FIELD THEORY ANOMALIES INVOLVING THE
ENERGY-MOMENTUM TENSOR

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Ph.D. Thesis Submitted to Iowa State University

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Date Transmitted: May 1975

PREPARED FOR THE U.S. ENERGY RESEARCH AND DEVELOPMENT
ADMINISTRATION UNDER CONTRACT NO. W-7405-eng-82

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Price: Microfiche \$2.25

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ABSTRACT

Generalized Ward-Takahashi identities (WTI's) of the energy-momentum tensor in a free spinor field theory with $SU(3)$ currents were analyzed for possible non-canonical behavior (anomalies). The WTI's, derived by a functional integration technique, are the divergence equations for the axial-vector, vector, and tensor currents in n -point functions (n -pf's), i.e. Green's functions, involving the energy-momentum tensor and all possible combinations of vector, axial-vector, scalar, and pseudo-scalar external currents (V , A , S , and P). In addition the trace identities (TI) of the n -pf's involving the energy-momentum tensor were investigated.

The WTI's and trace identities were first checked by naive manipulation of single loop Feynman diagrams, (i.e. ignoring divergences and surface terms in shifts of integration variables). Then defining "physical" amplitudes in terms of a Pauli-Villars type regularization and a self-consistent renormalization scheme (by addition of polynomial counterterms), possible anomalies of the WTI's and TI were defined. As a result of the regularization, $1/m$ -terms of n -pf's having a coefficient m in the axial-vector WTI and the trace identity were found to contribute to possible anomalies. Using the counterterms in a self-consistent manner, it was found that all the tensor and vector WTI's could be satisfied (thus satisfying Poincare and gauge invariance, respectively), while anomalies were found in the axial-vector WTI and trace identity.

The AWTI anomalies found were shown to be consistent with axial-vector anomalies found previously in n -pf's without the energy-momentum tensor, namely in abnormal parity n -pf's with up to five V and A currents but no S or P currents. The TI anomalies were found in a variety of normal parity n -pf's up to $n = 5$. The TI anomalies are consistent with anomalies found at zero momentum by other authors.

Supplementary material on the relation of S-matrix and mass matrix formalism for doubled resonances is also presented.

Implications and possible phenomenological applications are discussed.

CHAPTER I. INTRODUCTION

History of Ward Identities and Anomalies

The Ward (or Ward-Takahashi) identity¹ was introduced in quantum field theory² in the 1950's as a relation between the photon-electron-electron vertex, represented by

$$S_F^i(p) \Gamma_\mu(p, p') S_F^i(p')$$

$$= - \int d^4x \int d^4y e^{ip \cdot x} e^{-ip' \cdot y} \langle 0 | T(\psi(x) j_\mu^{\text{em}}(0) \bar{\psi}(y)) | 0 \rangle ,$$

and the electron propagator represented by

$$i S_F^i(p) = \int d^4x e^{ip \cdot x} \langle 0 | T(\psi(x) \bar{\psi}(0)) | 0 \rangle ,$$

where $j_\mu^{\text{em}}(x) = \bar{\psi}(x) \gamma_\mu \psi(x)$ is the electromagnetic current and $\psi(x)$ is the electron field in quantum electrodynamics. The expressions for $\Gamma_\mu(p, p')$ and $S_F^i(p)$ are to be calculated according to the Feynman graph rules² of quantum electrodynamics (QED). As given in Fig. 1, $\Gamma_\mu(p, p')$ represents the sum of all proper graphs (having no two parts connected by a single line), having an external photon line and two external electron lines, while $S_F^i(p)$ represents the sum $S_F^i(p) = \frac{1}{\not{p} - m - \Sigma(p)}$, where $\Sigma(p)$ is given by the sum of all proper graphs having two external electron lines.

The Ward-Takahashi identity states

$$(p - p')^\mu \Gamma_\mu(p, p') = S_F^i(p)^{-1} - S_F^i(p')^{-1} \quad (1.1a)$$

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Figure 1a. The photon electron-electron vertex in quantum electrodynamics. It is proper (cannot be divided into 2 diagrams by cutting a single line).

Figure 1b. The total electron propagator, an infinite sum of proper electron-electron vertices.

Figure 1c. Proper electron-electron vertex.

a) $\Gamma_{\mu}(p, p') =$

b) $i S_F^{-1}(p) =$

c) $-i \Sigma(p) =$

or

$$(p - p')^\mu S_F^I(p) \Gamma_\mu(p, p') S_F^I(p') = S_F^I(p') - S_F^I(p) \quad (1.1b)$$

which in the notation of this work is expressed as

$$\begin{aligned} k^\mu \langle j_\mu^{\text{em}}(k) \psi(-p) \bar{\psi}(p') \rangle \\ = -i \langle \psi(-p) \bar{\psi}(p) \rangle + i \langle \psi(-p') \bar{\psi}(p') \rangle \end{aligned} \quad (1.1c)$$

where

$$\langle \psi(-p) \bar{\psi}(p) \rangle = i S_F^I(p)$$

and

$$\langle j_\mu^{\text{em}}(k) \psi(-p) \bar{\psi}(p') \rangle = S_F^I(p) \Gamma_\mu(p, p') S_F^I(p')$$

with $k = p - p'$.

This Ward identity basically expresses current conservation in QED, namely $\partial^\mu j_\mu^{\text{em}}(x) = 0$. It plays a useful role in the renormalization of QED, in which the terms $\Gamma_\mu(p, p')$ and $S_F^I(p)$ (as well as the photon propagator) must have infinities removed into renormalization constants Z_1 and Z_2 . Visually

$$S_F^I(p) = Z_2 \tilde{S}_F^I(p)$$

$$\Gamma_\mu(p, p') = Z_1^{-1} \tilde{\Gamma}_\mu(p, p')$$

where the tilde terms are finite. Equation (1) leads to the relation $Z_1 = Z_2$, which is also known as Ward's identity.

The generalized Ward-Takahashi identity (WTI) is a generalization of Eq. (1) for any quantum field theory having some current with an index, e.g. $A_\mu(x)$, $V_\mu(x)$, $\theta_{\lambda\rho}(x)$. It is the divergence equation of that current coupled to other currents in a vertex function. Derived from Lagrangians using canonical methods of field theory (equations of motion, commutation relation), they generally give the divergence of one vertex function in terms of others.

In 1969 it was noted by Bell and Jackiw³ that the Ward identity for the axial-vector current $j_\mu^5(x)$ coupled to two photons formally gave a vanishing result for $\pi^0 \rightarrow \gamma\gamma$ decay when interpreted according to PCAC (partial conservation of axial-vector current) $\partial^\mu j_\mu^5 = 2im_0 j^5(x)$ as noted by Sutherland and Veltman,⁴ but that the lowest order calculations in perturbation theory (the triangle graphs) by e.g. J. Steinberger⁵ in 1949 gave finite results in agreement with experiment. Calculating in the σ -model, Bell and Jackiw saved the PCAC relation at the expense of modifying the σ -model. In the same year Adler,⁶ analyzed the problem in terms of QED with the inclusion of axial-vector and pseudoscalar currents. He found that in order to preserve gauge invariance of the photons that the PCAC relation must be modified to include an anomalous term, namely

$$\partial^\mu j_\mu^5(x) = 2im_0 j^5(x) + S \frac{\alpha_0}{4\pi} \epsilon_{\alpha\beta\gamma\delta} F^{\alpha\beta}(x) F^{\gamma\delta}(x) \quad (1.2)$$

where $F^{\alpha\beta}$ is the electromagnetic tensor

$$F_{\mu\nu}(x) = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x), \quad A_\mu(x) = j_\mu^{\text{em}}(x),$$

$$j_\mu^5(x) = \bar{\psi}(x)\gamma_\mu\gamma_5\psi(x), \quad j^5(x) = \bar{\psi}(x)\gamma_5\psi(x).$$

The anomalous term came from just the diagrams needed to give the correct experimental result for $\pi^0 \rightarrow \gamma\gamma$, namely the triangle diagram (see Fig. 2). It was later observed that Schwinger⁷ had done essentially the same analysis in 1951 using manifestly gauge invariant field equations, although the significance of the work was not observed at that time.

Subsequent work⁸ confirmed Adler's conjecture that the lowest order triangle graphs gave the complete form of the anomaly, with all higher order corrections vanishing (see Fig. 2b).

Various authors including Brown, Shih, and Young (BSY)⁹ and Bardeen¹⁰ extended the anomaly investigation to n-point functions (n-pf's), involving arbitrary numbers of vector, axial-vector, scalar, and pseudoscalar currents (V, A, S, and P respectively). They found that if gauge invariance (zero divergence of the vector current) were to hold, axial-vector anomalies could occur only in abnormal parity n-pf's having only V and A currents up to $n = 5$. However, there was an ambiguity present since the gauge invariance of the vector currents could not be taken as an absolute principle.

Figure 2a. The triangle loop graph contributions to the axial-vector Ward identity anomaly of $\langle \text{AVV} \rangle$.

Figure 2b. Some higher order triangle graphs whose contributions to the anomaly vanishes when summed.

Much work has been done on both the phenomenological aspects and theoretical implications of the axial-vector anomalies. They have been applied to many meson decays including $\pi^0 \rightarrow \gamma\gamma$, $\eta \rightarrow \gamma\gamma$, $\chi^0(960) \rightarrow \gamma\gamma$,¹¹ as well as inverse processes such as $\gamma\gamma \rightarrow \pi^0$, $\gamma\gamma \rightarrow 3\pi$, $\gamma \rightarrow 3\pi$, and $\gamma\pi \rightarrow \pi\pi$ which could be studied in e^+e^- colliding beams¹² (by annihilation of e^+e^- or by two photon annihilation, $e^+e^- \rightarrow e^+e^- \gamma^* \gamma^* \rightarrow e^+e^- 3\pi$, etc.) or by the Primakoff effect at high energy¹³ (e.g. $\pi^+ + (Z, A) \rightarrow \pi^+ + (Z, A) + 2\pi$ by photon exchange). Theoretically the coefficient S of the anomalous term is related to the number of constituent particles in the field theory (e.g. quarks). Furthermore the anomalies upset the renormalization of certain theories, requiring additional constraints and particles to cancel the anomalies in the overall renormalization of the theory.¹⁴

Recently interest has grown in theories of scale invariance (the invariance of theories with respect to spatial scale and conformal transformations) and the breaking of scale invariance.¹⁵⁻¹⁷ The stress-energy-momentum tensor which plays a central role in theories of scale invariance has been introduced in an "improved" version in quantum field theory by Callan, Coleman, and Jackiw.¹⁸ The trace relations of this tensor have been shown to have anomalies, similar in nature to the axial-vector Ward identity anomalies.¹⁹ Crewther²⁰ and Chanowitz and Ellis²¹ have related this anomaly to the ratio

$$R = \frac{\sigma(e^+e^- \rightarrow \text{hadron})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} \text{ at high energy,} \quad (1.3)$$

the asymptotic ratio for electron positron annihilations into hadrons as compared to annihilations into a $\mu^+\mu^-$ pair (a purely electromagnetic effect). This ratio gives further information on the constituents of field theories for strong interactions.

Motivation for Present Work

Suura and Young²² have derived generalized Ward-Takahashi and trace relations for the energy-momentum tensor coupled to arbitrary numbers of SU(3) currents V, A, S and P in a free spinor field theory using a functional integration technique. In this work we are interested in checking these canonically derived relations in a lowest order perturbation calculation to see how the anomalies appear. Besides the vector and axial-vector Ward identities, we have the divergence and trace of the energy-momentum tensor to work with. Hopefully this will shed some light on the previous calculations of anomalies.

This investigation will closely follow the work of Brown, Shih, and Young,⁹ using the results of their calculation, and checking them in the process.

CHAPTER II. WARD-TAKAHASHI AND TRACE IDENTITIES

Derivation

The Ward-Takahashi and trace identities considered here are derived by a functional integration technique from the free field Lagrangian

$$\mathcal{L} = \frac{1}{2} \bar{\psi}(x) (i\gamma^\mu \partial_\mu - m) \psi(x) + \frac{1}{2} (-i\partial_\mu \bar{\psi}(x) \gamma^\mu - m\bar{\psi}(x)) \psi(x)$$

where $\psi(x)$ is the free spinor field, a column matrix in the internal symmetry space, and γ^μ are the Dirac gamma matrices. The SU(3) currents are defined by

$$j_i^a(x) \equiv \bar{\psi}(x) \frac{\lambda^a}{2} \Gamma_i \psi(x) , \quad a = 0, 1, \dots, 8 , \quad (2.1)$$

where λ^a are the Gell-Mann matrices of SU(3) and $\Gamma_i = 1, i\gamma_5, \gamma_\mu$, and $\gamma_\mu \gamma_5$ for the currents $S^a(x)$, $P^a(x)$, $V_\mu^a(x)$, and $A_\mu^a(x)$ respectively. These are the scalar, pseudo-scalar, vector, and axial vector currents respectively. The energy-momentum tensor (of Callan, Coleman, and Jackiw¹⁸) as derived in Ref. 22 is

$$\theta_{\lambda\rho} = \frac{i}{4} [\bar{\psi} \gamma_\lambda \partial_\rho \psi - \partial_\rho \bar{\psi} \gamma_\lambda \psi + (\lambda \leftrightarrow \rho)] - g_{\lambda\rho} \mathcal{L} \quad (2.2)$$

with the trace

$$\theta = \theta_\lambda^\lambda = m \bar{\psi} \psi = m \sqrt{6} S^0 . \quad (2.3)$$

The last equation expresses the fact that the trace is proportional to the unitary singlet scalar current.

In the functional integration formalism vacuum expectation values of products of n currents $Q_i(x_i)$ are given by

$$\langle T^*(Q_1(x_1) \cdots Q_n(x_n)) \rangle_0 \equiv N^{-1} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} Q_1(x_1) \cdots Q_n(x_n) e^{iI} \quad (2.4)$$

where $N = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iI}$ and $I = \int \mathcal{L}(x) d^4x$. The Q_i represent Heisenberg operators on the left-hand side and functions of integration variables on the right-hand side of the above equation. The T^* indicates that the above n -point function is the time ordered (T) product, explicitly made covariant, (of Koba and Nishijima) for which

$$\langle T^*(\partial_\mu \psi(x) \partial_\nu \bar{\psi}(y) \psi(z) \cdots) \rangle_0 = \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \langle T(\psi(x) \bar{\psi}(y) \psi(z) \cdots) \rangle_0. \quad 22$$

Ward-Takahashi identities (WTI's) are derived by making certain variations on fields ψ and $\bar{\psi}$. For instance, the variation

$$\psi(x) \rightarrow (1 + i\omega(x)\gamma_5 \frac{\lambda_a}{2}) \psi(x)$$

$$\bar{\psi}(x) \rightarrow \bar{\psi}(x) (1 + i\omega(x)\gamma_5 \frac{\lambda_a}{2}),$$

where $\omega(x)$ is an arbitrary infinitesimal function with well-behaved derivatives, will induce the variations

$$I \rightarrow I + \int d^4x \omega(x) [\partial^\mu A_\mu^a(x) - 2m P^a(x)] + \mathcal{O}(\omega^2(x))$$

$$v_\nu^b(x) \rightarrow v_\nu^b(x) + \omega(x) f_{abc} A_\nu^c(x)$$

which when applied to the 2-pf

$$\langle T^*(v_\mu^b(y) v_\nu^c(z)) \rangle_0 \equiv N^{-1} \int d\psi d\bar{\psi} v_\mu^b(y) v_\nu^c(z) e^{iI}$$

leads to

$$\begin{aligned} \frac{\partial}{\partial x_\lambda} \langle T^*(A_\lambda^a(x) v_\mu^b(y) v_\nu^c(z)) \rangle_0 &= 2m \langle T^*(P^a(x) v_\mu^b(y) v_\nu^c(z)) \rangle_0 \\ &+ i f_{abd} \langle T^*(A_\mu^d(y) v_\nu^c(z)) \rangle_0 \delta^4(y-x) \\ &+ i f_{acd} \langle T^*(v_\mu^b(y) A_\nu^d(z)) \rangle_0 \delta^4(z-x). \end{aligned}$$

This can be transformed to momentum space to give

$$\begin{aligned} k^\lambda \langle A_\lambda^a(k) v_\mu^b(p) v_\nu^c(q) \rangle &= -2mi \langle P^a(k) v_\mu^b(p) v_\nu^c(q) \rangle \\ &+ f_{abd} \langle A_\mu^d(p+k) v_\nu^c(q) \rangle + f_{acd} \langle v_\mu^b(p) A_\nu^d(q+k) \rangle, \end{aligned} \quad (2.5)$$

the naive axial-vector WTI for the AVV vertex, where in general we define

$$\begin{aligned} (2\pi)^4 \delta(k+q_1+\dots+q_n) \langle \theta_{\lambda\rho}(k) j_1^a(q_1) \dots j_n^d(q_n) \rangle \\ \equiv \int dx dy_1 \dots dy_n e^{-ikx} e^{-iq_1 y_1} \dots e^{-iq_n y_n} \\ \langle T^*(\theta_{\lambda\rho}(x) j_1^a(y) \dots j_n^d(y_n)) \rangle_0. \end{aligned} \quad (2.6)$$

and the corresponding definition of

$$\langle j_1^a(q_1) \dots j_n^d(q_n) \rangle ,$$

the current n-pf. Similar procedures¹⁹ generate the vector and tensor WTI's and the trace identity.

The general form of the vector WTI with n-currents. (including one vector current) and the energy momentum tensor is

$$\begin{aligned} p^\mu \langle \theta_{\lambda\rho}(k) v_\mu^a(p) \dots j_\nu^b(q) \dots \rangle \\ = \frac{i}{2} (g_{\lambda\alpha} g_{\rho\beta} + g_{\lambda\beta} g_{\rho\alpha} - 2g_{\lambda\rho} g_{\alpha\beta}) p^\alpha \langle v^a(k+p)^\beta \dots j_\nu^b(q) \dots \rangle \\ + \dots + f^{abb'} \langle \theta_{\lambda\rho}(k) \dots j_\nu^{b'}(p+q) \dots \rangle + \dots \end{aligned} \quad (2.7)$$

The axial-vector WTI is

$$\begin{aligned} p^\mu \langle \theta_{\lambda\rho}(k) A_\mu^a(p) \dots j_\nu^b(q) \dots \rangle \\ = 2m g_{\lambda\rho} \langle P^a(p+k) \dots j_\nu^b(q) \dots \rangle - 2mi \langle \theta_{\lambda\rho}(k) P^a(p) \dots j_\nu^b(q) \dots \rangle \\ + \frac{i}{2} [g_{\lambda\alpha} g_{\rho\beta} + g_{\lambda\beta} g_{\rho\alpha} - 2g_{\alpha\beta} g_{\lambda\rho}] p^\alpha \langle A^a(k+p)^\beta \dots j_\nu^b(q) \dots \rangle \end{aligned}$$

$$+ \dots + \langle \theta_{\lambda\rho}(k) \dots \left\{ \begin{array}{l} f^{abb'} \left(\begin{array}{l} A_{\nu}^{b'}(p+q) \\ V_{\nu}^{b'}(p+q) \end{array} \right) \\ d^{abb'} \left(\begin{array}{l} P^{b'}(p+q) \\ -S^{b'}(p+q) \end{array} \right) \end{array} \right\} \dots \rangle + \dots$$

$$\text{for } j_{\nu}^b = \left\{ \begin{array}{l} V_{\nu}^b \\ A_{\nu}^b \\ S_{\nu}^b \\ P_{\nu}^b \end{array} \right\} \quad (2.8)$$

The identities with currents only can be obtained by removing the $\theta_{\lambda\rho}(k)$ in the above equation and dropping all terms with λ, ρ indices.

The tensor WTI is

$$\begin{aligned} & k^{\lambda} \langle \theta_{\lambda\rho}(k) j_{\mu}^a(p) \dots j_{\nu}^d(q) \rangle \\ &= -i(k+p)_{\rho} \langle j_{\mu}^a(p+k) \dots j_{\nu}^d(q) \rangle \\ &+ \dots -i(k+q)_{\rho} \langle j_{\mu}^a(p) \dots j_{\nu}^d(q+k) \rangle \\ &+ \frac{i}{2} k^{\lambda} (\bar{\Sigma}_{\lambda\rho})_{\mu}^{\tau} \langle j_{\tau}^a(k+p) \dots j_{\nu}^d(q) \rangle \\ &+ \dots + \frac{i}{2} k^{\lambda} (\bar{\Sigma}_{\lambda\rho})_{\nu}^{\tau} \langle j_{\mu}^a(p) \dots j_{\tau}^d(q+k) \rangle \end{aligned} \quad (2.9)$$

where

$$(\bar{\Sigma}_{\lambda\rho})_{\mu}^{\tau} = \begin{cases} g_{\lambda\mu} g_{\rho}^{\tau} - g_{\lambda}^{\tau} g_{\rho\mu} & \text{for V and A} \\ 0 & \text{for S and P} \end{cases}$$

and the trace identity is

$$\begin{aligned}
& g^{\lambda\rho} \langle \theta_{\lambda\rho}(k) j_{\mu}^a(p) \dots j_{\nu}^b(q) \rangle \\
&= \sqrt{6} m \langle S^0(k) j_{\mu}^a(p) \dots j_{\nu}^b(q) \rangle - 3i [\langle j_{\mu}^a(p+k) \dots j_{\nu}^b(q) \rangle \\
&+ \dots + \langle j_{\mu}^a(p) \dots j_{\nu}^b(q+k) \rangle] \quad (2.10)
\end{aligned}$$

In all the above n-pf's, 4-momentum conservation is understood, i.e.

$$k + p + \dots + q + \dots = 0 .$$

The Feynman rule for the energy-momentum tensor, as given in Ref. 19, is

$$\Gamma_{\lambda\rho}(p, q) = \frac{1}{4} \gamma_{\lambda} (p+q)_{\rho} + \frac{1}{4} \gamma_{\rho} (p+q)_{\lambda} - \frac{1}{2} g_{\lambda\rho} (\not{p} + \not{q} - 2m) , \quad (2.11)$$

where $\Gamma_{\lambda\rho}(p, q)$ is the $\theta_{\lambda\rho} - \bar{\psi} - \psi$ vertex as illustrated in Fig. 3. The Feynman rule for the trace of the energy-momentum tensor is proportional to that for the scalar current as indicated in Eq. (2.3).

Using the Feynman rules for spinor electrodynamics (see Ref. 2) as well as those given above, the n-pf's can be represented by their lowest order loop integrals for purposes of verifying the WTIs and trace identities. We assume that these lowest order terms of the perturbation expansion will show any anomalous behavior of the full n-pf's. This assumption is justified by the work previously cited on the absence of higher order corrections to the axial-vector triangle anomaly⁸

Figure 3. Feynman diagram for $\Gamma_{\lambda\rho}(p, q)$, the $\theta_{\lambda\rho} - \psi - \bar{\psi}$ vertex.

and on non-perturbative calculations of the axial anomaly^{15,23} as well as the trace relation.²⁰

For example, the single loop representation of the triangle graph PVV of Fig. 2a is

$$\begin{aligned} \langle P^a(p) V_\mu^b(q) V_\nu^c(s) \rangle &= \frac{i}{16} \text{Tr} \langle \lambda_a \lambda_b \lambda_c + C(3) \lambda_a^T \lambda_b^T \lambda_c^T \rangle \\ &\int_{\ell} \text{Tr} \{ i \gamma^5 S(\ell+q) \gamma_\mu S(\ell) \gamma_\nu S(\ell-s) \} \\ &+ \frac{i}{16} \text{Tr} (\lambda_a \lambda_c \lambda_b + C(3) \lambda_a^T \lambda_c^T \lambda_b^T) \int_{\ell} \text{Tr} \{ i \gamma^5 S(\ell+q) \gamma_\nu S(\ell) S(\ell-s) \gamma_\mu \} \end{aligned}$$

where $C(3) = C_P C_V C_V = +$ is the overall charge conjugation sign ($C_P = C_A = C_S = +$, $C_V = -$), $\int_{\ell} = \frac{1}{(2\pi)^4} \int d^4 \ell$ represents the loop integral, and $S(\ell+q) = (\ell-m)^{-1}$ is the fermion propagator. The Feynman graph representation has a factor $\frac{i}{2}$ for each fermion propagator, $\frac{1}{2}$ for charge conjugation invariance, and an overall-sign for the spinor loop.²

The θ PVVV loop graph, Fig. 4, is given by

$$\begin{aligned} \langle \theta_{\lambda\rho}(k) P^a(p) V_\mu^b(q) V_\nu^c(s) V_\sigma^d(t) \rangle \\ = - \frac{i}{32} w^{abcd} [J_{\lambda\rho P V_\mu V_\nu V_\sigma}(p, 0, -q, -q-s, -q-s-t) \\ + C(4) J_{\lambda\rho V_\sigma V_\nu V_\mu P}(t, 0, -s, -s-q, -s-q-p)] \end{aligned}$$

Figure 4. Feynman diagram for the first term of the single loop representation of $\langle \theta_{\lambda\rho}(k) P^a(p) v_\nu^b(q) v_\sigma^c(s) v_\tau^d(t) \rangle$.

+ 11 more pairs, giving in all 24 permutation of a, b, c, d
with $P, V_\mu, V_\nu, V_\sigma$ and p, q, s, t,

where $W^{abcd} = \text{Tr}(\lambda_a \lambda_b \lambda_c \lambda_d + C(4) \lambda_a^T \lambda_b^T \lambda_c^T \lambda_d^T)$, $C(4) = C_P C_V C_V C_V = -$ and

$$J_{\lambda p P V_\mu V_\nu V_\sigma}(a, b, c, d, e) \\ = \int_{\ell} \text{Tr} \{ \Gamma_{\lambda p}(\ell+a, \ell+e) S(\ell+a) i \gamma^5 S(\ell+b) \gamma_\mu S(\ell+c) \gamma_\nu S(\ell+d) \gamma_\sigma S(\ell+e) \}.$$

The general form for the n-pf's with tensor are given in Appendix A for $n = 2, 3, 4, 5, 6$. The corresponding forms for the n-pf's with currents only are given in Ref. 9.

Naive Manipulations of Feynman Graphs to Check

WTI's and Trace Identities

The WTI's and trace identities can be verified naively from the single loop Feynman graphs, that is by ignoring divergences and allowing changes of variables and translations which rigorously could give rise to surface terms in the Feynman integrals. When examined carefully, it is such divergences and surface terms which give rise to anomalies.

As an example of a naive verification of the trace identity we examine the 2-pf (ignoring the factor $\frac{1}{4} (\lambda_a + C_a \lambda_a^T)$)

$$\langle \theta_{\lambda p}(k) j_1(-k) \rangle = \int_{\ell} \text{Tr} \{ [\Gamma_{\lambda p}(\ell, \ell+k) S(\ell+k) \Gamma, S(\ell)] \} \\ = \int_{\ell} \text{Tr} \{ [\frac{1}{4} (2\ell+k)_{\lambda} \gamma_p + \frac{1}{4} (2\ell+k)_{\rho} \}$$

$$- \frac{1}{2} g_{\lambda\rho} (2\ell + k - 2m) \left] \frac{i}{\ell + k - m} \Gamma_1 \frac{i}{\ell - m} \right\}.$$

Taking the trace of this we get

$$\begin{aligned} & g^{\lambda\rho} \langle \theta_{\lambda\rho}(k) j_1(-k) \rangle \\ &= \int_{\ell} \text{Tr} \left\{ \left[\frac{1}{2} (2\ell + k) - 2(2\ell + k - 2m) \right] \frac{i}{\ell + k - m} \Gamma_1 \frac{i}{\ell - m} \right\} \\ &= \int_{\ell} \text{Tr} \left\{ \left[m - \frac{3}{2} (\ell - m + \ell + k - m) \right] \frac{i}{\ell + k - m} \Gamma_1 \frac{i}{\ell - m} \right\} \\ &= \int_{\ell} \text{Tr} \left\{ m \frac{1}{\ell + k - m} \Gamma_1 \frac{1}{\ell - m} \right\} - \frac{3}{2} i \int_{\ell} \text{Tr} \left\{ \Gamma_1 \frac{i}{\ell - m} \right\} \\ &\quad - \frac{3}{2} i \int_{\ell} \text{Tr} \left\{ \frac{i}{\ell + k - m} \Gamma_1 \right\} \\ &= \langle \theta(k) j_1(-k) \rangle - 3i \langle j_1(0) \rangle \end{aligned}$$

where we use the property

$$\text{Tr} \left\{ (\ell - m) \dots \frac{i}{\ell - m} \right\} = \text{Tr} \{ \dots \}$$

and we make the translation $\ell + k \rightarrow \ell$ in the last integral and add the last two terms for $\langle j_1(0) \rangle$. This is the potentially dangerous manipulation. Note that the above integrals go like $\int d^4\ell \ell^{-2}$ and $\int d^4\ell \ell^{-1}$ and are highly divergent.

The naive verification of the axial-vector WTI is discussed in Ref. 6 and relies on the identity⁹

$$S(\ell+p)\not{p}\gamma_5 S(\ell) = 2m S(\ell+p)\gamma_5 S(\ell) + \gamma_5 S(\ell) + S(\ell+p)\gamma_5 .$$

Similar manipulations can be used to give naive verification of the vector and tensor WTI's.

Before 1969, and except for the calculation of Schwinger, such Ward identities and trace relations were assumed to hold true. In current algebra calculations this assumption was equivalent to the free use of so-called canonical equal-time commutators. The presence of the anomalies reflects what is known as the failure of canonical field theory.²⁴

CHAPTER III. FAILURE OF CANONICAL FIELD THEORY

Adler-Bell-Jackiw-Schwinger Anomaly

The breakdown of canonical field theory was first discovered with the explicit calculations of the triangle graphs in the AVV Ward identity (Eq. (2.5)). The relevant diagrams are those of $\langle AVV \rangle$ and $\langle PVV \rangle$, since the $\langle AV \rangle$ terms can be shown to vanish by symmetry considerations. Adler explicitly shows that by requiring gauge invariance of the V (e.g. photon) vertices all divergences in the diagrams can be removed and the result Eq. (1.2), or (in our notation)

$$p^\mu \langle A_\mu^a(p) V_\nu^b(q) V_\sigma^c(s) \rangle = -2mi \langle P^a(p) V_\nu^b(q) V_\sigma^c(s) \rangle - 3\gamma d^{abc} \epsilon_{\nu\sigma\alpha\beta} q^\alpha s^\beta \quad (3.1)$$

follows, where $\gamma = 1/24 \pi^2$. Schwinger's analysis used the same assumptions but with a different calculation method.

K. Wilson¹⁵ has made a configuration-space analysis of this problem and found that the anomaly arises from a ninth order short distance singularity of the product $\langle T(A_\mu(x) V_\nu(y) V_\sigma(0)) \rangle_0$. That is, such a product of currents can be expanded around the short-distance region $x^2 < \epsilon$, $y^2 < \epsilon'$, $(x-y)^2 < \epsilon''$ with a leading term proportional to ϵ^{-9} where $\epsilon \sim \epsilon' \sim \epsilon''$. It is this singularity which leads to the anomaly in Eq. (3.1). This is now called a canonical singularity, since it arises from a careful consideration of field theory, and is hypothesized by Wilson to give the proper behavior of hadronic (strong

interaction) currents, even though perturbation theory cannot apply to strong interactions directly.

Other calculational techniques include the ϵ -separation method, which is a configuration space analysis similar to that of Wilson, and the regulator method used by Brown, Shih, and Young (BSY).⁹ Bardeen¹⁰ used the former method and obtained a functional form for all axial-vector anomalies of n-pf's with external currents, V, A, S and P. BSY used a regulator technique and obtained the same set of anomalies plus an alternative set allowing for violation of the vector Ward identity..

This work will use the method and results of BSY extended to the n-pf's with tensor.

Trace Anomalies

In the study of scale invariance of field theory, Callan¹⁷ and Symanzik²⁶ found trace identity anomalies in high orders of renormalizable models of strong interactions. These anomalies are of a logarithmic type and would cause a breakdown of Wilson's idea of canonical short distance behavior for hadronic currents. Coleman and Jackiw¹⁹ suggested a different type of anomaly, which Chanowitz and Ellis²¹ and Crewther²⁰ have found. This anomaly arises in a trace identity of the form

$$\langle \theta(0) v_{\mu}^a(p) v_{\nu}^b(-p) \rangle = i \left\{ 2 - p \cdot \frac{\partial}{\partial p} \right\} \langle v_{\mu}^a(p) v_{\nu}^b(-p) \rangle + 2i\gamma \delta^{ab} (g_{\mu\nu} p^2 - p_{\mu} p_{\nu}),$$

where the term on the right is the anomaly, and is the result of a canonical short distance singularity in analogy to the axial-vector Ward identity anomaly. Although our trace identity is of a different form than that above, it will be seen that the anomalies will be equivalent.

CHAPTER IV. CALCULATION OF ANOMALIES

Universal Regularization

The anomalies are calculated by the method of BSY (Ref. 9). It is a Pauli-Villars type of regularization scheme.²⁷ The loop integral representations of the n-pf's contain many divergences which must be removed. This is done by defining the universally regularized n-pf

$$\langle j_1(p_1) \dots j_n(p_n) \rangle_{VR} = \sum_{i=0}^N C_i \langle j_1(p_1) \dots j_n(p_n) \rangle_{m_i} \quad (4.1)$$

as a finite sum of n-pf's with m_i as the mass of the fermion in the loop integral. The C_i are functions of the set of masses m_i with $C_0 = 1$ and $m_0 = m$. It is understood that the limit $m_i \rightarrow \infty$, $i \neq 0$ is taken at the end of the calculation. The conditions

$$\sum_{i=0}^N C_i m_i^{\alpha} \ln^{\beta} m_i^2 = 0, \quad \alpha = 0, 1, 2, 3, \quad \beta = 0, 1$$

and

$$\sum_{i=1}^N \frac{|C_i|}{m_i} \rightarrow 0 \quad \text{as } m_i \rightarrow \infty$$

insure that the n-pf's will be free of ultraviolet divergences and integration displacement ambiguities.⁹ Equations (4.1) and (4.2) define the universal regularization.

Having removed the divergences, we then must allow terms polynomial in the momenta to be re-added to the n-pf's with some finite coefficient. These are called counter-terms.

For example, if we wish to calculate the anomaly in

$$k^\mu \langle A_\mu(k) V_\nu V_\sigma \rangle = -2mi \langle P(k) V_\nu V_\sigma \rangle + \Delta_A(AVV)$$

we replace $\langle A_\mu V_\nu V_\sigma \rangle \rightarrow \langle A_\mu V_\nu V_\sigma \rangle_R + \delta \langle AVV \rangle$, etc., where $\delta \langle AVV \rangle$ is the counter-term for $\langle A_\mu V_\nu V_\sigma \rangle$ and

$$\langle AVV \rangle = \sum_{i=0}^N C_i \langle AVV \rangle_{m_i}, \quad m_i \rightarrow \infty, \quad i \neq 0.$$

The anomaly is

$$\begin{aligned} \Delta_A(AVV) &= -2i \sum_{i=0}^N C_i (m_i - m) \langle P^a(k) V_\mu V_\nu \rangle + \delta(C) \\ &= 2i \langle PVV \rangle \Big|_{1/m} + \delta(C) \end{aligned}$$

where $\delta(C) = k^\mu \delta(A_\mu VV) + 2mi \delta(PVV)$, the sum total of the counter-terms.

As a result of the regularization the $1/m$ term of $\langle PVV \rangle$ contributes, along with the counterterms.

Similarly the trace anomaly of

$$g^{\lambda\rho} \langle \theta_{\lambda\rho} VV \rangle = \sqrt{6} m \langle S^0 VV \rangle - 3i [\langle VV \rangle + \langle \bar{V}\bar{V} \rangle] + \Delta_T(\theta VV)$$

is given by

$$\Delta_T(\theta VV) = -\sqrt{6} \langle S^0 VV \rangle \big|_{1/m} \\ + g^{\lambda\rho} \delta(\theta VV) - \sqrt{6} m \delta(S^0 VV) + 3i[\delta(VV) + \delta(VV)]$$

with appropriate momenta for the current.

In general the anomaly is given by the $1/m$ -term of all terms having coefficient m in the WTI plus a sum of counterterms given by the WTI or trace identity.

The anomalies for Eq. (2.8) and (2.10) are denoted by $\Delta_A(\theta_{\lambda\rho}(k) - A_\mu^a(p) \dots)$ and $\Delta_T(\theta_{\lambda\rho}(k) \dots)$ respectively and are understood to be added to the right hand side to give the "true" form of the equation, i.e. satisfying single loop perturbation theory. They are given by

$$\Delta_A(\theta_{\lambda\rho}(k) A_\mu^a(p) \dots) = 2i \langle \theta_{\lambda\rho}(k) P^a(p) \dots \rangle \big|_{1/m} \\ - 2g_{\lambda\rho} \langle P^a(k+p) \dots \rangle \big|_{1/m} + \delta(\text{LHS}) - \delta(\text{RHS}) \quad (4.3)$$

and

$$\Delta_T(\theta_{\lambda\rho}(k) \dots) = -\sqrt{6} \langle \theta_{\lambda\rho}(k) \dots \rangle \big|_{1/m} + \delta(\text{LHS}) - \delta(\text{RHS}) \quad (4.4)$$

where $\langle \dots \rangle \big|_{1/m}$ is the coefficient of the $1/m$ -term in a large m expansion of $\langle \dots \rangle$ and $\delta(\text{LHS})$ and $\delta(\text{RHS})$ are the sum of counterterms of the individual n -pf's entering respectively the left- and right-hand sides of Eqs. (2.8) and 2.10). Similar definitions are understood

for the vector and tensor Ward identities, which, of course, have no $1/m$ -term contributions. We will see that the vector and tensor anomalies vanish.

The counterterms for Ward identities with currents only have been determined by BSY⁹ and are used (and corrected in some cases) in this work. The counterterms are determined, up to certain arbitrary parameters, by choosing them in a self-consistent manner which minimizes the number of anomalies. The counterterms for the various current n -pf's are related in a chain-like manner from $n = 1$ up to $n = 4$. The $1/m$ -terms are possible for current n -pf's from $n = 1$ to $n = 5$ and for n -pf's with tensor from $n = 1$ to $n = 5$. This is due to the mass (or momentum) dimensionality of the n -pf's as exhibited in the momentum representation. For example, the 5-pf of Fig. 4 has a loop integral with a dimension 4 from $\int d^4\ell$, dimension 1 from $\Gamma_{\lambda\rho}$, and dimension -1 from each propagator $\frac{i}{\ell^2 + m^2}$ thus having an overall dimension $4 + 1 - n$. The current n -pf's have dimension $4 - n$.

The chains of n -pf's are divided into two classes, those with normal parity and those with abnormal parity. Parity is normal if the overall parity of the n -pf is $(-1)^r$ where r = rank of the n -pf, that is, the number of 4 vector indices. Now $\theta_{\lambda\rho}$, A_μ , and S have positive parity, while V_μ and P have negative parity. Of course, $\theta_{\lambda\rho}$ is rank 2, V_μ and A_μ are rank 1, and P and S are rank 0, so that A and P are the abnormal parity currents. An n -pf must have an odd number of A and P currents to have abnormal parity. Note that abnormal parity terms must contain the totally antisymmetric tensor $\epsilon_{\lambda\rho\mu\nu}$.

The Feynman diagram representations of n-pf's have the property that they are either even or odd under the transformation $m \rightarrow (-m)$, i.e. the change of sign of the fermion loop mass. This property is called mass parity. θ , V , and A have even mass parity, while P and S have odd mass parity. A mass expansion of an n-pf must have all even powers of m or odd powers of m corresponding to its mass parity. The counterterms are also further restricted by this consideration. The $1/m$ -term is necessarily of odd mass parity, so it can only contribute when its n-pf has odd mass parity. Thus $\langle PVV \rangle|_{1/m}$ and $\langle SVV \rangle|_{1/m}$ are non-zero, but $\langle PVS \rangle|_{1/m} = 0$ and $\langle SVS \rangle|_{1/m} = 0$.

The n-pf's, counterterms, and $1/m$ -terms must also obey charge conjugation invariance. $\theta_{\lambda\rho}$, A_μ , P , and S have $C = +$, positive charge conjugation, while V has $C = -$, negative charge conjugation. A term which has overall $C = +$ must be symmetric under interchange of space index, $SU(3)$ index, and momentum. For example, $\langle \theta_{\lambda\rho}(k) A_\mu^a(p) V_\nu^b(q) V_\sigma^c(s) \rangle$, must be symmetric under $\nu, b, q \leftrightarrow \sigma, c, s$, while $\langle \theta_{\lambda\rho}(k) V_\mu^a(p) V_\nu^b(q) V_\sigma^c(s) \rangle$ must be antisymmetric under $\mu, a, p \leftrightarrow \nu, b, q \leftrightarrow \sigma, c, s$. This further restricts the form of the counterterms and $1/m$ -terms.

Even with all these restrictions the possible counterterms are quite complicated with up to 15 arbitrary parameters, as for $\delta(\theta_{\lambda\rho}(k) V_\mu^a(p) V_\nu^b(q))$ which has rank 4, even mass parity, even charge conjugation, and normal parity.

We see already that no anomalies appear in the current n-pf's or tensor $(n+1)$ -pf's for $n \geq 6$.

Finite Range Integration

The calculation of the anomalies requires the explicit calculation of the $1/m$ -term of single loop Feynman graphs. This is done for the current n -pf's, $\langle j_1 \dots j_n \rangle$ in Ref. 9. The results are checked and used here. For the tensor $(n+1)$ -pf $\langle \theta j_1 \dots j_n \rangle$, we carry out a similar calculation with the addition of $\theta_{\lambda\rho}(k)$ at one vertex. The $SU(3)$ structure of the tensor $(n+1)$ -pf will be identical to that of the current n -pf containing the same currents. A summary of the n -pf loops is given in Appendix B.

We illustrate the general method of calculation with the 3-pf. We can separate the terms in $\Gamma_{\lambda\rho}(\ell+a, \ell+b)$ into separate integrals as follows:

$$\begin{aligned} \Gamma_{\lambda\rho 12}(a, 0, b) &= \int d^4\ell \text{Tr} \left\{ \left[\frac{1}{4} \gamma_\lambda (2\ell+a+b)_\rho + \frac{1}{4} \gamma_\rho (2\ell+a+b)_\lambda \right. \right. \\ &\quad \left. \left. - \frac{1}{2} g_{\lambda\rho} (2\ell+a+b-2m) \right] \cdot \frac{1}{\ell+a-m} \Gamma_1 \frac{1}{\ell-m} \Gamma_2 \frac{1}{\ell+b-m} \right\} \\ &= \frac{1}{2} [f_{\lambda\rho 12}(a, 0, b) + f_{\rho\lambda 12}(a, 0, b)] \\ &\quad + \frac{1}{4} [(a+b)_\lambda F_{\rho 12}(a, 0, b) + (a+b)_\rho F_{\lambda 12}(a, 0, b)] \\ &\quad - \frac{1}{2} g_{\lambda\rho} [D_{12}(0, b) + D_{12}(0, a)] \end{aligned}$$

where $F_{\lambda 12}$ and D_{12} are 3- and 2-vertex spinor loops from Ref. 9 and

$$f_{\lambda\rho 12}(a, 0, b) = \int d^4\ell \text{Tr} \left\{ \gamma_\lambda \ell_\rho \frac{1}{\ell+a-m} \Gamma_1 \frac{1}{\ell-m} \Gamma_2 \frac{1}{\ell+b-m} \right\}$$

We have calculated expressions for the integrals of the $1/m$ -terms of $f_{\lambda\rho 12}(a, 0, b)$ by two different methods. The first follows the method of Ref. 9 using Feynman parametrization:²⁸

$$f_{\lambda\rho 12}(a, 0, b) = \int_0^1 2y \, dy \int_0^1 dx \, \text{Tr} \left\{ \gamma_\lambda \int_{\ell}^{\ell} \frac{1}{(\ell^2 - 2\ell \cdot p_y - \Delta_y)^3} \right. \\ \left. (\ell + \not{x} + m) \Gamma_1 (\ell + m) \Gamma_2 (\ell + \not{b} + m) \right\} \quad (4.5)$$

where $p_y = -axy - b(1-y)$, $\Delta_y = m^2 - a^2xy - b^2(1-y) = m^2 + \bar{\Delta}_y$, and $\int_{\ell} \equiv \frac{d^4\ell}{(2\pi)^4}$ is an integral over momentum space ℓ . To obtain the $1/m$ -term, the above expression is expanded in odd powers of n with the factors of ℓ isolated. The ℓ integration is done by finite range integration (FRI).²⁹

The FRI formulae

$$J_{\mu\nu}^{(3)}(p, \Delta) \equiv \int_{\ell} \frac{\ell_\mu \ell_\nu}{(\ell^2 - 2\ell \cdot p - \Delta)^3}, \text{ etc.}$$

are substituted in at the proper place retaining only those terms which contribute to the $1/m$ -terms. The Feynman integration (over x and y) is then done to achieve the final result (see Eq. (B.1e)) for $f_{\lambda\rho 12}(a, 0, b) |_{1/m}$ in terms of vector a, b , indices λ, ρ , and γ -matrices $\gamma, \Gamma_1, \Gamma_2$.

An alternative approach, beginning from Eq. (4.5) is to expand the propagators using the identity

$$\frac{1}{\not{x} + \not{a} - m} = \frac{1}{\not{x} - m} - \frac{1}{\not{x} - m} \not{a} \frac{1}{\not{x} - m} + \frac{1}{\not{x} - m} \not{a} \frac{1}{\not{x} - m} \not{a} \frac{1}{\not{x} - m} \dots$$

Letting $s = (\not{x} - m)^{-1}$, we get

$$f_{\lambda\rho 12}(a, 0, b) = \int_0^1 \text{Tr} [\gamma_\lambda \not{\epsilon}_\rho (s - \not{a}s + s\not{a}s - s\not{a}s\not{a}s + \dots)]$$

$$\Gamma_1 s \Gamma_2 (s - s\not{a}s + s\not{a}s\not{a}s - s\not{a}s\not{a}s\not{a}s + \dots)]$$

By factoring out the momenta a, b from this integral, we see that only terms of order s^6 will contribute to the $1/m$ -term since $\int \not{\epsilon}_\rho s^6 \sim 1/m$, independent of momenta. Thus

$$\begin{aligned} f_{\lambda\rho 12}(a, 0, b) \Big|_{1/m} = & - [a^\alpha a^\beta a^\delta J_{\lambda\rho\alpha\beta\delta 12} + a^\alpha a^\beta b^\delta J_{\lambda\rho\alpha\beta 12\delta} \\ & + a^\alpha b^\beta b^\delta J_{\lambda\rho\alpha 12\beta\delta} + b^\alpha b^\beta b^\delta J_{\lambda\rho 12\alpha\beta\delta}] \end{aligned} \quad (4.6)$$

where

$$J_{\lambda\rho 12345} = \int \text{Tr} [\gamma_\lambda \not{\epsilon}_\rho \frac{1}{\not{x} - m} \Gamma_1 \frac{1}{\not{x} - m} \Gamma_2 \dots \Gamma_5 \frac{1}{\not{x} - m}] \Big|_{1/m} \quad (4.7)$$

The $1/m$ -term for the integral for Eq. (4.7) is given in (Eq. (B.4d)).

Although Eq. (4.6) was more easily arrived at than Eq. (B.1c), its use in a particular case requires the complicated trace calculation of Eq. (B.4d), whereas the trace of Eq. (B.1e) is easily calculated by

hand. Similar calculations up through the 6 vertex spinor loop, for which both methods give the same answer, are summarized in Appendix B.

Based on the $SU(3)$ structure, charge conjugation, dimension, and tensor structure of the $1/m$ -terms we conclude that $1/m$ -terms may contribute to axial-vector Ward identity anomalies for those 3-, 4-, 5-, and 6-pf's listed in Table I. The possible n -pf's giving trace identity anomaly contributions are listed in Table II.

The trace calculation of the $1/m$ -terms for specific currents V , A , S and P was done with the help of A. C. Hearn's REDUCE-2 program.³⁰ This computer program is a general algebraic program with the capability of doing Feynman trace calculations for high energy physics, as well as a wide variety of algebraic manipulations. It was compiled and placed on the IBM 360/65 for use in this calculation.

Counterterms

The counterterms, which renormalize the n -pf's, are local polynomials (in fermion mass m and momenta).⁹ To begin with the counterterms are required to satisfy the same structure as the n -pf's themselves. That is, they have the same dimension, mass, parity, tensor structure, $SU(3)$ structure, and charge conjugation symmetry as their n -pf's. There are two separate chains of counterterms, the normal parity and abnormal parity, the latter necessarily containing an $\epsilon_{\mu\nu\sigma\tau}$ in all of its terms.

We begin with the counterterms of Ref. 9, which were found by requiring all normal-parity Ward identities to be satisfied, but which

Table 1. Universally regularized n-pf's with tensor $\theta_{\lambda\rho}$ which have naive AWTI anomalies (i.e. $1/m$ term contributions).

3-pf	$\langle\theta AA\rangle$				
4-pf	$\langle\theta AAV\rangle$	$\langle\theta ASP\rangle$	$\langle\theta AVV\rangle$	$\langle\theta AAA\rangle$	
5-pf	$\langle\theta AAVV\rangle$	$\langle\theta APSV\rangle$	$\langle\theta AAAAA\rangle$	$\langle\theta AASS\rangle$	$\langle\theta AAPP\rangle$
	$\langle\theta AVVV\rangle$	$\langle\theta AAAV\rangle$			
6-pf	$\langle\theta AAVVVV\rangle$	$\langle\theta AAAAV\rangle$	$\langle\theta AAVSS\rangle$	$\langle\theta APVVS\rangle$	$\langle\theta AAAAPS\rangle$
	$\langle\theta AAVPP\rangle$	$\langle\theta APSSS\rangle$	$\langle\theta APPPS\rangle$		
	$\langle\theta AAAAA\rangle$	$\langle\theta AAADV\rangle$	$\langle\theta AVVVV\rangle$		

Table 2. Universally regularized n-pf's with tensor $\theta_{\lambda\rho}$ which have naive TI anomalies.

3-pf	$\langle\theta VV\rangle$	$\langle\theta AA\rangle$	$\langle\theta SS\rangle$	$\langle\theta PP\rangle$	
4-pf	$\langle\theta VVV\rangle$	$\langle\theta VAA\rangle$	$\langle\theta VSS\rangle$	$\langle\theta VPP\rangle$	$\langle\theta ASP\rangle$
5-pf	$\langle\theta VVVV\rangle$	$\langle\theta VVAA\rangle$	$\langle\theta VVSS\rangle$	$\langle\theta VVPP\rangle$	$\langle\theta AAAAA\rangle$
	$\langle\theta AASS\rangle$	$\langle\theta AAPP\rangle$	$\langle\theta VASP\rangle$	$\langle\theta SSSS\rangle$	$\langle\theta SSPP\rangle$
	$\langle\theta PPPP\rangle$				

could not satisfy all abnormal-parity Ward identities. We input these counterterms and the general forms of our tensor n-pf counterterms into the Ward-Takahashi and trace identities, seeking to eliminate as many anomalous terms as possible.

We find that all tensor and vector WTI's can be satisfied naively, i.e. without anomalies. Indeed, the satisfaction of the tensor WTI's, which can be thought of as an expression of Poincare invariance,¹⁹ requires the satisfaction of all vector WTI's. In particular, the possible abnormal parity counterterms of BSY are removed. This removes the previous ambiguity which allowed vector Ward identity anomalies. The counterterms are listed in Appendix C, along with a (corrected) list of the current n-pf counterterms from Ref. 9.

A few comments are called for.

1. All abnormal parity counterterms vanish.
2. All counterterms can be expressed in terms of four parameters a_1 , a_2 , a_3 , and a_4 introduced in Ref. 9.
3. The counterterms $\delta(\theta_{\lambda\rho})$ and $\delta(\theta_{\lambda\rho} S^a)$ were obtained by assuming that the trace identity is satisfied for these essentially trivial cases. The latter also implies that $a_4 = 3a_1$, eliminating a_4 in the counterterms.
4. Four new parameters were eliminated by assuming naive axial-vector WTI's for the normal parity n-pf's, $\langle\theta AA\rangle$, $\langle\theta VAA\rangle$, $\langle\theta AP\rangle$, and $\langle\theta ASP\rangle$.
5. For the abnormal parity series, the assumption of naive tensor WTI's implies that all abnormal parity counterterms vanish.

Thus the extension of the n-pf's to include the energy-momentum tensor and its trace does not further complicate the structure of the counterterms, but rather simplifies it.

The Anomalies

Having determined the counterterms and $1/m$ -terms we can compute the complete set of axial-vector WTI anomalies and trace identity anomalies according to Eqs. (4.3) and (4.4). Explicit calculation shows that all the normal parity n-pf's retain their naive forms of AWTI's. Anomalies occur only in the AWTI's of the following two sets of abnormal parity n-pf's: the set containing the energy-momentum tensor: $\langle \theta AVV \rangle$, $\langle \theta AAA \rangle$, $\langle \theta AAAV \rangle$, $\langle \theta AVVV \rangle$, $\langle \theta AAAAA \rangle$, $\langle \theta AAAAVV \rangle$, and $\langle \theta AVVVVV \rangle$; and the set containing internal symmetry currents only: $\langle AVV \rangle$, $\langle AAA \rangle$, $\langle AAAV \rangle$, $\langle AVVV \rangle$, $\langle AAAAA \rangle$, $\langle AAAAVV \rangle$, and $\langle AVVVVV \rangle$. Their anomalies arise simply from the $1/m$ -terms of $\langle \theta PVV \rangle$, etc. and $\langle PVV \rangle$, etc., as given in Eqs. (4.3) and (4.4) and in Ref. 9 (see also Appendix B).

The second set above is Bardeen's minimal set. We discard the second minimal set of Ref. 9 based on the fact that it contains anomalous vector WTI's and hence anomalous tensor WTI's. In the following, we list all the 14 anomalies:

$$\Delta_A(AVV) = -3\gamma d^{abc} \epsilon_{\nu\sigma\alpha\beta} q_s^\alpha s^\beta$$

$$\Delta_A(\theta AVV) = 3i\gamma d^{abc} \{g_{\lambda\rho} \epsilon_{\nu\sigma\alpha\beta} p^\alpha (q-s)^\beta$$

$$- \frac{1}{2} [g_{\nu\lambda} \epsilon_{\sigma\rho\alpha\beta} p_s^\alpha s^\beta + g_{\sigma\lambda} \epsilon_{\nu\rho\alpha\beta} p^\alpha q^\beta + (\lambda \leftrightarrow \rho)]$$

$$\Delta_A(AAA) = \frac{1}{3} \Delta_A(AVV)$$

$$\Delta_A(\theta AAA) = \frac{1}{3} \Delta_A(\theta AVV)$$

$$\Delta_A(AAAV) = -\frac{i\gamma}{8} \epsilon_{\nu\sigma\tau\alpha} X^\alpha$$

$$\begin{aligned} \Delta_A(\theta AAAV) = & -\frac{\gamma}{8} \{ g_{\lambda\rho} \epsilon_{\nu\sigma\tau\alpha} [2X^\alpha + 6\bar{w}_2 k^\alpha] + \frac{1}{2} [\epsilon_{\nu\sigma\tau\lambda} X_\rho + (\lambda \leftrightarrow \rho)] \\ & - \frac{1}{2} [g_{\lambda\nu} \epsilon_{\rho\sigma\tau\alpha} k^\alpha (-\bar{w}_1 + \bar{w}_2) + g_{\lambda\sigma} \epsilon_{\nu\rho\tau\alpha} k^\alpha (\bar{w}_3 + \bar{w}_2) \\ & + g_{\lambda\tau} \epsilon_{\nu\sigma\rho\alpha} k^\alpha (\bar{w}_1 + 4\bar{w}_2 - \bar{w}_3) + (\lambda \leftrightarrow \rho)] \} \end{aligned}$$

$$\Delta_A(AVVV) = -\frac{3}{8} i \gamma \epsilon_{\nu\sigma\tau\alpha} Y^\alpha$$

$$\begin{aligned} \Delta_A(\theta AVVV) = & -\frac{3}{8} \gamma \{ 2 g_{\lambda\rho} \epsilon_{\nu\sigma\tau\alpha} [Y^\alpha + (-\bar{w}_1 + \bar{w}_2 + \bar{w}_3) k^\alpha] \\ & + \frac{1}{2} [\epsilon_{\nu\sigma\tau\lambda} Y_\rho + (\lambda \leftrightarrow \rho)] \\ & - \frac{1}{2} [g_{\lambda\nu} \epsilon_{\sigma\tau\rho\alpha} k^\alpha (-\bar{w}_1 + \bar{w}_2) + g_{\lambda\sigma} \epsilon_{\nu\rho\tau\alpha} k^\alpha (\bar{w}_2 + \bar{w}_3) \\ & + g_{\lambda\tau} \epsilon_{\nu\sigma\rho\alpha} k^\alpha (\bar{w}_3 - \bar{w}_1) + (\lambda \leftrightarrow \rho)] \} \end{aligned}$$

$$\Delta_A(AAAAA) = \frac{\gamma}{32} \epsilon_{\nu\sigma\tau\eta} Z^{abcde}$$

+ 23 more terms with $\nu\sigma\tau\eta$ permuted with bcde.

$$\Delta_A(\theta AAAAA) = -3i g_{\lambda\rho} \Delta_A(AAAAA)$$

$$\Delta_A (AAAVV) = \frac{\gamma}{16} \epsilon_{\nu\sigma\tau\eta} [Z^{abcde} + Z^{abdce} + Z^{acbed} \\ + Z^{acebd} + 3Z^{abdec} + Z^{adcbe} - (d \leftrightarrow e)]$$

$$\Delta_A (\theta AAAVV) = -3i g_{\lambda\rho} \Delta_A (AAAVV)$$

$$\Delta_A (AVVVV) = -\frac{3}{32} \gamma \epsilon_{\nu\sigma\tau\eta} Z^{abcde} \\ + 23 \text{ more terms with } \nu\sigma\tau\eta \text{ permuted with } bcde.$$

$$\Delta_\lambda (\theta AVVV) = -3i g_{\lambda\rho} \Delta_A (AVVV). \quad (4.8)$$

where

$$\gamma \equiv \frac{1}{24\pi^2 i}$$

$$X_\mu \equiv \bar{W}_1 (t - q)_\mu + \bar{W}_2 (4t + q + s)_\mu + \bar{W}_3 (s - t)_\mu$$

$$Y_\mu \equiv -\bar{W}_1 (t + q)_\mu + \bar{W}_2 (q + s)_\mu + \bar{W}_3 (s + t)_\mu$$

$$W_1 \equiv \text{Tr}[\lambda_a \lambda_b \lambda_c \lambda_d - \lambda_a^T \lambda_b^T \lambda_c^T \lambda_d^T] \quad (4.9)$$

$$\bar{W}_2 \equiv \text{Tr}[\lambda_a \lambda_b \lambda_d \lambda_c - \lambda_a^T \lambda_b^T \lambda_d^T \lambda_c^T]$$

$$\bar{W}_3 \equiv \text{Tr}[\lambda_a \lambda_c \lambda_b \lambda_d - \lambda_a^T \lambda_c^T \lambda_b^T \lambda_d^T]$$

$$Z^{abcde} = \text{Tr}[\lambda_a \lambda_b \lambda_c \lambda_d \lambda_e + \lambda_a^T \lambda_b^T \lambda_c^T \lambda_d^T \lambda_e^T], \text{ etc.}$$

In the above expressions the energy-momentum tensor carries the tensor indices $\lambda\rho$ and momentum k , the vector and axial-vector currents carry the vector indices μ, ν, σ, τ , and η ; the momenta p, q, s, t , and u ; and internal symmetry indices a, b, c, d , and e in that order from left to right.

Since all abnormal parity counterterms vanish abnormal parity trace identity anomalies can only arise from the $1/m$ -terms. Explicit calculation shows, however, that all such terms vanish. Thus, only normal parity n -pf's have trace identity anomalies. These can arise from the $1/m$ -terms, counterterms, or both. The trivial cases $\langle\theta_{\lambda\rho}(0)\rangle$ and $\langle\theta_{\lambda\rho}(k)s^0(-k)\rangle$ have their counterterms adjusted to give vanishing TI anomalies. There are 25 remaining TI anomalies which are listed below:³¹

$n = 3$

$$\Delta_T(\theta VV) = 2iy\delta^{ab}(g_{\mu\nu}p \cdot q - p_\nu q_\mu)$$

$$\Delta_T(\theta AA) = \Delta_T(\theta VV) + 12iy\delta^{ab}m^2g_{\mu\nu}$$

$$\Delta_T(\theta AP) = -6y\delta^{ab}mq_\mu$$

$$\Delta_T(\theta PP) = -3iy\delta^{ab}p \cdot q$$

$$\Delta_T(\theta SS) = -iy\delta^{ab}[12m^2 + 3p \cdot q]$$

$$\underline{N = 4}$$

$$\Delta_T(\theta VVV) = 2iy f^{abc} [g_{\mu\nu}(q-p)_\sigma + g_{\mu\sigma}(p-s)_\nu + g_{\nu\sigma}(s-q)_\mu]$$

$$\Delta_T(\theta VAA) = \Delta_T(\theta VVV)$$

$$\Delta_T(\theta VAP) = 6yf^{abc} m g_{\mu\nu}$$

$$\Delta_T(\theta AAS) = -12y d^{abc} m g_{\mu\nu}$$

$$\Delta_T(\theta VSS) = 3iy f^{abc} (q-s)_\mu$$

$$\Delta_T(\theta VPP) = \Delta_T(\theta VSS)$$

$$\Delta_T(\theta ASP) = 3iy d^{abc} (q-s)_\mu$$

$$\Delta_T(\theta SSS) = 18y d^{abc} m$$

$$\Delta_T(\theta SPP) = \frac{1}{3} \Delta_T(\theta SSS)$$

$$\underline{n = 5}$$

$$\begin{aligned} \Delta_T(\theta VVVV) = & \frac{i}{4} y [g_{\mu\nu} g_{\sigma\tau} (w_1 + w_2 - 2w_3) + g_{\mu\tau} g_{\nu\sigma} (w_1 - 2w_2 + w_3) \\ & + g_{\mu\sigma} g_{\nu\tau} (-2w_1 + w_2 + w_3)] \end{aligned}$$

$$\Delta_T(\theta VVAA) = \Delta_T(\theta VVVV)$$

$$\Delta_T(\theta A A A A) = \Delta_T(\theta V V V V) \quad (4.10)$$

$$\Delta_T(\theta V V S S) = -\frac{3}{8} i y g_{\mu\nu} (w_1 + w_2 - 2w_3)$$

$$\Delta_T(\theta V V P P) = \Delta_T(\theta V V S S)$$

$$\Delta_T(\theta V A S P) = \frac{3}{8} y g_{\mu\nu} (\bar{w}_1 - \bar{w}_2 + 2\bar{w}_3)$$

$$\Delta_T(\theta A A S S) = -\frac{3}{8} i y g_{\mu\nu} (w_1 + w_2 + 2w_3)$$

$$\Delta_T(\theta A A P P) = \Delta_T(\theta A A S S)$$

$$\Delta_T(\theta S S S S) = \frac{3}{4} i y (w_1 + w_2 + w_3)$$

$$\Delta_T(\theta S S P P) = \frac{3}{4} i y (w_1 + w_2 - w_3)$$

$$\Delta_T(\theta P P P P) = \Delta_T(\theta S S S S)$$

where

$$w_1 \equiv \text{Tr}[\lambda_a \lambda_b \lambda_c \lambda_d + \lambda_a^T \lambda_b^T \lambda_c^T \lambda_d^T]$$

$$w_2 \equiv \text{Tr}[\lambda_a \lambda_b \lambda_d \lambda_c + \lambda_a^T \lambda_b^T \lambda_d^T \lambda_c^T]$$

$$w_3 \equiv \text{Tr}[\lambda_a \lambda_c \lambda_b \lambda_d + \lambda_a^T \lambda_c^T \lambda_b^T \lambda_d^T]$$

(4.11)

The assignments for the vector and internal symmetry indices and the momenta in the above are the same as those given below Eq. (4.9). The anomalies arising from the counterterms and from the mixture of the $1/m$ -terms and the counterterms are underlined with wavy and straight lines respectively. The rest are from the $1/m$ -terms only.

CHAPTER V. DOUBLE WARD IDENTITIES

Derivation

The anomalies calculated in Sec. IV can be checked against each other for internal consistency through the use of "double" Ward identities. Relations between various anomalies can be derived by assuming, as one would expect from the basic principles of quantum mechanics, that momentum operators commute. For instance, we can combine the tensor WTI and the axial-vector WTI in the relation

$$(p^\mu k^\lambda - k^\lambda p^\mu) \langle \theta_{\lambda\rho}(k) A_\mu^a(p) \dots j_\nu^b(q) \dots \rangle = 0 \quad (5.1)$$

where the momenta act on the n-pf in order from right to left. From (5.1), we derive the tensor-axial relation

$$\begin{aligned} k^\lambda \Delta_{A\lambda\rho}(k) A_\mu^a(p) \dots j_\nu^b(q) \dots &= -p_\rho \Delta_A(A_\mu^a(k+p) \dots j_\nu^b(q) \dots) \\ &+ \dots + \left[\frac{i}{2} k^\lambda (\bar{\Sigma}_{\lambda\rho})_\nu^\tau - i(k+q)_\rho g_\nu^\tau \right] \Delta_A(A_\mu^a(p) \dots j_\tau^b(k+q) \dots) \\ &+ \dots \end{aligned} \quad (5.2)$$

where

$$(\bar{\Sigma}_{\lambda\rho})_\mu^\tau = \begin{cases} g_{\lambda\mu} g_{\rho\tau} - g_{\lambda\tau} g_{\rho\mu} & \text{for V and A} \\ 0 & \text{for S and P} \end{cases}$$

Note the similarity of (5.2) to the tensor WTI, Eq. (2.9).

From the assumption

$$(p^\mu q^\nu - q^\nu p^\mu) \Delta_A(\theta_{\lambda\rho}(k) A_\mu^a(p) v_\nu^b(q) \dots j_\sigma^c(s) \dots) = 0 \quad (5.3)$$

we can derive the vector-axial relation

$$\begin{aligned} & q^\nu \Delta_A(\theta_{\lambda\rho}(k) A_\mu^a(p) v_\nu^b(q) \dots j_\sigma^c(s) \dots) \\ &= \frac{i}{2} [q_\lambda g_\rho^\alpha + q_\rho g_\lambda^\alpha - 2g_{\lambda\rho} q^\alpha] \Delta_A(A_\mu^a(p) v_\alpha^b(k+q) \dots j_\sigma^c \dots) \\ &+ f^{baa'} \Delta_A(\theta_{\lambda\rho}(k) A_\mu^{a'}(p+q) \dots j_\sigma^c(s) \dots) \\ &+ \dots + \Delta_A(\theta_{\lambda\rho}(k) A_\mu^a(p) \dots j_\sigma^{c'}(q+s) \dots) \end{aligned} \quad (5.4)$$

which appears similar to the vector WTI Eq. (2.7).

Of course, we also expect the trace operation to commute with the momenta. From

$$(g^{\lambda\rho} p^\mu - p^\mu g^{\lambda\rho}) \langle \theta_{\lambda\rho}(k) v_\mu^a(p) \dots j_\nu^b(q) \dots \rangle = 0, \quad (5.5)$$

we derive the trace vector relation

$$\begin{aligned} & p^\mu \Delta_T(\theta_{\lambda\rho}(k) v_\mu^a(p) \dots j_\nu^b(q) \dots) \\ &= \dots + f^{abb'} \Delta_T(\theta_{\lambda\rho}(k) \dots j_\nu^{b'}(p+q) \dots) + \dots \end{aligned} \quad (5.6)$$

which is analogous to the vector WTI without tensor. From

$$(g^{\lambda\rho}p^\mu - p^\mu g^{\lambda\rho}) \langle \theta_{\lambda\rho}(k) A_\mu^a(p) \dots j_\nu^b(q) \dots \rangle = 0 \quad (5.7)$$

a trace-axial relation is obtained between trace anomalies and axial-vector anomalies. Since the trace anomalies occur in normal parity series only and the axial-vector in abnormal series only, the equation can be separated into two parts which are the trace-axial relations

$$\begin{aligned} & g^{\lambda\rho} \Delta_A(\theta_{\lambda\rho}(k) A_\mu^a(p) \dots j_\nu^b(q) \dots) \\ &= \sqrt{6} m \Delta_A(S^0(k) A_\mu^a(p) \dots j_\nu^b(q) \dots) \\ &- 3i[\dots + \Delta_A(A_\mu^a(p) \dots j_\nu^b(k+q) \dots) + \dots] \end{aligned} \quad (5.8)$$

and

$$\begin{aligned} & p^\mu \Delta_T(\theta_{\lambda\rho}(k) A_\mu^a(p) \dots j_\nu^b(q) \dots) \\ &= -2mi \Delta_T(\theta_{\lambda\rho}(k) P^a(p) \dots j_\nu^b(q) \dots) \\ &+ \dots c \, \chi^{abb'} \Delta_T(\theta_{\lambda\rho}(k) \dots j_\nu^b(p+q) \dots) \end{aligned} \quad (5.9)$$

where

$$\chi^{abb'} j_\nu^{b'}(q) = \left\{ \begin{array}{c} f^{abb'} \begin{pmatrix} A_\nu^{b'}(q) \\ V_\nu^{b'}(q) \end{pmatrix} \\ d^{abb'} \begin{pmatrix} P^{b'}(q) \\ -S^{b'}(q) \end{pmatrix} \end{array} \right\} \text{ for } j_\nu^b = \left\{ \begin{array}{c} V_\nu^b \\ A_\nu^b \\ S^b \\ P^b \end{array} \right\} .$$

The first is a trace identity without the axial current as in Eq. (2.10), and the second is an axial-vector WTI without the tensor current.

Check of Anomalies

Using the double Ward identity relations (5.2), (5.4), (5.6), (5.8), and (5.9), we can check the internal consistency of the anomalies against each other. Note that since the n-pf's have been regularized in the derivation of the anomalies, no new anomalies should appear in the double Ward identities.

From Eq. (5.2) we can use the anomalies for n-pf's with currents only to compute the divergence of the anomaly for the (n+1)-pf with tensor. For example, from

$$\Delta_A(A_\mu^a(p)V_\nu^b(q)V_\sigma^c(s)) = -3yd^{abc}\epsilon_{\nu\sigma\alpha\beta}q^\alpha s^\beta$$

we can derive

$$\begin{aligned} & k^\lambda \Delta_A(\theta_{\lambda\rho}(k)A_\mu^a(p)V_\nu^b(q)V_\sigma^c(s)) \\ &= k^\lambda 3iy d^{abc} \{g_{\lambda\rho}\epsilon_{\nu\sigma\alpha\beta}k^\alpha(s^\beta - q^\beta) \\ &+ 2g_{\lambda\rho}\epsilon_{\nu\sigma\alpha\beta}q^\alpha s^\beta - \frac{1}{2}[g_{\lambda\nu}\epsilon_{\rho\sigma\alpha\beta}(k^\alpha + q^\alpha)s^\beta \\ &+ g_{\lambda\sigma}\epsilon_{\nu\rho\alpha\beta}q^\alpha(k^\beta + s^\beta) + \lambda \leftrightarrow \rho]\} . \end{aligned}$$

Using the identity

$$g_{\lambda\rho}\epsilon_{\mu\nu\sigma\tau} + g_{\lambda\mu}\epsilon_{\nu\sigma\tau\rho} + g_{\lambda\nu}\epsilon_{\sigma\tau\rho\mu} + g_{\lambda\sigma}\epsilon_{\tau\rho\mu\nu} + g_{\lambda\tau}\epsilon_{\rho\mu\nu\sigma} = 0$$

and the identity $\epsilon_{\nu\sigma\alpha\beta}k^\alpha k^\beta = 0$ extensively, the above result can be put into the form of the result given in Eq. (4.8). No "extra" term of the form $k_\lambda k_\rho - k^2 g_{\lambda\rho}$ appears in the calculation. Thus, the explicit calculation of the tensor n-pf anomalies contains no new anomalies, that is, terms unrelated to the anomalies of current-only n-pf's. The same holds true for $\Delta_A(\theta AAA)$ and for the 5- and 6-pf axial-anomalies whose structure allows no possible extra terms.

The relations (5.4) and (5.8) have also been used to check the axial anomalies against each other. A relation similar to Eq. (5.4), with the tensor $\theta_{\lambda\rho}$ omitted, can be used to relate the axial anomalies for current n-pf's.

The trace identities can be checked against each other through the use of Eqs. (5.6) and (5.9). The results are consistent.

CHAPTER VI. SUPPLEMENTARY MATERIAL ON DOUBLED RESONANCES IN THE MASS MATRIX FORMALISM

Background

The properties of the S-matrix when two resonances overlap have now been studied by several authors, the most thorough work appearing in the papers of Rebbi and Slansky (RS)³² and Dothan and Horn (DH).³³ The latter paper and numerous others³⁴ of less complete nature were inspired as efforts to correct an earlier Letter³⁵ in which misleading conclusions were reached.

In this note we wish to explore and establish the relation between the different possible viewpoints,^{32,33,36} for treating overlapping resonances. Specifically, the initial approach of DH and the present approach^{36,37} (particularly Ref. 36) is very similar even though the language and areas of emphasis differ. In both, the physical system under study is defined in terms of the Hamiltonian separated into an "unperturbed" part H_0 and the interaction part H_I , the former defining the states of the system and H_I describing the transitions between these states. Dothan and Horn³³ construct the S-matrix for such a system, solving the time dependent Schrödinger equation by making the Weisskopf-Wigner approximation.³⁸ They concentrated on the scattering problem and through its solution elegantly derived the unitarity (Bell-Steinberger³⁹) relations as the connection between resonance overlap and channel-vector overlap. In Ref. 36 the stationary scattering states are found as the eigenstates of the full Hamiltonian and used mainly to study decay properties of the resonance states.

On the other hand, Rebbi and Slansky³² give their entire treatment in terms of S-matrix theory in which resonances are introduced as poles of the S-matrix. The most general parametrization consistent with two body unitarity is found by imposing causality which requires certain factorization properties.

Derivation of a Unitary Double-Resonance

Formula in the Mass-Matrix Approach

In the Hamiltonian approach the scattering amplitude is written in the notation of Ref. 36 as

$$T_{fi} = g_f^+ M^{-1} (E + i\epsilon) g_i, \quad (6.1)$$

where $g_{i\nu}$ ($g_{f\nu}$) is the matrix element of the interaction H_1 between the intermediate resonance state $|\nu\rangle$, $\nu = 1, 2$, and the initial (final) state, multiplied by the appropriate phase space factors. The 2×2 matrix (in $\nu = 1, 2$ space) $\underline{m} - i\underline{\gamma}$ defined by

$$M(E \pm i\epsilon) = E - \underline{m} \pm i\underline{\gamma}$$

is called the mass matrix, or effective Hamiltonian (see, e.g., Eq. (5) of Ref. 36 or Eq. (46) of DH). The states $|\nu\rangle$ can be taken as eigenstates of H_0 , and if the time reversal operator commutes with H_0 the mass matrix $\underline{m} - i\underline{\gamma}$ is symmetric³³ since the couplings g are then real. The elements of this matrix are parametrized as

$$\tilde{m} - i\tilde{\gamma} = \begin{pmatrix} M - i\Gamma - \Delta e^{-i\bar{\theta}} & -\beta \Delta e^{-i(\bar{\theta} + \bar{\theta}')} \\ -i\beta \Delta e^{-i(\bar{\theta} + \bar{\theta}')} & M - \Gamma + \Delta e^{-i\bar{\theta}} \end{pmatrix}, \quad (6.2)$$

in terms of which the eigenvalues of the mass matrix are

$$M_{\pm} = M - i\Gamma \pm \Delta e^{-i\bar{\theta}} (1 - \beta^2 e^{-2i\bar{\theta}'})^{\frac{1}{2}}.$$

When $M_+ \neq M_-$ we may use the transformation which diagonalizes the mass matrix to write the scattering amplitude as

$$T_{fi} = \frac{\frac{1}{2}a_{fi}}{E - M_+} + \frac{\frac{1}{2}a_{fi}}{E - M_-} - \frac{\Delta e^{-i\bar{\theta}} (b_{fi} + i\beta e^{-i\bar{\theta}'} c_{fi})}{(E - M_+)(E - M_-)}. \quad (6.3)$$

The quantities a_{fi} , b_{fi} , and c_{fi} (elements of matrices in channel space) are defined in terms of couplings of particles 1 and 2 to channels i and f by

$$\begin{aligned} a_{fi} &= g_{f1}g_{i1} + g_{f2}g_{i2} \\ b_{fi} &= g_{f1}g_{i1} - g_{f2}g_{i2} \\ c_{fi} &= g_{f1}g_{i2} + g_{f2}g_{i1} \end{aligned} \quad (6.4)$$

The above two equations define the amplitude for scattering through two intermediate states (1 and 2) from initial channel i to the final channel f . This scattering amplitude of Eqs. (6.3) and (6.4) can be compared with that of RS, Eqs. (15) and (27). Equating these

two sets of equations leads to relations of which the obvious solutions are

$$g_{i1} = \sqrt{\Gamma} X_i \text{ and } g_{i2} = \sqrt{\Gamma} U_i, \quad (6.5)$$

to connect the mass matrix and S-matrix "couplings". For the remaining parameters we then have

$$\begin{aligned} \Delta \cos \bar{\theta} &= \Gamma(\xi \mathcal{K} - s) \cos \theta, \quad \beta \Delta \sin(\bar{\theta} + \bar{\theta}') = \xi \mathcal{K} \sqrt{1 - \frac{2s}{\xi \mathcal{K}}} \sin \theta \\ \Delta \sin \bar{\theta} &= \Gamma(\xi \mathcal{K} - s) \sin \theta, \quad \beta \Delta \cos(\bar{\theta} + \bar{\theta}') = \xi \mathcal{K} \sqrt{1 - \frac{2s}{\xi \mathcal{K}}} \cos \theta \end{aligned} \quad (6.6)$$

Enforcing a correspondence between the mass matrix and S-matrix parameters leads to $\bar{\theta}' = 0$ with $\bar{\theta} = \theta$. Thus, the RS parametrization of the amplitude implies that the phase of an off-diagonal element of the mass matrix is determined by the diagonal elements. This result can be questioned physically because either the RS S-matrix amplitude or Eq. (6.3) could be equally well applied to scattering of various pairs of mesons through electromagnetically mixed resonances, e.g. $\rho - \omega$ or $A_2^0 - f_0$, and $\bar{\theta}' = 0$ implies that the relative phase of the electromagnetic interaction is determined by the strong interaction part. From Eq. (6), besides $\bar{\theta}' = 0$ and $\bar{\theta} = \theta$ $\Gamma s = \Delta(1 - \beta^2)^{\frac{1}{2}}$ can be seen to be in accordance with the RS interpretation of s as the distance and θ as the angle of the line connecting the two poles. The relation between the remaining RS parameters and the mass matrix quantities is

$$\Gamma \xi \mathcal{K} = \Delta[1 + (1 - \beta^2)^{\frac{1}{2}}]$$

which implies a simple correspondence between ξ and Δ in the limit in which the two poles coincide.

The main emphasis of the RS treatment was utilization of all unitarity restrictions upon the S-matrix. In principle, in the Hamiltonian approach unitarity of the scattering amplitude is automatically satisfied due to Hermiticity of the Hamiltonian. However, if the formalism is used only as a guide to find the appropriate parametrization, unitarity has to be imposed on the parameters externally. We write the scattering amplitude as a matrix in channel space, the partial wave unitarity equation being

$$T^\dagger - T = 2iTpT^\dagger, \quad (6.7)$$

where p is the diagonal matrix of phase space factors. Substitution of the scattering amplitude from Eq. (6.1) into (6.7) leads to

$$2igpg^\dagger = M^\dagger(E+i\epsilon) - M(E+i\epsilon) = 2i\gamma. \quad (6.8)$$

With $\bar{\theta}^1 = 0$ in Eqs. (6.2), (6.5), and (6.6) and using these relations in the above one sees that this matrix Eq. (6.8) is the three equations (27d, 27e, 27f) of RS (which read $X \cdot X = 1 + [\xi\mathcal{K} - s]\sin\theta$, $U \cdot U = 1 - [\xi\mathcal{K} - s]\sin\theta$, and $X \cdot U = -\xi\mathcal{K}[1 - 2s/\xi\mathcal{K}]^{\frac{1}{2}}\cos\theta$). On the other hand when one recalls the connection of g_i to the interaction, one sees that Eq. (6.8) is the definition of the γ -matrix (Eqs. (11) and (46) of DH). Unitarity thus is seen to relate the resonance parameters to the coupling strengths. The physical content of unitarity becomes quite clear when Eq. (6.8) is considered in terms of eigenstates

of the mass matrix, but this is not necessary to show in view of the elegant derivation and discussion by DH of the resulting unitarity sum rule.

To close this section, we study the case when the two poles coincide. (This will also serve to amplify on the formalism behind two earlier letter contributions.³⁷⁾ Coincidence occurs when $\beta \rightarrow 1$ in the mass matrix parametrization of Eq. (6.2), or $\mathcal{K} \rightarrow 1$ in the S-matrix³² approach. The situation $\Delta = 0$ (or $\xi = 0$ in the S-matrix case) corresponds to two independent simple resonances and is not considered further. When $\beta \rightarrow 1$, then the RS parameters $\Gamma\xi = \Delta$ and the amplitude Eq. (6.3) becomes

$$T_{fi} = \frac{a_{fi}}{E - M + i\Gamma} - \Delta e^{-i\theta} \frac{(g_{f1} + ig_{f2})(g_{i1} + ig_{i2})}{(E - M + i\Gamma)^2} \quad (6.9)$$

Then, the mass matrix can be written in the Jordan form (see, e.g. Refs. 33 and 36) by transforming to an orthogonal basis

$$|d_1\rangle = (|1\rangle + i|2\rangle)/\sqrt{2}, \quad |d_2\rangle = (|1\rangle - i|2\rangle)/\sqrt{2}, \quad (6.10)$$

where $|d_1\rangle$ is the only eigenstate of the mass matrix. This transformation from $|1\rangle$, $|2\rangle$ to $|d_1\rangle$, $|d_2\rangle$ is given by a unitary matrix which we name V (defined by Eq. (6.12) below),

$$V = 2^{-\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$

The resulting mass matrix is

$$V^\dagger (\underline{m} - i\underline{\gamma}) V = (M - i\Gamma) \underline{1} - \begin{pmatrix} 0 & 2\Delta e^{-i\theta} \\ 0 & 0 \end{pmatrix} \quad (6.11)$$

where $\underline{1}$ stands for the 2×2 unit matrix. The couplings $g(d)$ for the states $|d_\nu\rangle$, $\nu = 1, 2$, are found by transforming the original couplings (i and f denote initial and final state, resp.)

$$g_i(d) = V^\dagger g_i, \quad g_f^\dagger(d) = g_f^\dagger V, \quad (6.12)$$

or more explicitly

$$\begin{aligned} g_{i1}(d) &= 2^{-\frac{1}{2}}(g_{i1} - ig_{i2}) \\ g_{i2}(d) &= 2^{-\frac{1}{2}}(g_{i1} + ig_{i2}) \end{aligned} \quad (6.13)$$

The expression for T_{fi} can likewise be transformed by inserting $V^\dagger V = 1$ into Eq. (6.1)

$$T_{fi} = g_f^\dagger V^\dagger V M^{-1} (E + i\epsilon) V^\dagger V g_i \quad (6.14)$$

to get the expressions employed earlier.³⁷ This same result was also derived later in the appendix of the DH paper where the parameter η in Eq. (A8) of DH must be associated with $-2\Delta e^{-i\theta}$, the off-diagonal element in Eq. (6.11) above. Finally, if in Eq. (6.14) the $g(d)$'s are

expressed in terms of the other couplings, g_i , then Eq. (6.14) leads directly to Eq. (6.4) for the amplitude.

Comparison to S-matrix Approaches

Comparing our result with that of RS shows that the phase $\bar{\theta}'$ in the off-diagonal elements of the mass matrix Eq. (6.2) must be zero for agreement between the two approaches. Thus, it appears that the parametrization obtained here is more general than the solution found by RS. In fact, one may check that this is so (with considerable algebra) by taking our amplitude from Eq. (6.3) and showing that that parametrization of the (numerator) matrix elements satisfies the matrix unitarity equations of RS (Eq. 11 a-f of Ref. 32).

When the two resonance poles coincide (the "double-resonance" limit, RS phraseology) the mass matrix is non-diagonalizable. The best that can be done is to transform the mass matrix to Jordan canonical form which leads readily to the definition of generalized eigenvectors with non-exponential time dependence. The so-called dipole formula $T \propto (E - M)/(E - M + i\Gamma)^2$ follows from Eq. (6.9) when the parameters Δ and θ are related to the couplings by the following equations ($\bar{\theta}'$ has already been set as zero),

$$\tan\theta (g_{f2}g_{i1} + g_{f2}g_{i2}) = g_{f2}g_{i2} - g_{f1}g_{i1}$$

and

$$\Gamma \cos\theta (g_{f1}g_{i1} + g_{f2}g_{i2}) = \Delta(g_{f2}g_{i1} + g_{f1}g_{i2}) .$$

Thus, the "dipole" or double pole intensity distribution which gives the same symmetrically twin-peaked intensity distribution in all channels of the one⁴⁰ partial wave amplitude can be derived from a multitude of possible couplings of the resonances to the initial and final channels.

Application to Data

The only physical example of this double pole intensity distribution in a scattering experiment constructible in the laboratory still appears to be that suggested in atomic physics some years ago.³⁷ The A_2 meson, as has always been clear even when the earliest split mass distribution was found, could not be a "dipole" of this type despite efforts by various experimenters to fit everything with the "dipole" shape. The consensus of opinion on the A_2 now, based mainly on experiments involving three pion final states, favors the single Breit-Wigner interpretation.⁴¹ However, the $K\bar{K}$ mass distributions obtained in $\pi^- p \rightarrow A_2^- p \rightarrow (K^- K^0) p$ are background free and exhibit as much as six standard deviation peaks and dips depending on the experiment and energy at which it is done.⁴² No two experiments on the $K\bar{K}$ system have been done at the same energy: Baud et al. at 7 GeV/c see the symmetric twin peak shape; Grayer et al. at 17.2 GeV/c appears to have a broad bump with a narrow spike on top; Foley et al. at 20.3 GeV/c have a somewhat distorted single peak; and, Crennell et al. at 4.5 GeV/c discover a relatively symmetric single peak. Although all of these four $K\bar{K}$ mass distributions can not be fit simultaneously with a

Breit-Wigner curve or with the "dipole" shape, they can readily be fit with the amplitude of Eq. (6.9) which is Eq. (28) of Rebbi and Slansky.³² This $K\bar{K}$ amplitude is written as

$$T = Az + Be^{-i\theta} z^2, \quad (6.15)$$

where $z = \frac{1}{2}\gamma / (E - M + i\frac{1}{2}\gamma)$, $M = 1303$ MeV, and $\gamma = 36$ MeV are fixed for all energies. The resulting (high quality) fits of $|T|^2$ to $K\bar{K}$ mass distributions at each energy determine θ versus A/B to be a straight line (within errors) varying in θ from -4.7 to -4.9 radians as A/B goes from $.7$ (at 4.5 GeV/c) to 2.3 (at 20.3 GeV/c).⁴³ No special physical significance is claimed for this rather simple dependence with θ being practically $-3\pi/2$ and barely changing over a 16 GeV/c interval. However, it is clear from this that none of the non-split distributions at other energies prove that the 7 GeV/c $K\bar{K}$ mass distribution is not split. From the experimental point of view to disprove the 7 GeV/c data, it is necessary to do an experiment at essentially the same energy with similar kinematical restrictions.

To summarize the present results: We begin with the mass matrix description of scattering through more than one resonance in the same partial wave and derive an expression for the scattering amplitude. Unitarity is used to place restrictions upon the couplings. This result is compared with the amplitude found from an S-matrix derivation by Rebbi and Slansky, and we conclude that we have generalized the parametrization to include an additional, physically reasonable phase and suppress the interdependences in ξ , $\mathcal{K}$ and s . The resulting $\xi \rightarrow 1$,

or double resonance, limit is used to derive a relatively few parameter amplitude which for illustrative purposes does very well in accommodating measured $K\bar{K}$ mass distributions at ~ 1.3 GeV produced in $\pi^- p$ collisions between 4.5 and 20.3 GeV/c. Predictions for the $K\bar{K}$ mass distribution at other energies in the context of this particular double-resonance formula of Eq. (6.15) can readily be supplied.

CHAPTER VII. CONCLUSIONS

Relation to Previous Work

Our results show that the inclusion of the symmetric energy-momentum tensor into the n-pf's of a free spinor field theory with SU(3) currents leads to no new structure in the anomalous axial-vector Ward identities. The use of the tensor WTI does, however, remove the previous ambiguity of Brown, Shih, and Young⁹ in regard to possible vector Ward identity anomalies. The conclusion is that the satisfaction of the tensor WTI's, which is a statement of Poincare invariance, implies the satisfaction of the vector WTI's, and forces all current anomalies to occur in the axial-vector WTI's. Axial anomalies occur in the WTI's of the n-pf's $\langle AVV \rangle$, $\langle AAA \rangle$, $\langle AVVV \rangle$, $\langle AAVV \rangle$, $\langle AVVVV \rangle$, $\langle AAAVV \rangle$, and $\langle AAAAA \rangle$, known as Bardeen's minimal set,¹⁰ as well as in the corresponding set with the tensor current.

We also find a large number of trace anomalies, 25 in all, which are restricted to the normal parity n-pf's, and thus completely disjoint from the axial anomalies. In contrast to the axial anomalies which arise from the $1/m$ -terms only and are independent of fermion mass m , the trace anomalies arise from the $1/m$ -terms, the counterterms, or both.

Chanowitz and Ellis²¹ have calculated trace anomalies for a zero momentum equation of the form

$$\begin{aligned}
& \langle \theta(0) j_{\mu_1}^{a_1}(p_1) \dots j_{\mu_n}^{a_n}(p_n) \rangle \\
&= i \left\{ 4 - n - \sum_{\ell=1}^{n-1} p_{\ell} \frac{\partial}{\partial p_{\ell}} \right\} \langle j_{\mu_1}^{a_1}(p_1) \dots j_{\mu_n}^{a_n}(p_n) \rangle \\
&- \Delta_T(\theta j_{\mu_1}^{a_1} \dots j_{\mu_n}^{a_n})
\end{aligned} \tag{7.1}$$

where $p_n = -(p_1 + \dots + p_{n-1})$.

Equation (7.1) can be obtained from the tensor WTI and the trace identity as follows: differentiate Eq. (2.9) with respect to k_p , and set $k_p = 0$; then eliminate the term $g^{\lambda\rho} \langle \theta_{\lambda\rho} j_{\mu_1}^{a_1} \dots j_{\mu_n}^{a_n} \rangle$ by its corresponding TI. The result shows that the canonical trace anomaly of Ref. 21 is given by the negative of our anomaly evaluated at $k = 0$, i.e. zero 4-momentum carried by the energy-momentum tensor.³¹

Possible Phenomenological Applications

In regard to the possible phenomenological applications of the WTI's and trace identities of the n-pf's with energy-momentum tensor, the anomalous trace has already been applied to a low energy theorem for the decay $\epsilon^0 \rightarrow \gamma\gamma$,^{20,21} the ϵ^0 being a $J^P = 0^+$ unitary singlet state dominating the low mass (≤ 1000 MeV/c) region of $\pi\pi$ scattering. This dominance of the low energy tensor current by scalar mesons is known as "partially conserved dilation current" (PCDC)⁴⁴ in analogy of PCAC ("partially conserved axial-vector current"). The presence of the trace anomaly allows the decay $\epsilon^0 \rightarrow \gamma\gamma$ to occur, and relates

its rate to the parameter R defined in Eq. (1.3).^{20,21,45} It has also been suggested that the traceless part of the stress energy momentum tensor could be dominated by $J^P = 2^+$ mesons such as the f .⁴⁶ The possibility of explaining the decay of the newly discovered massive J or ψ particles exists. If there exist charmed mesons related to these, the mass-matrix formalism of Sec. VI can be used to describe mixing of such states.

Theoretical Questions

Since there is no well-established model for strong interactions (in the sense of QED for electrodynamics), the apparent success of anomalous PCAC and (to a lesser extent) of anomalous PCDC raises the exciting possibility of guiding the construction of field theory models for strong interactions. For example, Freund and Nandi⁴⁷ have conjectured on the basis of PCAC and vector-meson dominance that the strength of strong interactions is inversely proportional to the number of postulated quarks (fundamental fermion fields in some field theory). They further extend this idea to the strength of electromagnetic, weak, and gravitational interactions.

As mentioned in the introduction, the presence of the anomalies can upset the renormalization scheme of a field theory.¹⁴ In order to regain renormalizability it is necessary to introduce additional particles or fields with appropriate couplings which cancel the original anomaly. For instance Gross and Jackiw¹⁴ outline a scheme in which lepton anomalies are canceled by hadron anomalies in a unified model of

electromagnetic, weak, and strong interactions. It should be noted that such theories retain the PCAC anomaly as applied to $\pi^0 \rightarrow \gamma\gamma$, for instance, even though they are said to be "anomaly free".

There remain many open questions as to the nature of the anomalies and their relation to elementary particle physics.

LITERATURE CITED

1. J. C. Ward, Phys. Rev. 78, 1824 (1950) and Y. Takahashi, Nuovo Cimento 6, 370 (1957).
2. J. D. Bjorken and S. D. Drell, Relativistic Quantum Fields, (McGraw-Hill Book Co., New York, 1965).
3. J. S. Bell and R. Jackiw, Nuovo Cimento 60A, 47 (1969).
4. D. G. Sutherland, Nucl. Phys. B2, 433 (1967) and M. Veltman, Proc. Roy. Soc. A301, 107 (1967).
5. J. Steinberger, Phys. Rev. 76, 1180 (1949).
6. S. L. Adler, Phys. Rev. 177, 2426 (1969).
7. J. Schwinger, Phys. Rev. 82, 664 (1951).
8. S. L. Adler and W. A. Bardeen, Phys. Rev. 182, 1517 (1969); B.-L. Young, T. F. Wong, G. Gounaris, and R. W. Brown, Phys. Rev. D4, 348 (1971); S. L. Adler, R. W. Brown, T. F. Wong, and B.-L. Young, Phys. Rev. D4, 1787 (1971); E. S. Abers, D. A. Dicus, and V. L. Teplitz, Phys. Rev. D3, 485 (1971); D. Corrigan, Phys. Rev. D8, 1123 (1973); M. S. Chanowitz, Phys. Rev. D9, 503 (1974).
9. R. W. Brown, C.-C. Shih, and B.-L. Young, Phys. Rev. 186, 1491 (1969).
10. W. A. Bardeen, Phys. Rev. 184, 1848 (1969).
11. S. Okubo, Phys. Rev. 179, 1629 (1969); C. S. Lai, Nucl. Phys. B14, 348 (1969); S. L. Glashow, R. Jackiw, and S. S. Shei, Phys. Rev. 187, 1916 (1969); G. J. Gounaris, Phys. Rev. D1, 1426 (1970); B.-L. Young, Phys. Rev. D2, 606 (1970); C. S. Lai and S. Y. Lo, Nuovo Cimento, 4A, 198 (1971); F. Strocchi and R. V. Caffarelli, Phys. Lett. 35B, 595 (1971); M. D. Scadron and R. L. Thews, Phys. Rev. D6, 844 (1972); H. Terazawa, Phys. Rev. D7, 3123 (1973); V. M. Raval and R. Ramachandran, Phys. Rev. D8, 1144 (1973); G. Furlan, F. Legovini, and N. Paver, Nuovo Cimento 17A, 635 (1973).
12. S. L. Adler, B. W. Lee, S. B. Treiman, and A. Zee, Phys. Rev. D4, 3497 (1971); R. Aviv, N. D. Hari Dass, and R. F. Sawyer, Phys. Rev. Lett. 26, 591 (1971); N. D. Hari Dass, Phys. Rev. D5, 1542 (1972); R. Aviv and A. Zee, Phys. Rev. D5, 2372 (1972); A. Zee, Phys. Rev. D6, 900 (1972).

13. M. V. Terentiev, JETP Lett. 14, 94 (1971) and Phys. Lett. 38B, 419 (1972); J. Smith and N. Stanko, Phys. Rev. D6, 1332 (1973); J. Smith and Z. E. S. Uy, Phys. Rev. D8, 3056 (1973).
14. D. J. Gross and R. Jackiw, Phys. Rev. D6, 477 (1972); C. Bouchiat, J. Iliopoulos, and Ph. Meyer, Phys. Lett. 38B, 519 (1972); H. Fritzsch, M. Gell-Mann, and H. Leutwyler, Phys. Lett. 47B, 365 (1973).
15. K. Wilson, Phys. Rev. 179, 1499 (1969).
16. K. Wilson, Phys. Rev. D2, 1478 (1970).
17. C. G. Callan, Jr., Phys. Rev. D2, 1541 (1970).
18. C. G. Callan, Jr., S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) 59, 42 (1970).
19. S. Coleman and R. Jackiw, Ann. Phys. (N.Y.), 67, 552 (1970).
20. R. J. Crewther, Phys. Rev. Lett. 28, 1421 (1972).
21. M. S. Chanowitz and J. Ellis, Phys. Rev. D7, 2490 (1973), and Phys. Lett. 40B, 397 (1972).
22. H. Suura and B.-L. Young, Phys. Rev. D8, 4354 (1973).
23. A. Zee, Phys. Rev. Lett. 29, 1198 (1972).
24. For excellent review articles see S. L. Adler, "Lecture on Elementary Particles and Quantum Field Theory," Brandeis Summer Institute, (Massachusetts Institute of Technology Press, Cambridge, Mass., 1970), and R. Jackiw in S. B. Treiman, R. Jackiw, and D. J. Gross, Lectures on Current Algebra and Its Applications (Princeton University Press, Princeton, N. J., 1972).
25. J. Ellis and R. Jaffe, SLAC-PUB-1353, Dec. 1973 (unpublished).
26. K. Symanzik, Comm. Math. Phys. 18, 227 (1971).
27. W. Pauli and F. Villars, Rev. Mod. Phys. 21, 434 (1949).
28. J. M. Jauch and F. Rohrlich, The Theory of Photon and Electrons (Addison-Wesley Publishing Co., Inc., Cambridge, Mass., 1955), pp. 454-460.
29. A. I. Akhiezer and V. B. Berestetsky, Quantum Electrodynamics (State Technico-Theoretical Literature Press, Moscow, 1953) (English translation by Consultants Bureau Enterprises, Inc., New York, 1953).

30. A. C. Hearn, REDUCE 2 User's Manual, 2nd. ed., Univ. of Utah, March 1973).
31. Comparison with the trace anomalies listed in Ref. 21 shows agreement with our result except for $\langle \theta_{\lambda p}^{AA} \rangle$. We get

$$\frac{R}{4\pi^2} (p^2 g_{\mu\nu} - p_\mu p_\nu) - \frac{g_{\mu\nu}}{\pi^2} m^2,$$
 instead of

$$\frac{R}{8\pi^2} (p^2 g_{\mu\nu} - p_\mu p_\nu) - \frac{g_{\mu\nu}}{\pi^2} m^2$$
 as given in Eq. (14.12), Ref. 21, where $R = 2/3$.
32. C. Rebbi and R. Slansky, Phys. Rev. 185, 1838 (1969) and D1, 1499 (1970). We use the abbreviation RS for this paper.
33. Y. Dothan and D. Horn, Phys. Rev. D1, 916 (1970). We refer to this paper as DH in the text.
34. See, e.g., K. W. McVoy, Phys. Rev. Lett. 23, 56 (1969); G. C. Wick, Phys. Lett. 30B 126 (1969); K. Yamamoto, Prog. of Theor. Phys. 43, 162 (1970).
35. W. D. McGlinn and D. Polis, Phys. Rev. Lett. 22, 908 (1969).
36. P. V. Ruuskanen, Nucl. Phys. B22, 253 (1970). We follow this notation in the present work. A sign difference in definition of the amplitude in terms of the S-matrix should be noted, $S = 1 - 2iT$ whereas RS use $S = 1 + 2iA$ (where $A = T$).
37. For earlier work with a (unitary) double-resonance amplitude, see K. E. Lassila and P. V. Ruuskanen, Phys. Rev. Lett. 19, 762 (1967) and 17, 490 (1966). The present work in part amplifies on the theoretical background leading to the general model-independent formulas in these letter articles.
38. V. Weisskopf and E. Wigner, A. Physik 63, 54 (1930); V. Weisskopf, Ann. Physik 9, 23 (1931).
39. J. S. Bell and J. Steinberger, Proceedings of the Oxford International Conference on Elementary Particles, 1965 (Rutherford High Energy Laboratory, Chilton, Berkshire, 1966), p. 195.
40. This corresponds to the $\xi = 1$, $s = 0$ situation in RS Ref. 1. It probably is not necessary to point out that this specific intensity shape does not occur in "all partial wave cross sections" as stated by RS, but is still in the same partial wave for which the S-matrix was being studied.

41. R. Diebold, Proceedings of the XVIth Interaction Conference on High Energy, Batavia, Illinois (to be published).
42. Experiments on $\pi^- p \rightarrow (KK)^- p$ are: at 4.5 GeV/c: D. I. Crennell et al., Phys. Lett. 35B, 185 (1971); at 7.0 GeV/c: R. Baud et al., ibid. 31B, 397 (1970); at 17.2 GeV/c: G. Grayer et al., ibid. 34B, 333 (1971); at 20.3 GeV/c: K. J. Foley et al., Phys. Rev. Lett. 26, 413 (1971).
43. When the absolute value squared of Eq. (15) is fit to a given mass distribution, although A/B and θ found to vary very little, at each mass value a quadratic equation for A/B results which has two solutions. The one given in the text has A/B increasing with energy. The other solution likewise has a linear θ vs. A/B plot, but A/B decreases as incident bombarding energy is changed. Neither solution is preferable over the other; both are mathematically possible results of an application of Eq. (9).
44. G. Mack, Nucl. Phys. B5, 499 (1968); P. Carruthers, Phys. Rev. D2, 2265 (1970) and Phys. Reports 1C, 1 (1971); R. J. Crewther, Phys. Lett. 33B, 305 (1970) and Phys. Rev. D3, 3152 (1971); and J. Ellis, Nucl. Phys. B22, 478 (1970).
45. See also H. Terazawa, Phys. Rev. Lett. 32, 694 (1974) and S. Eliezer, "Approximate Scale Invariance and $\lim_{s \rightarrow \infty} [\sigma(e^+e^- \rightarrow \text{hadrons}) / \sigma(e^+e^- \rightarrow \mu^+\mu^-)]$ in the SU(3) Limit," to be published in Phys. Rev. D.
46. B. Renner, Phys. Lett. 33B, 599 (1971); H. Jones and A. Salam, Phys. Lett. 34B, 149 (1971); and H. Kleinert, L. P. Staunton, and P. H. Weisz, Nucl. Phys. B38, 87 (1972).
47. P. G. O. Freund and S. Nandi, Phys. Rev. Lett. 32, 181 (1974).

ACKNOWLEDGMENTS

I would like to thank my adviser, Dr. K. E. Lassila for his help, advice, and tolerance during my graduate student years. Special thanks go to Dr. Bing-Lin Young for suggesting and collaborating in the anomaly calculations. Thanks also to Dr. Robert B. Clark for introducing me to the REDUCE program, and to A. C. Hearn and his group at Utah for providing the REDUCE-2 program which made the trace calculations possible. Financial support from the National Science Foundation and Ames Laboratory is gratefully acknowledged.

APPENDIX A. LOOP INTEGRALS

Here we summarize the Feynman graph loop integral representations of tensor n-pf's.

2-pf

$$\langle \theta_{\lambda\rho}(k) j_1^a(p) \rangle = \frac{1}{4} \text{Tr} (\lambda_a + c_a \lambda_a^T) \int_{\ell} \text{Tr} \{ \Gamma_{\lambda\rho}(\ell, \ell+k) S(\ell+k) \Gamma_1 S(\ell) \}$$

(by comparison $\langle j_1^a(p) \rangle = -\frac{i}{4} \text{Tr} (\lambda_a + c_a \lambda_a^T) \int_{\ell} \text{Tr} \{ \Gamma_1 S(\ell+a) \}$), where $\Gamma_{\lambda\rho}(p, q)$ is given by Eq. (2.11) and $S(k) = (\not{k} - m)^{-1}$.

3-pf

$$\langle \theta_{\lambda\rho}(k) j_1^a(p) j_2^b(q) \rangle = \frac{i}{8} U^{ab} [F_{\lambda\rho 12}(p, 0, -q) + c(2) F_{\lambda\rho 21}(q, 0, -p)]$$

where

$$U^{ab} = \text{Tr} (\lambda_a \lambda_b + c_a c_b \lambda_a^T \lambda_b^T), \quad c(2) = c_a c_b$$

and

$$F_{\lambda\rho 12}(a, b, c) = \int_{\ell} \text{Tr} \{ \Gamma_{\lambda\rho}(\ell+a, \ell+c) S(\ell+a) \Gamma_1 S(\ell+b) \Gamma_2 S(\ell+c) \}$$

4-pf

$$\langle \theta_{\lambda\rho}(k) j_1^a(p) j_2^b(q) j_3^c(s) \rangle$$

$$= \frac{-1}{16} v^{abc} [H_{\lambda\rho 123}(p, 0, -q, -q-s) + c(3)H_{\lambda\rho 321}(s, 0, -q, -q-p)]$$

$$\frac{-1}{16} v^{acb} [H_{\lambda\rho 132}(p, 0, -s, -s-q) + c(3)H_{\lambda\rho 231}(q, 0, -s, -s-q)]$$

$$\frac{-1}{16} v^{bac} [H_{\lambda\rho 213}(q, 0, -p, -p-s) + c(3)H_{\lambda\rho 312}(s, 0, -p, -p-q)]$$

where

$$U^{abc} = \text{Tr}(\lambda_a \lambda_b \lambda_c + c_a c_b c_c \lambda_a^T \lambda_b^T \lambda_c^T)$$

and

$$H_{\lambda\rho 123}(a, b, c, d) = \int_{\ell} \text{Tr} \{ \Gamma_{\lambda\rho}(\ell+a, \ell+d) S(\ell+a) \Gamma_1 S(\ell+b)$$

$$\Gamma_2 S(\ell+c) \Gamma_3 S(\ell+d) \} .$$

5-pf

$$\langle \theta_{\lambda\rho}(k) j_1^a(p) j_2^b(q) j_3^c(s) j_4^d(t) \rangle$$

$$= - \frac{i}{32} w^{abcd} [J_{\lambda\rho 1234}(p, 0, -q, -q-s, -q-s-t)$$

$$+ c(4) J_{\lambda\rho 4321}(t, 0, -s, -s-q, -s-q-p)]$$

+ 11 more pairs giving all 24 permutations of a,b,c,d with
1,2,3,4 and p,q,s,t.

6-pf

$$\langle \theta_{\lambda\rho}(k) j_1^a(p) j_2^b(q) j_3^c(s) j_4^d(t) j_5^e(u) \rangle$$

$$= \frac{1}{64} z^{abcde} K_{\lambda\rho 12345}(p, 0, -q, -q-s, -q-s-t, -q-s-t-u)$$

+ 119 more terms, giving all 120 permutations of a,b,c,d,e with 1,2,3,4,5 and p,q,s,t,u.

where W^{abcd} , Z^{abcde} , $c(4)$, $H_{\lambda\rho 1234}$, and $J_{\lambda\rho 12345}$ have definitions analogous to those for U^{abc} , $c(3)$, $H_{\lambda\rho 123}$, etc. Figure 3 gives the first term for a specific example of the 5-pf, $\langle \theta P V V V V \rangle$.

Throughout this paper the tensor current has momentum k with indices λ, ρ . The $SU(3)$ currents are generally given momenta p, q, s, t, u , space-time indices $\mu, \nu, \sigma, \tau, \eta$ and $SU(3)$ indices a, b, c, d, e in order from left to right in the n -pf's, counterterms, and anomalies. Exceptions are of course necessary at times.

APPENDIX B. 1/M-TERMS

Here we summarize the expressions for the 1/m-terms obtained from the spinor loop-integrals for the n-pf's containing the energy-momentum tensor. Expressions for the n-pf's having currents only can be found in Appendix A, Ref. 9. Let us remark that since the fermion loop-integrals possess a definite mass parity (i.e. evenness or oddness in m), all the n-pf's having even numbers of S and P have vanishing 1/m-terms.

Let $\gamma \equiv 1/24\pi^2$

3-pf

$$\begin{aligned} \langle \theta_{\lambda\rho}(k) j_1^a(p) j_2^b(q) \rangle |_{1/m} = & \frac{i}{8} \text{Tr}[\lambda_a \lambda_b + c(2) \lambda_a^T \lambda_b^T] \{ F_{\lambda\rho 12}(p, 0, -q) \\ & + c(2) F_{\lambda\rho 12}(q, 0, -p) \} \end{aligned} \quad (\text{B.1a})$$

where $c(2) = c_1 c_2$.

$$\begin{aligned} F_{\lambda\rho 12}(a, 0, b) = & \frac{1}{2} [f_{\lambda\rho 12}(a, 0, b) + (\lambda \leftrightarrow \rho)] \\ & + \frac{1}{4} [(a+b) F_{\rho\lambda 12}(a, 0, b) + (\lambda \leftrightarrow \rho)] \\ & - \frac{1}{2} g_{\lambda\rho} [D_{12}(0, b) + D_{12}(0, a)] \end{aligned} \quad (\text{B.1b})$$

$$D_{12}(0, a) = -\frac{\gamma}{8} a^2 \text{Tr}[\Gamma_1 \not{a} \Gamma_2 - \Gamma_1 \Gamma_2 \not{a}] \quad (\text{B.1c})$$

$$\begin{aligned} F_{123}(a, 0, b) = & \frac{\gamma}{8} \text{Tr}\{(a^2 + b^2 - a \cdot b) [\Gamma_1 \Gamma_2 \Gamma_3 - \frac{1}{2} (\Gamma_1 \gamma_\alpha \Gamma_2 \gamma^\alpha \Gamma_3 \\ & + \Gamma_1 \gamma_\alpha \Gamma_2 \Gamma_3 \gamma^\alpha + \Gamma_1 \Gamma_2 \gamma_\alpha \Gamma_3 \gamma^\alpha)]\} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} (a^\alpha a^\beta + b^\alpha b^\beta + a^\alpha b^\beta) [\{\Gamma_1, \gamma_\alpha\} \Gamma_2 \gamma_\beta \Gamma_3 + \Gamma_1 \{\Gamma_2, \gamma_\alpha\} \Gamma_3 \gamma_\beta \\
& + \Gamma_1 \gamma_\beta \Gamma_2 \{\Gamma_3, \gamma_\alpha\} + 6[\Gamma_1 \not\Gamma_2 \Gamma_3 \not\gamma + \Gamma_1 \not\Gamma_2 \{\Gamma_3, \gamma \cdot \Sigma_f\} \\
& + \Gamma_1 \{\Gamma_2, \gamma \cdot \Sigma_f\} \Gamma_3 \not\gamma] \}
\end{aligned} \tag{B.1d}$$

$$\text{with } \Sigma_f = -\frac{1}{3} (a+b) .$$

$$f_{\lambda\rho 12}(a,0,b) =$$

$$\begin{aligned}
& \frac{\gamma}{4} \text{Tr} \{ -\frac{1}{20} v_\rho \gamma_\lambda \Gamma_1 \Gamma_2 \\
& - (a+b) \rho \gamma_\lambda \not\Gamma_1 \Gamma_2 \not\gamma \\
& + \frac{1}{4} [2a_\rho a_\alpha + a_\rho b_\alpha + b_\rho a_\alpha + 2b_\rho b_\alpha - g_{\rho\alpha} (a^2 - a \cdot b + b^2)] \gamma_\lambda [\not\Gamma_1 \{\gamma_\alpha, \Gamma_2\} \\
& + \{\gamma_\alpha, \Gamma_1\} \Gamma_2] \\
& - \frac{1}{10} [3a_\rho a_\alpha a_\beta + a_\rho a_\alpha b_\beta + a_\rho b_\alpha a_\beta + b_\rho a_\alpha a_\beta + a_\rho b_\alpha b_\beta + b_\rho a_\alpha b_\beta \\
& + b_\rho b_\alpha a_\beta + 3b_\rho b_\alpha b_\beta] \\
& + \frac{1}{40} [g_{\alpha\beta} v_\rho + g_{\rho\alpha} v_\beta + g_{\rho\beta} v_\alpha] \}
\end{aligned}$$

with

$$v = 4a^2 a + 4b^2 b + 3a^2 b + 3b^2 a - 4a \cdot ba - 4a \cdot bb . \tag{B.1e}$$

4-pf

$$\begin{aligned}
 & \langle \theta_{\lambda\rho}(k) j_1^a(p) j_2^b(q) j_3^c(s) \rangle |_{1/m} \\
 &= \frac{-1}{16} \{ v^{abc} [H_{\lambda\rho 123}(p,0,-q,-q-s) + c(3)H_{\lambda\rho 321}(s,0,-q,-q-p)] \\
 &+ v^{abc} [H_{\lambda\rho 132}(p,0,-s,-s-q) + c(3)H_{\lambda\rho 231}(q,0,-s,-s-p)] \\
 &+ v^{bac} H_{\lambda\rho 213}(q,0,-p,-p-s) + c(3)H_{\lambda\rho 312}(s,0,-p,-p-q)] \} \quad (B.2a)
 \end{aligned}$$

where

$$c(3) = c_1 c_2 c_3$$

$$v^{abc} = \text{Tr}[\lambda_a \lambda_b \lambda_c + c(3)\lambda_a^T \lambda_b^T \lambda_c^T] \quad (B.2b)$$

$$\begin{aligned}
 & H_{\lambda\rho 123}(a,0,b,c) \\
 &= \frac{1}{2} [h_{\lambda\rho 123}(a,0,b,c) + (\lambda \leftrightarrow \rho)] \\
 &+ \frac{1}{4} [(a+c)\rho H_{\lambda 123}(a,0,b,c) + (\lambda \leftrightarrow \rho)] \\
 &- \frac{1}{2} g_{\lambda\rho} [F_{312}(c,0,b) + F_{312}(a,0,b)] \quad (B.2c)
 \end{aligned}$$

$$\begin{aligned}
 & H_{1234}(a,b,c,d) = -\frac{\gamma}{8} \text{Tr}[h_{1234}(a) + h_{2341}(b) + h_{3412}(c) \\
 &+ h_{4123}(d)] \quad (B.2d)
 \end{aligned}$$

$$h_{1234}(\rho) = \Gamma_1(\rho + \Sigma_h) \Gamma_2 [2\Gamma_3 \Gamma_4 - \gamma_\alpha \Gamma_3 \gamma^\alpha \Gamma_4 - \gamma_\alpha \Gamma_3 \Gamma_4 \gamma^\alpha - \Gamma_3 \gamma_\alpha \Gamma_4 \gamma^\alpha] \quad (B.2e)$$

$$\text{with } \Sigma_h = -\frac{1}{4} (a + b + c + d)$$

$$\text{Let } f_1(\rho) = -\frac{1}{4} (a + b + c) \rho$$

$$f_2(\rho, \alpha) = \frac{1}{20} (2a_\rho a_\alpha + 2b_\rho b_\alpha + 2c_\rho c_\alpha + a_\rho b_\alpha + b_\rho a_\alpha + a_\rho c_\alpha + c_\rho a_\alpha + b_\rho c_\alpha + c_\rho b_\alpha)$$

$$f_3 = -\frac{1}{20} (3a^2 + 3b^2 + 3c^2 - 2a \cdot b - 2a \cdot c - 2b \cdot c) \quad (B.2f)$$

then we have

$$h_{\lambda\rho 123}(a, 0, b, c)$$

$$= -\frac{\gamma}{4} \text{Tr}\{f_2(\rho, \alpha) \gamma_\lambda [\{\gamma_\alpha, \Gamma_1\} \Gamma_2 \Gamma_3 + \Gamma_1 \Gamma_2 \{\gamma_\alpha, \Gamma_3\}]\}$$

$$+ f_1(\rho) \gamma_\lambda (\not{\Gamma}_1 \Gamma_2 \Gamma_3 + \Gamma_1 \not{\Gamma}_2 \Gamma_3 + \Gamma_1 \Gamma_2 \not{\Gamma}_3)$$

$$+ \frac{1}{2} f_3 \gamma_\lambda [\{\gamma_\rho, \Gamma_1\} \Gamma_2 \Gamma_3 + \Gamma_1 \Gamma_2 \{\gamma_\rho, \Gamma_3\}]$$

$$- \frac{1}{2} \gamma_\lambda (\not{\Gamma}_1 \gamma_\rho \not{\Gamma}_2 \not{\Gamma}_3 + \not{\Gamma}_1 \Gamma_2 \not{\Gamma}_3 \gamma_\rho + \not{\Gamma}_1 \gamma_\rho \not{\Gamma}_2 \Gamma_3 + \not{\Gamma}_1 \Gamma_2 \gamma_\rho \not{\Gamma}_3)$$

$$+ \gamma_\rho \not{\Gamma}_1 \Gamma_2 \not{\Gamma}_3 + \Gamma_1 \gamma_\rho \not{\Gamma}_2 \not{\Gamma}_3)$$

$$- \frac{1}{2} h_{40}(f_1, \rho, \alpha, \beta) \gamma_\lambda (\not{\Gamma}_1 \gamma_\alpha \Gamma_2 \gamma_\beta \Gamma_3 + \not{\Gamma}_1 \gamma_\alpha \Gamma_2 \Gamma_3 \gamma_\beta + \not{\Gamma}_1 \Gamma_2 \gamma_\alpha \Gamma_3 \gamma_\beta)$$

$$\begin{aligned}
& + \gamma_\alpha \Gamma_1 \gamma_\beta \Gamma_2 \gamma_\gamma \Gamma_3 + \gamma_\alpha \Gamma_1 \Gamma_2 \gamma_\beta \Gamma_3 \gamma_\gamma + \Gamma_1 \gamma_\alpha \Gamma_2 \gamma_\beta \Gamma_3 \gamma_\gamma \\
& + \gamma_\alpha \Gamma_1 \gamma_\beta \Gamma_2 \Gamma_3 \gamma_\gamma + \gamma_\alpha \Gamma_1 \Gamma_2 \gamma_\beta \Gamma_3 \gamma_\gamma + \Gamma_1 \gamma_\alpha \Gamma_2 \gamma_\beta \Gamma_3 \gamma_\gamma) \\
& - \left[\frac{1}{4} h_{40}(\rho, \alpha, \beta, \delta) f_3 + \frac{1}{2} (g_{\rho\alpha} f_2(\beta, \delta) + 5 \text{ more}) \right] \gamma_\lambda \\
& (\gamma_\alpha \Gamma_1 \gamma_\beta \Gamma_2 \gamma_\delta \Gamma_3 + \gamma_\alpha \Gamma_1 \gamma_\beta \Gamma_2 \Gamma_3 \gamma_\delta + \gamma_\alpha \Gamma_1 \Gamma_2 \gamma_\beta \Gamma_3 \gamma_\delta + \Gamma_1 \gamma_\alpha \gamma_\beta \Gamma_3 \gamma_\delta) \} \quad (B.2g)
\end{aligned}$$

where

$$h_{40}(\rho, \alpha, \beta, \delta) = g_{\rho\alpha} g_{\beta\delta} + g_{\rho\beta} g_{\alpha\delta} + g_{\rho\delta} g_{\alpha\beta}$$

5-pf

$$\begin{aligned}
& \langle \theta_{\lambda\rho}(k) j_1^a(p) j_2^b(q) j_3^c(s) j_4^d(t) \rangle |_{1/m} \\
& = - \frac{i}{32} w^{abcd} [J_{\lambda\rho 1234}(p, 0, -q, -q-s, -q-s-t) \\
& + c(4) J_{\lambda\rho 4321}(t, 0, -s, -s-q, -s-q-p)]
\end{aligned}$$

+ 11 more pairs, giving totally 24 permutations of 1234, synchronized with abcd and p,q,s,t. (B.3a)

where

$$c(4) = c_1 c_2 c_3 c_4$$

$$w^{abcd} = \text{Tr}[\lambda_a \lambda_b \lambda_c \lambda_d + c(4) \lambda_a^T \lambda_b^T \lambda_c^T \lambda_d^T] \quad (B.3b)$$

$$\begin{aligned}
J_{\lambda\rho 1234}(a,0,b,c,d) &= \frac{1}{2} [j_{\lambda\rho 1234}(a,0,b,c,d) + (\lambda \leftrightarrow \rho)] \\
&+ \frac{1}{4} [(a+d) j_{\lambda 1234}(a,0,b,c,d) + (\lambda \leftrightarrow \rho)] \\
&- \frac{1}{2} g_{\lambda\rho} [H_{1234}(0,b,c,d) + H_{1234}(0,b,c,a)] \quad (B.3c)
\end{aligned}$$

$$\begin{aligned}
J_{12345}(a,b,c,d,e) &= \frac{\gamma}{16} \text{Tr}\{2\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5 - \frac{1}{2} (\Gamma_1\gamma_\alpha\Gamma_2\gamma^\alpha\Gamma_3\Gamma_4\Gamma_5 \\
&+ \Gamma_1\gamma_\alpha\Gamma_2\Gamma_3\gamma^\alpha\Gamma_4\Gamma_5 + \text{eight more perms.}) \\
&+ \frac{1}{4} h_{40}(\alpha,\beta,\delta,\epsilon) (\Gamma_1\gamma_\alpha\Gamma_2\gamma_\beta\Gamma_3\gamma_\delta\Gamma_4\gamma_\epsilon\Gamma_5 \\
&+ \Gamma_1\gamma_\alpha\Gamma_2\gamma_\beta\Gamma_3\gamma_\delta\Gamma_4\Gamma_5\gamma_\epsilon + \text{three more perms.}) \} \quad (B.3d)
\end{aligned}$$

$$\text{Let } W_\rho = -\frac{1}{5}(a+b+c+d)\rho.$$

$$j_{\lambda\rho 1234}^{a,0,b,c,d} =$$

$$\begin{aligned}
&\frac{\gamma}{8} \text{Tr}\{W_\rho\gamma_\lambda\Gamma_1\Gamma_2\Gamma_3\Gamma_4 \\
&- \frac{1}{4} \gamma_\lambda [(\Gamma_1\gamma_\rho\Gamma_2\Gamma_3\Gamma_4 + \Gamma_1\Gamma_2\gamma_\rho\Gamma_3\Gamma_4 + \Gamma_1\Gamma_2\Gamma_3\gamma_\rho\Gamma_4 + \Gamma_1\Gamma_2\Gamma_3\Gamma_4\gamma_\rho) \\
&+ \gamma_\rho\Gamma_1\Gamma_2\Gamma_3\Gamma_4 + \Gamma_1\gamma_\rho\Gamma_2\Gamma_3\Gamma_4 + \Gamma_1\Gamma_2\gamma_\rho\Gamma_3\Gamma_4 + \Gamma_1\Gamma_2\Gamma_3\gamma_\rho\Gamma_4 \\
&+ \gamma_\rho\Gamma_1\Gamma_2\Gamma_3\Gamma_4 + \Gamma_1\gamma_\rho\Gamma_2\Gamma_3\Gamma_4 + \Gamma_1\Gamma_2\gamma_\rho\Gamma_3\Gamma_4 + \Gamma_1\Gamma_2\Gamma_3\gamma_\rho\Gamma_4]
\end{aligned}$$

$$\begin{aligned}
& + (\gamma_\rho \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 + \Gamma_1 \gamma_\rho \Gamma_2 \Gamma_3 \Gamma_4 + \Gamma_1 \Gamma_2 \gamma_\rho \Gamma_3 \Gamma_4 + \Gamma_1 \Gamma_2 \Gamma_3 \gamma_\rho \Gamma_4) d] \\
& - \frac{1}{4} h_{40}(\rho, \alpha, \beta, w) \gamma_\lambda (\gamma_\alpha \Gamma_1 \gamma_\beta \Gamma_2 \Gamma_3 \Gamma_4 + 9 \text{ more perms.}) \\
& + \frac{1}{8} h_{40}(\rho, \alpha, \beta, \delta) \gamma_\lambda [\gamma_\alpha (\Gamma_1 \gamma_\beta \Gamma_2 \gamma_\delta \Gamma_3 \Gamma_4 + 3 \text{ more perms.}) \\
& + (\gamma_\alpha \Gamma_1 \gamma_\beta \Gamma_2 \gamma_\delta \Gamma_3 \Gamma_4 + 3 \text{ more perms.}) \\
& + (\gamma_\alpha \Gamma_1 \gamma_\beta \Gamma_2 \gamma_\delta \Gamma_3 \Gamma_4 + 3 \text{ more perms.}) \\
& + (\gamma_\alpha \Gamma_1 \gamma_\beta \Gamma_2 \gamma_\delta \Gamma_3 \Gamma_4 + 3 \text{ more perms.}) d] \\
& + \frac{1}{8} h_{60}(\rho, \alpha, \beta, \delta, \epsilon, w) \gamma_\lambda (\gamma_\alpha \Gamma_1 \gamma_\beta \Gamma_2 \gamma_\delta \Gamma_3 \gamma_\epsilon \Gamma_4 + 4 \text{ more perms.}) \}
\end{aligned}$$

where

$$\begin{aligned}
h_{60}(\rho, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) & \equiv \\
& g_{\rho \alpha_1} h_{40}(\alpha_2, \alpha_3, \alpha_4, \alpha_5) + \dots + g_{\rho \alpha_5} h_{40}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)
\end{aligned}$$

6-pf

$$\langle \theta_{\lambda \rho}(k) j_1^a(p) j_2^b(q) j_3^c(s) j_4^d(t) j_5^e(u) \rangle |_{1/m}$$

$$= \frac{1}{64} z^{abcde} [k_{\lambda \rho 12345} + c(5) k_{\lambda \rho 54321}]$$

+ 59 more pairs giving totally 120 permutation of 12345,
synchronized with abcde,

(B.4a)

where

$$c(5) = c_1 c_2 c_3 c_4 c_5$$

(B.4b)

$$z^{abcde} = \text{Tr}\{\lambda_a \lambda_b \lambda_c \lambda_d \lambda_e + c(5) \lambda_a^T \lambda_b^T \lambda_c^T \lambda_d^T \lambda_e^T\}$$

Note that the 6-pf $1/m$ term is dimensionless.

$$K_{\lambda\rho 12345}$$

$$= \frac{1}{2} [k_{\lambda\rho 12345} + (\lambda \leftrightarrow \rho)] - g_{\lambda\rho} [j_{12345}] \quad (\text{B.4c})$$

$$k_{\lambda\rho 12345}$$

$$= J_{\lambda\rho 12345}$$

$$= \frac{\gamma}{80} \text{Tr}\{\gamma_\lambda (\gamma_\rho \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5 + \Gamma_1 \gamma_\rho \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5 + 4 \text{ more perms.})$$

$$- \frac{1}{4} h_{40}(\rho, \alpha, \beta, \delta) \gamma_\lambda (\gamma_\alpha \Gamma_1 \gamma_\beta \Gamma_2 \gamma_\delta \Gamma_3 \Gamma_4 \Gamma_5 + 19 \text{ more perms.})$$

$$+ \frac{1}{8} h_{60}(\rho, \alpha, \beta, \delta, \epsilon, \zeta) \gamma_\lambda (\gamma_\alpha \Gamma_1 \gamma_\beta \Gamma_2 \gamma_\delta \Gamma_3 \gamma_\epsilon \Gamma_4 \gamma_\zeta \Gamma_5 + 5 \text{ more perms.}) \}$$

(B.4d)

Alternative solution for the 3-, 4-, and 5-pf vertex loops are available in terms of $J_{\lambda\rho 12345}$ given in Eq. (B.4d). They are

$$f_{\lambda\rho 12}(a, 0, b)$$

$$= - [a^\alpha a^\beta a^\delta J_{\lambda\rho\alpha\beta\delta 12} + a^\alpha a^\beta b^\delta J_{\lambda\rho\alpha\beta 12\delta}$$

$$+ a^\alpha b^\beta b^\delta J_{\lambda\rho\alpha 12\beta\delta} + b^\alpha b^\beta b^\delta J_{\lambda\rho 12\alpha\beta\delta}]$$

(B.5a)

$$h_{\lambda\rho 123}(a,0,b,c)$$

$$= a^\alpha a^\beta J_{\lambda\rho\alpha\beta 123} + b^\alpha b^\beta J_{\lambda\rho 12\alpha\beta 3} + c^\alpha c^\beta J_{\lambda\rho 123\alpha\beta} \\ + a^\alpha b^\beta J_{\lambda\rho\alpha 12\beta 3} + a^\alpha c^\beta J_{\lambda\rho 123\beta} + b^\alpha c^\beta J_{\lambda\rho 12\alpha 3\beta} \quad (B.5b)$$

$$j_{\lambda\rho 1234}(a,0,b,c,d)$$

$$= - [a^\alpha J_{\lambda\rho\alpha 1234} + b^\alpha J_{\lambda\rho 12\alpha 34} + c^\alpha J_{\lambda\rho 123\alpha 4} \\ + d^\alpha J_{\lambda\rho 1234\alpha}] \quad (B.5c)$$

Although the forms of (B.5) are conceptually simpler, they may require far more trace calculation than the previous forms.

APPENDIX C. COUNTERTERMS²⁷

Here we list the counterterms for the n-pf's, including both those from Ref. 9 and the new ones involving $\theta_{\lambda\rho}$, which we indicate by θ . For the Lorentz index and momentum assignments, see the discussion of Appendix A. Let $\gamma = 1/24\pi^2 i$. The normal parity set is

(1) $n = 1$

$$\delta(S) = -6^{\frac{1}{2}} i a_1 \delta^{ao} m^3, \quad (C.1a)$$

$$\delta(\theta) = -\frac{3}{2} i a_1 m^4 g_{\lambda\rho}; \quad (C.1b)$$

(2) $n = 2$

$$\delta(VV) = a_2 \delta^{ab} (p^2 g_{\mu\nu} - p_\mu p_\nu),$$

$$\delta(AA) = \delta(VV) + \delta^{ab} (\gamma p^2 g_{\mu\nu} + 4a_3 m^2 g_{\mu\nu}),$$

$$\delta(AP) = -2i a_3 \delta^{ab} m q_\mu, \quad (C.2a)$$

$$\delta(PP) = \delta^{ab} (a_3 p^2 + a_1 m^2),$$

$$\delta(SS) = \delta^{ab} [(a_3 - \gamma) p^2 + 3a_1 m^2],$$

$$\delta(OS) = \sqrt{2/3} \delta^{0a} m (a_3 - \gamma) (k^2 g_{\lambda\rho} - k_\lambda k_\rho); \quad (C.2b)$$

(3) $n = 3$

$$\delta(VVV) = a_2 f^{abc} [g_{\mu\nu} (p - q)_\lambda + g_{\mu\lambda} (s - p)_\nu + g_{\nu\lambda} (q - s)_\mu],$$

$$\delta(V\Lambda\Lambda) = \delta(VVV) + \gamma f^{abc} [3g_{\mu\nu} p_\lambda - 3g_{\mu\lambda} p_\nu + g_{\nu\lambda} (q-s)_\mu] ,$$

$$\delta(VAP) = -2ia_3 f^{abc} g_{\mu\nu} ,$$

$$\delta(AAS) = i(4a_3 - 6\gamma) d^{abc} g_{\mu\nu} ,$$

$$\delta(VPP) = a_3 f^{abc} (q-s)_\mu ,$$

$$\delta(VSS) = (a_3 - \gamma) f^{abc} (q-s)_\mu , \quad (C.3a)$$

$$\delta(\Lambda PS) = d^{abc} [a_3 (s-q)_\mu + 3\gamma q_\mu] ,$$

$$\delta(PPS) = ia_1 d^{abc} ,$$

$$\delta(SSS) = 3i(a_1 + 2\gamma) d^{abc} ;$$

$$\delta(\theta VV) = ia_2 \delta^{ab} \{ g_{\lambda\rho} (p_\mu p_\nu + q_\mu q_\nu + q_\mu p_\nu) + g_{\mu\nu} (p_\lambda p_\rho + q_\lambda q_\rho) \}$$

$$- g_{\mu\nu} g_{\lambda\rho} [p^2 + q^2 + p \cdot q] + \frac{1}{2} (y_{\lambda\mu} y_{\rho\nu} + g_{\lambda\nu} g_{\rho\mu}) k^2$$

$$- (g_{\lambda\mu} q_\rho p_\nu + g_{\rho\mu} q_\lambda p_\nu + g_{\lambda\nu} p_\rho q_\mu + g_{\rho\nu} p_\lambda q_\mu)$$

$$- \frac{1}{2} (g_{\lambda\mu} q_\rho q_\nu + g_{\rho\mu} q_\lambda q_\nu + g_{\lambda\nu} p_\rho p_\mu + g_{\rho\nu} p_\lambda p_\mu) \} \quad (C.3b)$$

$$\delta(\theta AA) = \delta(\theta VV) + \delta^{ab} \{ -i\gamma g_{\mu\nu} (p_\lambda p_\rho + q_\lambda q_\rho) + g_{\lambda\rho} g_{\mu\nu} (-4ia_3 m^2 + i\gamma p \cdot q) \}$$

$$+ i \frac{\gamma}{2} [-g_{\lambda\mu} (p-q)_\rho k_\nu - g_{\rho\mu} (p-q)_\lambda k_\nu \\ + g_{\lambda\nu} (p-q)_\rho k_\mu + g_{\rho\nu} (p-q)_\lambda k_\mu] \}$$

$$\delta(\theta AP) = \delta^{ab} a_3^m (-2g_{\lambda\rho} p_\mu + g_{\lambda\mu} q_\rho + g_{\rho\mu} q_\lambda)$$

$$\delta(\theta PP) = i\delta^{ab} [-a_1^m g_{\lambda\rho} - \frac{1}{3} (2a_3 + \gamma) (k_\lambda^2 g_{\lambda\rho} - k_\lambda k_\rho) \\ + a_3 (p \cdot q g_{\lambda\rho} - p_\lambda p_\rho - q_\lambda q_\rho)]$$

$$\delta(\theta SS) = i\delta^{ab} [-3a_1^m g_{\lambda\rho} - \frac{1}{3} (2a_3 + \gamma) (k_\lambda^2 g_{\lambda\rho} - k_\lambda k_\rho) \\ + (a_3 - \gamma) (p \cdot q g_{\lambda\rho} - p_\lambda p_\rho - q_\lambda q_\rho)]$$

$$(4) \quad n = 4$$

$$\delta(VVVV) = \frac{1}{8} a_2 [g_{\mu\nu} g_{\lambda\sigma} (-w_1 - w_2 + 2w_3) + g_{\mu\lambda} g_{\nu\sigma} (2w_1 - w_2 - w_3) \\ + g_{\mu\sigma} g_{\nu\lambda} (-w_1 + 2w_2 - w_3)] ,$$

$$\delta(AAAA) = \delta(VVVV) + \frac{1}{2} \gamma [g_{\mu\nu} g_{\lambda\sigma} (-w_1 - w_2 + w_3) \\ + g_{\mu\lambda} g_{\nu\sigma} (w_1 - w_2 - w_3) + g_{\mu\sigma} g_{\nu\lambda} (-w_1 + w_2 - w_3)] , \quad (C.4a)$$

$$\delta(AAVV) = \delta(VVVV) + \frac{1}{2} \gamma [g_{\mu\nu} g_{\lambda\sigma} (-w_1 - w_2 + 2w_3)$$

$$+ g_{\mu\lambda} g_{\nu\sigma} (3w_1 - 3w_2) + g_{\mu\sigma} g_{\nu\lambda} (-3w_1 + 3w_2)] ,$$

$$\delta(VVPP) = -\frac{1}{8} a_3 g_{\mu\nu} (w_1 + w_2 - 2w_3) ,$$

$$\delta(VVSS) = -\frac{1}{8} (a_3 - \gamma) g_{\mu\nu} (w_1 + w_2 - 2w_3)$$

$$\delta(AAPP) = -\frac{1}{8} a_3 g_{\mu\nu} (w_1 + w_2 + 2w_3) + \frac{3}{8} \gamma g_{\mu\nu} (w_1 + w_2)$$

$$\delta(AASS) = -\frac{1}{8} a_3 g_{\mu\nu} (w_1 + w_2 + 2w_3) + \frac{3}{4} \gamma g_{\mu\nu} (w_1 + w_2 + w_3)$$

$$\delta(\Lambda VSP) = -\frac{1}{8} i g_{\mu\nu} [a_3 (\bar{w}_1 - \bar{w}_2 - 2\bar{w}_3) + 3\gamma (\bar{w}_2 + \bar{w}_3)] ,$$

$$\delta(PPPP) = -\frac{1}{8} a_1 (w_1 + w_2 + w_3) ,$$

$$\delta(SSPP) = -\frac{1}{8} a_1 (w_1 + w_2 - w_3) - \frac{3}{8} \gamma (w_1 + w_2) ,$$

$$\delta(SSSS) = -\frac{1}{8} (a_1 + 8\gamma) (w_1 + w_2 + w_3) ,$$

$$\delta(\theta VVV) = i f^{abc} a_2 \{ -2g_{\lambda\rho} [g_{\mu\nu} (p-q)_\sigma + g_{\nu\sigma} (q-t)_\mu + g_{\mu\sigma} (t-p)_\nu]$$

$$+ \frac{1}{2} [g_{\mu\nu} (g_{\lambda\sigma} (q-p)_\rho + g_{\rho\sigma} (g-p)_\lambda)$$

$$+ g_{\mu\sigma} (g_{\lambda\nu} (p-t)_\rho + g_{\rho\nu} (p-t)_\lambda)$$

$$+ g_{\nu\sigma} (g_{\lambda\mu} (t-q)_\rho + g_{\mu\rho} (t-q)_\lambda)] \} ,$$

(C.4b)

$$\begin{aligned}
\delta(\theta VAA) = & \delta(\theta VVV) + if^{abc} \gamma \{ g_{\lambda\rho} [3g_{\mu\nu} (q+s-p)_\sigma \\
& - 3g_{\mu\sigma} (q+s-p)_\nu + 2g_{\nu\sigma} (s-q)_\mu] \\
& + \frac{1}{2} [(g_{\lambda\mu} g_{\rho\nu} + g_{\rho\mu} g_{\lambda\nu}) k_\sigma - (g_{\lambda\sigma} g_{\rho\mu} + g_{\rho\sigma} g_{\lambda\mu}) k_\nu] \\
& - \frac{1}{2} g_{\nu\sigma} [g_{\rho\mu} (q-s)_\lambda + g_{\lambda\mu} (q-s)_\rho] \\
& + \frac{3}{2} [g_{\mu\sigma} (p_\lambda g_{\rho\nu} + p_\rho g_{\lambda\nu}) - g_{\mu\nu} (p_\lambda g_{\rho\sigma} + p_\rho g_{\lambda\sigma})] \}
\end{aligned}$$

$$\delta(\theta VAP) = -4a_3 f^{abc} g_{\lambda\rho} g_{\mu\nu}$$

$$\delta(\theta AAS) = 4(2a_3 - 3\gamma) d^{abc} g_{\lambda\rho} g_{\mu\nu}$$

$$\delta(\theta VSS) = -i(a_3 - \gamma) f^{abc} \{ 2g_{\lambda\rho} (q-s)_\mu + \frac{1}{2} [g_{\lambda\mu} (q-s)_\rho + g_{\rho\mu} (q-s)_\lambda] \}$$

$$\delta(\theta VPP) = -ia_3 f^{abc} \{ 2g_{\lambda\rho} (q-s)_\mu + \frac{1}{2} [g_{\lambda\mu} (q-s)_\rho + g_{\rho\mu} (q-s)_\lambda] \}$$

$$\delta(\theta ASP) = d^{abc} \{ g_{\lambda\rho} [3i\gamma p_\mu - i(2a_3 - 3\gamma)(q-s)_\mu]$$

$$- \frac{1}{2} a_3 [g_{\lambda\mu} q_\rho + g_{\rho\mu} q_\lambda] + \frac{i}{2} (a_3 - 3\gamma) (g_{\lambda\mu} s_\rho + g_{\rho\mu} s_\lambda) \}$$

$$\delta(\theta SSS) = 6(a_1 + 2\gamma) d^{abc} g_{\lambda\rho}$$

$$\delta(\theta SPP) = 2a_1 d^{abc} g_{\lambda\rho}$$

(5) $n = 5$

$$\delta(\theta VVVV) = \frac{3}{8} i a_2 g_{\lambda\rho} [g_{\mu\nu} g_{\sigma\tau} (w_1 + w_2 + 2w_3) \\ + g_{\mu\sigma} g_{\nu\tau} (2w_1 + w_2 + w_3) + g_{\mu\tau} g_{\nu\sigma} (w_1 + 2w_2 + w_3)] ,$$

$$\delta(\theta VVAA) = \delta(\theta VVVV) + \frac{3}{8} i \gamma g_{\lambda\rho} [g_{\mu\nu} g_{\sigma\tau} (w_1 + w_2 - 2w_3) \\ + g_{\mu\sigma} g_{\nu\tau} (-3w_1 + 3w_2) + g_{\mu\tau} g_{\nu\sigma} (3w_1 - 3w_2)] ,$$

$$\delta(\theta AAAA) = \delta(\theta VVVV) + \frac{3}{2} i \gamma g_{\lambda\rho} [g_{\mu\nu} g_{\sigma\tau} (w_1 + w_2 - w_3) \\ + g_{\mu\sigma} g_{\nu\tau} (-w_1 + w_2 + w_3) + g_{\mu\tau} g_{\nu\sigma} (w_1 - w_2 + w_3)] ,$$

$$\delta(\theta VVSS) = \frac{3}{8} i (a_3 - \gamma) g_{\lambda\rho} g_{\mu\nu} (w_1 + w_2 - 2w_3) ,$$

$$\delta(\theta VVPP) = \frac{3}{8} i a_3 g_{\lambda\rho} g_{\mu\nu} (w_1 + w_2 - 2w_3) ,$$

$$\delta(\theta V\Lambda SP) = \frac{3}{8} g_{\lambda\rho} g_{\mu\sigma} [-\bar{w}_1 (a_3 - 3\gamma) + \bar{w}_2 a_3 - \bar{w}_3 (2a_3 - 3\gamma)] ,$$

$$\delta(\theta AAASS) = \frac{3}{8} i g_{\lambda\rho} g_{\mu\nu} [(a_3 - 6\gamma) (w_1 + w_2) + 2(a_3 - 3\gamma) w_3] ,$$

$$\delta(\theta AAAPP) = \frac{3}{8} i g_{\lambda\rho} g_{\mu\nu} [(a_3 - 3\gamma) (w_1 + w_2) + 2a_3 w_3] ,$$

$$\delta(\theta SSSS) = \frac{3}{8} i (a_1 + 8\gamma) g_{\lambda\rho} (w_1 + w_2 + w_3) , \quad (C.5)$$

$$\delta(\theta SSPP) = \frac{3}{8} i g_{\lambda\rho} [(w_1 + w_2)(a_1 + 3\gamma) - w_3 a_1]$$

$$\delta(\theta PPPP) = \frac{3}{8} i a_1 g_{\lambda\rho} (w_1 + w_2 + w_3).$$