n=1}^m \frac{\mathcal{R}_1(a_n)}{\mathcal{R}_2'(a_n)} e^{-t/\tau_n} \quad (6.18)$$

where a_n is the n^{th} zeros of $\mathcal{R}_2(s)$ and $\tau_n = -1/a_n$. The series on the right hand side of (6.18) represents an asymptotic solution of $R(r, t)$ for large time regime.

6.5 Results

Equation (6.1) is reformulated for a truncated domain to give

$$v(s) = \frac{\pi}{h} s I(s) \mu_0 \sum_{i=1}^{\infty} \mathcal{J}_1(\kappa_i) \mathcal{J}_2(\kappa_i) \cdot \frac{\kappa_i \mu_r I_0(\kappa_i r) I_1(\gamma_i r) - \gamma_i I_1(\kappa_i r) I_0(\gamma_i r)}{\kappa_i \mu_r K_0(\kappa_i r) I_1(\gamma_i r) + \gamma_i K_1(\kappa_i r) I_0(\gamma_i r)}. \quad (6.19)$$

where h is the length of rod being truncated. Note that κ_i satisfies $\sin(\kappa_i h) = 0$, $\kappa_i = i\pi/h$ and $\gamma_i = \sqrt{\kappa_i^2 + s\mu_r\mu_0\sigma}$. For a step electric current excitation,

$$v(s) = \frac{\pi\mu_0}{h} \sum_{i=1}^{\infty} \mathcal{J}_1(\kappa_i) \mathcal{J}_2(\kappa_i) \mathcal{R}(\kappa_i r, \gamma_i r). \quad (6.20)$$

A comparison with (6.19) shows how $\mathcal{R}(\kappa_i r, \gamma_i r)$ is defined. Using the shift theorem gives

$$V(t) = \frac{\pi}{h\mu_r\sigma} \sum_{i=1}^{\infty} \kappa_i^2 \mathcal{J}_1(\kappa_i) \mathcal{J}_2(\kappa_i) R(\kappa_i r, tr/\tau_i) e^{-t/\tau_i}. \quad (6.21)$$

Figure 6.2 Zeros of $\mathcal{R}_2(s)$ for aluminum rod

Numerical calculations have been carried out to verify the short time series and large time series using FFT results. The zeros of $\mathcal{R}_2(s)$ are found iteratively using a subroutine “FindRoot” provided by Mathematica. Starting from the largest κ_i , the initial guess of zeros are found visually by a plot of $\mathcal{R}_2(s)$ verse s . When the zeros of $\mathcal{R}_2(s)$ for κ_i are found they are input as the initial guess of the zeros of $\mathcal{R}_2(s)$ for κ_{i-1} until i reaches 2, see Figure 6.2 and 6.3. The zeros of $\mathcal{R}_2(s)$ for smaller κ spread wider as κ increases. The time constants for each κ are partially listed in Table 6.3 and 6.4.

When the driver coil is excited by a step current the induced voltage $V(t)$ in the pick-up coil is evaluated using (6.21), where the series is truncated at 60 terms. The infinite

Table 6.1 Probe parameters.

	Driver coil (mm)	Pick-up coil (mm)
Inner radius	16.0	12.5
Outer radius	20.0	15.0
Length	10.0	10.0
Number of turns	400	800

Table 6.2 Rod parameters.

Material	σ (MS/m)	μ_r	r (mm)
Aluminum	16.45	1	10.0
Steel	0.85	80	10.0

Table 6.3 Partial list of the time constant τ_n for aluminum rod.

n	1	2	3	4	5	6
κ_1	1.705E-04	3.238E-05	1.318E-05	7.098E-06	4.427E-06	3.022E-06
κ_2	6.799E-04	1.295E-04	5.270E-05	2.839E-05	1.771E-05	1.209E-05
κ_3	1.524E-03	2.911E-04	1.185E-04	6.386E-05	3.984E-05	2.720E-05
κ_4	2.698E-03	5.170E-04	2.107E-04	1.135E-04	7.081E-05	4.835E-05
κ_5	4.194E-03	8.071E-04	3.290E-04	1.773E-04	1.106E-04	7.553E-05
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
κ_{58}	4.098E-01	1.000E-01	4.272E-02	2.340E-02	1.470E-02	1.008E-02
κ_{59}	4.237E-01	1.033E-01	4.417E-02	2.420E-02	1.521E-02	1.043E-02
κ_{60}	4.348E-01	1.066E-01	4.564E-02	2.502E-02	1.573E-02	1.078E-02

rod is truncated to 100 times the coil length, $h = 1m$. The coil and rods parameters are shown in Table 6.1 and 6.2. For short time regime, the first three terms in the series (6.12) are taken and two terms in $\Delta(z)$ are used. For large time series, the first 33 zeros are used for aluminum rod and first 42 zeros are used for steel rod. Note that $V(t)$ can also be evaluated through the Fourier transform, which is used to verify the short and large time series solutions described in this paper. The time step used in the Fourier

Table 6.4 Partial list of the time constant τ_n for steel rod.

n	1	2	3	4	5	6
κ_1	1.561E-04	3.180E-05	1.308E-05	7.070E-06	4.416E-06	3.017E-06
κ_2	5.364E-04	1.224E-04	5.147E-05	2.803E-05	1.756E-05	1.202E-05
κ_3	1.048E-03	2.630E-04	1.134E-04	6.231E-05	3.922E-05	2.691E-05
κ_4	1.666E-03	4.460E-04	1.969E-04	1.093E-04	6.911E-05	4.754E-05
κ_5	2.392E-03	6.670E-04	3.003E-04	1.683E-04	1.070E-04	7.378E-05
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
κ_{58}	2.300E-01	6.861E-02	3.263E-02	1.902E-02	1.245E-02	8.775E-03
κ_{59}	2.379E-01	7.097E-02	3.375E-02	1.968E-02	1.288E-02	9.077E-03
κ_{60}	2.460E-01	7.338E-02	3.489E-02	2.034E-02	1.331E-02	9.385E-03

Figure 6.3 Zeros of $\mathcal{R}_2(s)$ for steel rod

transform is $0.1\mu s$ and a total of $2^{17} - 1$ points are taken. The Fourier transform was performed on a PC with Pentium IV 1.6GHz processor and 640M bytes memory and took approximately 54 minutes. Most of the time is spent on evaluating function values at each frequencies.

Figure 6.4 shows an excellent agreement between the short time series and FFT results at short time regime while the large time series is consistent with FFT results over all the time range considered. The two series agree with each other over a considerable time range, from $1\mu s$ to $1000\mu s$. The time cost for the results shown in Figure 6.4 calculated by short and large time series is typically less than 1 minute given all the zeros of $\mathcal{R}_2(s)$. Therefore, it is advantageous to use the series solution rather than FFT when the values of only few points on the curve are required. Figure 6.5 plots the relative error of the two series against FFT results for the aluminum rod. The error of short time series solution decreases rapidly as the time decrease, which agrees with the fact that the short time series is accurate at short time limit. The error of large time series

decreases as time increases since the series is more accurate at long time limit. Figure 6.6 show the comparison for a steel rod where the curves maintain the similar shape to those of the aluminum rod, but with much higher magnitude. Figure 6.7 plots the errors of the curves in Figure 6.6 relative to FFT results.

Figure 6.8 and 6.10 show the induced EMF in the pick-up coil for the aluminum and steel rod when the fields is excited by an exponentially rising current in the driver coil. Again, the analytical solution of short time and large time series are verified.

6.6 Conclusion

Two asymptotic series have been developed to evaluate the transient electromagnetic force induced in a pick-up coil encircling an infinite rod with the fields excited by a step electric current in a driver coil. Numerical calculations show that both series agree with FFT results very well at their valid regimes respectively and have a considerable overlap with each other over intermediate time range. For limiting case, only one or two terms in the series are enough to give out accurate results. The work in this paper made a step forward towards the derivation of time domain dyadic Green's function in the cylindrical coordinate.

Figure 6.4 Aluminum rod: induced EMF in a pick-up coil. The fields is excited by a step current in a driver coil.

Figure 6.5 Aluminum rod: relative error of the short and large time series solutions in Figure 6.4 compared with FFT results.

Figure 6.6 Steel rod: induced EMF in a pick-up coil. The fields is excited by a step current in a driver coil.

Figure 6.7 Steel rod: relative error of the short and large time series solutions in Figure 6.6 compared with FFT results

Figure 6.8 Aluminum rod: induced EMF in a pick-up coil. The fields is excited by an exponentially rising current $I(t) = I_0(1 - e^{-t/\tau_0})$ with $I_0 = 1$ and a time constant $\tau_0 = 275\mu s$ in a driver coil.

Figure 6.9 Aluminum rod: relative error of the short and large time series solutions in Figure 6.8 compared with FFT results.

Figure 6.10 Steel rod: induced EMF in a pick-up coil. The fields is excited by an exponentially rising current $I(t) = I_0(1 - e^{-t/\tau_0})$ with $I_0 = 1$ and a time constant $\tau_0 = 275\mu s$ in a driver coil.

Figure 6.11 Steel rod: relative error of the short and large time series solutions in Figure 6.10 compared with FFT results.

CHAPTER 7. CONCLUSIONS AND FUTURE WORK

The method of asymptotic series expansion has been developed to formulate the transient response due to a conductive plate and a cylindrical rod for short and large time regime. Numerical calculations based on the two asymptotic series have been validated by the results computed using FFT and experimental measurements. A quasi-static time domain dyadic Green's function with its partial reflection part in asymptotic series form has been derived for an electric source in a plate by transverse potential decomposition. The acquisition of the Green's function enables us to evaluate the transient response due to a subsurface crack in a conductive plate using integral equation method. Predictions of the external magnetic field are in good agreement with experimental data.

The work conducted in this dissertation could be refined and extended in a number of ways. Special elements are desired on the top edge of the crack, which ensures the vanishing of dipole density there. For a benchmark problem, it would be useful to compute the transient response due to a curved crack instead of a rectangular one. An efficient inversion algorithm that recovers the crack position and shape could be the next step along this line. The analytical formulation of the transient response due to a rod is considered as a preliminary step in solving transient eddy current problems in structures with cylindrical symmetry. In practical situation, a derivation of the transient response due to a pipe would be more useful. To evaluate the time domain response of defects in the rod or pipe a dyadic Green's function tailored to corresponding geometry is required.

APPENDIX A. THE EFFECT OF OPENING ON EDDY CURRENT PROBE RESPONSE FOR AN IDEALIZED THROUGH CRACK

A paper published in *Review of Progress in Quantitative Nondestructive Evaluation*

Fangwei Fu¹, J. R. Bowler¹ and T. P. Theodoulidis²

¹Iowa State University, Center for Nondestructive Evaluation,
1915 Scholl Road, Ames, IA 50011, USA

²Department of Engineering and Management of Energy Resources, University of West
Macedonia, Kastorias and Fleming, 50100 Kozani, Greece.

A.1 Abstract

A structure representing an idealized through crack was formed by placing two coplanar aluminum rectangular plates next to one another with their edges separated by a small distance. The coil impedance variation with position was measured as a coil was moved over the adjacent plate edges. An analytical theory is used to evaluate the coil impedance change due to the gap between the plates. This theory is based on the truncated region eigenfunction expansion method. The difference between the eddy current probe signal due to a notch compared with that of a crack can be partly accounted for by the difference in the opening. We have investigated the effect of varying the opening of the simulated crack and shown theoretically and experimentally how the coil impedance

changes with position, opening and frequency.

A.2 Introduction

For practical purposes, it is of great interest to evaluate the effect of crack opening in eddy current nondestructive evaluation. A number of models have been proposed to predict the impedance change of an eddy current probe due to cracks with or without opening. Often, a numerical scheme is used based on an integral equation formulation [57]- [65] or differential equation formulation [66]. In the case of an ideal crack, its opening is assumed to be zero and no current is allowed to flow across it. Based on these assumptions, the crack has been represented as a surface layer of current dipoles, which is determined by an electric integral equation (EIFE) with dyadic Green's function kernel. An approximate numerical solution was found by using a boundary element method [57], [67] and the impedance change of an eddy current probe due to the crack was found using reciprocity theorem. In this approach, the crack is the only region that needs to be meshed. Alternatively, in the work of Albanese et al [68], an integral formulation in terms of a two component vector potential was used and a solution found using an edge element scheme where the whole of conductive plate was discretized. Again, a reciprocity theorem was used to calculate the crack signal. Later, this work was extended to consider cracks with finite opening and the effect of crack opening was investigated [69]. The differential equation formulation is often associated with finite element solution, which approximates Maxwell equations in their differential form by a sparse matrix equation [66]. The discretization involves both the crack and conductive parts. Such numerical schemes that require a discrete approximation of the field over a large region require large computational resources.

Recently, a method based on eigenfunction expansion and region truncation has been proposed to determine the impedance of an eddy current probe over a right angle

conductive wedge [70]. This new technique has a number of advantages. First, it allows the match of interface conditions at two perpendicular surfaces simultaneously. Second, it significantly reduces the computational cost while the accuracy is well under control. Based on this new method, an alternative to traditional numerical methods has been developed to compute the impedance change of an eddy current probe due to through cracks simulated by using two coplanar plates.

A.3 Formulation

Figure A.1(a) shows the geometry of the problem to be investigated in this paper. An idealized through crack was formed by placing two coplanar aluminum rectangular plates next to one another with their edges separated by a small distance $2c$. A coil, Figure A.1(a), is placed on the plates with its axis normal to the plate faces and d denotes the distance between the axis of coil and z axis. The dimensions of plates in x and y direction are large compared with the coil radius such that end effects are negligible. The plates have a thickness of $2b$ in z direction and the plane $x = 0$ bisects the gap between the plates. An analytical model is outlined here to evaluate the impedance variation of the coil as it moves across the gap.

A.3.1 Scalar Decomposition

In solving electromagnetic problems, it is often desirable to express the vector fields in terms of scalar quantities. It is well known [71] that the magnetic flux density can be written as

$$\mathbf{B} = \nabla \times \nabla \times \mathbf{W} \quad (\text{A.1})$$

and

$$\mathbf{W} = \hat{x}W_a + \hat{x} \times \nabla W_b \quad (\text{A.2})$$

Figure A.1 An eddy current coil over two coplanar conductive plates next to one another with their edges separated by a small distance. Note that in (c), (d), (e), (f), the arrow pointing to the left indicates the current in the coil flows in clockwise direction viewed from above and the arrow pointing to the right indicates current flows in counterclockwise direction.

with the x axis as the preferred direction. Note that $\mathbf{W}$ has two components, the transverse electric potential W_a and the transverse magnetic potential W_b , both of which satisfy the Laplace equation in a nonconductive source free region in the quasi-static limit.

With the Fourier transformation with respect to y defined as

$$\widetilde{W}_{a,b}(x, v, z) = \int_{-\infty}^{\infty} W_{a,b}(x, y, z) e^{-vy} dy, \quad (\text{A.3})$$

and its inverse as

$$W_{a,b}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widetilde{W}_{a,b}(x, v, z) e^{vy} dv, \quad (\text{A.4})$$

$\widetilde{W}_a$ and $\widetilde{W}_b$ satisfy the Helmholtz equation,

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - (v^2 + k^2) \right] \widetilde{W}_{a,b} = 0, \quad (\text{A.5})$$

in the Fourier domain for the region of a homogeneous plate having a conductivity σ and a permeability μ_0 . Note that $k^2 = j\omega\mu_0\sigma$ and $k^2 = 0$ for air.

It is helpful to consider the electromagnetic field in this problem as the sum of a predefined source field generated by the coil and the field due to the induced current in the conductive plate. To complete the formulation, the source field in the absence of the plate is specified next.

A.3.2 Source Field

The problem can be largely simplified by considering four identical coils at the image positions of the original coil with respect to $x = 0$ plane and $z = 0$ plane, see Figure A.1(c). A solution is said to be of even parity if the current in the symmetrically placed coils flows in the same azimuthal direction. On the other hand, a solution is said to be of odd parity if the current in an image coils flows in the opposite azimuthal direction. Thus, the electromagnetic field produced by the coil in Figure A.1(a) is the sum of the four fields whose sources and parity are depicted in Figure A.1(c), (d), (e) and (f). For

each combination of even and odd parity, we only need to work out the solution in the region $x > 0$ and $z > 0$, as it is shown in Figure 1(b), and write down the solutions in other regions using symmetry considerations. Therefore, we focus on deriving the solution of the even ($x = 0$) - even ($z = 0$) case, Figure A.1(c). A truncation boundary is imposed at $x = a$ where the normal component of magnetic flux density vanishes. To solve for the electromagnetic field, the boundary conditions of transverse electric and transverse magnetic potential must be identified. The magnetic flux density components are given by

$$B_x = \frac{\partial^2 W_a}{\partial x^2} - k^2 W_a, \quad B_y = \frac{\partial^2 W_a}{\partial x \partial y} + k^2 \frac{\partial W_b}{\partial z} \quad \text{and} \quad B_z = \frac{\partial^2 W_a}{\partial x \partial z} - k^2 \frac{\partial W_b}{\partial y}. \quad (\text{A.6})$$

It can be inferred from the continuity of magnetic flux density that the transverse electric potential satisfies :

$$\left. \frac{\partial^2 W_a}{\partial x^2} \right|_{x=0} = 0, \quad \text{and} \quad \left. \frac{\partial^2 W_a}{\partial z^2} \right|_{z=0} = 0 \quad (\text{A.7})$$

for even parity boundaries at $x = 0$ and $z = 0$ and

$$\left. \frac{\partial W_a}{\partial x} \right|_{x=0} = 0, \quad \text{and} \quad \left. \frac{\partial W_a}{\partial z} \right|_{z=0} = 0 \quad (\text{A.8})$$

for odd parity boundary. With the parity boundary and truncation boundary condition in view, the transverse electric potential representing the even-even source solution in the absence of the plate is a solution of (A.5) having the form

$$\frac{\partial \tilde{W}_a^{(0)}}{\partial x} = \frac{1}{2} \sinh(vz) C_0^{(0)} + \frac{1}{2} \sum_{i=1}^{\infty} \cos(u_i x) \sinh(\kappa_i z) C_i^{(0)}, \quad (\text{A.9})$$

where $\kappa_i = \sqrt{u_i^2 + v^2}$ with a positive real part. This form guarantees that the normal component of the magnetic flux density on the domain boundary at $x = 0$ is zero since $\frac{\partial^2 \tilde{W}_a^{(0)}}{\partial x^2}$ is zero on this boundary. Choosing the eigenvalues $u_i = i\pi/a$ ensures that the normal magnetic flux density at the boundary plane $x = a$, is zero, see Figure A.1(b). The coefficients $C_i^{(0)}$ are the same as those found for a single coil given [70]. Equation

(A.9) represents a quarter of the field due to a coil with N turns and carrying a current I . For a coil whose geometrical parameters are given in Figure A.1(a),

$$C_i^{(0)} = \epsilon_i \pi \mu_0 I \frac{N \cos(u_i d)}{a(r_2 - r_1)(h_2 - h_1)} \frac{\chi(\kappa_i r_1, \kappa_i r_2)(e^{-\kappa_i h_1} - e^{-\kappa_i h_2})e^{-\kappa_i b}}{\kappa_i^4}, \quad (\text{A.10})$$

where $\epsilon_0 = 1$ and $\epsilon_i = 2$ for $i = 1, 2, 3 \dots$. Also we have defined

$$\chi(a_1, a_2) = \int_{a_1}^{a_2} J_1(r) r dr, \quad (\text{A.11})$$

which can be computed from a numerical representation of a Meijer G-function or using numerical routines for Struve and Bessel functions [72].

A.3.3 Impedance

The region above the plane $z = b$ is designated region 1 and the region in which $0 \leq z \leq b$ is region 2, Figure 1(b). Correspondingly, potentials due to induced current and their associated expansion coefficients are given superscripts “(1)” whereas the superscript “(0)” refers to the prescribed whole domain coil solution for a non-conductive region defined in previous section.

The solution of (A.5) for the region above the plane $z = b$ but below the upper coil ($b \leq z \leq b + h_1$), Figure 1(b), is written

$$\frac{\partial \widetilde{W}_a^{(1)}}{\partial x} = \frac{1}{2} \sinh(vz) C_0^{(0)} + e^{-v(z-b)} C_0^{(1)} + \frac{1}{2} \sum_{i=1}^{\infty} \cos(u_i x) \left[\sinh(\kappa_i z) C_i^{(0)} + e^{-\kappa_i(z-b)} C_i^{(1)} \right], \quad (\text{A.12})$$

where the $C_i^{(0)}$ are prescribed in terms of the source parameters by (A.10). To ensure that the x -component of the magnetic flux at the boundary $x = a$ is zero, we retain the discrete values $u_i = i\pi/a$. In satisfying (A.5), the correct z -dependence again requires that $\kappa_i = \sqrt{u_i^2 + v^2}$, where the root with positive real part is taken. The coefficient $C_0^{(1)}$ and $C_i^{(1)}$ for an even source have been found by applying the continuity condition of tangential magnetic field at $z = b$ and $x = c$ interfaces [70].

The impedance change due to the presence of induced current in a conductor is derived by calculating the rate of change of flux linkage to the coil from the z -component of magnetic flux density due to eddy currents:

$$I\Delta Z = -\frac{j2\pi\omega N}{(r_2 - r_1)(h_2 - h_1)} \times \left[\int_{-\infty}^{\infty} \frac{(e^{-vh_1} - e^{-vh_2})}{v^3} \chi(vr_1, vr_2) C_0^{(1)} dv + \int_{-\infty}^{\infty} \sum_{i=1}^{\infty} \frac{(e^{-\kappa_i h_1} - e^{-\kappa_i h_2})}{\kappa_i^3} \cos(u_i d) \chi(\kappa_i r_1, \kappa_i r_2) C_i^{(1)} dv \right]. \quad (\text{A.13})$$

The contributions from Figure A.1(d), (e) and (f) can be worked out in a similar way using different boundary conditions at $x = 0$ and $z = 0$. Finally, the sum of these four parts gives the coil impedance change due to the gap.

A.4 Result

The change in coil impedance $\Delta Z = Z(d) - Z_0$ was measured as a function of coil the coordinate d when a cylindrical air-cored coil was scanned across the idealized through cracks formed by placing two coplanar aluminum 2024 plates parallel to each other. The axis of the coil bisects the gap when $d = 0$. The reference impedance Z_0 is the coil impedance measured above the plate, which is obtained by averaging the impedance measurements at the start and end points at each scan. Constant width air gaps were produced by inserting appropriate color coded shims with uniform thickness between the two plates. The parameters of coil and plates are shown in Table 1, where the conductivity of the plates was measured by a commercial conductivity meter Zetec MIZ-21A. The dimension of the coil was significantly larger than the width of air gap which means the effects of the imperfections of the plate edge are small. The impedance of coil was measured by an Agilent 4294A Precision Impedance Analyzer. The experiments were carried at frequencies far below the resonant frequency of the coil such that no correction is required for the effect of stray capacitance.

Figure A.2 and Figure A.3 show good agreements between theoretical predictions

Figure A.2 Change in coil resistance and inductance as a function of coil center position for coil C5 scanned across two gaps between two coplanar Al 2024 plates at 480Hz with constant coil liftoff. The circles represent the experimental measurement and the lines are theory prediction. (a) Resistance change. (b) Inductance change.

Figure A.3 Change in coil resistance and inductance as a function of coil center position for coil C5 scanned across two gaps between two coplanar Al 2024 plates at 9980Hz with constant coil liftoff. The circles represent the experimental measurement and the lines are theory prediction. (a) Resistance change. (b) Inductance change.

and experimental results at low and high frequencies. Equation (A.13) was coded in Mathematica and it took less than two minutes for the algorithm to compute the coil impedance at 40 positions. It can be seen that the coil resistance variation is very different at low and high frequencies. At low frequency, the resistance change increases to a peak, then decreases to a minimum as the coil is moved towards and across the crack. At high frequency, the resistance increases monotonically to a maximum at $d = 0$. Figure 4 shows the effect of crack opening on the coil impedance change with coil C5 centered over the crack. The figure shows that both the resistance and reactance vary almost linearly with the width of the gap. This result indicates that crack opening may be deduced from eddy current probe impedance measurements.

A.5 Conclusion

The truncated region eigenfunction expansion method has been used to compute the impedance of an eddy current probe due to through cracks formed by two coplanar plates. The derived formulae have been validated by demonstrating a good agreement between theoretical predictions and experimental results. The effect of crack opening was investigated and it was found that coil impedance varies linearly with crack opening. Coil impedance scans measured at different frequencies show that the resistance change

Table A.1 Parameters for coil C5 and conductive plate

Coil			
Outer radius	18.04 mm	Inner radius	9.33 mm
Coil length	10.05 mm	Coil lift-off	1.92 mm
Free space inductance (Exp)	84.8 mH	Number of turns	1910
Free space resonant frequency	125 kHz		
Plate			
Conductivity	17.40 MS/m	Permeability	1.0
Dimensions	203.20 x 152.40 x 6.57 mm

Figure A.4 Normalized coil resistance change with coil C5 centering over the crack. (a) Resistance change. (b) Reactance change.

varies differently at the low and high frequency limit.

A.6 Acknowledgement

Thanks are due to Dr. Steve Burke for providing the coil and helpful communications. This work was performed at the Center for NDE at Iowa State University with funding from the Air Force Research Laboratory on contract FA8650-04-C-5528.

BIBLIOGRAPHY

- [1] R. A. Smith and G. R. Hugo, "Transient eddy current NDE for ageing aircraft - capabilities and limitations," *Insight*, vol. 43, no. 1, pp. 14–25, Jan. 2001.
- [2] G. R. Hugo and R. A. Smith, "Deep corrosion and crack detection in aging aircraft using transient eddy-current NDE," in *Proc. 5th Joint NASA/FAA/DoD Conference on Aging Aircraft*, Orlando, USA, Sept. 2001.
- [3] R. Ludwig and W. D. Xiao, "Numerical and analytical modeling of pulsed eddy currents in a conducting half-space," *IEEE Trans. Magn.*, vol. 26, no. 1, pp. 299–307, Jan. 1990.
- [4] O. A. Mohammed and F. G. Üler, "Comparison of two techniques for the solution of the 3-D nonlinear transient eddy current problem," *IEEE Trans. Magn.*, vol. 28, no. 4, pp. 87–90, July 1992.
- [5] W. D. Xiao, R. Ludwig, and P. Palanisamy, "Numerical simulation of pulsed eddy-current nondestructive testing phenomena," *IEEE Trans. Magn.*, vol. 26, no. 6, pp. 3089–3096, Nov. 1990.
- [6] B. Allen, N. Ida, and W. Lord, "Finite element modelling of pulsed eddy currents ndt phenomena," *IEEE Trans. Magn.*, vol. 21, no. 6, pp. 2250–2253, Nov. 1985.
- [7] U. Patel and D. Rodger, "Finite element modelling of pulsed eddy currents for nondestructive testing," *IEEE Trans. Magn.*, vol. 32, no. 3, pp. 1593–1596, May 1996.

- [8] C. Gaier and H. Haas, "Edge element analysis of magnetic and transient eddy current fields using $\vec{H}$ -formulation," *IEEE Trans. Magn.*, vol. 31, no. 3, pp. 1460–1463, May 1995.
- [9] H. Tsuboi, N. Seshima, I. Sebestyen, J. Pávó, S. Gyimothy, and A. Gasparics, "Transient eddy current analysis of pulsed eddy current testing by finite element method, *TEAM Workshop Problem 27*," *IEEE Trans. Magn.*, vol. 40, no. 2, pp. 1330–1333, Mar. 2004.
- [10] H. Tsuboi, M. Tanaka, and T. Misaki, "Computation accuracies of boundary element method and finite element method in transient eddy current analysis," *IEEE Trans. Magn.*, vol. 24, no. 6, pp. 3174–3176, Nov. 1988.
- [11] J. Pávó, "Numerical calculation method for pulsed eddy-current testing," *IEEE Trans. Magn.*, vol. 38, no. 2, pp. 1169–1172, Mar. 2002.
- [12] S. M. Berg, "Modeling and inversion of pulsed eddy current data," Ph.D. dissertation, Delft University, Netherlands, Dec. 2003.
- [13] J. R. Bowler, "Prediction and analysis of transient eddy-current probe signals," in *Review of Progress in Quantitative Nondestructive Evaluation*, D. O. Thompson and D. E. Chimenti, Eds., vol. 10. New York, USA: Plenum Press, 1991, pp. 287–293.
- [14] J. R. Bowler and M. Johnson, "Pulsed eddy-current response to a conducting half-space," *IEEE Trans. Magn.*, vol. 33, no. 3, pp. 2258–2262, May 1997.
- [15] J. Bowler, "Time domain half-space dyadic green's functions for eddy-current calculations," *J. Appl. Phys.*, vol. 86, no. 11, pp. 6494–6500, 1999.

- [16] J. R. Bowler and F. Fu, "Transient eddy current interaction with an open crack," in *Review of Progress in Quantitative Nondestructive Evaluation*, D. O. Thompson and D. E. Chimenti, Eds., vol. 23. New York, USA: AIP Press, Feb. 2004, pp. 329–335.
- [17] M. Clemens and T. Weiland, "Transient eddy-current calculation with the FI-method," *IEEE Trans. Magn.*, vol. 35, no. 3, pp. 1163–1166, May 1999.
- [18] K. R. Davey and L. Turner, "Prediction of transient eddy current fields using surface impedance methods," *IEEE Trans. Magn.*, vol. 25, no. 5, pp. 4156–4158, Sept. 1989.
- [19] D. Netter, J. Leveque, P. Masson, and A. Ressoug, "Monte carlo method for transient eddy-current calculations," *IEEE Trans. Magn.*, vol. 40, no. 5, pp. 3450–3456, Sept. 2004.
- [20] J. C. Moulder, M. W. Kubovich, E. Uzal, and J. H. Rose, "Pulsed eddy-current measurements of corrosion-induced metal loss: theory and experiment," in *Review of Progress in Quantitative Nondestructive Evaluation*, D. O. Thompson and D. E. Chimenti, Eds., vol. 15. New York, USA: Plenum Press, 1996, pp. 2065–2072.
- [21] J. A. Bieber, S. K. Shaligram, J. H. Rose, and J. C. Moulder, "Time-gating of pulsed eddy current signals for defect characterization and discrimination in aircraft lap-joints," in *Review of Progress in Quantitative Nondestructive Evaluation*, D. O. Thompson and D. E. Chimenti, Eds., vol. 17. New York, USA: Plenum Press, 1998, pp. 1915–1921.
- [22] J. A. Bieber, C. C. Tai, and J. C. Moulder, "Quantitative assessment of corrosion in aircraft structures using scanning pulsed eddy current," in *Review of Progress in Quantitative Nondestructive Evaluation*, D. O. Thompson and D. E. Chimenti, Eds., vol. 18. New York, USA: Kluwer Academic, 1999, pp. 315–322.

- [23] S. Giguère and S. J. M. Dubios, “Pulsed eddy current: finding corrosion independently of transducer lift-off,” in *Review of Progress in Quantitative Nondestructive Evaluation*, D. O. Thompson and D. E. Chimenti, Eds., vol. 20. New York, USA: AIP Press, Apr. 2001, pp. 449–456.
- [24] B. A. Lepine, S. Giguère, D. S. Forsyth, and A. Chahbaz, “Interpretation of pulsed eddy current signals for locating and quantifying metal loss in thin skin lap slices,” in *Review of Progress in Quantitative Nondestructive Evaluation*, D. O. Thompson and D. E. Chimenti, Eds., vol. 22. New York, USA: AIP Press, Mar. 2003, pp. 418–422.
- [25] S. Giguère, B. A. Lepine, and J. M. S. Dubios, “Detection of cracks beneath rivet heads via pulsed eddy current technique,” in *Review of Progress in Quantitative Nondestructive Evaluation*, D. O. Thompson and D. E. Chimenti, Eds., vol. 22. New York, USA: AIP Press, Mar. 2003, pp. 1968–1975.
- [26] S. Giguère, B. A. Lepine, and J. M. S. Dubios, “Pulsed eddy current (PEC) characterization of material loss in multi-layer structures,” *Research in Nondestructive Evaluation*, vol. 13, pp. 119–129, 2001.
- [27] S. K. Burke, G. R. Hugo, and D. J. Harrison, “Transient eddy-current NDE for hidden corrosion in multilayer structures,” in *Review of Progress in Quantitative Nondestructive Evaluation*, D. O. Thompson and D. E. Chimenti, Eds., vol. 18. New York, USA: Kluwer Academic, 1999, pp. 307–314.
- [28] M. J. Johnson and J. R. Bowler, “Inversion of transient eddy-current signals for the determination of conducting plate parameters,” in *Review of Progress in Quantitative Nondestructive Evaluation*, D. O. Thompson and D. E. Chimenti, Eds., vol. 15. New York, USA: Plenum Press, 1996, pp. 849–856.

- [29] C. C. Tai, J. H. Rose, , and J. C. Moulder, “Thickness and conductivity of metallic layers from pulsed eddy current measurement,” in *Review of Progress in Quantitative Nondestructive Evaluation*, D. O. Thompson and D. E. Chimenti, Eds., vol. 16. New York, USA: Plenum Press, 1997, pp. 409–416.
- [30] J. A. Bieber, J. H. Rose, and N. Nakagawa, “Pulsed eddy current method: An overview,” in *Review of Progress in Quantitative Nondestructive Evaluation*, D. O. Thompson and D. E. Chimenti, Eds., vol. 19. New York, USA: AIP Press, Apr. 2000, pp. 469–475.
- [31] M. J. Johnson, “Pulsed eddy current measurements for materials characterization and flaw detection,” Ph.D. dissertation, The Univ. of Surrey, Surrey, UK, Mar. 1997.
- [32] C. V. Dodd and W. E. Deeds, “Analytical solutions to eddy current probe-coil problems,” *J. Appl. Phys.*, vol. 39, no. 6, pp. 2829–2838, 1968.
- [33] T. Theodoulidis and E. Kriezis, “Series expansions in eddy current nondestructive evaluation models,” *J. Mater. Process. Tech.*, vol. 161(1-2), pp. 343–347, 2005.
- [34] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. New York, USA: Dover publications, Inc., 1972.
- [35] T. E. Copson, Ed., *Theory of Functions of a Complex Variable*. Oxford, UK: Clarendon Press, 1962.
- [36] E. C. Titchmarsh, Ed., *The Theory of Functions*. Oxford, UK: Oxford University Press, 1935.
- [37] S. K. Burke, G. R. Hugo, and D. J. Harrison, “Transient eddy-current NDE for hidden corrosion in multilayer structures,” in *Review of Progress in Quantitative Nondestructive Evaluation*, D. O. Thompson and D. E. Chimenti, Eds., vol. 18. New York, USA: Kluwer Academic, 1999, pp. 307–314.

- [38] T. V. Yioultsis, K. S. Charitou, C. S. Antonopoulos, and T. D. Tsiboukis, "A finite difference time domain scheme for transient eddy current problems," *IEEE Trans. Magn.*, vol. 37, no. 5, pp. 3145–3149, Sept. 2001.
- [39] V. O. deHaan and P. A. deJong, "Analytical expressions for transient induction voltage in a receiving coil due to a coaxial transmitting coil over a conducting plate," *IEEE Trans. Magn.*, vol. 40, no. 2, pp. 371–378, Mar. 2004.
- [40] S. K. Burke and M. E. Ibrahim, "Mutual impedance of air-cored coils above a conducting plate," *J. Phys. D: Appl. Phys.*, vol. 37, no. 13, pp. 1857–1868, 2004.
- [41] F. Fu and J. R. Bowler, "Transient eddy current driver pick-up probe response due to a conductive plate," *IEEE Trans. Magn.*, Accepted.
- [42] T. Theodoulidis, "Model of ferrite core probes for eddy current nondestructive evaluation," *J. Appl. Phys.*, vol. 93, no. 5, pp. 3071–3078, 2003.
- [43] L. B. Felsen and N. Marcuvitz, Eds., *Radiation and Scattering of Waves*. Englewood, New Jersey: Prentice-Hall, Inc., 1973.
- [44] S. Barkeshli and P. H. Pathak, "On the dyadic green's function for a planar multilayered dielectric/magnetic media," *IEEE Trans. Microwave Theory Tech.*, vol. 40, no. 1, pp. 128–142, Jan. 1992.
- [45] G. W. Hanson, "Dyadic green's function for a multilayered planar medium—a dyadic eigenfunction approach," *IEEE Trans. Antennas Propagat.*, vol. 52, no. 12, pp. 3350–3356, Dec. 2004.
- [46] W. A. SanFilipo and G. W. Hohmann, "Integral equation solutions for the transient electromagnetic response of a three-dimensional body in a conducting half-space," *Geophysics*, vol. 50, no. 5, pp. 798–809, 1985.

- [47] I. V. Lindell, “TE/TM decomposition of electromagnetic sources,” *IEEE Trans. Antennas Propagat.*, vol. 36, no. 10, pp. 3350–3356, Oct. 1988.
- [48] Y. Yoshida and J. R. Bowler, “A vector potential integral formulation for eddy-current probe response to cracks,” *IEEE Trans. Magn.*, vol. 36, no. 2, pp. 461–469, Mar. 2000.
- [49] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes, The Art of Scientific Computing*. Cambridge, UK: CUP, 1988.
- [50] T. Wang and G. W. Hohmann, “A finite difference time domain solution for three dimensional electromagnetic modelling,” *Geophysics*, vol. 58, no. 6, pp. 797–809, 1993.
- [51] O. Bíró, K. Preis, and K. R. Richter, “Various FEM formulations for the calculation of transient 3d eddy currents in nonlinear media,” *IEEE Trans. Magn.*, vol. 31, no. 3, pp. 1307–1312, May 1995.
- [52] R. Albanese and G. Rubinacci, “Analysis of a transient nonlinear 3-d eddy current problem with differential and integral methods,” *IEEE Trans. Magn.*, vol. 32, no. 3, pp. 776–779, May 1996.
- [53] D. Albertz and G. Henneberger, “On the use of the new edge based $\vec{A} - \vec{A}$, $\vec{T}$ formulation for the calculation of time-harmonic, stationary and transient eddy current field problems,” *IEEE Trans. Magn.*, vol. 36, no. 4, pp. 818–822, July 2000.
- [54] H. Tanaka, H. Tsuboi, F. Kobayashi, and T. Misaki, “Transient eddy current analysis by the boundary element method using fourier transforms,” *IEEE Trans. Magn.*, vol. 29, no. 2, pp. 1722–1725, Mar. 1993.

- [55] Y. Zhang, K. R. Shao, Y. Guo, and J. D. Lavers, “A boundary meshless method for transient eddy current problems,” *IEEE Trans. Magn.*, vol. 41, no. 10, pp. 4090–4092, Oct. 2005.
- [56] J. R. Bowler, “Pulsed eddy-current interaction with subsurface cracks,” in *Review of Progress in Quantitative Nondestructive Evaluation*, D. O. Thompson and D. E. Chimenti, Eds., vol. 18. New York, USA: Kluwer Academic, 1999, pp. 477–483.
- [57] J. Bowler, “Eddy current interaction with an ideal crack, Part I: The forward problem,” *J. Appl. Phys.*, vol. 75, no. 12, pp. 8128–8137, 1994.
- [58] J. A. Stratton, *Electromagnetic Theory*. New York, USA: McGraw Hill Companies, 1941.
- [59] J. R. Bowler, “The application of tensor green’s functions to the calculation of eddy-current distributions,” The Univ. of Surrey, Farnborough, United Kingdom, Report to DRA, June 1994.
- [60] J. R. Bowler and F. Fu, “Time domain dyadic green’s function for an electric source in a conductive plate,” *IEEE Trans. Magn.*, to be published.
- [61] R. F. Harrington, *Field Computation by Moment Methods*. New York, USA: Macmillan, 1968.
- [62] F. Fu and J. R. Bowler, “Evaluation of transient eddy-current signals due to a plate using asymptotic solutions,” in *Electromagnetic Nondestructive Evaluation (IV)*, L. Udpa and N. Bowler, Eds. East Lansing, USA: IOS Press, June 1–2, 2004, pp. 15–22.
- [63] J. A. Tegopoulos and E. E. Kriezis, *Eddy currents in linear conductive media*. Netherlands: Elsevier Science, 1985.

- [64] R. F. Jardim, B. Laks, and C. Santos, “Eddy current decay method applied to a new geometry,” *J. Appl. Phys.*, vol. 61, no. 12, pp. 5237–5242, 1987.
- [65] H. Lee and D. Kim, “Impedance calculation for a plate with crack in eddy current NDT using 3D indirect BIEM,” *IEEE Trans. Magn.*, vol. 36, no. 5, pp. 3131–313, Sept. 2000.
- [66] M. Tanaka, H. Tsuboi, K. Ooshima, and M. Hashimo, “Finite element model for crack with narrow gap in ECT problem,” in *Electromagnetic Nondestructive Evaluation (VI)*, F. Kojima, T. Takagi, S. S. Udpa, and P. J, Eds. Kobe, Japan: IOS Press, May 17–19, 2001, pp. 68–73.
- [67] D. Chen, K. R. Shao, J. Sheng, and W. Yan, “Eddy current interaction with a thin-opening crack in a plate conductor,” *IEEE Trans. Magn.*, vol. 36, no. 4, pp. 1745–1749, July 2000.
- [68] R. Albanese, G. Rubinacci, A. Tamburrino, and F. Villone, “Reconstruction of cracks with integral methods,” in *Electromagnetic Nondestructive Evaluation (II)*, R. Albanese, G. Rubinacci, T. Takagi, and S. S. Udpa, Eds. Reggio, Italy: IOS Press, Sept. 14–16, 1997, pp. 107–113.
- [69] G. Rubinacci, R. Albanese, A. Tamburrino, and F. Villone, “Effects of crack thickness in eddy current testing,” in *Electromagnetic Nondestructive Evaluation (IV)*, S. S. Udpa, T. Takagi, P. J, and R. Albanese, Eds. Des Moines, USA: IOS Press, Aug. 1–3, 1999, pp. 25–31.
- [70] T. P. Theodoulidis and J. R. Bowler, “Eddy current coil interaction with a right-angled conductive plate,” *Proc. Roy. Soc. Ser. A.*, Aug. 17, 2005.
- [71] W. R. Smythe, Ed., *Static and Dynamic Electricity*. McGraw Hill, New York: Hemisphere Pub, 1968.

- [72] I. S. Gradshteyn and I. M. Ryzhik, Eds., *Table of integrals series and products*.
New York, USA: Academic Press, 1980.

ACKNOWLEDGEMENTS

I would like to take this opportunity to convey my deepest gratitude to the many individuals who have helped, encouraged and seen me through the completion of this dissertation. First and foremost, a special and most sincere appreciation is devoted to my major professor Dr. John Bowler for his guidance, patience and support throughout this research and the writing of this thesis. His insights and words of encouragement have often inspired me and renewed my hopes for completing my graduate education.

I would like to thank Dr. Aleksandar Dogandžić, Dr. Mani Mina, Dr. Ronald Roberts and Dr. Jiming Song for kindly serving as my graduate study committee member. Your review of this manuscript and insightful suggestions have made it a better document. I would additionally thank Dr. Aleksandar Dogandžić and Dr. Jiming Song for your inspirational teaching style.

I am very grateful to Dr. Marcus Johnson for providing invaluable assistance on laboratory work. I would like to thank Dr. Steve Burke for providing the experimental data in Chapter 4 and Dr. Theodoros Theodoulidis for providing the numerical results in Appendix A. Thanks are due to other members in Dr. John Bowler's research group with who I have worked, including, Dr. Nicola Bowler, Dr. Yongqiang Huang, Dr. Haiyan Sun, Vipul Katyal, Wei Zhang, Waqar Habib, Sanyi Zhan, Raza Ali and Brent Eaton. You have made my graduate study an extremely enjoyable experience.

This work was performed at the Center for Nondestructive Evaluation with funding from NASA program.