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WITHOUT VISCOSITY

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An MHD Theory of Divertors Without Viscosity

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ABSTRACT

A simple model of a divertor is studied with the classical two-fluid MHD equations. The ion temperature is assumed zero, which has the simplifying effect of giving zero ion viscosity. The electrons are shown to leave the divertor much closer to the main body of the plasma than the ions. The divertor width is found comparable to the ion gyroradius calculated with the electron temperature, which is substantially wider than some classical estimates.

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I. INTRODUCTION

The importance of divertors for controlling impurities in high temperature plasma confinement systems has been recognized over many generations of experiments.^{1,2} However, very little theory exists on plasma flow in a divertor so discussions of divertors³ are generally based on the theory of planar discharges. Since the distinction between a planar discharge and a divertor is important for the physics, we will define the terms.

A planar discharge is uniform except in one dimension. Continuously heated electrons interact with a background neutral gas to create plasma at a steady rate. The plasma flows to neutralizing walls which are two parallel infinite planes. The theory has concentrated on the electrostatic sheath which occurs to equalize the electron and ion currents to the walls.^{4,5,6} With the usual assumption that the electron temperature, T_e , is large compared to the ion temperature, there are two primary results. The ions enter the sheath at a velocity $C_s = (T_e/m_+)^{1/2}$ where m_+ is the mass associated with a positive charge. The jump in the electrostatic potential across the sheath is approximately $3.5 T_e/e$.

A divertor, unlike a planar discharge, is intrinsically two dimensional. Plasma diffuses from the main plasma body onto open field lines and then flows along the open lines to a neutralizing wall (see Fig. 1). The theory of divertors as developed in this paper has several major points.

1. The plasma in the divertor forces the electric field to be constant over distances comparable to $\rho_+(T_e)$, the ion gyro-radius calculated with the electron temperature. Consequently, the requirement that $\phi \approx 3.5 T_e/e$ for equal electron and ion currents cannot be satisfied on each field line. If we assume the main plasma body is stationary, the electron outflow occurs on a scale of $\rho_e L/\lambda_e$ and is near the main plasma. Here, ρ_e is the electron gyroradius, L is the divertor length, and λ_e is the electron mean free path. We have assumed $(m_+/m_e)^{1/2} \gg L/\lambda_e \gg 1$.

2. The ion outflow occurs on a scale of $\rho_+(T_e)$, which is also the divertor width. This is substantially wider than the simple classical estimate $(D_\perp L/C_s)^{1/2} \approx (\rho_+ \rho_e L/\lambda_e)^{1/2}$.

3. The ion flow velocity into the electrostatic sheath near the neutralizing walls is $C_s = (T_e/m_+)^{1/2}$. The density drops by a factor of $1/2^{1/2}$ along the field lines from the center of the divertor to the electrostatic sheath.

4. In the portions of the divertor where there is little electron flow to the walls the pressure and density are related by $\partial \ln(p)/\partial x = 3 \partial \ln(n)/\partial x$ where the x direction points across the divertor.

Even though the electron and ion currents to the walls are concentrated in different regions of the divertor, the integrated flux is the same for the two species. In particular, quasi-neutrality is assumed to hold throughout the divertor except for the Debye length scale electrostatic sheath near the neutralizing walls.

The divertor calculations in this paper are carried out with the two fluid MHD equations. These equations have not formed the basis for planar discharge theory. The standard methods for calculating the flow in a planar discharge have been based on the Bohm stability criterion^{5,7} or the assumption of collisionless zero temperature ions and collisional finite temperature electrons.^{4,6} The equivalence of the MHD theory of a planar discharge and a generalized Bohm stability criterion has been demonstrated.⁸

Since divertor theory is intrinsically complicated, we will consider the very simple divertor depicted in Fig. 1. This model has no magnetic field in the symmetry direction, which would be required for a direct comparison with tokamak type plasmas. The plasma is also assumed to be at low β . The ion temperature is considered sufficiently small to be assumed zero. The most severe approximation involved in the zero ion temperature model is the neglect of ion viscosity. The effect of viscosity is to accentuate the differences between divertor and planar discharge theories. Since the purpose of this paper is to illustrate, as simply as possible, the importances of the differences, the zero viscosity theory is justified by its relative tractability.

In Section II, the two-fluid equations needed for describing the divertor will be given. Section III is a calculation of the fluid flow. The equation for the electrostatic potential is developed in Section IV. In Section V, the transport is

evaluated and the widths of the regions of electron and ion flow to the walls are calculated. The results are summarized in Section VI and the approximations are discussed.

II. THE EQUATIONS

One interesting feature of the divertor problem is that a very complete set of two-fluid equations must be used for a solution to exist. In particular, the force the electrons and ions exert on each other must include the thermal force and not just the drag force, $n\vec{j}$, as is customary in plasma physics. A rather general set of equations have been given by Braginskii,⁹ which have been used in this paper. There are three basic equations, the kinetic or Navier-Stokes, the continuity, and the energy balance equations.

Since there are two species, the kinetic equation has six components. Rather than use equations for the two types of particles, it is more convenient to consider the normal MHD equation--the sum of the ion and electron equations--plus one of the particle equations. The magnetic field is assumed to be in the $\hat{z}$ direction, which is also the direction of the neutralizing walls. The symmetry direction is $\hat{y}$, and $\hat{x}$ points across the divertor (see Fig. 1). The velocity of the ions and electrons is $\vec{v}$ and $\vec{u}$ respectively; and the current is determined by their difference $\vec{j} = en(\vec{v} - \vec{u})$. The components of the kinetic equations are

$\hat{x}$ component

$$(Ions) \quad en(E_x + v_y B/c) = 0 \quad (1)$$

$$(MHD) \quad \partial P / \partial x - j_y B/c = 0 \quad (2)$$

$\hat{y}$ component

$$(MHD) \quad m_+ n \vec{v} \cdot \vec{\nabla} v_y + j_x B/c = 0 \quad (3)$$

$$(Electrons) \quad - enu_x B/c = R_y \quad (4)$$

$\hat{z}$ component

$$(MHD) \quad \partial(m_+ n v_x v_z) / \partial x + \partial(m_+ n v_z^2 + P) / \partial z = 0 \quad (5)$$

$$(Electrons) \quad \partial P / \partial z + enE_z = 0 \quad (6)$$

The vector $\vec{R}$ is the electron-ion drag and thermal force. It is negligible in the $\hat{x}$ and $\hat{z}$ directions provided the mean free path of the electrons λ_e is greater than $(m_e/m_+)^{1/2}$ times the divertor length L . In the $\hat{y}$ direction, this force is

$$R_y = ne[j_y - \frac{3}{2} (nc/B) \partial T_e / \partial x] / \sigma_{\perp} \quad (7)$$

with $\sigma_{\perp}$ being the orthogonal conductivity, $\sigma_{\perp} = e^2 n \tau_e / m_e$. The term $\vec{v} \cdot \vec{\nabla} v_x$ was neglected since ion conservation implies $v_x \sim (\delta/L) v_z$ when δ is the divertor width. The velocity v_z is of order $C_s = (T_e/m_+)^{1/2}$. If we assume the electrostatic

potential ϕ is of order T_e/e , Eq. (1) implies $v_y \sim (\rho_+/\delta)C_s$. Since we wish to describe divertors with $\delta \sim \rho_+$ the term $\vec{v} \cdot \vec{\nabla} v_y$ was retained.

To insure particle conservation, there are two continuity equations

$$\text{(Ions)} \quad \partial(nv_x)/\partial x + \partial(nv_z)/\partial z = 0 \quad (8)$$

$$\text{(Electrons)} \quad \partial(nu_x)/\partial x + \partial(nu_z)/\partial z = 0 \quad (9)$$

The electron flux in the $\hat{z}$ direction is controlled by the electrostatic sheath which forms within the last few Debye lengths of the walls at $z = \pm L$. Assuming Maxwellian electrons ($\lambda_e \ll L$), the flux to the wall, $(nu_z)_L$, is⁶

$$(nu_z)_L = n_L \left(\frac{T_e}{2\pi m_e} \right)^{1/2} \exp [-e\phi_L/T_e] \quad (10)$$

The electrostatic potential is zero on the wall and ϕ_L on the plasma side of the electrostatic sheath. The density is n_L on the plasma edge of the sheath.

Since we have assumed zero ion temperature, only the energy balance equation for the electrons is required.

$$\vec{v} \cdot \left[\frac{5}{2} n T_e \vec{u} + \vec{q} \right] = -en\vec{E} \cdot \vec{u} + \vec{R} \cdot \vec{v} - q_{eq} \quad (11)$$

The energy flux due to random motion is $\vec{q}$. We will only require an explicit expression for q_x . Letting $\kappa_{\perp}$ be the orthogonal thermal conductivity, $\kappa_{\perp} = 4.66 n T_e / (m_e \omega_{ce}^2 \tau_e)$

$$q_x = - \kappa_{\perp} \partial T / \partial x + \frac{3}{2} \frac{T_e}{e \omega_{ce} \tau_e} j_y . \quad (12)$$

For $\lambda_e \gg (m_e/m_+)^{1/2} L$, the parallel conductivity is so high that $\partial T_e / \partial z = 0$ can be assumed. The term q_{eq} is the energy loss of the electrons to the ions through thermal equilibration

$$q_{eq} = 3 \frac{m_e}{m_i} \frac{n}{\tau_e} T_e . \quad (13)$$

Finally, an expression is required for the electron energy flux from the plasma into the electrostatic sheath⁷.

$$\left(\frac{5}{2} n T_e u_z + q_z \right)_L = (2 T_e + e \phi)_L (n u_z)_L . \quad (14)$$

III. THE FLOW

The fluid mechanics problem presented by a divertor closely resembles that of a nozzle.¹⁰ The flow velocity in a nozzle is limited by the sound speed provided the flux monotonically increases down stream. The sound speed is reached at the end of the nozzle provided the flow is into a vacuum or equivalently a perfectly absorbing wall. These results will apply to our model of a divertor.

The flow of the plasma out of the divertor is controlled by the $\hat{z}$ component of the MHD Eq. (5) and the ion continuity Eq. (8). The transport Eqs. (4) and (11) basically give the dependence of the density and temperature on x . The z dependence of n as well as $v_x(x,z)$ and $v_y(x,z)$ are determined by the flow. The boundary condition at $x = 0$ provides one other equation. Namely, the flux into the divertor, $(nv_x)_{x=0}$, is a given function of z . It is assumed this inward flux is controlled by the main body of the plasma--the divertor must take whatever flux is given it.

The equations which determine the divertor flow are non-linear, coupled, partial differential equations. Remarkably, there is an exact solution which is of interest. To obtain the exact solution, the transport equation results must be of the form

$$n(x, z) = n_1(z) \exp(-x/\delta) \quad (15)$$

$$P(x, z) = P_1(z) \exp(-\gamma x/\delta) \quad (16)$$

with γ and δ constants. These are not the exact forms that will be derived in the section on transport. However, they are quite reasonable physically and will be assumed.

To obtain the exact solution, we try

$$\Gamma(x, z) = nv_z = \Gamma_1(z) \exp\left[-\left(\frac{1+\gamma}{2}\right) x/\delta\right] \quad (17)$$

One finds the x dependence of both the continuity equation and the $\hat{z}$ component of the MHD equation cancels, giving

$$d\Gamma_1/dz = [(1 + \gamma)/(2\delta)] (nv_x)_1 \quad (18)$$

where $(nv_x)_1 = (nv_x)_{x=0}$ which is a known function of z . One also finds that

$$nv_x = (nv_x)_1 \exp\left[-\left(\frac{1+\gamma}{2}\right) x/\delta\right] \quad (19)$$

$$d(m_+ \Gamma_1^2/n_1 + P_1)/dz = [(2\gamma)/(\gamma + 1)] m_+ \frac{\Gamma_1}{n_1} \frac{d\Gamma_1}{dz} \quad (20)$$

Equations (18) and (20) are the ordinary differential equations which must be solved. First, consider Eq. (20). If Γ_1 is considered the independent variable, it can be written as

$$d(m_+ \Gamma_1^2 / n_1 + P_1) / d\Gamma_1 = \frac{2\gamma}{\gamma+1} m_+ \Gamma_1 / n_1 . \quad (21)$$

Defining $C_s = [T_e(x=0)/m_+]^{1/2}$ so C_s is a constant; $M(z) = v_z(x=0)/C_s$, the Mach number; and noting that $n_1 = \Gamma_1 / (MC_s)$, Eq. (21) reduces to

$$\frac{1}{\Gamma_1} \frac{d\Gamma_1}{dM} = \frac{1}{M} \frac{(1 - M^2)}{1 - \frac{\gamma-1}{\gamma+1} M^2} . \quad (22)$$

Equation (22) implies that Γ_1 has its maximum at $M = 1$. As is well known in nozzle theory,¹⁰ if the flow is into a perfectly absorbing wall, the maximum of Γ_1 occurs at the wall. The flow becomes sonic, $M = 1$, at the sheath near the neutralizing wall, $z = L$. If the density at the point $x = 0$, $z = 0$ is denoted by n_c , the solution to Eq. (22) is

$$\Gamma_1 = n_c C_s M \left(1 - \frac{\gamma-1}{\gamma+1} M^2 \right)^{\frac{1}{\gamma-1}} . \quad (23)$$

The density is

$$n_1 = n_c \left(1 - \frac{\gamma-1}{\gamma+1} M^2 \right)^{\frac{1}{\gamma-1}} . \quad (24)$$

The greatest interest is, of course, attached to finding the values of Γ_1 and n_1 at the neutralizing walls, Γ_L and n_L , that is at $M = 1$.

$$\Gamma_L / (n_C C_S) = n_L / n_C = \left(\frac{2}{\gamma+1} \right)^{\frac{1}{\gamma-1}} . \quad (25)$$

If we assume the temperature and density both drop across the sheath so $1 < \gamma < \infty$, one finds that n_L/n_C lies in narrow limits. For $\gamma = 1$, $n_L/n_C = 1/e^{1/2}$ or $\ln(n_C/n_L) = 1/2$. For γ approaching infinity, $n_L/n_C = 1$. The case of most interest, $\gamma = 3$, gives $n_L/n_C = 1/2^{1/2}$ or $\ln(n_C/n_L) = 0.3466$.

To obtain the z dependence of Γ_1 and hence M and n_1 , Eq. (18) must be used. It gives

$$\Gamma_1(z) = \frac{1+\gamma}{2\delta} \int_0^z (nv_x)_{x=0} dz . \quad (26)$$

Remember $(nv_x)_{x=0}$ is a specified boundary condition, the flux into the divertor being determined by the body of the plasma.

The density at $x = 0$, $z = 0$, which is n_C , can be found using Eqs. (25) and (26)

$$n_C = \left(\frac{1+\gamma}{2} \right)^{\frac{\gamma}{\gamma-1}} \frac{Q_+(0)}{C_S \delta} \quad (27)$$

$Q_+(0)$ is the total flux of positive charge entering the divertor between $z = 0$ and L . The quantity $\left(\frac{1+\gamma}{2}\right)^{\frac{\gamma}{\gamma-1}}$ is $e^{1/2}$ for $\gamma = 1$, $2\sqrt{2}$ for $\gamma = 3$, and infinite for infinite γ .

One final note, the Mach number M was defined for $x_1 = 0$. However, the form of $\Gamma(x,z)$, $n(x,z)$, and $C_s(x)$ are such that the Mach number at a given value of z is independent of x .

III. THE ELECTROSTATIC POTENTIAL

The flow of electrons to the wall from the divertor is exponentially dependent on the parameter $\eta = e\phi/T_e$. Consequently, the electrostatic potential is one of the most important quantities in the theory. If the electrons and ions are to leave the divertor together, η must be a constant which has a value of about 3. That is $\phi \propto T_e$. Even a modest variation in η across the divertor means the bulk of the electrons will leave in a narrow region about the minimum of η . In this section, we will show that unless $\delta \gg \rho_+(T_e)$, with δ the divertor width, the differential equation for the electrostatic potential is insensitive to the temperature distribution or the relative distribution of ion and electron currents to the wall.

The kinetic equations give expressions for $E_x = -\partial\phi/\partial x$ and $E_z = -\partial\phi/\partial z$. The equation for E_x is derived from Eqs. (1) and (3)

$$m_+ n \vec{v} \cdot \vec{\nabla} E_x = j_x B^2/c^2 \quad (28)$$

Equation (6) coupled with $\partial T_e/\partial z = 0$ implies

$$\phi = \phi_0(x) + (T_e/e) \ln(n/n_0) \quad (29)$$

where $n_0 = n(x, z=0)$. Substituting this result for ϕ in the equation for E_x and letting

$$\varepsilon = - \partial [(T_e/e) \ln(n/n_0)] / \partial x \quad (30)$$

gives

$$nv_x d^2 \phi_0 / dx^2 + j_x B^2 / m_+ c^2 - nv_x \partial \varepsilon / \partial x \quad (31)$$

$$- nv_z \partial \varepsilon / \partial z = 0$$

This equation can be greatly simplified by integrating over z to obtain an ordinary differential equation for $\phi_0(x)$. To do this, we define the total flux of positive and negative charge across the divertor, Q_+ and Q_e .

$$Q_+(x) = \int_0^L n v_x dz \quad (32)$$

$$Q_e(x) = \int_0^L n u_x dz \quad (33)$$

The continuity equations imply that $-dQ_+/dx$ and $-dQ_e/dx$ are the flux of positive and negative charge to the wall.

We now integrate Eq. (31) from $z = 0$ to L . The integration of the first two terms is trivial; so only the integration of the last two terms will be given. The continuity equation implies

$$\int_0^L [nv_x \partial \epsilon / \partial x + nv_z \partial \epsilon / \partial z] dz = \int_0^L [\partial (nv_x \epsilon) / \partial x + \partial (nv_z \epsilon) / \partial z] dz$$

$$= \epsilon_L + d(\bar{\epsilon} Q_+) / dx \quad (34)$$

with

$$\epsilon_L(x) = \epsilon(x, z = L) \quad (35)$$

$$\bar{\epsilon}(x) = [\int_0^L nv_x \epsilon dz] / [\int_0^L nv_x dz] \quad (36)$$

By $z = L$ we mean just on the plasma side of the electrostatic sheath. Consequently,

$$d^2 \phi_0 / dx^2 + (eB^2 / m_+ c^2) (1 - Q_e / Q_+) = d\bar{\epsilon} / dx - (\epsilon_L - \bar{\epsilon}) \frac{d \ln Q_+}{dx} \quad (37)$$

This is the differential equation for determining $\phi_0(x)$, the electrostatic potential at $z = 0$. Coupled with Eq. (29), it determines the electrostatic potential everywhere in the divertor plasma.

The differential equation for $\phi_0(x)$ is not as difficult as it appears. The complicated part is written on the right side of Eq. (37) and depends in some manner on ϵ . The results of the fluid flow indicate $|\ln(n/n_0)| \lesssim 1/2$. This means the terms which depend on ϵ can change $e\phi_0$ by an amount comparable to $1/2 T_e(x)$ and η by an amount roughly $\pm 1/2$. This correction is small compared to the η 's of

interest, $\eta \geq 3$. The terms on the right side of Eq. (37) will, therefore, be neglected.

The term in Eq. (37) which depends on the magnetic field makes a contribution to ϕ_0 comparable to $(x/\rho_+)^2 (1 - Q_e/Q_+) T_e$. This contribution is negligible over distances small compared to ρ_+ and Eq. (37) predicts a constant electric field over such small scales. However, for broad divertors the magnetic field term forces $Q_e \approx Q_+$. If δ is the divertor width $|(Q_+ - Q_e)/Q_+| \approx (\rho_+/\delta)^2$ for $\delta \gg \rho_+$.

The important parameter in divertor theory is η , not ϕ_0 . Values for η will be calculated in the next section on transport. However, we have shown that unless $\delta \gg \rho_i$ the electrostatic potential is rather independent of T_e and not strongly affected by different electron and ion flows in the divertor, $Q_e \approx Q_+$.

V. TRANSPORT

The features, which most distinguish divertor and planar discharge theory, involve particle and energy transport. For example, we will find that η can vary from $\ln[L/\lambda_e]$ at the inner edge of the divertor to infinity at the outer edge. This is in contrast to $\eta \approx 3.5$ in a planar discharge. In this section, the regions of electron and ion escape will be calculated, and the interesting relation $\partial \ln(p)/\partial x = 3\partial \ln(n)/\partial x$ will be shown to hold in the outer parts of the divertor. Particle transport, being simpler, will be considered first; then energy transport will be discussed.

Equation (4) gives the electron transport across the divertor. Using Eq. (7) for R_y and defining an effective adiabatic index γ

$$(\gamma - 1)\partial \ln(n)/\partial x = \partial \ln(T)/\partial x \quad (38)$$

the electron flux across the divertor, $Q_e(x)$, is

$$Q_e(x) = - \int_0^L \left(\frac{3-\gamma}{2} \frac{nTC^2}{\sigma_{\perp} B^2} \frac{\partial n}{\partial x} \right) dz \quad (39)$$

If $\eta = e\phi/T_e$ is large throughout the outer part of a divertor, then Eq. (10) implies $Q_e(x)$ must be negligibly small in this

region. Under these circumstances, we call the electrons stagnant. Equation (39) says a region of stagnant electrons has an average effective adiabatic index, $\bar{\gamma} = 3$.

The electron temperature distribution is determined by the energy equation (11). The term $\vec{R} \cdot \vec{v}$ in this equation must be put in a better form. The most important part of $\vec{R} \cdot \vec{v}$ is $R_y v_y$. The term $R_z v_z$ makes a contribution which scales like q_{eq} but is generally numerically smaller--of order $Z/10$ smaller where Z is the charge of the ions--and will be neglected. The term $R_x v_x$ is negligible due to the smallness of v_x . Using Eq. (1) for v_y and Eq. (4) for R_y , we have

$$\vec{R} \cdot \vec{v} = enE_x u_x \quad (40)$$

Substituting, the energy equation reduces to

$$\vec{\nabla} \cdot \left(\frac{5}{2} n T_e \vec{u} + \vec{q} \right) = - enE_z u_z - q_{eq} \quad (41)$$

The energy equation is used to determine the electron temperature. Since the parallel thermal conductivity is high, T_e is only a function of x . An integration of Eq. (41) from $z = 0$ to L yields an ordinary differential equation for $T_e(x)$. The integration is straightforward except for the energy flux into the sheath, which is evaluated with Eq. (14), and the term involving q_x . To evaluate the q_x term, we define the average effective adiabatic index $\bar{\gamma}$.

$$\bar{\gamma} = \int_0^L \gamma \left(\frac{1}{\tau_e} \frac{\partial n}{\partial x} \right) dz / \int_0^L \left(\frac{1}{\tau_e} \frac{\partial n}{\partial x} \right) dz . \quad (42)$$

Using this expression, Eq. (12) for q_x , Eq. (2) for j_y and Eq. (38) for γ one finds

$$\int_0^L q_x dz = -LK_e dT_e/dx \quad (43)$$

where

$$K_e = \left(\frac{(k-3)\bar{\gamma} - k}{2(\bar{\gamma}-1)} \right) \left(\frac{1}{L} \int_0^L \frac{nT_e}{m_e \omega_{ce}^2 \tau_e} dz \right) \quad (44)$$

with $k = (2)(4.66) = 9.32$. The coefficient in front of the effective thermal conductivity would be 4.66 if the effect of the j_y term on q_x were neglected. Inclusion of this term reduces the coefficient to 2.41 for $\bar{\gamma} = 3$ and to zero for $\bar{\gamma} = 1.47$. Letting $\bar{q}_{eq}$ be the average over z of q_{eq} , the integrated energy equation is

$$\frac{d}{dx} \left(LK_e \frac{dT_e}{dx} \right) = -e\phi_0 \frac{dQ_e}{dx} + L\bar{q}_{eq} + \left\{ \frac{5}{2} \frac{dT_e}{dx} Q_e + T_e \int_0^L \left[\frac{1}{2} + \ln \left(\frac{n_0}{n} \right) \right] \frac{\partial n u_x}{\partial x} dz \right\} . \quad (45)$$

The terms in curly brackets contribute significantly only for $\eta \lesssim 2$. This essentially comes from the exponential dependence of dQ_e/dx on η . The solution to the divertor problem will be

found to violate the assumption of negligible electron inertia for $\eta \lesssim 2$, and the equation we used for the heat flow into the sheath is questionable for such small values of η . Consequently, the terms in curly brackets can be neglected without changing the validity of the theory. The energy or heat equation we will study is, therefore,

$$\frac{d}{dx} \left(L K_e \frac{dT_e}{dx} \right) = - e \phi_0 \frac{dQ_e}{dx} + L \bar{q}_{eq} \quad (46)$$

If the main plasma is assumed stationary so $v_y(x=0) = 0$, then ϕ can be considered a constant in the energy equation. The reason is $v_y(x=0) = 0$ implies $E_x(x=0) = 0$. The equation for the electrostatic potential allows the electric field E_x to change significantly only on an ion gyroradius scale. However, we will find that $Q_e(x)$ goes to zero in a much shorter distance $\rho_e L / \lambda_e$. The assumption $v_y(x=0) = 0$ will be made during the transport calculation.

The two terms on the right side of Eq. (46) represent cooling through electrons flowing to the walls and through thermal equilibration with the ions. A comparison of the two terms using Eq. (10) for electron flow to the walls and Eq. (13) for thermal equilibration shows a switch in importance occurs at $\eta = e\phi/T_e$ equal to η_s .

$$\eta_s \approx \ln \left(\frac{m_i}{m_e} \frac{\lambda_e}{L} \right) \quad (47)$$

The dominant term is electron flow to the walls for $\eta < \eta_s$ and thermal equilibrium for $\eta > \eta_s$. Assuming $(m_i/m_e)^{1/2} \gg L/\lambda_e \gg 1$, η_s lies between 4 and 8 for hydrogen.

To proceed further an expression is required for $\bar{\gamma}$.

Integrating Eq. (46) from x to ∞ and letting

$$\epsilon(x) = \left(\int_x^\infty L \bar{q}_{eq} dx \right) / [e\phi_0 Q_e(x)] \quad (48)$$

we have

$$LK_e \frac{dT_e}{dx} = -e\phi_0 (1 + \epsilon) Q_e \quad (49)$$

One can show that ϵ changes exponentially from being small to large at η_s . Equation (49) together with Q_e from Eq. (39) gives

$$\bar{\gamma} = [k + 3(1 + \epsilon)\eta] / [(k - 3) + (1 + \epsilon)\eta] \quad (50)$$

where $k = 9.32$. This equation predicts a small variation in $\bar{\gamma}$. For $\eta = 2$, $\bar{\gamma} = 1.84$ while the maximum value of $\bar{\gamma}$ is 3. The limited variation of $\bar{\gamma}$ means the variation of density and temperature across the divertor are closely related.

In the inner region of the divertor, $\eta < \eta_s$, the energy equation predicts a temperature dependence of roughly

$$T_e(x) \approx \left(\frac{e\phi}{2} \right) / \ln[(x + \delta_e)/x_0] \quad (51)$$

where $x_0 \approx \rho_e (L/\lambda_e)^{1/2}$, $\delta_e = x_0 \exp(\eta_0/2)$, $\rho_e = (T_e/m_e)^{1/2} \omega_{ce}$, and η_0 is the value of η at $x = 0$. The electron temperature drops rapidly for small x , $(dT_e/dx)_{x=0} = (2T_e)/(\eta\delta_e)$ but within a distance of a few δ_e changes very slowly.

An expression for $Q_e(x)$ can be derived from Eq. (10).

$$Q_e(x) = \int_x^\infty n \left(\frac{T_e}{2\pi m_e} \right)^{1/2} \exp(-e\phi/T_e) dx. \quad (52)$$

Equation (29) implies this equation for $Q_e(x)$ is independent of the value of z used to evaluate n and ϕ .

The temperature given by Eq. (51) predicts

$$Q_e(0) \approx n (T_e/m_e)^{1/2} \rho_e (L/\lambda_e)^{1/2} \exp(-\eta_0/2). \quad (53)$$

and that

$$Q_e(x) \approx Q_e(0) \frac{\delta_e}{x + \delta_e}. \quad (54)$$

The positive charge flux, Q_+ , is determined by the fluid flow

$$Q_+(x) = \int_x^\infty (nv_z)_L dx \approx \int_x^\infty \left(\frac{2}{\gamma-1} \right)^{\frac{1}{\gamma-1}} n (T_e/m_+)^{1/2} dx. \quad (55)$$

This integral does not converge with the temperature given by Eq. (51). However, thermal equilibration will cause T_e to go to zero on a scale $\rho_+(T_e)$. This means $Q_+(0) \approx n (T_e/m_+)^{1/2} \rho_+$. Equating $Q_+(0)$ and $Q_e(0)$ gives

$$\eta_0 \approx \ln(L/\lambda_e)$$

$$\delta_e \approx \rho_e (L/\lambda_e) . \quad (56)$$

Since we have assumed $L/\lambda_e \ll (m_+/m_e)^{1/2}$, $\delta_e \ll \rho_+$. This means the electrons flow to the wall in a very narrow region, $\sim \delta_e$, compared to the width of the divertor, $\sim \rho_+$.

Equation (51) can also be considered an equation for η , $\eta \approx 2\ln[(x + \delta_e)/x_0]$. For large enough values of x , η becomes larger than η_s and thermal equilibration dominates the cooling. This occurs at $x_s \approx Z^{1/2}\rho_+$ where Z is the charge of the ions. In the region dominated by thermal equilibration, the energy equation has an exact solution for the appropriate boundary conditions. Letting $x_1 = x - x_s$, the boundary conditions are $T_e(x_1 = 0) = T_s$ and $dT_e/dx = 0$ at the point $x_1 = \delta_0$ where the electron temperature vanishes. The exact solution assuming $\gamma = 3$ is

$$T(x_1) = T_s [(x_1 - \delta_0)/\delta_0]^2$$

$$\delta_0^2 = (k - \frac{9}{2}) \rho_i^2 (T_s) \quad (57)$$

where $k = (2)(4.66)$ and $\rho_i = Z^{1/2}\rho_+$. Numerically $\delta_0 = 2.20 \rho_i (T_s)$.

If the small terms which involve ϵ are neglected, Eq. (37) for the electrostatic potential implies that in the stagnant

electron region, $\eta > \eta_s$,

$$\phi_0(x_1) = \phi_0(x_1 = 0) - \epsilon_s x_1 - \frac{1}{2} \frac{T_s}{e} (x_1/\rho_+)^2 \quad (58)$$

with ϵ_s the electric field at $x_1 = 0$ and $\rho_+ = \rho_+(T_s)$.

Combining (57) and (58), η can be evaluated

$$\eta = \left[\eta_s - \left(\frac{e\epsilon_s}{T_s} \right) x_1 - \frac{1}{2} \left(\frac{x_1}{\rho_+} \right)^2 \right] / [(x_1 - \delta_0)/\delta_0]^2 \quad (59)$$

The character of this equation changes depending on the value of ϵ_s . The critical ϵ_s is

$$\epsilon_c = (T_s/e\rho_+) [\eta_s(\rho_+/\delta_0) - (\delta_0/2\rho_+)] \quad (60)$$

For $\epsilon_s \leq \epsilon_c$, η goes to infinity at $x_1 = \delta_0$ which is the location of $T_e = 0$. For $\epsilon_s > \epsilon_c$, η vanishes at some value of x_1 less than δ_0 , and the stagnant electron model is not quite right.

For the divertor model we are considering with $x_s \approx \rho_i$, one has $\epsilon_s \sim \epsilon_c$; so it is not clear whether $\epsilon_s \leq \epsilon_c$ or not. Actually the divertor model is not radically changed for $\epsilon_s > \epsilon_c$. As the electrostatic potential becomes small on the outside of the divertor due to the term $-\frac{1}{2} \frac{T_s}{e} (x_1/\rho_+)^2$, electrons again flow to the wall. Each electron which flows to the wall carries away energy which must be replaced by electron conduction through a distance ρ_+ . As the potential falls so will the electron

temperature, so the divertor will still have a distance scale of approximately ρ_+ . Because of the low efficiency of electron conduction across the divertor, the fraction of electrons lost from the outer part of the divertor is approximately $L/[(m_+/m_e)^{1/2}\lambda_e]$ of the total number. This number has been assumed small; so the bulk of the electrons are lost from a small region of the divertor ($\sim \delta_e$) as before.

VI. CONCLUSION

Though there are similarities between the divertor and the planar discharge, the theories have numerous differences. Both theories predict an ion flow velocity to the wall of $C_s = (T_e/m_+)^{1/2}$. However, the divertor width is very different from that predicted by a simple extension of discharge results. The classical theory of divertors predicts the ions will flow out over a width of order the ion gyroradius calculated with the electron temperature, $\rho_+(T_e)$. The electrons will flow to the wall over a much narrower region of order $\rho_e L/\lambda_e$, where ρ_e is the electron gyroradius, L is the length of the divertor, and λ_e is the electron mean-free path. A simple extension of discharge theory to divertors, with the ions and electrons leaving together, would give a divertor width of roughly $(\rho_+ \rho_e L/\lambda_e)^{1/2}$.

The electric field--and under some boundary conditions the electrostatic potential--are relatively constant across the divertor compared to the result for planar discharges, $\phi \approx 3.5 T_e/e$. In the outer regions of the divertor $e\phi/T_e$ becomes so large electrons cannot flow in significant numbers to the wall. In this region of stagnant electrons, one finds $\bar{\gamma}$ defined so $\bar{\gamma} \partial \ln n / \partial x = \partial \ln P / \partial x$ has a value of 3. For $\bar{\gamma} = 3$, the density in the divertor drops by a factor $1/\sqrt{2} = 0.7071$ along any field line from its

maximum in the center ($z = 0$) to just on the plasma side of the electrostatic sheath which forms within a few Debye lengths of the wall.

The fact that electrons and ions leave the divertor in separate regions does not contradict the assumption of quasi-neutrality in the divertor. Equal integrated fluxes of electrons and ions are assumed to enter and leave the divertor. There will be currents in the conducting neutralizing walls to short out any charge buildup due to the electron and ion flows occurring in different regions.

This paper has assumed $\lambda_e \ll 1$ so the electrons form a Gaussian distribution. Actually, the equations essentially remain valid for $\lambda_e \ll (m_+/m_e)^{1/2} L$. However, the factor $(T_e/m_e)^{1/2}$ in Eq. (10) goes to something like L/τ_e for $\lambda_e \sim L$. This affects formulas with factors of L/λ_e . These factors would become a number of order unity for $\lambda_e > L$. The assumption $\lambda_e \ll (m_+/m_e)^{1/2} L$ is required to assume the electrons form a truncated Maxwellian over most of the divertor. Of course, a truncated Gaussian will have somewhat different transport coefficients than assumed in the paper.

Viscous terms have been neglected more for mathematical simplicity rather than for small size. Such terms would primarily affect Eqs. (3) and (5). The effect through Eq. (3) would be to hold the electric field even more constant than in the zero viscosity model. If the boundary condition $v_z = 0$ were taken on the inner divertor edge, equation (5) with viscosity

would move the ion flow further from the body of the plasma and its average would be somewhat slower than C_s . In the viscous free theory, the boundary condition $v_z(x=0) = 0$ cannot be imposed without overdetermining the equations. This boundary condition problem always occurs in viscous free fluid models.

The most important conclusion of this paper is that electrons and ions need not leave a divertor on the same field lines. On this point there is strong experimental evidence from the FMI Spherator at Princeton. FMI divertor experiments¹¹ gave $e\phi/T_e \lesssim 5$ in a small region about the separatrix--that is on field lines adjacent to the main body of the plasma. However, for field lines further from the main body of the plasma $e\phi/T_e$ was measured as large as 20. A measurement was also made of the velocity of plasma flow to the neutralizing plate.¹² The measurement was $V \approx \frac{1}{3} (T_e/m_+)^{1/2}$. The reduction of the flow velocity from the predicted $(T_e/m_+)^{1/2}$ is probably a viscosity effect.

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Fig. 1. The Divertor Model. The main body of the plasma is surrounded by a uniform z directed magnetic field. It is assumed to be of infinitesimal extent in the x direction but of a size comparable to L in the z direction. All quantities are independent of y . Equal numbers of electrons and positive charges diffuse across the plasma surface onto the open z directed magnetic field lines. Calculations are carried out in the $x \geq 0, z \geq 0$ quadrant.

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