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A Practical Review of the Kompaneets Equation and its Application to Compton Scattering

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ABSTRACT

In this study, we explore both inverse Compton and Compton scattering processes using the Chang and Cooper scheme to form a deterministic solution of the Kompaneets equation. We examine the individual terms of the Kompaneets equation and illustrate their effect on the equilibrium solution. We use two examples (a Gaussian line profile and a Planck profile) to illustrate the advective and diffusive properties of the Kompaneets operator. We also explore both inverse Compton scattering and Compton scattering, and discuss and illustrate the Bose-Einstein condensation feature of the Compton scattering spectrum.

INTRODUCTION

The Kompaneets equation is a specialized form of a Fokker-Planck equation. This equation exhibits both an advective and diffusive character in its solutions. Kompaneets derived his equation specifically for use in inverse Compton Scattering as applied to nonrelativistic astrophysics problems.¹ The equation contains a diffusion term, a recoil term, and an induced emission term (nonlinear). In this study, we describe the general form of the solution admitted by including each term, and the physical consequences of each advance in complication. We also provide four examples of Compton scattering that illuminate the properties of the Kompaneets equation and its limitations. Some of these examples are drawn from the astrophysics literature.

The Kompaneets equation, as discussed by Katz,² is shown in Eqn. (1).

$$\frac{\partial n}{\partial y} = \left(\frac{1}{x^2} \right) \cdot \frac{\partial}{\partial x} \left[x^4 \left(\frac{\partial n}{\partial x} + n + n^2 \right) \right] . \quad (1)$$

The variable n is the photon number density. The Compton parameter y and the reduced energy variable x are defined to be

$$y \equiv \left(\frac{k_B T}{m_e c^2} \right) N_e \sigma_T c t . \quad (2)$$

The constant m_e is the mass of the electron, N_e is the electron number density, σ_T is the Thompson scattering cross section, c is the speed of light, and t is time. The variable x is defined as

$$x \equiv \frac{h\nu}{k_B T} . \quad (3)$$

The variable $h\nu$ is the photon energy (in keV) and $k_B T$ is the electron temperature (in keV). The full derivation (without typographical errors) may be found in Rybicki and Lightman³ and Katz. We will not dwell on the derivation; however, it is important to understand the conditions under which the Kompaneets is valid and accurate. The original Kompaneets equation is applied only in the nonrelativistic energy regime with the photon energy $h\nu \ll k_B T$ and the electron temperature $k_B T \ll m_e c^2$. This expresses the fact that the Kompaneets equation is formulated and (much) more accurate for inverse Compton scattering.

We now turn to the examination of the quasi-equilibrium solutions for the Kompaneets equation. The quasi equilibrium solution to Eqn. (1) can be found by equating the terms in brackets on the RHS (right-hand side) of the equation to zero. By taking only the first two terms, $\left(\frac{\partial n}{\partial x} + n = 0\right)$, it is readily verifiable that the solution to this subset is

$$n = A \cdot e^{-h\nu / k_B T} \quad . \quad (4)$$

This solution leads to a Wien spectrum. By including the last term (nonlinear) term in brackets, $\left(\frac{\partial n}{\partial x} + n + n^2 = 0\right)$, the solution can be verified to be

$$n = \frac{1}{Ce^{\frac{h\nu}{k_B T}} - 1} \quad . \quad (5)$$

This solution leads to a Bose-Einstein spectrum. The constant C is the *fugacity*. The constant C is 1 for a Planck photon distribution. An important point to note here is that an equilibrium photon spectrum (Planck) can not be reached by Compton scattering alone. The photon distribution will relax to a Bose-Einstein distribution with a fugacity of C at a time t . This means that Compton scattering preserves particles. The numerical solution to the Kompaneets equation then must also posses this important property.

There are several approaches to differencing the Kompaneets equation discussed by Larson et al.⁴ Park and Petrosian⁵ also evaluate the Chang and Cooper⁶ approach along with those examined by Larson et al. Park and Petrosian conclude that the Chang and Cooper approach (fully implicit) is the most stable over a wide range of parameters. Of these differencing methods, we choose the approach originated by Chang and Cooper. Their approach provides two important properties: (1) spectrum positivity, and (2) particle number conservation.

Spectrum positivity is strictly true with the method for the linear Kompaneets equation. Therefore, Chang and Cooper formulate the nonlinear term $n(n + I)$ by using the “old” time for the n term and the “new” time for the $(n+I)$ term. This lagging of the time-step effectively creates a linear problem from the nonlinear equation. One must be careful, however, not to use too large a time-step in the simulation as this time-step can introduce significant error in the resultant spectrum.

The differencing scheme of Chang and Cooper formulates a generalized particle flux that results in a first order method in space and time while conserving particle number. In order to guarantee a positive definite spectrum, a weighting factor δ (to weight cell fluxes across boundaries) is chosen such that the quasi-equilibrium solution to Eqn. (1) is exactly satisfied at any given time t . The differencing scheme also results in a diagonally dominant tri-diagonal matrix that is easily solved by Gaussian elimination and backward-forward substitution. Because of the diagonal dominance, no pivoting strategy is required. The radiation-energy density exchange rate is of principal importance to a multi-temperature, radiation-hydrodynamics simulation code. We briefly derive the simple result for the exchange rate assuming a Planck distribution. Then we follow this by the generalization for a nonPlanckian radiation profile. The derivation follows the treatment by Katz. In this treatment, we omit the nonlinear term in n . The radiation energy density is calculated as

$$E_r = \frac{2(k_B T)^4}{h^3 c^3} \int_0^\infty n 4\pi x^3 dx ; \quad (6)$$

Then the energy density exchange rate is

$$\frac{h^3 c^3}{8\pi (k_B T)^4} \frac{dE_r}{dy} = \frac{\partial}{\partial y} \int_0^\infty n x^3 dx ; \quad (7)$$

$$= \int_0^\infty dx x^3 \frac{\partial n}{\partial y} ; \quad (7a)$$

$$= \int_0^\infty dx x \frac{\partial}{\partial x} \left[x^4 \left(\frac{\partial n}{\partial x} + n \right) \right] ; \quad (7b)$$

$$= \int_0^\infty x d \left[x^4 \left(\frac{\partial n}{\partial x} + n \right) \right] ; \quad (7c)$$

We now integrate by parts and assume that $n(x)$ drops off sufficiently rapidly (it will usually decline exponentially) as $x \rightarrow \infty$, but does not rise too rapidly as $x \rightarrow 0$, so that we have

$x^5 \left(\frac{\partial n}{\partial x} + n \right) \rightarrow 0$ in both these limits. Then

$$\frac{h^3 c^3}{8\pi (k_B T)^4} \frac{dE_r}{dy} = - \int_0^\infty x^4 \left(\frac{\partial n}{\partial x} + n \right) dx ; \quad (7d)$$

$$= - \int_0^\infty n x^4 dx - \int_0^\infty x^4 \frac{\partial n}{\partial x} dx ; \quad (7e)$$

Now integrating by parts again, we have

$$\frac{h^3 c^3}{8\pi (k_B T)^4} \frac{dE_r}{dy} = - \int_0^\infty n x^4 dx + 4 \int_0^\infty n x^3 dx ; \quad (7f)$$

and reverting to the energy exchange rate at time t , we have

$$\frac{dE_r}{dt} = \left(\frac{N_e \sigma_T \cdot (k_B T)}{m_e c^2} \right) \cdot \left(\frac{8\pi (k_B T)^4}{(hc)^3} \right) \cdot \left[4 \int_0^\infty n x^3 dx - \int_0^\infty n x^4 dx \right] . \quad (8)$$

We know that for a Planckian $n(x) = \frac{1}{e^x - 1}$ and we know that

$$E_r = \left(8\pi k^4 T^4 / h^3 c^3 \right) \cdot \int_0^\infty x^3 e^{-x} (1 + e^{-x} + e^{-2x} + \dots) dx .$$

By integrating the kernel term by term we have: $E_r = \left(8\pi^5 k^4 / 15 h^3 c^3 \right) T^4$.

The integral reduces to $6\zeta(4)$ where $\zeta(4)$ is the Riemann-Zeta function of order 4.⁷ We can now write the radiation energy exchange rate as

$$\frac{dE_r}{dt} = \left(4 \frac{N_e \sigma_T \cdot c}{m_e c^2} \right) \cdot E_r \cdot \left[(k_B T) - (k_B T) \cdot \left(\frac{1}{4} \right) \frac{\int_0^\infty nx^4 dx}{\int_0^\infty nx^3 dx} \right]; \quad (9)$$

$$\frac{dE_r}{dt} = \left(4 \frac{N_e \sigma_T c k_B}{m_e c^2} \right) \cdot E_r \cdot [T - T_r]; \quad (10)$$

where we have defined

$$T_r \equiv T \cdot \left(\frac{1}{4} \right) \frac{\int_0^\infty nx^4 dx}{\int_0^\infty nx^3 dx}; \quad (11)$$

Equation (10) expresses the same result as found in Bowers and Wilson.⁸ Equation (8) and the expression for T_r in Equation (11) are general so that $n(x)$ can be any well-behaved photon number density distribution function and may be numerically integrated. Becker⁹ has developed a Green's function approach for solving the linear Kompaneets equation and thus permits analytic calculations of Wien spectra and energy exchange rates.

DISCUSSION

We illustrate the behavior of and the solutions to the Kompaneets equation for both a Gaussian line profile and a Planckian spectrum characterization of the radiation field with four examples. For the first three examples (inverse Compton scattering), we will use an initial 10 keV temperature for the electron field. The electron field energy density is calculated using

$$E_e = \frac{3}{2} N_e k_B T; \quad (12)$$

The exchange of energy between the electron and radiation fields conserves energy. The fourth example considers Compton scattering with the initial electron field temperature of 1 keV and the initial radiation field temperature of 2 keV.

The first example considers inverse Compton scattering using a narrow Gaussian profile photon number density to simulate an emission line. A small simulation code was executed that solves the Kompaneets Eqn. (1). The red curve in Fig. 1 shows the initial photon number density and the green curve shows the Bose-Einstein equilibrium of the photon density at an arbitrary time t . This figure illustrates both the advective and diffusive nature of the Kompaneets operator. The initial photon profile is both translated upward and broadened in energy.

Fig. 1. The initial profile (red curve) was a narrow Gaussian of width 0.03 keV and centroid at 0.5 keV. The green curve represents the photon distribution at an arbitrary time t . The electron number density was $N_e = 6.0 \cdot 10^{22}$. One thousand photon groups, logarithmically spaced, were used to cover the energy space. The time step was $\Delta t = 5.0 \cdot 10^{-11}$ s.

Figure 2 illustrates the change in photon intensity from the initial distribution to the equilibrium state. The red curve represents the initial photon profile intensity, whereas the green curve represents the photon profile intensity at the end of the simulation. The blue curve represents the Bose-Einstein equilibrium solution for the intensity of the initial photon profile. Note that the Bose-Einstein intensity curve (blue) completely overlays the green Kompaneets evolution curve.

Fig. 2. The initial photon intensity profile (red curve) was a narrow Gaussian of width 0.03 keV and centroid at 0.5 keV. The electron number density was $N_e = 6.0 \cdot 10^{22}$. The green curve represents the evolving equilibrium Bose-Einstein intensity solution. One thousand photon groups, logarithmically spaced, were used to cover the energy space. The time step was $\Delta t = 5.0 \cdot 10^{-11}$ s.

Figure 3 illustrates the change in both the electron and radiation field energy densities. As expected for inverse Compton scattering, the radiation field (red curve) gains energy at the expense of the electron field (green curve). As seen in the figure, there are no instabilities in the solution fields.

Fig. 3. The electron energy density (green curve) declines as expected by transferring energy to the radiation field. The radiation energy density (red curve) gains as expected by receiving energy from the electron field.

The last figure for Example 1, Fig. 4, shows the evolution of the electron and radiation field temperatures as a function of simulation time. A smooth approach to equilibrium is illustrated in the figure and no instabilities were observed in the temperature solution fields.

Fig. 4. The red curve shows the electron field temperature declining as expected while the radiation field temperature, the green curve, is increasing. Both the electron and radiation field temperatures vary smoothly with time and exhibit no instabilities.

The second example considers inverse Compton scattering using a Planckian profile photon number density. A small simulation code was executed that solves the Kompaneets Eqn. (1). The red curve in Fig. 5 shows the initial photon number density and the green curve shows the Bose-Einstein equilibrium of the photon density at an arbitrary time t . This graphic illustrates both the advective and diffusive nature of the Kompaneets operator. The initial photon profile is both translated upward and broadened in energy. The broadening in energy is much less, however, than for the line profile case of Example 1.

Fig. 5. The initial profile (red curve) was a Planckian with a 1 keV temperature. The green curve represents the solution at an arbitrary time t . The electron number density was $N_e = 6.0 \cdot 10^{22}$. One thousand photon groups, logarithmically spaced, were used to cover the energy space. The time step was $\Delta t = 5.0 \cdot 10^{-11}$ s.

Figure 6 illustrates the change in photon intensity from the initial distribution to the equilibrium state. The red curve represents the initial Planckian photon profile intensity, whereas the green curve represents the photon profile intensity at the end of the simulation. The blue curve represents the Bose-Einstein equilibrium solution for the intensity of the initial Planckian photon profile. Note that the Bose-Einstein intensity curve (blue) completely overlays the green Kompaneets evolution curve.

Fig. 6. The initial photon intensity profile (red curve) was Planckian with a 1 keV temperature. The green curve represents the evolving equilibrium Bose-Einstein intensity solution. The electron number density was $N_e = 6.0 \cdot 10^{22}$. One thousand photon groups, logarithmically spaced, were used to cover the energy space. The time step was $\Delta t = 5.0 \cdot 10^{-11}$ s.

Figure 7 illustrates the change in both the electron and radiation field energy densities. As expected for inverse Compton scattering, the radiation field (red curve) gains energy at the expense of the electron field (green curve). As seen in the figure, there are no instabilities in the solution fields.

Fig. 7. The electron energy density (green curve) declines as expected by transferring energy to the radiation field. The radiation energy density (red curve) gains as expected by receiving energy from the electron field. The time step was $\Delta t = 5.0 \cdot 10^{-11}$ s.

The last figure for Example 2, Fig. 8, shows the evolution of the electron and radiation field temperatures as a function of simulation time. A smooth approach to equilibrium is illustrated in the figure and no instabilities were observed in the temperature solution fields.

Fig. 8. The red curve shows the electron field temperature declining as expected while the radiation field temperature, the green curve, is increasing. Both the electron and radiation field temperatures vary smoothly with time and exhibit no instabilities. The time step was $\Delta t = 5.0 \cdot 10^{-11}$ s.

Example 3 uses the same initial Gaussian profile as Example 1. We now examine the evolution (in time) of the initial line profile intensity. The results are shown for early time evolution in Fig. 9 and for later time evolution in Fig. 10. Castor¹⁰ shows two similar figures [Fig. 12.8 (a) and Fig. 12.8 (b)] in his text. Our results show the full time-dependent solution behavior for the Kompaneets operator on a Gaussian line profile photon number density distribution. Castor also discusses various approximations to the Kompaneets equation that admit analytic solutions.

Fig. 9. The Gaussian line profile, with a temperature of 0.5 keV, is shown evolving in a time-dependent simulation. The electron field temperature was held constant at 10 keV. The time interval between curves was $\Delta t = 5.0 \cdot 10^{-9} \text{ s}$. Note that both the advective and diffusive nature of the Kompaneets operator produces a smooth solution with no instabilities or other pathological behavior.

Fig. 10. The Gaussian line profile, T_i , with a temperature of 0.5 keV, is shown evolving in a time-dependent simulation. The simulation time starts at $t = 5.0 \cdot 10^{-8} \text{ s}$. The electron field temperature, T_e , was held constant at 10 keV. The time interval between curves was $\Delta t = 5.0 \cdot 10^{-8} \text{ s}$. The evolution behavior at later times show that the population of the lower energy groups is markedly changed by advection to the equilibrium solution, whereas the higher energy frequency groups show little advection.

We now consider our last example applying Kompaneets equation to Compton scattering. In the previous three examples, the inverse Compton process has been analyzed and illustrated. In Example 4 we consider the case of Compton scattering in which the radiation field loses energy to the electron field. As stated in the introduction to this study, the Kompaneets equation is less accurate for this process. For this example we will restrict our attention to the change in the intensity of the radiation field and the character of the equilibrium solution for Compton (down) scattering.

The initial conditions for this example are that the electron field is initially at a 1 keV temperature, whereas the radiation field is described by a Planck distribution whose initial temperature is 2 keV. The electron number density was $N_e = 2.0 \cdot 10^{16}$. Figure 11 contains four representations of radiation field intensity. The initial Planck intensity is represented by the red curve. The green curve represents an evolution of the radiation field after 0.1 seconds. The peak of the radiation intensity curve has moved downward as expected. The Bose-Einstein equilibrium solution for these physical conditions is represented by the blue curve. A narrow but prominent peak in the Bose-Einstein solution can be seen in the lowest photon energy group. The character of this solution was noted by Larsen et al.⁴ This character is the result of the physical fact that Compton scattering alone will not permit the relaxation of a radiation field intensity to a pure Planck distribution. The Bose-Einstein solution is “Planck-like” in shape but also exhibits the peak at the spectrum origin. The magenta curve represents the intensity of the equivalent Planck distribution and lies below the Bose-Einstein solution curve; it is shown here for comparison purposes only and does not represent any time evolution. Figures 12 and 13 show the Kompaneets solutions approaching the Bose-Einstein solution at later simulation times.

Liu et al.¹¹ explored Compton scattering with these same physical conditions. They have extended the traditional formulation of the Kompaneets equation to include relativistic effects. In our simulation code, we have chosen to apply the relativistic corrections developed by Cooper.¹² This model is the same approach used by Morel¹³ as a follow-on to the Larsen et al. paper. The Compton amplitude, shown in Fig. 11 by the green curve, agrees with the results of Liu et al. (as shown in their Fig. 2a) to within $\sim 1.5\%$. A rigorous quantitative measure is difficult to determine from data extraction based on the figure in their paper.

Fig. 11. The initial radiation Planckian intensity of temperature 2 keV and the Kompaneets solution at a time of $t = 0.1s$ are represented by the red and green curves respectively. The Bose-Einstein equilibrium solution (with the peak at the photon origin) is represented by the blue curve. The magenta curve represents the equivalent Planck distribution with a temperature of 1 keV. One hundred photon groups, linearly space, spanned the spectrum.

Fig. 12. The initial radiation Planckian intensity of temperature 2 keV and the Kompaneets solution at a time of $t = 0.2s$ are represented by the red and green curves respectively. The Bose-Einstein equilibrium solution (with the peak at the photon origin) is represented by the blue curve. The magenta curve represents the equivalent Planck distribution with a temperature of 1 keV. One hundred photon groups, linearly space, spanned the spectrum.

Fig. 13. The initial radiation Planckian intensity of temperature 2 keV and the Kompaneets solution at a time of $t = 0.5s$ are represented by the red and green curves respectively. The Bose-Einstein equilibrium solution (with the peak at the photon origin) is represented by the blue curve. The magenta curve represents the equivalent Planckian distribution with a temperature of 1 keV. One hundred photon groups, linearly space, spanned the spectrum.

SUMMARY

In this study, we have explored both inverse Compton and Compton scattering processes using the Chang and Cooper scheme to form a deterministic solution of the Kompaneets equation. The individual terms of the Kompaneets equation were examined and their effect on the equilibrium solution was illustrated. The linear form of the Kompaneets equation admits a Wien equilibrium spectral solution while the nonlinear form of the Kompaneets equation admits the Bose-Einstein equilibrium solution. The advective and diffusive properties of the Kompaneets operator have been illustrated by two examples using different initial radiation profiles: (1) Gaussian line profile and (2) Planck profile.

Although most of this study explored inverse Compton scattering, some examination of Compton scattering was conducted. The “piling up” of photons in the Bose-Einstein solution spectrum was discussed along with the physical reasons for this behavior. As normally formulated, solutions to the Kompaneets equation are strictly valid only in the nonrelativistic regime; however, relativistic correction factors have been applied to the Chang and Cooper scheme to permit a more accurate determination of spectral quantities in higher energy regimes.

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