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TOPICAL REPORT
FSEC-NSG-217-74/37B
AEC/SNS 3063-6

COMPRESSIBLE FLOW THROUGH A TUBE CONNECTING A CONTINUUM AND A VACUUM

Prepared by
A. Schock

For

SPACE NUCLEAR SYSTEMS DIVISION
ATOMIC ENERGY COMMISSION

Under Contract No.
AT (49-15) 3063

August 5, 1974

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FAIRCHILD
SPACE & ELECTRONICS COMPANY
Germantown, Maryland 20767

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Yours truly,

Alfred Schock
Manager, Nuclear Systems Group

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Part I: STEADY-STATE SOLUTIONS

1. Introduction

Certain fluid flow applications require the analysis of flow through a long tube whose upstream end operates in the continuum regime and whose downstream end is in the molecular-flow regime. For example, in radioisotope power sources used in space vehicles, prevention of excessive pressure build-up in the heat source usually requires that the helium generated by alpha decay be vented to space vacuum. In doing so, it is generally desirable that the vent tube have a very low conductance, so as to maintain a reasonably high helium pressure (e.g., several psi) in the heat source. This promotes heat transfer across internal gaps, which is beneficial because it reduces the fuel operating temperature. Maintenance of such upstream pressures in spite of the very low helium flow rates (typically less than 1 gm/yr) may require vent tubes with extremely high length-to-diameter ratios. Over the length of these tubes, the flow changes from continuum to free-molecular.

Numerous solutions for compressible continuum flow and for molecular or Knudsen flow have been published, although the latter are often of limited validity, because they are derived from a momentum balance which omits the acceleration term resulting from fluid expansion. It is this term which leads to the Mach number limit for flow leaving the tube. As will be shown, that limit can apply not only to continuum flow, as generally recognized in compressible flow solutions, but also to Knudsen flow. The purpose of this paper, then, is to derive a general solution for flow in the continuum, Knudsen, and transition regimes, taking proper account of the Mach number limit in all regimes.

2. Steady-State Analysis

To demonstrate that the Mach number limit follows simply from the basic momentum balance, even in Knudsen flow, let us start our derivation from the fundamental relation $F = ma$. Consider a fluid element of length dx at position x in a tube of inner diameter D . The element's mass is $\frac{1}{4}\pi D^2 \rho dx$, where ρ is the fluid density at x . The fluid pressure produces an accelerating force $\frac{1}{4}\pi D^2 p$ at the x -face, and a retarding force $\frac{1}{4}\pi D^2 (p+dp)$ at the $x+dx$ face. In addition, it is retarded at the tube wall by the shear force $(\pi D dx) \tau_w$, where τ_w is the shear stress, or rate of momentum loss per unit wall area. The resultant of these forces equals the product of the element's mass and the rate of change of its mean velocity v :

$$\frac{1}{2} \pi D^2 [p - (p + dp)] - \pi D \tau_w dx = \left(\frac{1}{4} \pi D^2 \rho dx \right) (dv/dt). \quad (1)$$

But $\frac{dv}{dt} = \left(\frac{dv}{dx} \right) \left(\frac{dx}{dt} \right) = v \frac{dv}{dx}.$ (2)

Inserting this in Eq. (1), we obtain

$$-dp - (4\tau_w/D) dx = \rho v dv. \quad (3)$$

To follow convention, let us define a friction factor *

$$f \equiv \tau_w / \frac{1}{2} \rho v^2, \quad (4)$$

in terms of which Eq. (3) can be expressed as

$$dp + \rho v dv + (2f \cdot v^2/D) dx = 0. \quad (5)$$

* The friction factor f defined here is commonly referred to as the Fanning friction factor in continuum flow theory. Some textbooks express their equations in terms of the Weisbach or Darcy friction factor f' , which differs from f by a factor of four ($f' = 4f$).

Since we are concerned with simple, low-pressure gases, far from their critical point, the gas density ρ is defined by the perfect gas law,

$$\rho = p / RT, \quad (6)$$

where T is the absolute temperature, and R is the gas constant for the fluid; i.e., the universal gas constant divided by the molecular weight.

The gas velocity v can be expressed in terms of the Mach number M :

$$v = M \sqrt{\gamma RT}, \quad (7)$$

where γ is the specific heat ratio C_p / C_v , which generally has a value of 5/3 for monatomic gases and 7/5 for diatomic gases. The values of v and M are both averages over the fluid cross-section.

Next, we note that the mass flow rate $\dot{m}$ is given by

$$\dot{m} = \frac{1}{4} \pi D^2 \rho v = \frac{1}{4} \pi D^2 (\gamma / RT)^{1/2} M p. \quad (8)$$

We must now decide whether to analyze isothermal or adiabatic flow.

Neither is strictly correct, but since the problem of primary interest involves extremely long and thin tubes, with very low mass flow rates, let us examine the isothermal case. With $\dot{m}$, D , γ , and T assumed constant in Eq.(8), it follows that the product Mp is also constant, so that

$$dp = -(\gamma / M) dM. \quad (9)$$

From Eqs. (6), (7), and (8), we find that

$$\rho v dv = -\gamma M^2 dp = \gamma M^2 dM, \quad (10)$$

$$\rho v^2 = \gamma M^2. \quad (11)$$

Inserting Eqs. (9), (10), and (11) into (5), we obtain

$$(1 - \gamma M^2) dM = 2\gamma M^2 f (dx/D). \quad (12)$$

It is convenient in isothermal problems to introduce a variable $\mathcal{M}$, defined by

$$\mathcal{M} \equiv \sqrt{\gamma} M = v/\sqrt{RT} = 4\pi \sqrt{RT}/\dot{m}. \quad (13)$$

Equation (12) may therefore be written as

$$\left(\frac{1 - \mathcal{M}^2}{\mathcal{M}^3} \right) d\mathcal{M} = 2f \left(\frac{dx}{D} \right). \quad (14)$$

As can be seen, Eqs. (12) and (14) show that the derivatives dM/dx and $d\mathcal{M}/dx$ go to infinity at $M = \gamma^{-1/2}$ or $\mathcal{M} = 1$. Since the derivative $d\mathcal{M}/dx$ is positive when $\mathcal{M} < 1$ and negative when $\mathcal{M} > 1$, it is clear that $\mathcal{M} = 1$ is an uncrossable limit for isothermal flow. (The corresponding limit for adiabatic flow is $M = 1$.)

Note that the Mach number limit comes about from the term $\mathcal{M}^2$ in Eq. (14) or γM^2 in Eq. (12). That term can in turn be traced back to the acceleration term $\rho v dv$ in Eq. (3). Clearly, omission of that acceleration term can lead to serious errors, particularly for problems with high inlet-to-outlet pressure ratios.

It should be emphasized that in the derivation thus far we have made no assumption about whether the flow is in the continuum or in the Knudsen regime. Starting from a simple momentum balance, we have shown that the Mach number limit applies to either regime. No expression for the friction factor f has been inserted yet. Before doing so, it is convenient

to introduce two well-known dimensionless ratios: The Reynold's number $\mathcal{R}$, which is a relative measure of dynamic and viscous forces, and is defined by

$$\mathcal{R} \equiv 2v_p r / \mu = 4\dot{m} / \pi D \mu, \quad (15)$$

where μ is the fluid's viscosity; and the Knudsen number $\mathcal{K}$, which is the ratio of the molecules' mean free path λ_m to the tube diameter D :

$$\mathcal{K} \equiv \lambda_m / D = \sqrt{1/\pi RT} \mu / D p. \quad (16)$$

When the Knudsen number is much smaller than unity ($\mathcal{K} < 0.01$), continuum flow exists. When $\mathcal{K}$ is greater than one, intermolecular collisions are negligible, placing us in the free-molecular flow regime. The transition region ($0.01 < \mathcal{K} < 1$) is referred to as "slip flow", because the usual assumption of zero velocity at the tube wall is no longer valid.

From Eqs. (13), (15), and (16), it is clear that the dimensionless numbers $\mathcal{K}$, $\mathcal{R}$, and $\mathcal{M}$ are related by

$$\mathcal{K} \mathcal{R} = \sqrt{1/\pi} \mathcal{M}. \quad (17)$$

In continuum flow ($\mathcal{K} < 0.01$), the friction factor f in a smooth tube is only a function of the Reynold's number $\mathcal{R}$. Since we are primarily interested in low mass flow rates, and hence low Reynold's numbers, let us confine our attention to the laminar flow regime ($\mathcal{R} < 2000$). For laminar continuum flow, it can be shown that the friction factor and Reynold's number are related by

$$f = 16 / \mathcal{R}. \quad (18)$$

The derivation of Eq. (18) assumed zero fluid velocity at the tube wall. Since this assumption is not valid when $K > 0.01$, a correction factor Φ must be added to cover the slip-flow and free-molecular-flow regimes:

$$f = 16 / R \Phi \quad (19)$$

To compute the correction factor Φ , Maxwell^[1] derived the theoretical expression:

$$\Phi = 1 + 8 \left(\frac{2}{\beta} - 1 \right) K, \quad (20)$$

where β is the fraction of molecules striking the wall which is diffusely reflected (while the fraction $1-\beta$ is specularly reflected). Extensive flow tests^[2] of various gases through glass capillaries with diameters down to 0.003 cm yielded the empirical expression

$$\Phi = 1 + 11.3 K, \quad (21)$$

which implies a value of 0.84 for β .

As shown in Figure 1, which is reproduced from McAdams^[3], Eq.(21) gives excellent agreement with experimental results over a wide range of Knudsen numbers. Combining Eqs. (17), (19), and (21), we obtain a simple expression for the friction factor f , applicable to (laminar) continuum flow, slip flow, and free molecular flow:

$$f = \frac{16}{R (1 + 11.3 K)} = \frac{16}{R + 13.8 M}, \quad (22)$$

Inserting this into Eq. (14) gives

$$\frac{dx}{D} = \frac{(R + 13.8 M)(1 - M^2)}{32 M^3} dM \quad (23)$$

As can be seen from Eq. (15), for a tube with constant diameter, mass flow rate, and fluid viscosity, the Reynold's number R is also constant. Equation (23) can therefore be integrated to solve for the tube length $L = x_2 - x_1$:

$$\frac{L}{D} = \frac{R}{64} \left[\frac{1}{M_1^2} - \frac{1}{M_2^2} - \log_8 \left(\frac{M_2^2}{M_1^2} \right) \right] + 0.431 \left[\frac{1}{M_1} - \frac{1}{M_2} - (M_2 - M_1) \right]. \quad (24)$$

To aid in understanding this and later solutions, it is convenient to employ a dimensionless pressure $\mathcal{P}$, which is defined as the reciprocal of the Knudsen number K :

$$\mathcal{P} \equiv \frac{1}{K} = \frac{D}{\lambda_m} = \frac{P}{\sqrt{\frac{1}{2}\pi RT} \mu / D} \quad (25)$$

From Eqs (17) and (25), it is clear that the dimensionless variables $\mathcal{P}$, R , and M are related by

$$R = \sqrt{\frac{1}{2}\pi} M \mathcal{P}, \quad (26)$$

so that Eq. (24) can be expressed as

$$\frac{L}{D} = \frac{1}{32} \left\{ \frac{\pi}{4R} (\mathcal{P}_1 - \mathcal{P}_2)(\mathcal{P}_1 + \mathcal{P}_2 + 22) - R \left[\log_8 \left(\frac{\mathcal{P}_1}{\mathcal{P}_2} \right) + 11(\mathcal{P}_2^{-1} - \mathcal{P}_1^{-1}) \right] \right\}, \quad (27)$$

Parenthetically, it may be noted that in the special case when the pressure difference $\mathcal{P}_1 - \mathcal{P}_2$ is small compared to $\mathcal{P}_2$, Eq. (27) is approximated by

$$\frac{L}{D} = \frac{1}{32} (\mathcal{P}_1 - \mathcal{P}_2)(\mathcal{P}_2 + 11) \left[\frac{\pi}{2R} - \frac{R}{\mathcal{P}_2^2} \right], \quad (28)$$

and that $\mathcal{P}_2 + 11 \approx \mathcal{P}_2$ in the continuum-flow regime, where $K < 0.01$ and $\mathcal{P} > 100$.

3. Problems with Unknown Tube Length L :

In this type of problem, the mass flow rate $\dot{m}$ and inlet pressure p_1 are known. Therefore, R can be computed from Eq. (15), and $\mathcal{P}_1$ obtained from Eq. (25):

$$\mathcal{P}_1 = \frac{P_1}{\sqrt{\frac{1}{2}\pi RT} \mu / D}. \quad (29)$$

In computing $\mathcal{P}_2$, however, we must bear in mind that $\mathcal{M}_2$ cannot exceed unity, no matter how low the pressure p_3 of the space to which the tube is discharged. Therefore,

$$\mathcal{M}_2 = \left\{ \frac{1}{4 \dot{m} \sqrt{RT} / \pi D^2 p_3} \right\}, \quad \text{whichever is less.} \quad (30)$$

But if $\mathcal{M}_2 \leq 1$, it follows from Eq. (26) that $\mathcal{P}_2$ cannot be less than $R/\sqrt{\frac{1}{2}\pi}$. Therefore,

$$\mathcal{P}_2 = \left\{ \begin{array}{l} R/\sqrt{\frac{1}{2}\pi} \\ p_3 / [\sqrt{\frac{1}{2}\pi RT} \mu / D] \end{array} \right\}, \quad \text{whichever is greater.} \quad (31)$$

Thus, the outlet pressure p_2 at the end of the tube is not necessarily equal to the discharge pressure p_3 . When $\mathcal{M}_2 < 1$, then $p_2 = p_3$; but when $\mathcal{M}_2 = 1$, then $\mathcal{P}_2 = R/\sqrt{\frac{1}{2}\pi}$, and $p_2 = 4 \dot{m} \sqrt{RT} / \pi D^2 > p_3$.

Once R , $\mathcal{P}_1$, and $\mathcal{P}_2$ have been computed, the tube length L is obtained from Eq. (27).

4. Graphical Solutions

The problem discussed in the preceding section was straightforward, since Eq. (27) is an explicit solution for L . This is not the case in other types of problems. For example, when the inlet pressure p_1 is unknown, Eq. (27) is a transcendental equation in $\mathcal{P}_1$, which cannot be solved explicitly. Therefore, it is very desirable to provide generalized graphical solutions for this and other problem types, in order to eliminate or at least minimize the need for tedious trial-and-error solutions.

To enable us to depict a general solution in a single figure, let χ_0 be defined as the value of χ where $\mathcal{M} = 1$. At sufficiently low discharge pressures, this will always occur at the exit end of the tube ($\chi_0 = \chi_2$, $\mathcal{M}_2 = 1$).

When the discharge pressure p_3 exceeds $4m\sqrt{RT}/\pi D^2$, the Mach limit is never reached ($M_2 < 1$). In that case, the point x_0 (at which $M=1$) is a theoretical point, beyond the end of the tube ($x_0 > x_2$). In either case, we can define a dimensionless distance ξ from x_0 :

$$\xi \equiv (x_0 - x) / L. \quad (32)$$

From Eq. (23) we see that

$$\int_0^\xi \frac{dM}{M^3} = - \int_0^\xi \frac{(R + 12.5 M^2)(1 - M^2)}{M^3} dM, \quad (33)$$

which can be integrated to yield

$$\xi = \frac{1}{64} (M^{-6} - 1 + \log M^2) R + 0.421 (1 - M^2) / M. \quad (34)$$

Solutions of Eqs. (26) and (34) are illustrated in Figures 2, 3, and 4. These figures are really cross-plots of each other. Figure 2 presents plots of M^{-1} versus ξ , for constant values of K and R ; and Figure 3 shows curves of P versus ξ , for constant values of M and R . In any given tube, the value of R is uniform. The constant- R curves in Figure 3 may therefore be interpreted as log-log plots of typical pressure profiles. Flow on these curves is from right to left (increasing x , decreasing ξ). As flow proceeds, the Mach and Knudsen numbers both increase, until the Mach number limit $M=1$ is reached at $\xi=0$ ($x=x_0$).

The regions of continuum flow ($K < 0.01$), slip flow ($0.01 < K < 1$), and free molecular flow ($K > 1$) have been identified in Figure 3. As can be seen, even when the discharge pressure p_3 is zero, the outlet pressure p_2 at the tube exit has a finite value. Depending on the Reynold's number, a tube discharging to a perfect vacuum can have either molecular flow, slip flow, or even continuum flow at its exit end.

This is illustrated more clearly in Figure 4, which shows curves of R versus P_1 for constant ξ_1 (solid curves) and for constant M_1 (dashed curves).

Each solid curve may be interpreted as a plot of mass flow rate $\dot{m}$ versus inlet pressure p_1 , for a tube of length $L = \frac{4}{3}D$ discharging to a pressure $p_2 < 4 \dot{m} \sqrt{RT} / \pi D^2$. For that case, $\mathcal{M}_2 = 1$, and $\mathcal{P}_2 = \mathcal{R} / \sqrt{\frac{1}{2}\pi}$ (from Eq. 26). It was therefore possible to add a $\mathcal{P}_2$ scale at the right edge of the figure, and to identify the regions of molecular flow ($\mathcal{P} < 1$), slip flow ($1 < \mathcal{P} < 100$), and continuum flow ($\mathcal{P} > 100$), both for the inlet ($\mathcal{P}_1$) and outlet ($\mathcal{P}_2$) of the tube. As can be seen, even though a tube is discharged to an absolute vacuum, any of these three flow regimes can exist at its outlet end.

Figure 4 illuminates another interesting point. It will be recalled that the equations derived thus far applied only to laminar flow ($\mathcal{R} < 2100$). How, it may be wondered, does one analyze a tube with turbulent continuum flow at its inlet end, changing to molecular flow at its outlet? The Knudsen-flow correction factor Φ (Eq. 21) was a correction to the laminar-flow friction factor (Eq. 19). There is no valid reason to assume that it can also be applied to the equation for the turbulent-flow friction factor. In fact, the very concept of turbulent molecular flow seems ludicrous, since the molecules are essentially unaware of each other's existence in this regime.

Fortunately, Figure 4 demonstrates that the problem posed here is merely academic, without practical significance. As can be seen, when $\mathcal{R} > 125$ then $\mathcal{P}_2 > 100$, and continuum flow exists throughout the tube, regardless of how low the discharge pressure is. The postulated case of turbulent flow at the inlet and molecular or slip flow at the outlet is seen to be impossible. When the inlet flow is turbulent ($\mathcal{R} > 2100$), then $\mathcal{P}_2 > 1680$, which is well into the continuum-flow regime. Therefore, the commonly used turbulent-flow friction factor correlations may be used without any correction, even for the case of vacuum discharge.

One suggested correlation, reportedly ^[4] accurate over a very wide range of Reynold's numbers ($3000 < \mathcal{R} < 3,000,000$) is

$$f = 0.00140 + 0.125 \mathcal{R}^{-0.32} \quad (35)$$

Inserting this into Eq. (14), we obtain

$$\frac{1-M^2}{M^3} dM = (0.00280 + 0.250 R^{-0.32}) \frac{dx}{D}, \quad (36)$$

which can be integrated to yield

$$\frac{L}{D} = \frac{2(M_1^{-2} - M_2^{-2}) - 4 \log(M_2/M_1)}{0.0112 + R^{-0.32}}, \quad (37)$$

from which M_1 and M_2 can be eliminated by use of Eq. (26):

$$\frac{L}{D} = \frac{\pi(P_1^2 - P_2^2)/R^2 - 4 \log(P_1/P_2)}{0.0112 + R^{-0.32}}. \quad (38)$$

For the case when $M_2 = 1$, $\gamma_0 = \gamma_2$, and $\beta_2 = 0$, we obtain

$$\xi_1 = \frac{L}{D} = \frac{\pi(P_1/R)^2 - 2 - 4 \log(\sqrt{\frac{1}{2}\pi} P_1/R)}{0.0112 + R^{-0.32}}. \quad (39)$$

Figure 5 depicts the relation between the variables ξ_1 , P_1 , and R , both for laminar flow (Eq. 26 and 34) and for turbulent flow (Eq. 39). The fold-out chart is large enough to permit readings with a fair degree of accuracy. It is most useful for the case when $M_2 = 1$; i.e., when the discharge pressure P_3 is less than $\mu \sqrt{RT} R/D$. For that case, it can be used to find the unknown tube length L , mass flow rate $\dot{m}$, or inlet pressure P_1 , when the other two are known.

Figure 5A displays a small copy of Figure 5, in which the regions where the inlet and outlet flows are molecular **M**, slip **S**, laminar **L**, and turbulent **T** are mapped. As shown, the subsonic domain contains seven regions. In four of these (i.e., $T \rightarrow T$, $L \rightarrow L$, $S \rightarrow S$, and $M \rightarrow M$) the same type of flow extends throughout the tube. In two regions ($L \rightarrow S$, and $S \rightarrow M$) there is a transition from one flow regime to a second. And in one region ($L \rightarrow S \rightarrow M$) there will be zones of laminar, slip, and molecular flow within a single tube. As can be seen, this can only occur in long, thin tubes ($L/D > 500$).

5. Problems with Unknown Inlet Pressure p_1

In this type of problem, the mass flow rate $\dot{m}$ is known, enabling us to compute R (Eq. 15), M_2 (Eq. 30), and

$$\xi_1 = \frac{1}{64} (M_2^{-2} - 1 + \log M_2^2) R + 0.431 (1 - M_2)^2 / M_2 + L/D. \quad (40)$$

In the special case when L/D is very small compared to ξ_1 , the pressure drop $P_1 - P_2$ is small compared to P_2 . For that case, we can solve Eq. (31) for P_2 , Eq. (28) for P_1 , and Eq. (29) for p_1 .

For the general case, knowing R and ξ_1 , we can use Figure 5 to find P_1 and therefore p_1 . If this graphical interpolation procedure is not accurate enough, a more precise value of p_1 can be obtained by solving the transcendental equation,

$$\xi_1 = \frac{1}{64} (M_1^{-2} - 1 + \log M_1^2) R + 0.431 (1 - M_1)^2 / M_1, \quad (41)$$

for M_1 . Fortunately, for a great many cases of interest it is possible to compute an accurate value of M_1 without tedious trial and error solutions. When $M_1 \ll 1$, Eq. (41) can be approximated by

$$\xi_1 \approx \frac{R}{64 M_1^2} + 0.431 \frac{1 - 2 M_1}{M_1}, \quad (42)$$

from which we can obtain an explicit solution for M_1 :

$$M_1 \approx \frac{1 + [1 + 0.290 R (1.16 \xi_1 + 1)]^{\frac{1}{2}}}{4 (1.16 \xi_1 + 1)} \quad (43)$$

By combining Eqs. (26) and (43), it is also possible to obtain an explicit approximate solution for P_1 :

$$P_1 \approx \frac{3.20 (1.16 \xi_1 + 1) R}{1 + [1 + 0.290 (1.16 \xi_1 + 1) R]^{\frac{1}{2}}}, \quad (44)$$

Equations (43) and (44) are quite accurate, except for values of $\mathcal{M}_1$ above 0.05. Since the domain where Eq. (43) is not accurate is only a small fraction of the total domain of interest, the required graphical solutions can be considerably more accurate than Figure 5. This is illustrated by Figure 6, which shows plots of $\mathcal{M}_1$ versus $\mathcal{E}_1$, and $\mathcal{R}$, for $\mathcal{M}_1 > 0.01$.

The dotted curves in Figure 6 show the percent error in computing $\mathcal{M}_1$ from the approximate Eq. (43) instead of the exact Eq. (41). For values of $\mathcal{E}_1$ and $\mathcal{R}$ above these curves, the value of $\mathcal{M}_1$ should be read from the figure. For values of $\mathcal{E}_1$ and $\mathcal{R}$ below the dotted curves, $\mathcal{M}_1$ can be computed from Eq. (43). Once $\mathcal{M}_1$ is known, the inlet pressure p_1 is obtained from Eq. (13):

$$p_1 = 4 \dot{m} \sqrt{RT} / \pi D^2 \mathcal{M}_1. \quad (45)$$

6. Problems with Unknown Mass Flow Rate $\dot{m}$:

For this type of problem, we employ Eq. (27), which relates the unknown $\mathcal{R}$ to the variables $\mathcal{Q}_1$, $\mathcal{Q}_2$, and L/D . While $\mathcal{Q}_1$ can be computed from Eq. (29), it is not possible to use Eq. (31) for an "a priori" solution for $\mathcal{Q}_2$, since $\mathcal{R}$ is not known. The simplest way around this difficulty is to first assume that $\mathcal{M}_2 = 1$; and then, for the known values of $\mathcal{Q}_1$ and L/D , use Figure 5 to determine what the value of $\mathcal{R}$ would be if the initial assumption ($\mathcal{M}_2 = 1$) were correct. If it were correct, then $\mathcal{R} = \sqrt{\frac{1}{2}\pi} \mathcal{Q}_2$, $p_2 \geq p_3$, and

$$\mathcal{R} \geq \frac{D p_3}{\mu \sqrt{RT}}. \quad (46)$$

Thus, if the tentative value of $\mathcal{R}$ read from Figure 5 satisfies the Inequality (46), then it is the correct value of $\mathcal{R}$. If the inequality is not satisfied, then $\mathcal{M}_2 < 1$, $p_2 = p_3$, and

$$\mathcal{Q}_2 = \frac{p_3}{\sqrt{\frac{1}{2}\pi RT} \mu / D}. \quad (47)$$

With $\mathcal{P}_1$, $\mathcal{P}_2$, and L/D known, Eq. (27) can be explicitly solved for $\mathcal{R}$:

$$\mathcal{R} = \frac{\left[1 + \frac{\frac{1}{4}\pi [\log(\mathcal{P}_1/\mathcal{P}_2) + 11(\mathcal{P}_2^{-1} - \mathcal{P}_1^{-1})](\mathcal{P}_1 + \mathcal{P}_2 + 22)(\mathcal{P}_1 - \mathcal{P}_2)}{(16L/D)^2} \right]^{\frac{1}{2}} - 1}{\frac{\log(\mathcal{P}_1/\mathcal{P}_2) + 11(\mathcal{P}_2^{-1} - \mathcal{P}_1^{-1})}{16L/D}} \quad (48A)$$

When the argument of the square root is close to 1, Eq. (48A) reduces to the much simpler form,

$$\mathcal{R} = \frac{\pi(\mathcal{P}_1 - \mathcal{P}_2)(\mathcal{P}_1 + \mathcal{P}_2 + 22)}{128L/D} \quad (48B)$$

For the case when the Inequality (46) is satisfied and $\mathcal{M}_2 = 1$, Figure 5 gives the correct value of $\mathcal{R}$, but is only readable with limited precision, because it spans many orders of magnitude for each variable. If $\mathcal{M}$ must be determined with greater accuracy, it is convenient to combine Eqs. (26) and (41):

$$\mathcal{S}_1 = \frac{L}{D} = \frac{\sqrt{\frac{1}{2}\pi}}{64} \left[(\mathcal{M}_1^{-1} - \mathcal{M}_1 + 2\mathcal{M}_1 \log \mathcal{M}_1) \mathcal{P}_1 + 22(1 - \mathcal{M}_1)^2 / \mathcal{M}_1 \right] \quad (49)$$

Equation (49) can be solved for $\mathcal{M}_1$ by trial and error. For many problems of interest, however, it is again possible to derive an explicit expression for $\mathcal{M}_1$ which will be shown to approximate the true answer very closely. To do so, we note that for small values of $\mathcal{M}_1$ (i.e., $\mathcal{M}_1 \ll 1$, $\mathcal{M}_1$ negligible compared to $\mathcal{M}_1^{-1}$) Eq. (49) reduces to

$$2.32 \mathcal{S}_1 \approx \frac{0.0454 \mathcal{P}_1 + (1 - \mathcal{M}_1)^2}{\mathcal{M}_1} \quad (50)$$

which can be explicitly solved for $\mathcal{M}_1$:

$$\mathcal{M}_1 = (1.16 \mathcal{S}_1 + 1) \left\{ 1 - \left[1 - \frac{1 + \mathcal{P}_1/22}{(1.16 \mathcal{S}_1 + 1)^2} \right]^{\frac{1}{2}} \right\} \quad (51A)$$

When the second term in the square bracket above is very small compared to 1, Eq. (51A) can be approximated by

$$\mathcal{M}_1 \approx \frac{1 + P_1/22}{2(1.16 \mathfrak{K}_1 + 1)} \quad (51B)$$

which can also be expressed in the dimensional form:

$$\dot{m} \approx \frac{D^3 P_1 / \sqrt{K T} + D^4 g_1^2 / 27.6 R T_1}{8(1.16 L + D) / \pi} \quad (52)$$

Figure 7 presents working graphs of Eq. (49), showing $\mathcal{M}_1$ as a function of $\mathfrak{K}_1$ and $\mathfrak{K}_1$. The dotted curves A and B show the locus of points at which the approximate Eqs. (51A) and (51B), respectively, result in an error of 1% in $\mathcal{M}_1$. For values of $\mathfrak{K}_1$ and $\mathfrak{K}_1$ above Curve-A, $\mathcal{M}_1$ should be read from Figure 7. For values of $\mathfrak{K}_1$ and $\mathfrak{K}_1$ between Curves A and B, $\mathcal{M}_1$ can be computed from Eq. (51A); and below Curve B, the simplified Eq. (51B) is accurate to better than 1%. Once $\mathcal{M}_1$ is known, the unknown mass flow rate $\dot{m}$ is obtained from Eq. (13):

$$\dot{m} = \frac{1}{4} \pi D^2 P_1 \mathcal{M}_1 / \sqrt{K T} \quad (53)$$

7. Summary of Working Equations

The equations derived in this paper are general solutions for isothermal flow of a perfect gas through a uniform tube. Although the paper's title refers to flow between a continuum and a vacuum, the solutions also apply to flow between two continua or between two vacua.

Before summarizing the sequence of equations to be used in solving a specific problem, a few words about units. Any dimensionally consistent set of units may be used for any equation in this report. When checking for dimensional consistency of English engineering units, care must be taken to distinguish between pounds-mass (lbm) and pounds-force (lbf). When neces-

sary to balance an equation, the conversion relation $1 \text{ lbf} = (1 \text{ lbm})(32.2 \text{ ft/sec}^2)$ may be employed. Elementary engineering textbooks, with little faith in their readers' ability to keep track of units, frequently insert a factor g_c into their equations:

$$g_c = 32.2 \frac{\text{lbm ft}}{\text{lbf sec}^2} = 1.$$

Insertion of g_c is valid, since it has a dimensionless value of unity.

For example, the gas constant R is given by:

$$R = \frac{1544}{W} \frac{\text{ft lbf}}{\text{lbm deg R}} = \frac{49,724}{W} \frac{\text{ft}^2}{\text{sec}^2 \text{ deg R}} = \frac{8.31 \times 10^7}{W} \frac{\text{cm}^2}{\text{sec}^2 \text{ deg K}} = \frac{8.31 \times 10^7}{W} \frac{\text{erg}}{\text{gm deg K}},$$

where W is the molecular weight of the gas.

One other point to note is that the symbol "log" appearing in this report always denotes the natural logarithm.

The working equations summarized in this section are for three types of problems. In all three, it is assumed that the tube's inside diameter D , the discharge pressure p_3 , and the fluid's gas constant R , temperature T , and viscosity μ are known.

This leaves three variables: the tube length L , the upstream pressure p_1 , and the mass flow rate $\dot{m}$.

- To find L when p_1 and $\dot{m}$ are known, go to 7.1.
- To find p_1 when $\dot{m}$ and L are known, go to 7.2.
- To find $\dot{m}$ when L and p_1 are known, go to 7.3.

7.1 Finding the unknown tube length L :

7.11 Solve Eq. (15) for R .

7.12 Solve Eq. (29) for p_1 .

7.13 Solve Eq. (31) for p_2 .

7.14 If $R < 2100$, solve Eq. (27) for L .

7.15 If $R > 2100$, solve Eq. (38) for L .

7.2 Finding the unknown inlet pressure p_1 :

7.21 Solve Eq. (15) for Q .

7.22 Solve Eq. (30) for M_2 .

7.23 Solve Eq. (40) for ξ_1 .

7.24 If $Q > 2100$, solve Eq. (37) for M_1 , and go to 7.28.

7.25 If $L/D \ll \xi_1$, solve Eq. (31) for $\mathcal{P}_2$, Eq. (28) for $\mathcal{P}_1$, and Eq. (29) for p_1 .

7.26 If the point (ξ_1, Q) lies above the dotted curves in Figure 6, read M_1 from that figure.

7.27 If not, compute M_1 from Eq. (43).

7.28 Solve Eq. (45) for p_1 .

7.3 Finding the unknown mass flow rate $\dot{m}$:

7.31 Solve Eq (29) for $\mathcal{P}_1$.

7.32 For $\mathcal{P}_1$ and L/D , obtain the tentative value of Q from Figure 5.

7.33 If $Q > D p_3 / \mu \sqrt{RT}$, then $M_2 = 1$. Go to 7.35*.

7.34 If not, solve Eq. (47) for $\mathcal{P}_2$ and Eq. (48) for Q .

7.35 Solve Eq. (15) for $\dot{m}$.

* For a more precise answer for the case $M_2 = 1$ and $Q < 2100$, compute $\xi_1 = L/D$ and $\mathcal{K}_1 = 1/\mathcal{P}_1$, obtain M_1 from Figure 7, and solve Eq.(53) for $\dot{m}$.

Part II. TRANSIENT APPLICATIONS

8. Introduction

Let us now apply the general steady-state solutions derived in Part I to three specific problems connected with helium venting from an alpha-emitting radioisotope heat source:

- (1) Designing the vent tube to maintain a specified steady-state pressure p_v in the heat source when the alpha-decay helium is discharged to space vacuum.
- (2) For a specific vent tube design (L and D) and initial heat source pressure p_0 , determining the rate of pressure decay when the helium charge is vented to 1 atmosphere (e.g., at the launch pad).
- (3) Calculating the subsequent rate of pressure decay, after launch, when helium is discharged to space vacuum.

Problems 2 and 3 require transient analyses, using the steady-state solutions derived in Part I. Problem 2 is important, because accessibility of the RTG (e.g., for opening vent valves) may end thirty days prior to launch. Because of helium loss through the vent tube during the prescribed 30-day hold period, the internal heat source pressure will decay towards one atmosphere. It is essential that the pressure difference (i.e., inside minus outside) at the end of this period be large enough to avoid the danger that minor fluctuations of the heat source temperature or the barometric pressure will cause a pressure reversal. If that should occur, air would be sucked back into the heat source, causing oxidation of its internal graphite and refractory metal components. It is therefore important to determine whether the pressure decay transient is slow enough so that the pressure difference at the end of thirty days is still significantly above its steady-state value.

Problem 3 is important for the following reason: At the time of launch (i.e., when the shroud is jettisoned), there is a pressure differential

of more than 1 atmosphere pushing outward on the heat source can. Because of its thin walls (e.g., 0.010") and high operating temperatures ($>1000^{\circ}\text{C}$), the resultant stresses cause significant creep rates. It is therefore important to know how quickly the internal pressure decays, in order to determine the integrated creep deformation and to assess the resultant danger of mechanical damage or electrical shorting.

9. Designing the Vent Tube

Consider the 2400-watt Pu-238 heat source, with a steady gas release rate $\dot{m}$ of 1.76×10^{-8} gm/sec of helium, which has a gas constant R of 2.08×10^7 cm²/sec² deg K. Let us assume a uniform vent tube temperature of 1000°C . At that temperature, helium has a viscosity of 0.055 centipoise, or 5.5×10^{-4} gm/cm sec.

The vent tube is designed to discharge to space, so that $p_3 = 0$, $M_2 = 1$, $\gamma_0 = \gamma_2$, $\xi_2 = 0$, and $\xi_1 = L/D$. Because of the very small mass flow rate $\dot{m}$, it can be shown that M_1 is much smaller than unity, so that the value of M_1 can be computed from Eq. (51B), or from its dimensional equivalent, Eq. (52). The latter equation can therefore be used to compute the relation between the tube length L and the steady-state pressure p_v of the heat source operating in space vacuum.

Figure 8 illustrates that relationship, for tube diameters of 0.005, 0.010, 0.020, and 0.030 inch. For example, for a tube diameter D of 0.010", and a vacuum-discharge pressure p_v of 1 psi (6.89×10^4 dynes/cm²), the required tube length L is 7.4 ft (226 cm). While larger diameters lessen the danger of vent plugging by condensible deposits, they greatly increase the tube length required to maintain a given inlet pressure.

10. Transient Analysis

Sections 10 to 12 deal with transient analyses. Let m denote the mass of free helium in the heat source at time τ , and let V denote the void volume in the heat source. These are related by

$$m = V p_1 / RT_1, \quad (54)$$

where T_1 is the average temperature of the gas in the heat source, which may or may not be the same as the vent tube temperature T . The rate of change of the helium mass m is given by

$$dm/dt = (V/RT_1) dp_1/dt = -(\dot{m} - \dot{m}_\alpha), \quad (55)$$

where $\dot{m}$ is the mass flow rate through the vent tube, and $\dot{m}_\alpha$ represents the helium generation rate due to alpha decay (assuming 100% release).

In carrying out the transient analyses, it will be convenient to employ the previously defined Reynold's number,

$$R = 4 \dot{m} / \pi D \mu,$$

as well as the asymptotic limit R_∞ of the Reynold's number as $t \rightarrow \infty$,

$$R_\infty = 4 \dot{m}_\alpha / \pi D \mu. \quad (56)$$

R_∞ is the value of R when the pressure discharge transient has died out, and we are left with the steady-state condition $\dot{m} = \dot{m}_\alpha$. In actuality, of course, $\dot{m}_\alpha$ and R_∞ are both diminishing, as the result of isotope decay. But Pu-238 has a half-life of 87.4 years, which is much longer than the time scale of the transients of interest here. For purposes of the present analysis, it is therefore permissible to treat R_∞ as a constant.

Making use of the dimensionless pressure $\mathcal{P}$ defined earlier in Eq. (25), and of a dimensionless time Θ , defined by

$$\Theta \equiv \frac{D^2}{V} \left(\frac{\pi R T_1}{8 T} \right)^{1/2} t, \quad (57)$$

Eq. (55) can be expressed in the dimensionless form

$$d\mathcal{P}_1/d\Theta = - (R - R_\infty). \quad (58)$$

In Sections 11 and 12, we employ the steady-state solutions derived in Part I to express the dimensionless mass flow rate R as functions of the inlet pressure $\mathcal{P}_1$ and tube geometry ratio L/D , to permit integration of Eq. (58).

9. Discharge to One Atmosphere

The dimensionless inlet pressure $\mathcal{P}_1$ is given by

$$\mathcal{P}_1 = \frac{P_1}{\sqrt{\frac{1}{2}\pi RT} \mu / D} \quad (59)$$

Because of the very small mass flow ($\dot{m} < 1$ gm/yr) and the relatively high discharge pressure ($P_2 = 1$ atm), it can be shown that $\mathcal{M}_2 \ll 1$. Therefore, the exit and discharge pressures are the same ($P_2 = P_3$), so that

$$\mathcal{P}_2 = \frac{P_2}{\sqrt{\frac{1}{2}\pi RT} \mu / D} \quad (60)$$

and a solution for the dimensionless mass flow rate $\mathcal{Q}$ is given explicitly by Eq. (48A). Inserting that solution into Eq. (58), we obtain:

$$\frac{d\mathcal{P}_1}{d\theta} = - \frac{\left\{ (16L/D)^2 + \frac{1}{4}\pi(\mathcal{P}_1 - \mathcal{P}_2)(\mathcal{P}_1 + \mathcal{P}_2 + 22) \left[\log(\mathcal{P}_1/\mathcal{P}_2) + 11(\mathcal{P}_2^{-1} - \mathcal{P}_1^{-1}) \right] \right\}^{\frac{1}{2}} - 16L/D}{\log(\mathcal{P}_1/\mathcal{P}_2) + 11(\mathcal{P}_2^{-1} - \mathcal{P}_1^{-1})} + \mathcal{Q}_\infty \quad (61)$$

Equation (61) is a general relation between the dimensionless inlet pressure $\mathcal{P}_1$ and time θ . Unfortunately, it can only be integrated numerically. To seek an at least approximate analytical solution, let us examine the relative magnitudes of the various terms in the above equations. For the case discussed in Section 9, with a tube diameter D of 0.010", it can be shown that $\mathcal{P}_2 = 229$. The inlet pressure $\mathcal{P}_1$ of course is larger than $\mathcal{P}_2$. Therefore,

$$\mathcal{P}_1 + \mathcal{P}_2 + 22 \approx \mathcal{P}_1 + \mathcal{P}_2 \quad (62)$$

Similarly, if we expand the term in the square brackets in Eq. (61),

$$\left[\log\left(\frac{\mathcal{P}_1}{\mathcal{P}_2}\right) + 11(\mathcal{P}_2^{-1} - \mathcal{P}_1^{-1}) \right] = \left(\frac{\mathcal{P}_1 - \mathcal{P}_2}{\mathcal{P}_2}\right) \left[1 - \frac{1}{2}\left(\frac{\mathcal{P}_1 - \mathcal{P}_2}{\mathcal{P}_2}\right) + \frac{1}{3}\left(\frac{\mathcal{P}_1 - \mathcal{P}_2}{\mathcal{P}_2}\right)^2 + \frac{11}{\mathcal{P}_1} \right] \quad (63)$$

it is clear that the term $11/\mathcal{P}_1$ can be dropped, since it is small compared to unity. Equation (61) therefore reduces to

$$\frac{d\mathcal{P}_1}{d\theta} = - \frac{16L/D}{\log(\mathcal{P}_1/\mathcal{P}_2)} \left\{ \left[1 + \frac{\frac{1}{4}\pi(\mathcal{P}_1^2 - \mathcal{P}_2^2) \log(\mathcal{P}_1/\mathcal{P}_2)}{(16L/D)^2} \right]^{\frac{1}{2}} - 1 \right\} + \mathcal{Q}_\infty \quad (64)$$

The two terms we have dropped, 22 and $11 P_1^{-1}$, both derived from the slip-flow correction to the continuum-flow friction factor equation (see Eq. 22). As would be expected, this correction term becomes small when the discharge pressure is high enough to put us into the continuum-flow regime ($P > 100$).

Next, we note that the inlet pressure P_1 cannot be greater than a few atmospheres without exceeding the yield strength of the heat source wall. Therefore, P_1 is of the order 10^3 , and the term $(P_1^2 - P_2^2)$ in Eq. (64) is of the order 10^6 . The term $\log(P_1/P_2)$ is of the order unity or less, and the term $(16L/D)^2$ is of order $(10^5)^2 = 10^{10}$, as shown in Section 9. The second term in the square bracket of Eq. (64) is therefore of order 10^{-4} or less. Since $(\sqrt{1+\epsilon} - 1) \approx \frac{1}{2}\epsilon$ for small values of ϵ , Eq. (64) reduces to

$$\frac{dP_1}{dt} = - \left[\frac{\pi(P_1^2 - P_2^2)}{128 L/D} - \dots \right] \quad (65)$$

As $t \rightarrow \infty$ and the transient dies out, the right side of Eq. (65) must approach zero. For a tube vented to atmosphere, the asymptotic limit P_A of the inlet pressure P_1 is therefore given by

$$P_A = \left[P_2^2 + \frac{128 L}{\pi D} Q_\infty \right]^{1/2} = \left[\frac{D^2 P_2^2}{\frac{1}{2} \pi P T \mu^2} + \frac{5.12 L \dot{m}_a}{\pi^2 D^2 \mu} \right]^{1/2} \quad (66)$$

so that Eq. (65) can be expressed as

$$\frac{dP_1}{dt} = - \frac{\pi D}{128 L} (P_1^2 - P_A^2) \quad (67)$$

This can be integrated to yield

$$\left(\frac{P_1 - P_A}{P_1 + P_A} \right) = \left(\frac{P_0 - P_A}{P_0 + P_A} \right) \exp \left[- \frac{\pi D P_A}{64 L} \theta \right], \quad (68)$$

where P_0 is the initial value of P_1 , at time $t = \theta = 0$.

Since Φ is proportional to p , the dimensionless pressure ratios in Eq. (68) can be replaced by actual pressure ratios. Moreover, if we let

$$\tau = \frac{64(\log 2) L t}{\pi D^3 \Phi_A} = \frac{22.5 V L}{D^3 T_1 \Phi_A} \left(\frac{T}{R} \right)^{\frac{1}{2}} \quad (69)$$

Eq. (68) can be written as

$$\left(\frac{p_1 - p_A}{p_1 + p_A} \right) = \left(\frac{p_0 - p_A}{p_0 + p_A} \right) \left(\frac{1}{2} \right)^{\tau/\tau} \quad (70)$$

In any interval $\Delta t = \tau$, the pressure ratio on the left side of the equation diminishes by a factor of two. The variable τ may therefore be regarded as the half-life for pressure decay, when the vent is discharging to atmospheric pressure.

Equation (70) can be solved for p_1 to yield

$$p_1 = p_A \left[1 + \frac{2}{\left(\frac{p_0 + p_A}{p_0 - p_A} \right) (2)^{\tau/\tau} - 1} \right] \quad (71)$$

where the asymptotic pressure limit p_A for the case of atmospheric discharge is obtained from Eq. (25) and (66):

$$p_A = \left[p_0^2 + \frac{256 \mu K T L \dot{m}_0}{\pi D^4} \right]^{\frac{1}{2}} \quad (72)$$

Figure 9 shows plots of the pressure ratio p_1/p_A versus the time ratio t/τ , for $p_0/p_A = 2, 3, 4$, and 5. Since p_A must be larger than the discharge pressure p_3 (i.e., 1 atmosphere), the pressure ratio p_1/p_A can also be interpreted as a lower limit of the heat source pressure (in atmospheres) at time t . Thus, it can be seen from Figure 9 that, even if the ultimate pressure drop $p_A - p_3$ is very small, the pressure drops at times $t < 3\tau$ are tenths of atmospheres, which is high enough to prevent backstreaming due to thermal or barometric fluctuations. If possible, it would therefore be desirable to have a long half-life τ , to produce a slow transient.

Combining Eqs. (66) and (69), we find that τ is given by

$$\tau = \frac{28.1 \mu V}{p_3} \left(\frac{T}{T_1} \right) \left[1 + \left(\frac{256 \mu R T \dot{m}_a}{\pi p_3^2} \right) \left(\frac{L}{D^4} \right) \right]^{-\frac{1}{2}} \left(\frac{L}{D^4} \right). \quad (73)$$

Thus, the tube geometry enters only as the ratio L/D^4 . For the illustrative parameters presented earlier in Section 9, Figure 10 presents a plot of τ versus L/D^4 ; and Figure 11 presents curves of $\Delta p = p_1 - p_3$ versus time, for an initial value Δp_0 of 2 atmospheres, and for L/D^4 values from 0.3 to $3 \times 10^{10} \text{ cm}^{-3}$. As can be seen, pressure decay half-lives of days, weeks, and even months are possible with large enough values of L/D^4 . While such slow transients would be beneficial, it is not clear whether the required geometry ratios L/D^4 are practical, or whether they would lead to excessive pressure drops during steady-state operation in space vacuum. Figure 12 attempts to answer this question.

The solid curves in Figure 12 depict atmospheric-discharge solutions, while the dashed curves present vacuum-discharge solutions. In all cases, the pressure differential Δp is plotted against the tube geometry ratio L/D^4 . The bottom solid curve represents steady-state discharge to one atmosphere, and the other solid curves are the transient solutions at $\tau = 30$ days, for initial pressure differentials of 2, 3, and 4 atmospheres. The dashed curves depict steady-state discharge to vacuum, for tube diameters of 0.005, 0.010, and 0.020 inch.

As can be seen, for small values of L/D^4 the transient dies out too quickly to have any significant effect on the heat source pressure at the end of the prescribed 30-day hold period. For large values of L/D^4 , the half-life τ is large enough to have a pronounced effect at $\tau = 30$ days, although the benefit of the transient is really not needed, since the steady-state pressure p_A is high enough to prevent backstreaming. These high values of L/D^4 may be impractical, however, because the resultant vacuum-discharge pressure differential would cause excessive creep during long-term operation.

There is a rather narrow interval $(0.7 \times 10^{10} < L/D^4 < 1.2 \times 10^{10})$ where the steady-state pressure p_v for vacuum discharge appears low enough to prevent excessive creep, where the steady-state pressure differential in air is too low for comfort, but where the transient effect offers a significant benefit in reducing the danger of backstreaming.

10. Transient Discharge to Vacuum

The pressure decay is again governed by dimensionless Eq. (58):

$$\alpha \dot{P}_1 / d\theta = - (R - \alpha_\infty).$$

For vacuum discharge, $M_2 = 1$. Therefore, from Eq. (24), M_1 , R , and L/D are related by

$$64 L/D = (M_1^{-2} - 1 + 2 \log M_1) R + 22 \sqrt{\frac{1}{2} \pi} (1 - M_1)^2 / M_1. \quad (74)$$

Because of the very small mass flow rate $\dot{m}$, it can be shown that $M_1 \ll 1$, so that Eq. (74) can be approximated by

$$64 L/D \approx R M_1^{-2} + 22 \sqrt{\frac{1}{2} \pi} (M_1^{-1} - 2), \quad (75)$$

from which we can eliminate M_1 by means of Eq. (26):

$$64 L/D = \frac{1}{2} \pi P_1 (P_1 + 22) / R - 55. \quad (76)$$

The term "55" can be dropped, since for problems of interest it is clearly much smaller than $64 L/D$. Solving the remaining equation for R and inserting the result into Eq. (58), we obtain

$$\frac{dP_1}{d\theta} = - \left[\frac{\pi P_1 (P_1 + 22)}{128 L/D} - R_\infty \right]. \quad (77)$$

In the steady state, after the transient has died out, the right side of Eq. (77) must vanish. Therefore P_v , the steady-state value of P_i for vacuum discharge, must satisfy the quadratic equation

$$P_v^2 + 22 P_v = \frac{128 L}{\pi D} R_\infty = \frac{512 L \dot{m}_a}{\pi^2 D^2 \mu}, \quad (78)$$

and Eq. (77) may be rewritten as

$$\frac{dP_i}{dt} = - \frac{\pi D}{128 L} (P_i + P_v + 22)(P_i - P_v). \quad (79)$$

This can be integrated to yield

$$\left(\frac{P_i - P_v}{P_i + P_v + 22} \right) = \left(\frac{P_o - P_v}{P_o + P_v + 22} \right) \exp \left[- \frac{\pi (P_v + 11)}{64 L / D} t \right], \quad (80)$$

where P_o is the initial value of P_i .

11. Illustrative Application of Transient Solutions

To illustrate the use of the atmospheric-discharge and vacuum-discharge pressure decay equations derived in Sections 9 and 10, let us examine a specific case. As before, we assume a free heat source volume V of 6210 cm^3 , at an average temperature T_i of 1100°C . The 2400-watt Pu-238 heat source emits helium at a rate $\dot{m}_a$ of $1.78 \times 10^{-8} \text{ gm/sec}$. At the assumed vent tube temperature T of 1000°C , helium has a viscosity μ of 5.5×10^{-4} poise.

For illustrative purposes, let us choose a tube diameter D of $0.010'' = (0.0254 \text{ cm})$, and an L/D^4 ratio of $0.8 \times 10^{10} \text{ cm}^{-3}$. This corresponds to a tube length L of 109 ft or 3330 cm, and an L/D ratio of 1.31×10^5 . Let us further assume that the heat source is charged to an initial pressure P_o of 3 atmospheres, and is then vented to the atmosphere for a time t of 15 days. At the end of this period it is launched into space, where further pressure decay occurs as the result of venting to vacuum.

From the definitions of the dimensionless parameters, we find that $\mathcal{P} = 229 \times (\bar{p} \text{ in atm})$, $\mathcal{Q} = 0.00289 \times (\dot{m} \text{ in gm/yr})$, $\mathcal{M} = (\bar{v} \text{ in km/sec})/163$, and $\mathcal{T} = 987 \times (\bar{t} \text{ in days})$. For the specified value $\dot{m}_\alpha = 0.561 \text{ gm/yr}$, we can compute the steady-state values: $\mathcal{Q}_\infty = 0.00162$, $\mathcal{P}_A = 247 (1.081 \text{ atm})$, $\mathcal{P}_V = 82.7 (0.362 \text{ atm})$, $\mathcal{M}_{1,\infty} = 1.56 \times 10^{-5} (2.55 \text{ cm/sec})$, and $\mathcal{M}_{2,\infty} = 1 (163 \text{ km/sec})$. From the initial pressure $\bar{p}_0 = 3 \text{ atm}$, we compute $\mathcal{P}_0 = 686$, $\mathcal{P}_2 = 229$, $\mathcal{Q}_0 = 0.00803 (2.78 \text{ gm/yr})$, $\mathcal{M}_{1,0} = 0.936 \times 10^{-4} (15.2 \text{ cm/sec})$, and $\mathcal{M}_{2,0} = 2.808 \times 10^{-4} (45.7 \text{ cm/sec})$. The half-life τ for pre-launch pressure decay is 7.56 days.

From the value of $\mathcal{Q}_\infty$ and $\mathcal{P}_V$, we can compute the steady-state value of $\mathcal{M}_1$ in vacuum, $\mathcal{M}_{1,\infty} = 1.565 \times 10^{-5}$. It is of parenthetical interest to use this to compute the mean transit time $\bar{T}$ of the gas passing through the tube:

$$\bar{T} = \int_{x_1}^{x_2} \frac{dx}{v} = \frac{D}{\sqrt{RT}} \int_0^{\mathcal{M}_1} \frac{d\mathcal{M}}{\mathcal{M}} = \frac{D}{\sqrt{RT}} \int_{\mathcal{M}_1}^1 \frac{(\mathcal{R} + 13.8\mathcal{M})(1 - \mathcal{M}^2)}{32\mathcal{M}^4} d\mathcal{M}, \quad (81)$$

which integrates to yield

$$\bar{T} = \frac{D}{96\sqrt{RT}} \left[\mathcal{R}(\mathcal{M}_1^{-3} - 3\mathcal{M}_1^{-1} + 2) + 20.7(\mathcal{M}_1^{-2} - 1 + 2 \log \mathcal{M}_1) \right]. \quad (82)$$

For the case at hand, the steady-state value of $\bar{T}$ in vacuum is 19.5 minutes.

The transient solutions, both for the assumed 15-day period before launch and for an illustrative 45-day period after launch, are displayed in Figures 13, 14, and 15. Each figure shows the time variation of dimensionless and the corresponding dimensional variables. Figure 13 displays the heat source pressure ($\mathcal{P}_1, \bar{p}_1$) and discharge pressure ($\mathcal{P}_2, \bar{p}_2$). Figure 14 depicts the mass flow rate ($\mathcal{Q}, \dot{m}$) through the vent tube. And Figure 15 shows the inlet and outlet velocities ($\mathcal{M}, \bar{v}$). As can be seen, the flow rate and rate of pressure drop increase discontinuously at launch. This is because of the 1-atmosphere step increase of the pressure differential Δp . For the same Δp , the flow rate is substantially higher in atmospheric discharge than in vacuum discharge.

One variable not plotted in Figure 13 is the outlet pressure (P_2, p_2) after launch. This varies from $P_2 = 0.0158$ (0.052 torr) at launch to $P_2 = 0.00129$ (0.00429 torr) at steady state. Thus, the outlet flow after launch is always free-molecular. The inlet flow changes from continuum at launch to slip-flow at steady state.

Finally, the three figures verify our earlier assumptions: Before launch, $M_1 \ll 1$, $M_2 \ll 1$, $P_2 > 100$ (continuum flow), and $Re < 2100$ (laminar flow). After launch, $M_1 \ll 1$, and $Re < 2100$. This confirms the validity of the approximations used in deriving the transient solutions for this type of problem.

References

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Figure 1:
EXPERIMENTAL FIT FOR CORRECTION FACTOR Φ

$$\Phi = \frac{16}{fR}$$

Figure 2:
GENERAL SOLUTION OF EQUATIONS 17 AND 34.

Figure 3:
GENERAL SOLUTION OF EQUATIONS 17 AND 34.

Figure 4
MAP OF INLET AND OUTLET FLOW REGIMES .

INLET , $\mathcal{K}_1 = \frac{1}{\mathcal{K}_1} = \frac{p_1}{\sqrt{\frac{1}{2}\pi RT} \mu/D}$

FIG. 5A
 GENERAL SOLUTION FOR GAS FLOW TO VACUUM
 Also valid when discharge pressure is less than $\mu\sqrt{RT}\dot{A}/D$

FIG. 5:
GENERAL SOLUTION FOR GAS FLOW TO VACUUM
 Also valid when discharge pressure is less than $\mu\sqrt{RT}\mathcal{R}/D$

$$\sqrt{\xi_1} = \left[\frac{1}{D} + \frac{R}{64} \left(\mathcal{M}_2^{-2} - 1 + \log_e \mathcal{M}_2^2 \right) + 0.431 \left(1 - \mathcal{M}_2 \right)^2 / \mathcal{M}_2 \right]^{\frac{1}{2}}, \text{ where } \mathcal{M}_2 = \frac{4 \dot{m} \sqrt{RT}}{\pi D^2 p_3} \text{ or } 1, \text{ whichever is less}$$

$$\mathcal{M}_1 = \frac{4 \dot{m} \sqrt{RT}}{\pi D^2 p_1}$$

$$R = \frac{\dot{m}}{\frac{1}{4} \pi D \mu}$$

Laminar flow limit

Figure 6:
SOLUTION FOR UNKNOWN INLET PRESSURE p_1

$$\sqrt{\xi_1} = \left(\frac{x_0 - x_1}{D} \right)^{\frac{1}{2}}, \left[= \sqrt{\frac{L}{D}} \text{ when } M_2 = 1 \right]$$

Figure 8:
 LENGTH OF VENT TUBE REQUIRED
 FOR A GIVEN BACK PRESSURE IN SPACE (VACUUM DISCHARGE)

Figure 9 : HEAT SOURCE PRESSURE DECAY
ATMOSPHERIC DISCHARGE

Figure 10:
EFFECT OF TUBE GEOMETRY
ON PRESSURE DECAY HALF-LIFE τ

Figure 11 : TIME VARIATION OF
PRESSURE DIFFERENCE ACROSS HEAT SOURCE WALL

Figure 12: PRESSURE DIFFERENCE Δp ACROSS HEAT SOURCE WALL
VERSUS TUBE GEOMETRY RATIO (L/D^4)

Figure 13:
HEAT SOURCE PRESSURE DECAY
BEFORE AND AFTER LAUNCH

Figure 14:
HELIUM VENTING RATE
BEFORE AND AFTER LAUNCH

Figure 15:
INLET AND OUTLET FLOW VELOCITIES
BEFORE AND AFTER LAUNCH

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