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THE SCATTERING AND POLARIZATION
OF SPIN ONE PARTICLE

G. R. Satchler

OAK RIDGE NATIONAL LABORATORY

operated by

UNION CARBIDE CORPORATION

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Oak Ridge, Tennessee
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I. INTRODUCTION

The present notes were motivated by a need to review and extend the published theory of the scattering and polarization of spin one particles, and to rewrite some of the formulae in a way more suitable for machine computation. This need arose in preparation for a study of deuteron elastic scattering in terms of a phenomenological optical model with spin-orbit coupling. The wave functions given by such a model may then be used in the distorted wave theory of direct reactions involving deuterons.⁽¹⁾ The theory is equally applicable to the scattering of other spin one particles, such as N^{14} ions.⁽²⁾

The elastic scattering of deuterons, for example, may serve as

- (i) an analyzer for the polarization of deuterons produced in a previous reaction,
- (ii) a source of polarized deuterons for use in a subsequent reaction, or
- (iii) a tool for studying the interaction of deuterons with complex nuclei and testing various models of this interaction.

Section II discusses the optical models which may be used to describe the scattering, and the possible forms of spin-orbit coupling. The scattering amplitudes are defined and described in Section III, including the effects of an isolated resonance, and used to give expressions for the differential cross-section and polarization in Section IV. The latter results are extended to include the change of polarization induced on scattering a polarized beam. In Section V we give formulae for the transformation of the polarization from center-of-mass to the laboratory system; this is necessary for the double scattering described in Section VI. Finally the change in polarization produced

by deflection in a magnetic field is described in Section VII. A more detailed discussion of possible tensor spin-orbit couplings is given in an appendix.

A number of authors have discussed the scattering and reaction of spin one particles, both from general principles^(3,4,5), and explicitly in terms of partial reaction amplitudes and scattering phase shifts^(6,7,8,9,10,11). Other authors have discussed special cases of scattering^(12,13) and reactions^(14,15) involving polarized deuterons.

II. OPTICAL MODELS FOR SPIN ONE PARTICLES

(a) The Central Potential

An optical model replaces the interaction between a particle and a complex nucleus by a simple complex potential well. The imaginary part of the potential accounts for inelastic events, whether they occur by absorption into a compound nucleus or by some direct mechanism. Following its initial success in describing the average scattering of nucleons, it has been applied to the scattering of deuterons,⁽¹⁶⁾ alphas, and even nitrogen ions.⁽²⁾ Attempts have been made to relate the deuteron potential to the optical potentials of the constituent neutron and proton,^(3,5,17) but they have been confined to an impulse approximation only suitable to high energies (of the order of 100 Mev or greater). Thus at lower energies the deuteron optical potential remains phenomenological, and we write it in the form

$$V(r) = -V f(r) - i W g(r). \quad (1)$$

V and W are the "strengths" of the real and imaginary parts, respectively, and f and g the corresponding "form factors." Fits to the low energy data have

so far used the Woods-Saxon form for both f and g ,⁽¹⁶⁾

$$f(r) = g(r) = \frac{1}{1 + \exp(\frac{r-R}{\alpha})}, \quad (2)$$

with V typically about 55 Mev while W ranges between 10 and 20 Mev. These figures correspond to a mean free path in nuclear matter between 1 and 3f. The data is insensitive to the potential in the interior, the scattering being dominated by the reflection and absorption near the surface. In view of the ease with which deuterons can initiate a direct reaction such as stripping, there may be some reason for peaking the imaginary potential $g(r)$ (and hence the absorption) at the nuclear surface.

(b) Spin-Orbit Coupling

Scattering by the central potential (1) will not induce any spin polarization, so the next step is to include some spin-orbit coupling in the optical potential. With a spin one particle this may take the form of a vector or tensor coupling. The vector coupling is the familiar one

$$[a_v(r) + i b_v(r)] \underline{L} \cdot \underline{S}. \quad (3)$$

For each partial wave with orbital angular momentum L , the total angular momentum of the particle may have the values $J = L, L \pm 1$. Then the potential due to (3) seen by the particle in these 3 states is

$$\left. \begin{array}{lll} [a_v + i b_v] L & \text{if} & J = L+1, \\ -[a_v + i b_v] & \text{if} & J = L, \\ -[a_v + i b_v] (L+1) & \text{if} & J = L-1. \end{array} \right\} \quad (4)$$

Of course, both J and L are conserved by this interaction. The form factors a_v , b_v are often chosen proportional to the gradient of the real and imaginary parts, respectively, of the central potential (1), so that they peak at the nuclear surface.

With spin one we may have in addition to (3) a coupling which is the scalar product of two tensors which are of rank two in the separate spin and orbit spaces. (18) This we may write

$$[a_T(r) + i b_T(r)] T \quad (5)$$

In spin space we have only one operator, $\underline{s}$, but in coordinate space we have three vectors, the position vector $\underline{r}$, the momentum $\underline{p} = -i\nabla$, and the orbital momentum $\underline{L} = -i\underline{r} \times \nabla$. The requirements of time reversal invariance and parity conservation do not allow us to mix these in constructing the tensor operator T. (25) So we can form three irreducible tensors T, one using $\underline{r}$,

$$T = \frac{(\underline{s} \cdot \underline{r})^2}{r^2} - \frac{2}{3}, \quad (6)$$

another using $\underline{P} = -i\nabla$,

$$T = (\underline{s} \cdot \underline{P})^2 - \frac{2}{3} P^2, \quad (7)$$

and finally one using L,

$$T = (\underline{L} \cdot \underline{s})^2 + \frac{1}{2} (\underline{L} \cdot \underline{s}) - \frac{2}{3} L^2. \quad (8)$$

A detailed discussion of the properties of these three and their matrix element analogs of (4) is relegated to Appendix III. Only the first, (6), has been discussed previously, in a special form appropriate to deuterons. (11) An unfortunate property (for computation purposes) of this and the momentum tensor (7) is that both have matrix elements off-diagonal in L, coupling partial

waves with angular momentum differing by two units (not one, because of parity conservation). Until we have definite form factors $a_T(r)$ and $b_T(r)$ we have no means of assessing the importance of these inter-channel transitions. There is no obvious reason why the off-diagonal elements should be appreciably smaller than the diagonal ones, especially if the interaction were concentrated at the nuclear surface, although the total effect of such transitions could be small. The answers to these questions can only be obtained by numerical studies. However, as a practical expedient, the first attempts to include the effects of a tensor coupling would presumably ignore the off-diagonal terms.

Apart from this, the multiplicity of possible tensor couplings is regrettable, especially as we have no reliable model for the interaction at lower energies. It means it may be difficult to get unambiguous results even within the framework of the optical model, although we may hope the vector coupling (3) is the most important; indeed for deuterons in the lowest (impulse) approximation it is the only non-zero term. (17)

The treatment so far has assumed we may neglect any coupling to the spin of the target nucleus. While there is undoubtedly some coupling to the target spin (when this is non-zero), for deuterons there is no reason to believe this is important for the average behavior of the scattering which is all an optical model describes. This last remark may not be true for heavy ions like N^{14} , where the target and incident particles may be comparable. Then, for example, the coupling of the target spin to the relative orbital angular momentum of the pair may be as strong as the coupling of the incident spin (as will be the case for the scattering of N^{14} by N^{14}). However, differential cross-sections

are usually not very sensitive to the spin-orbit coupling, and unless we wish to measure the polarization of the recoiling target this neglect should not cause much error.

III. SCATTERING AMPLITUDES

The wave function Ψ describing the scattering from spinless target nuclei is a solution of the Schrödinger equation

$$\nabla^2 \Psi(\underline{r}) + \frac{2\mu}{\hbar^2} \left[E_0 - \frac{Z_1 Z_2 e^2}{r} - V(r) \right] \Psi(\underline{r}) = 0. \quad (9)$$

E_0 is the energy of relative motion, (that is, in the center-of-mass system), $Z_1 e$, $Z_2 e$ are the charges of the incident and target particles, and $V(r)$ is the nuclear interaction potential between them (which may include the spin-orbit coupling described in the previous section). In the absence of any interaction ($Z_1 = V = 0$), for a particle of spin s with z -component m , Ψ would have the form

$$\Psi_m(\underline{r}) = \frac{1}{\sqrt{v}} e^{i \underline{k} \cdot \underline{r}} \chi_{sm} ; \quad k^2 = \frac{2\mu E_0}{\hbar^2}, \quad (10)$$

normalized to unit flux (v is the particle's speed). χ_{sm} is the spin "function" for spin s . With interactions, Ψ_m asymptotically ($r \rightarrow \infty$) has the form $\Psi_m(\text{incident}) + \Psi_m(\text{scattered})$ where

$$\Psi_m(\text{scattered}) = \sum_{m'} A_{m'm}(\theta) \chi_{sm'} \frac{e^{i(kr - \eta \log 2kr + \sigma_0)}}{r\sqrt{v}} \quad (11)$$

We now adhere to the notation and definitions used by Lane and Thomas.⁽¹⁹⁾

$A_{m'm}(\theta)$ is the amplitude for scattering through an angle θ when the emergent particle has spin z-component m' ($m' \neq m$ may be called "spin-flip" terms, and will arise only when spin-orbit coupling is included). The usual Coulomb parameter $\eta = Z_1 Z_2 e^2 / \hbar v$, and σ_0 is the S-wave Coulomb phase shift, $\sigma_0 = \arg \Gamma(1 + i\eta)$. Then by definition the differential cross-section for unpolarized particles is just

$$\frac{d\sigma(\theta)}{d\omega} = \frac{1}{2s+1} \sum_{m,m'} |A_{m'm}(\theta)|^2 \quad (12)$$

We may express $A_{m'm}$ in terms of the contributions from the various partial waves. The solution Ψ_m of (9) corresponding to incoming particles with spin component m may be expanded in partial waves of total angular momentum J , orbital momentum L ,

$$\Psi_m = \sum_{J L L'} a_{J L m} \frac{1}{r} \bar{F}_{L'L}^J(r) \psi_{J L' m} \quad (13)$$

where $\psi_{J L m}$ is the spin and angle function⁽²⁰⁾

$$\psi_{J L m} = \sum_{m'} i^L Y_L^{m-m'}(\theta, \phi) \chi_{s m'} (L s m-m' m | J m). \quad (14)$$

We have chosen the z-axis along the incident direction, so the total z-component of angular momentum is also m , and the coefficients in (13) are

$$a_{J L m} = \sqrt{4\pi(2L+1)} (L s 0 m | J m) / k \sqrt{v} \quad (15)$$

The terms $L' \neq L$ in (13) are the transitions between different orbital momenta induced by the tensor spin orbit coupling, as discussed in the previous section. Inserting (13) in the equation (9) we obtain the coupled radial equations⁽²¹⁾

$$\left[\frac{d^2}{dr^2} - \frac{L'(L'+1)}{r^2} + k^2 - \frac{\gamma k}{r} - \frac{2\mu}{\hbar^2} V_L^J \right] \bar{F}_{L'L}^J(r) = \sum_{L'' \neq L'} \frac{2\mu}{\hbar^2} \langle L'JM | L''J \rangle \bar{F}_{L''L}^J(r) \quad (16)$$

$V_L^J(r)$ is the diagonal value of $V(r)$, $\langle LJ | V | LJ \rangle$, for a particle with J and L , for example given by equation (4). The only dependence on J enters here (and on the right hand side). In the "external" region, where $V(r)$ is negligible, $\bar{F}_{L'L}^J$ is expressible in terms of the regular F_L , and irregular G_L , Coulomb functions,⁽¹⁹⁾ which are solutions of (16) with $V = 0$. Using the collision matrix $U_{L'L}^J$ as defined by Lane and Thomas,⁽¹⁹⁾

$$\bar{F}_{L'L}^J(r) = e^{i\omega_L} F_L(r) \delta_{L'L} + \frac{i}{2} e^{-i\omega_L} [e^{2i\omega_L} \delta_{LL'} - U_{L'L}^J] H_{L'}(r), \quad (17)$$

where $H_L = G_L + i F_L$ is the outgoing wave combination, and $\omega_L = \sigma_L - \sigma_0 =$

$\sum_{n=1}^L \tan^{-1}(\gamma/n)$ is the Coulomb phase shift. In the absence of any nuclear interaction, $V = 0$, U becomes diagonal, $U_{L'L}^J = e^{2i\omega_L} \delta_{LL'}$, and only the first term ("pure Coulomb") of (17) survives. Of course, if also the particle is uncharged, $\gamma = 0$, $\omega_L = 0$, and $F_L(r)/kr$ becomes $j_L(kr)$, the spherical Bessel component of a plane wave, with no scattering at all. The collision matrix U takes account of the additional scattering induced by the nuclear interaction V in (9) or (16). To be specific, let us ignore the possibility of terms off-diagonal in L ; this is the case of most interest. Then writing $\bar{F}_{LL}^J = \bar{F}_L^J$,

where now

$$\left[\frac{d^2}{dr^2} - \frac{L(L+1)}{r^2} + k^2 - \frac{2\gamma k}{r} - \frac{2\mu}{\hbar^2} V_L^J(r) \right] \bar{F}_L^J(r) = 0, \quad (18)$$

and putting $U_{LL}^J = \exp[2i(K_L^J + \omega_L)]$, (17) reduces to

$$\bar{F}_L^J(r) = e^{i\omega_L} \left[F_L(r) + e^{iK_L^J} \sin K_L^J H_L(r) \right]. \quad (19)$$

The significance of (19) is that the nuclear interaction V has introduced an additional phase shift K_L^J relative to the Coulomb waves. If the potential V is complex (to simulate absorption) then K_L^J is also complex. The experimental observables may be expressed in terms of the K_L^J . In practice we compute K_L^J by integrating the radial equation (18) out from the origin (where $\bar{F}_L^J(0) = 0$) and then matching on to the solution (19) when $V(r)$ has become negligible.

We now write down the partial wave expansion of the amplitudes in (11), corresponding to (13). If particles are incident along $\underline{k}_1$ and emerge along $\underline{k}_2$ (at angle θ), with arbitrary quantization axes we have

$$A_{m'm}(\underline{k}_1 \rightarrow \underline{k}_2) = -\frac{\sqrt{\pi}}{k} C(\theta) \delta_{mm'} + \frac{2\pi i}{k} \sum_{JLL'MM'} Y_L^M(\underline{k}_1)^* Y_{L'}^{M'}(\underline{k}_2) \quad (20) \\ \times \left[e^{2i\omega_L} \delta_{LL'} - U_{LL'}^J \right] (L S M_m | J M+m) (L' S M_{m'} | J M+m).$$

(a vector $\underline{k}$ as argument of a spherical harmonic denotes its polar angles θ_k, ϕ_k).

If we choose the z -axis along $\underline{k}_1$ and y -axis along $\underline{k}_1 \times \underline{k}_2$ (corresponding to (15)), (20) becomes⁽¹⁹⁾

$$A_{m'm}(\theta) = \frac{\sqrt{\pi}}{k} \left\{ -C(\theta) \delta_{mm'} + i \sum_{JL} \sqrt{2L+1} Y_L^{m-m'}(\theta, 0) \right. \\ \left. \times \left[e^{2i\omega_L} \delta_{LL'} - U_{LL'}^J \right] (Ls0m|Jm)(L's_{m-m'm'}|Jm) \right\} \quad (21)$$

$C(\theta)$ is the purely Coulomb contribution to the scattering, being the sum of the contributions from the F_L of (17).

$$C(\theta) = \frac{1}{\sqrt{4\pi}} \eta \operatorname{cosec}^2\left(\frac{1}{2}\theta\right) e^{-2i\gamma \log(\sin \frac{1}{2}\theta)} \quad (22)$$

When the spin $S = 1$, $J = L, L \pm 1$ only, and we can only have $L' = L$, or $L \pm 2$ ($L \pm 1$ being excluded because of parity). If we neglect the off-diagonal, $L \neq L'$, terms, we may put

$$\alpha_L^J = \frac{i}{2} \left[e^{2i\omega_L} - U_{LL}^J \right] = e^{2i\omega_L} \left[e^{iK_L^J} \sin K_L^J \right], \quad (23)$$

call $-\sqrt{\pi} C(\theta) = \gamma(\theta)$, and define the dimensionless amplitude

$$a_{m'm}(\theta) = k A_{m'm}(\theta) \\ = \gamma(\theta) \delta_{mm'} + \sum_{JL} \sqrt{4\pi(2L+1)} Y_L^{m-m'}(\theta, 0) \alpha_L^J (Ls0m|Jm)(Ls_{m-m'm'}|Jm). \quad (24)$$

It obeys the symmetry relation

$$a_{-m'-m}(\theta) = (-)^{m-m'} a_{m'm}(\theta). \quad (25)$$

There is another symmetry relation we shall use later. If $\hat{a}_{m'm}(\theta)$ denotes the amplitude for scattering through θ , but with z-components m, m' measured

with z-axis along $\underline{k}_2$, y axis along $\underline{k}_2 \times \underline{k}_1$, it follows that

$$\hat{a}_{m'm}(\theta) = a_{m'm}(\theta). \quad (26)$$

With spin $S = 1$, and the explicit values for the Clebsch-Gordan coefficients we soon get from (24)

$$a_{00} = \gamma + \sum_L P_L(\cos \theta) [(L+1)\alpha_L^+ + L\alpha_L^-]$$

$$a_{11} = \gamma + \sum_L P_L(\cos \theta) \frac{1}{2} [(L+2)\alpha_L^+ + (2L+1)\alpha_L^0 + (L-1)\alpha_L^-]$$

$$a_{10} = \sum_L P_L^1(\cos \theta) \frac{1}{\sqrt{2}} [\alpha_L^+ - \alpha_L^-]$$

$$a_{01} = \sum_L P_L^1(\cos \theta) \frac{1}{\sqrt{2}} \frac{1}{L(L+1)} [-L(L+2)\alpha_L^+ + (2L+1)\alpha_L^0 + (L+1)(L-1)\alpha_L^-]$$

$$a_{1-1} = \sum_L P_L^2(\cos \theta) \frac{1}{2} \frac{1}{L(L+1)} [L\alpha_L^+ - (2L+1)\alpha_L^0 + (L+1)\alpha_L^-]$$

$$= -\sqrt{2} \cot \theta (a_{01} + a_{10}) + (a_{11} - a_{00}). \quad (27)$$

We have used the superscripts +, 0, - to denote $J = L + 1, L, L-1$, respectively.

The other amplitudes are related by the symmetry (25). If there is no spin-orbit coupling, $a_{m'm}$ vanishes unless $m = m'$ (no spin-flip terms) and a_{mm} becomes independent of m . This is reflected in the partial wave amplitudes

α_L^J or U_{LL}^J which become independent of J . Then $a_{10} = a_{01} = a_{1-1} = 0$, and

$$a_{11} = a_{00} = \gamma + \sum_L (2L+1) \alpha_L P_L(\cos \theta). \quad (28)$$

Although our main interest here is in using an optical potential to represent the nuclear interaction V , the above formulae may be applied to other situations by taking the appropriate collision matrix elements.⁽¹⁹⁾ Equations (20) and (21) are quite general, except in so far as we have taken the target to have zero spin (or, equivalently, have neglected any coupling to the target spin). Equations (24) and (27) also neglect transitions between different L values. One problem of interest^(11,12,13) is to include an isolated resonance superimposed on a smooth background of "potential" scattering. Following Lane and Thomas⁽¹⁹⁾ (Section XII), the collision matrix U may then be written as a sum of two terms, $U = U^{(o)} + U^{(r)}$, $U^{(o)}$ describing the background scattering due to far-off levels and $U^{(r)}$ arising from the resonance level nearby. In the approximation that the background contribution R^o to the R matrix is considered diagonal, $U^{(o)}$ is the same as (18),

$$U_{LL}^{(o) J} = e^{2i(\kappa_L^J + \omega_L)},$$

while the resonant contribution (for a resonance of definite spin J_o and total width Γ_o) has the Breit-Wigner form

$$U_{L'L}^{(r) J} = \pm i \frac{[\Gamma_L^{J_o} \Gamma_{L'}^{J_o}]^{\frac{1}{2}}}{E_o - E - \frac{i}{2} \Gamma_o} e^{i(\omega_{L'} + \omega_L + \kappa_{L'}^{J_o} + \kappa_L^{J_o})} \quad (29)$$

(the level shift Δ_0 has been absorbed into the resonance energy E_0). ~~It~~ occurs according to the relative signs of the reduced widths. Note that instead of the usual "hard sphere" phase, $-\phi_L = -\tan^{-1}(F_L/G_L)$, there appears the same potential phase K_L^J in $U^{(r)}$ as in $U^{(o)}$. In the limit that the far-off levels do not contribute to the scattering ($R^0 = 0$), $K_L^J = -\phi_L$. Otherwise one may use a phenomenological optical potential to calculate the $U^{(o)}$ scattering by integrating (16) as before, then adding a term (29). Of course, as it stands (29) is only appropriate to a spin zero target; the generalization for non-zero spin is straightforward^(10,19) but will not be repeated here. For spin 1 particles, $L^0 = L$ or $L \pm 2$, only. So, if the resonance has parity $(-)^{J_0}$, only $L^0 = L = J_0$ can enter (if the target has even parity) and $U^{(\dagger)}$ becomes diagonal. Even without this condition, it is likely that the partial width $\Gamma_L^{J_0}$ for one L will be dominant, with the same effect.

The form (29) might be applied also when the compound nucleus levels are overlapping, but the cross-section is showing fluctuations as a function of energy. If we write the collision matrix as $U = \bar{U} + (U - \bar{U})$, its average behavior $\bar{U}$ is described by the optical model, or $U^{(o)}$, as above. One might attempt to fit a strong fluctuation from $(U - \bar{U})$ by a resonant term like (29). However, unless such a fluctuation were due to the contribution from one abnormally strong level (one term in the R matrix), the potential phase in (29) would have to be treated as an adjustable parameter.

IV. CROSS-SECTION AND POLARIZATION

Experimental observables depend upon quadratic combinations of the amplitudes $a_{m'm}$. By using the properties of angular momenta,⁽¹⁸⁾ the various

sums over magnetic quantum numbers may be carried out, and the results presented in compact form.⁽¹⁰⁾ However, these expressions contain double sums over partial waves L , etc., each of which may run to 20 or more terms at a typical energy of around 10 Mev (for deuterons). Moreover, the expressions for different observables weight the terms in these sums differently. Hence for our purpose it is more convenient to compute directly each amplitude $a_{m'm}(\theta)$ for each angle of interest, using equations (27), then express observables in terms of these.

The differential cross-section for unpolarized particles is given immediately from (12) and (24),

$$\begin{aligned} \frac{d\sigma(\theta)}{d\omega} &= \frac{\sum_{m m'} |a_{m'm}(\theta)|^2}{(2s+1) k^2} \\ &= \frac{1}{3k^2} \left[|a_{00}|^2 + 2(|a_{11}|^2 + |a_{10}|^2 + |a_{1-1}|^2 + |a_{01}|^2) \right] \quad (30) \end{aligned}$$

for spin $s = 1$, using the symmetry (25).

The polarization of particles of spin 1 is conveniently described by certain spin tensors t_{KQ} , with $K = 0, 1$ and 2 . These are the expectation values of operators T_{KQ} , which are covariant combinations of the components of the spin operator $\underline{s}$,

$$\begin{aligned} T_{00} &= 1 \quad ; \quad T_{1Q} = \sqrt{\frac{3}{2}} S_Q \quad ; \\ T_{20} &= \sqrt{\frac{1}{2}} (3 S_0^2 - 2) \quad ; \\ T_{2\pm 1} &= \sqrt{\frac{3}{2}} (S_0 S_{\pm 1} + S_{\pm 1} S_0) \quad ; \\ T_{2\pm 2} &= \sqrt{3} S_{\pm 1}^2 \quad , \end{aligned} \quad (31)$$

where $S_0 = S_z$, $S_{\pm 1} = \mp \sqrt{\frac{1}{2}} (S_x \pm i S_y)$. Then $t_{KQ} = \langle T_{KQ} \rangle$ is the same quantity that Lakin and Wolfenstein define.⁽³⁾ The $K = 1$ tensor is proportional to the usual vector polarization $\underline{P}$,

$$t_{1Q} = \sqrt{\frac{3}{2}} P_Q. \quad (32)$$

The tensors have the conjugation property

$$t_{KQ} = (-)^Q t_{K,-Q}^* \quad (33)$$

These particular combinations are chosen because of their properties under rotation of the coordinate axes. If the polarization is described by tensors t_{KQ} in some coordinate system, then in a new coordinate system the same polarization is described by tensors $\hat{t}_{KQ'}$, where

$$\hat{t}_{KQ'} = \sum_Q t_{KQ} D_{QQ'}^K(\alpha\beta\gamma) \quad (34)$$

$(\alpha\beta\gamma)$ are the Euler angles of the rotation which takes the old coordinate axes into the new, and the $D_{QQ'}^K$ are elements of the rotation matrix.⁽¹⁸⁾ The elements required for spin 1 ($K = 0, 1, 2$) are displayed in Appendix I. We shall need the transformation (34) in the following sections.

There are limitations on the magnitudes of the t_{KQ} ,

$$\sum_{KQ} |t_{KQ}|^2 \leq 2S+1,$$

while for spin $S = 1$, $-\sqrt{2} \leq t_{20} \leq \sqrt{1/2}$, and the real and imaginary parts of t_{21} , t_{22} lie between $\pm \frac{1}{2}\sqrt{3}$. Of course, the vector polarization $\underline{P}$ always lies between ± 1 .

Suppose the incident particles of spin S have Z -component of spin m with a probability amplitude b_m , normalized so that for an unpolarized beam

$\langle b_{m m'}^* \rangle = \frac{1}{2S+1} \delta_{mm'}$. Then the spin density matrix for the incident beam is $\rho_{mm'} = \langle b_{m m'}^* \rangle$, with $\text{tr } \rho = 1$. The associated spin tensors t_{kq}^{in} are

$$t_{kq}^{\text{in}} \equiv \langle T_{kq} \rangle_{\text{incident}} = \sum_{mm'} \rho_{mm'} \sqrt{2S+1} (-)^{k+m+s} (SS-m m' | kq). \quad (35)$$

If the incident beam is unpolarized, $\rho_{mm'} = \delta_{mm'}/(2S+1)$, and only $t_{00}^{\text{in}} = 1$ is non-zero. After scattering, described by the amplitudes $a_{m'm}$, the outgoing particles have polarization tensors t_{KQ}^{out} given by

$$\left(k^2 \frac{d\sigma(\theta)}{d\omega} \right) t_{KQ}^{\text{out}}(\theta) = \sum_{mm'm''m'''} \rho_{mm'} a_{m''m} a_{m'''m'}^* \sqrt{2S+1} (-)^{K+m''+s} (SS-m' m''' | KQ) \quad (36)$$

$(d\sigma/d\omega)$ is the unpolarized cross-section (30) and, ^{with k^2 ,} appears so that t_{00}^{out} is normalized to unity when unpolarized particles are incident. Using (35),

(36) may be put in the form

$$t_{KQ}^{\text{out}}(\theta) = \sum_{kq} \alpha_{kq}^{KQ}(\theta) t_{kq}^{\text{in}} \quad (37)$$

The coefficients α_{kq}^{KQ} are given in terms of the $a_{m'm}(\theta)$ in Appendix II. The input and output tensors in (37), of course, are referred to the same set of axes.

The case of practical importance is with unpolarized particles incident, so $k = q = 0$. Then $t_{00}^{\text{out}} = 1$. It is also convenient to have the polarization of the outgoing particles referred to their own direction of motion $\underline{k}_2$ (in center of mass) as z-axis, since it will probably be studied by a second

scattering or reaction. Choosing y-axis along $\underline{k}_2 \times \underline{k}_1$, we may use the relation (26) to express the corresponding tensor components $\hat{t}_{KQ}^{\text{out}}$ in terms of the amplitudes (27), (we use a "hat" to distinguish quantities referred to these axes). The only non-vanishing component of the vector polarization is $\hat{P}_y$, where, for $S = 1$, $\hat{P}_y = P_y = i \frac{2}{\sqrt{3}} \hat{t}_{11}^{\text{out}}$, and

$$k^2 \left(\frac{d\sigma(\theta)}{d\omega} \right) \hat{P}_y(\theta) = \frac{2\sqrt{2}}{3} \text{Im} \left[a_{00} a_{01}^* + a_{10} (a_{11}^* - a_{1-1}^*) \right]. \quad (38)$$

Using (33) we only need 3 of the 5 components t_{2Q} , (which are all real with this choice of axes)

$$k^2 \left(\frac{d\sigma}{d\omega} \right) \hat{t}_{20}(\theta) = \sqrt{\frac{1}{2}} k^2 \left(\frac{d\sigma}{d\omega} \right) - \sqrt{\frac{1}{2}} \left[2|a_{10}|^2 + |a_{00}|^2 \right]$$

$$k^2 \left(\frac{d\sigma}{d\omega} \right) \hat{t}_{21}(\theta) = \sqrt{\frac{2}{3}} \text{Re} \left[a_{00} a_{01}^* + a_{10} (a_{11}^* - a_{1-1}^*) \right]$$

$$k^2 \left(\frac{d\sigma}{d\omega} \right) \hat{t}_{22}(\theta) = \sqrt{\frac{4}{3}} \text{Re} \left[a_{1-1} a_{11}^* \right] - \sqrt{\frac{1}{3}} |a_{01}|^2.$$

(39)

We note that $\hat{t}_{21}$ depends upon the real part of the same quantity whose imaginary part determines $\hat{P}_y$. These quantities (38) and (39) completely determine the polarization induced by scattering unpolarized particles. They also determine the inverse problem, the differential cross-section for scattering of polarized particles, as discussed in Section VI.

Of course, if there is no spin-orbit coupling $a_{m'm}$ vanishes unless $m' = m$, and $a_{mm} = a$ is independent of m , being given by (28). Then

$$\frac{d\sigma(\theta)}{d\omega} = \frac{1}{k^2} |a|^2 \quad (40)$$

and the outgoing polarization is identical to the incoming,

$$t_{kq}^{\text{out}} = t_{kq}^{\text{in}}, \quad (41)$$

being undisturbed by the scattering. Perhaps it should also be stressed that it is not necessary to have a tensor spin-orbit potential (5) in order to have non-zero polarization tensors t_{2Q} ; they will also arise from the usual vector spin-orbit potential (3).

V. TRANSFORMATION OF POLARIZATION FROM CENTER-OF-MASS TO LABORATORY

The polarization produced by the scattering may be analyzed by a second scattering, or used as a source of polarized particles with which to study some nuclear reaction. We were careful in the previous section to compute the polarization referred to the direction of motion of the outgoing particles as z-axis, but still in the center-of-mass system. However, the target nuclei for the second scattering will be at rest in the laboratory system. In this section we discuss the transformation of the polarization into the laboratory system.

The differential cross-section transforms in the usual way, ⁽²²⁾ but the polarization tensors t_{KQ} , being ratios of intensities, do not change in magnitude. However, they are referred to the direction of motion in center-of-mass as z-axis, which is not the same as the direction of motion in the laboratory, while the particle spin directions are unchanged by the transformation from moving axes.

Let the spin tensors $t_{KQ}^{\text{lab}}(\psi)$ be referred to the same y axis ($\underline{k}_2 \times \underline{k}_1$) but with the laboratory direction $\underline{k}'_2$ as z'-axis, where ψ is the laboratory angle corresponding to a center-of-mass θ^{22} . Then from (34),

$$t_{KQ}^{\text{lab}}(\psi) = \sum_{Q'} \hat{t}_{KQ'}(\theta) D_{Q'Q}^K(0, \theta - \psi, 0). \quad (42)$$

The polarization $\underline{P}$ is perpendicular to both $\underline{k}_1$ and $\underline{k}_2$, hence it is unaffected by the rotation about $\underline{k}_1 \times \underline{k}_2$,

$$P_y^{\text{lab}}(\psi) = \hat{P}_y(\theta). \quad (43)$$

For $K = 2$, the rotation matrix elements are simple functions of $\sin(\theta - \psi) = x \sin \psi$, where x is the ratio of the speed of the center-of-mass to the speed of the observed particle in center-of-mass. ($x = \text{mass of particle} / \text{mass of target}$, for elastic scattering.)

$$\begin{aligned} t_{20}^{\text{lab}}(\psi) = & \hat{t}_{20}(\theta) \left(1 - \frac{3}{2} x^2 \sin^2 \psi\right) + \hat{t}_{21}(\theta) \sqrt{6} x \sin \psi \sqrt{1 - x^2 \sin^2 \psi} \\ & + \hat{t}_{22}(\theta) \sqrt{\frac{3}{2}} x^2 \sin^2 \psi, \end{aligned}$$

$$\begin{aligned}
 t_{21}^{\text{lab}}(\psi) &= -\hat{t}_{20}(\theta) \sqrt{\frac{3}{2}} x \sin \psi \sqrt{1-x^2 \sin^2 \psi} + \hat{t}_{21}(\theta) (1-2x^2 \sin^2 \psi) \\
 &\quad + \hat{t}_{22}(\theta) x \sin \psi \sqrt{1-x^2 \sin^2 \psi}, \\
 t_{22}^{\text{lab}}(\psi) &= \hat{t}_{20}(\theta) \sqrt{\frac{3}{8}} x^2 \sin^2 \psi - \hat{t}_{21}(\theta) x \sin \psi \sqrt{1-x^2 \sin^2 \psi} \\
 &\quad + \hat{t}_{22}(\theta) \left(1 - \frac{1}{2} x^2 \sin^2 \psi\right). \quad (44)
 \end{aligned}$$

That these corrections are not always negligible may be seen by considering deuterons scattered by carbon, where $x = \frac{1}{6}$. For a laboratory angle $\psi = 90^\circ$, $\theta \approx 100^\circ$, and, for example,

$$t_{22}^{\text{lab}}(90^\circ) \approx 0.99 \hat{t}_{22}(100^\circ) - 0.17 \hat{t}_{21}(100^\circ) + 0.02 \hat{t}_{20}(100^\circ).$$

The contribution of $\hat{t}_{21}$ to t_{22}^{lab} could well be important. Of course, for heavy ions, x may approach unity and the transformations (44) become essential.

VI. DOUBLE SCATTERING

The double scattering experiment presents the problem of computing the differential cross-section for the scattering of polarized particles, which is the inverse of the problem discussed at the end of Section IV. Suppose the first scattering took place with the incident (unpolarized) particles having energy E_1 (in the lab), with differential cross-section $d\sigma^{(1)}(\theta_1, E_1)/d\omega_1$ (in CMS). These particles will reappear at angle ψ_1 (the lab angle corresponding to CMS angle θ_1) with lab energy E_2 . ($E_2 \neq E_1$ because of the recoil of the target.) In the lab, the corresponding differential cross-section is

$$\frac{d\sigma^{(1)lab}(\psi_1, E_1)}{d\omega_1^{lab}} = G^{-1}(x_1, \psi_1) \frac{d\sigma^{(1)}(\theta_1, E_1)}{d\omega_1}, \quad (45)$$

where the correction factor $G(x, \psi)$ is tabulated in reference 22. These particles also have a polarization described in the lab system by tensors $t_{KQ}^{(1)lab}(\psi_1, E_1)$, as discussed in the previous section. The particles emergent along ψ_1 are now scattered a second time through angle θ_2 (in the CMS for the second scattering), for which the differential cross-section for unpolarized particles would be $d\sigma^{(2)}(\theta_2, E_2)/d\omega_2$, with the relation corresponding to (45),

$$\frac{d\sigma^{(2)lab}(\psi_2, E_2)}{d\omega_2^{lab}} = G^{-1}(x_2, \psi_2) \frac{d\sigma^{(2)}(\theta_2, E_2)}{d\omega_2}$$

We now consider the plane of the second scattering to be inclined at angle ϕ to the plane of the first (both in the lab system). Scattering first right then left in the same plane defines $\phi = 0$, so that right-right or left-left corresponds to $\phi = \pi$.

It is then straightforward, either by direct computation using (26) or by time-reversal arguments,⁽²³⁾ to show the differential cross-section for the double scattering is

$$\frac{d^2\sigma^{lab}(\psi_1\psi_2\phi;E_1E_2)}{d\omega_1^{lab}d\omega_2^{lab}} = \frac{d\sigma^{(1)lab}(\psi_1E_1)}{d\omega_1^{lab}} \frac{d\sigma^{(2)}(\psi_2E_2)}{d\omega_2^{lab}} \times \sum_{KQ} t_{KQ}^{(1)lab}(\psi_1E_1) \hat{t}_{KQ}^{(2)}(\theta_2E_2) \cos(Q\phi). \quad (46)$$

In this expression, the $t_{KQ}^{(1)lab}$ describe the polarization before the second scattering, and the $\hat{t}_{KQ}^{(2)}$ describe the effect each component has on the differential cross-section of the second scattering. Since the t_{KQ} are normalized so that $t_{00} = 1$, the unpolarized cross-sections $d\sigma^{lab}/d\omega^{lab}$ must appear as normalizing factors. It is as well to repeat that the energies E_1 and E_2 are measured in the lab system, whereas the $\hat{t}_{KQ}$ are to be computed, according to Sections III and IV, at the corresponding CMS energies of relative motion for each scattering. $t_{KQ}^{(1)lab}$, in addition, has to be transformed according to Section V from its CMS to the lab system.

As it stands, (46) does not assume the two scatterings to be from similar target nuclei. If they are, the superscripts (1) and (2) may be dropped. If we assume further that we may neglect the target nuclei recoil (so $E_1 = E_2$, $\theta_1 = \psi_1$, $x = 0$, $G(0, \psi) = 1$), (46) appears much less complicated,

$$\begin{aligned}
 \frac{d^2 \sigma(\theta_1, \theta_2, \phi)}{d\omega_1 d\omega_2} &= \frac{d\sigma(\theta_1)}{d\omega_1} \frac{d\sigma(\theta_2)}{d\omega_2} \sum_{\mathbf{k}\mathbf{Q}} \hat{t}_{\mathbf{k}\mathbf{Q}}(\theta_1) \hat{t}_{\mathbf{k}\mathbf{Q}}(\theta_2) \cos(\mathbf{Q}\phi) \\
 &= \frac{d\sigma(\theta_1)}{d\omega_1} \frac{d\sigma(\theta_2)}{d\omega_2} \left\{ 1 + \hat{t}_{20}(\theta_1) \hat{t}_{20}(\theta_2) \right. \\
 &\quad + 2 \left[\hat{t}_{11}(\theta_1) \hat{t}_{11}(\theta_2) + \hat{t}_{21}(\theta_1) \hat{t}_{21}(\theta_2) \right] \cos \phi \\
 &\quad \left. + 2 \hat{t}_{22}(\theta_1) \hat{t}_{22}(\theta_2) \cos 2\phi \right\} \quad (47)
 \end{aligned}$$

However, it is not always safe to ignore the corrections for center-of-mass motion.

VII. EFFECT OF MAGNETIC DEFLECTION

This has been discussed briefly by Lakin for deuterons.⁽³⁾ When a magnetic field H is applied to particles of charge Ze and mass m , their direction of motion rotates about H with a frequency $\omega = ZeH/mc$. Their spins, however, precess with a frequency $\omega_s = g e H / 2 m_p c = \omega g m / 2 Z m_p$, where g is the particle's g -factor and m_p the proton mass. The polarization tensors, whose components are the $t_{\mathbf{k}\mathbf{Q}}$, also rotate with ω_s ; that is to say, referred to axes rotating about H with frequency ω_s , the $t_{\mathbf{k}\mathbf{Q}}$ do not change. Thus if the beam direction is rotated by an angle α around H , the spins lag behind by an angle $[1 - (g m / 2 Z m_p)] \alpha$. For deuterons this is $(1 - g) \alpha \approx \frac{1}{7} \alpha$, so the spins almost follow the direction of motion. For N^{14} nuclei, for example, it is $(1 - g_N) \alpha \approx 0.6 \alpha$, so their spins precess much more slowly. However, we must be careful with complex ions if they are not fully ionized. Even after scattering, a heavy ion may pick up several electrons. If the ions carry a charge ne ($n \leq Z$), n must replace Z in the expression above for the rotation of the direction of motion. Further, unless the effect of the magnetic field on the

nucleus is strong enough to overcome its hyperfine coupling to the electrons (requiring $\geq 10^6$ gauss) or the electronic magnetic moment is zero, the nuclear spin motion will depend on the relative strengths of the hyperfine coupling and the interaction of the electrons with the field. For weak fields ($H \ll 10^3$ gauss) the ion's total angular momentum $\underline{F} = \underline{s} + \underline{j}$ (where $\underline{j}$ is the electronic angular momentum) will precess around $\underline{H}$. Furthermore, even in the absence of a magnetic field, the hyperfine coupling to any electrons picked up will tend to partially depolarize the nuclear spins.

We are unlikely to encounter these difficulties with deuterons, and we shall now ignore them. To find the change induced by this rotation in the polarization tensors t_{KQ} when we continue to refer them to the direction of motion of the particles as z-axis, we may use their rotational properties (34). To simplify the geometry we will only consider $\underline{H}$ perpendicular or parallel to the direction of motion $\underline{k}$. If parallel, $\underline{k}$ is unchanged, but the spin tensor rotates about $\underline{k}$ with frequency w_s , (anticlockwise, looking along $\underline{H}$, if g is positive). This merely multiplies each t_{KQ} by a phase $e^{iQ\beta}$, if β is the angle of rotation after time t , $\beta = w_s t$. If this field were applied along the direction of motion (in the lab) between the two scatterings described in Section VI, it has the effect of replacing ϕ in equations (46) and (47) by $(\phi + \beta)$, as though the angle between the two scattering planes was increased by β .

Now suppose the magnetic field is applied perpendicular to the direction of motion $\underline{k}$, so it has polar angles $(\frac{\pi}{2}, \chi)$ in the coordinate system with z-axis along $\underline{k}$, to which the t_{KQ} are referred. In time t the direction of

motion rotates through angle $\alpha = \omega t$ about $\underline{H}$, to $\underline{\tilde{k}}$, while the t_{KQ} remain unchanged in a coordinate system which is rotated through angle $\beta = \omega_s t$ about $\underline{H}$. So, as far as the values of the t_{KQ} are concerned, the magnetic deflection is equivalent to a rotation of the spin polarization through angle $(\alpha - \beta)$ about $\underline{H}$, but in the opposite sense. If we denote by $\tilde{t}_{KQ}$ the tensor components after deflection referred to z-axis along the new direction of motion $\underline{\tilde{k}}$ and y-axis along $\underline{H}$, we can use (34) to give the $\tilde{t}_{KQ}$ in terms of the original t_{KQ} . The rotation of axes is achieved by first rotating through $(\chi - \frac{\pi}{2})$ about the z-axis, to bring the y-axis parallel to $\underline{H}$, then rotating through $\delta = \alpha - \beta$ about $\underline{H}$ to bring the z-axis parallel to $\underline{\tilde{k}}$.

$$\tilde{t}_{KQ} = \sum_{Q'} t_{KQ'} D_{Q'Q}^K \left(\chi - \frac{\pi}{2}, \delta, 0 \right). \quad (48)$$

The transformation (48) has the same explicit forms as (44), with $x \sin \psi$ there replaced by $\sin \delta$, and each t_{KQ} , on the right of (48) multiplied by $(i)^{Q'} e^{-iQ'\chi}$. If the field had been applied along the y-axis, so $\chi = \frac{\pi}{2}$, this last factor, of course, would be unity.

In this way, the magnetic deflection will convert some components t_{KQ} into others. In an ensuing scattering, t_{KQ} gives rise to a $\cos(Q\phi)$ azimuthal dependence. Hence, for example, a t_{20} component which would give no left-right scattering asymmetry, may be rotated by a magnetic field so that it does introduce azimuthal anisotropy and is thus more easily observed. If deuterons, for example were deflected through $\alpha = \frac{\pi}{2}$ by a field along the y-axis (so

$\chi = \frac{\pi}{2}$), then $\delta \approx \frac{1}{7}\alpha \approx 13^\circ$, and the new polarization tensor components are, from (48) and (44).

$$\tilde{t}_{20} = 0.92 t_{20} + 0.54 t_{21} + 0.06 t_{22}$$

$$\tilde{t}_{21} = -0.27 t_{20} + 0.90 t_{21} + 0.22 t_{22}$$

$$\tilde{t}_{22} = 0.03 t_{20} - 0.22 t_{21} + 0.98 t_{22}$$

(49)

Thus if deuterons were produced aligned along their direction of motion, so only t_{20} were non-zero, an appreciable amount of t_{21} is introduced by the magnetic deflection, (although the effect is not so large as one might wish because w_s/w is so close to unity for deuterons).

If the field $\underline{H}$ were applied perpendicular to the particle's direction of motion (in the lab system) between the two scatterings described in Section VI, we must replace in (46) or (47) the tensor components $t_{KQ}^{(1)\text{lab}}$ by their values after the deflection,

$$\tilde{t}_{KQ}^{(1)\text{lab}} = \sum_{Q'} t_{KQ'}^{(1)\text{lab}} e^{iQ'(\frac{\pi}{2}-\chi)} d_{Q'Q}^K(\delta), \quad (50)$$

where we have used the property $D_{Q'Q}^K(\alpha\beta) = e^{-iQ'\alpha} d_{Q'Q}^K(\beta)$, (Appendix I).

The formulae (46) or (47) then describe the double scattering angular distribution with ϕ being the angle of the second scattering plane now measured from the plane of magnetic deflection instead of the first scattering plane; i.e. $H \cos \phi = \underline{H} \cdot \underline{n}$ where $\underline{n}$ is the unit vector along $\underline{k}_2' \times \underline{k}_3'$, the normal to

the second scattering plane (in the lab system). The diagram in Section VI is then altered to

The net effect of (50) on (46) and (47) is to replace

$$\sum_{\kappa Q} t_{\kappa Q}^{(1)} t_{\kappa Q}^{(2)} \cos(Q\phi)$$

by

$$\sum_{\kappa Q Q'} t_{\kappa Q'}^{(1)} d_{Q'Q}^{\kappa}(\delta) t_{\kappa Q}^{(2)} \cos(Q\phi - Q'\eta) \quad (51)$$

where $\eta = \frac{\pi}{2} - \chi$. Little can be said in general about the $K = 2$ terms, but the vector $K = 1$ terms have simple properties. Our choice of axes and the mechanics of the scattering ensure that for $K = 1$, Q and Q' take values ± 1 only,

and that $t_{11} = t_{1-1}$. Then if the field is perpendicular to the first scattering plane ($\chi = \frac{\pi}{2}$, $\eta = 0$), $K = 1$ contribution to the double scattering is unchanged by it because the polarization vector $\underline{P}$ produced in the first scattering is parallel to the field and does not precess. But if the field is parallel to the first scattering plane ($\chi = 0$, $\eta = \frac{\pi}{2}$), its effect on the $K = 1$ terms in (46) or (47) is to replace $\cos \phi$ by $\sin \phi \cos \delta$ ($\cos \phi$ is replaced by $\sin \phi$ because the reference plane from which ϕ is measured has been rotated through $\frac{\pi}{2}$ in this case).

Finally, we may add that the discussion of this section applies quite generally to non-relativistic particles of arbitrary spin S . Of course, if $S = \frac{1}{2}$, the $K = 2$ tensors vanish, and if $S > 1$, tensors with $K > 2$ are also required to describe their polarization.

APPENDIX I

Rotation Matrix Elements

The transformation matrix elements for a rotation of coordinate axes needed in (34), (42), (48), (50) and (51) factorize in the following way, (18)

$$D_{Q'Q}^K(\alpha\beta\gamma) = e^{-iQ'\alpha} d_{Q'Q}^K(\beta) e^{-iQ\gamma} \quad (52)$$

The $d_{Q'Q}^K$ are real, and the following symmetry properties have been used,

$$d_{Q'Q}^K = (-1)^{Q-Q'} d_{-Q'-Q}^K = (-1)^{Q-Q'} d_{QQ'}^K \quad (53)$$

$d_{00}^0 = 1$, while for $K = 1$ we have

$$d_{00}^1(\beta) = \cos\beta$$

$$d_{10}^1 = d_{0-1}^1 = -d_{-10}^1 = -d_{01}^1 = -\sqrt{\frac{1}{2}} \sin\beta$$

$$d_{11}^1 = d_{-1-1}^1 = \cos^2(\frac{1}{2}\beta)$$

$$-d_{1-1}^1 = d_{-11}^1 = \sin^2(\frac{1}{2}\beta) \quad (54)$$

when $K = 2$ we have

$$d_{00}^2 = \frac{1}{2} (3 \cos^2\beta - 1)$$

$$d_{10}^2 = d_{0-1}^2 = -d_{-10}^2 = -d_{01}^2 = -\sqrt{\frac{3}{2}} \sin\beta \cos\beta$$

$$d_{11}^2 = d_{-1-1}^2 = \frac{1}{2} (2 \cos\beta - 1)(\cos\beta + 1)$$

$$d_{1-1}^2 = d_{-11}^2 = -\frac{1}{2}(2\cos\beta + 1)(\cos\beta - 1)$$

$$d_{20}^2 = d_{02}^2 = d_{-20}^2 = d_{0-2}^2 = \sqrt{\frac{3}{8}} \sin^2\beta$$

$$d_{21}^2 = d_{-1-2}^2 = -d_{-2-1}^2 = -d_{12}^2 = -\frac{1}{2}\sin\beta(\cos\beta + 1)$$

$$d_{2-1}^2 = d_{1-2}^2 = -d_{-21}^2 = -d_{-12}^2 = \frac{1}{2}\sin\beta(\cos\beta - 1)$$

$$d_{22}^2 = d_{-2-2}^2 = \cos^4\left(\frac{1}{2}\beta\right)$$

$$d_{2-2}^2 = d_{-22}^2 = \sin^4\left(\frac{1}{2}\beta\right).$$

(55)

APPENDIX II

Polarization from Scattering Polarized Particles

The polarization produced by a scattering when the incident particles are polarized is given by equation (37).

$$t_{KQ}^{\text{out}}(\theta) = \sum_{kq} \alpha_{kq}^{KQ}(\theta) t_{kq}^{\text{in}} \quad (56)$$

where in terms of the scattering amplitudes $a_{m'm}(\theta)$

$$\left(k^2 \frac{d\sigma(\theta)}{d\omega} \right) \alpha_{kq}^{KQ}(\theta) = \sum_{m'm''m'''} a_{m''m}(\theta) a_{m''m'}^*(\theta) (-)^{K-k+m''m} \\ \times (SS-m''m''' | KQ)(SS-mm' | kq) \quad (57)$$

with $m' - m = q$ and $m''' - m'' = Q$. In particular,

$$\alpha_{00}^{00} = 1,$$

$$\left(k^2 \frac{d\sigma(\theta)}{d\omega} \right) \alpha_{00}^{KQ} = \sum_{mm''} a_{mm''} a_{m'm''}^* \frac{(-)^{K+m+s}}{\sqrt{2s+1}} (SS-mm' | KQ). \quad (58)$$

Since $t_{00}^{\text{in}} = 1$, these last are just the t_{KQ}^{out} given in equations (36), (38) and (39). The general α_{kq}^{KQ} is a 9×9 matrix, but we do not need to compute all its elements. From (25) and (57) we find

$$\alpha_{kq}^{KQ} = (-)^{Q-q} \alpha_{k-q}^{K-Q} = (-)^{K-k} \alpha_{kq}^{KQ*} \quad (59)$$

when the x,z plane is chosen to coincide with the scattering plane. The last relation shows that, with this choice of axes, the α_{kq}^{KQ} are pure real/imaginary as K-k is even/odd. Also, α_{KQ}^{kq} has the same explicit form as α_{kq}^{KQ} but with all $a_{m,m}(\theta)$ replaced by $a_{mm}(\theta)$.

Both incident and outgoing polarization tensors in (56) are referred to the same set of coordinate axes. If the scattering is from $\underline{k}_1$ to $\underline{k}_2$, and α_{kq}^{KQ} relates tensors referred to $\underline{k}_1$ as z-axis, $\underline{k}_1 \times \underline{k}_2$ as y-axis, then the symmetry property (26) shows the $\hat{\alpha}_{kq}^{KQ}$ relating tensors referred to $\underline{k}_2$ as z-axis, $\underline{k}_2 \times \underline{k}_1$ as y-axis are given by

$$\hat{\alpha}_{kq}^{KQ} = \alpha_{KQ}^{kq} \quad (60)$$

When the scattering interaction contains no spin-orbit coupling, so $a_{m'm} = a_{mm}$, we have $\alpha_{KQ}^{kq} = \delta_{kK} \delta_{qQ}$ and

$$t_{KQ}^{out} = t_{KQ}^{in}$$

that is, there is no change in the polarization (in the center-of-mass system), although, of course, the direction of motion of the particles has changed.

Rather than writing down all the coefficients α_{kq}^{KQ} , we give the Clebsch-Gordan coefficients for $S = 1$ which appear in (57):

$$(1 \ 1 \ -m \ q+m \mid 1q) =$$

$m \backslash q$	1	0	-1
1	0	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$
0	$-\sqrt{\frac{1}{2}}$	0	$\sqrt{\frac{1}{2}}$
-1	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	0

$$(1 \ 1 \ -m \ q+m \mid 2q) =$$

$m \backslash q$	2	1	0	-1	-2
1	0	0	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{2}}$	1
0	0	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{2}}$	0
1	1	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{6}}$	0	0

APPENDIX III

Tensor Spin-Orbit Coupling

The interaction potential of a particle of spin s with a zero spin target can be written as a sum of scalar products,

$$V = \sum_k v_k \underline{S}_k \cdot \underline{R}_k = \sum_k v_k \sum_q (-)^q \underline{S}_{kq} \underline{R}_{k-q} \quad (61)$$

$\underline{S}_k$ is a tensor in the spin-space of the particle, while $\underline{R}_k$ operates in coordinate space. Tensors $\underline{S}_k$ can be defined with rank $k \leq 2s$. The scalar term $k = 0$ corresponds to the usual central potential $V(r) = v_0(r)$ if we normalize $\underline{S}_0 = \underline{R}_0 = 1$, while $k = 1$ gives the familiar vector spin-orbit coupling with $\underline{S}_1 = \underline{s}$, $\underline{R}_1 = \underline{L}$. Spin $s = 1$ allows $k = 2$ also. In Section II we wrote $v_2(r)$ as $[a_T(r) + ib_T(r)]$, and $\underline{S}_2 \cdot \underline{R}_2$ as T , in equation (5). We can define the spin tensor $\underline{S}_2$ uniquely in terms of the components of the spin operator $\underline{s}$,⁽²⁴⁾

$$S_{2q} = \sum_{\mu} s_{\mu} s_{q-\mu} (s s \mu q - \mu / 2q), \quad (62)$$

where $s_0 = s_z$, $S_{\pm 1} = \mp(s_x \pm is_y)/\sqrt{2}$. For $s = 1$ the S_{2q} are just the operators T_{2q} already introduced in (31) in Section IV to describe the polarization, $T_{2q} = \sqrt{3} S_{2q}$.

We have three vectors out of which to construct the coordinate tensor $\underline{R}_2$, the position vector $\underline{r}$, the momentum $\underline{p} = -i\nabla$, and the orbital angular momentum $\underline{L} = -i\mathbf{r} \times \nabla$. They have the following parity and time reversal properties,

	<u>r</u>	<u>p</u>	<u>L</u>
parity	odd	odd	even
time reversal	even	odd	odd

while the spin tensor $\underline{S}_2$ has even parity and is even under time reversal. Hence, if we demand that the potential (61) conserve parity and be invariant under time reversal, we cannot mix the vectors. (These requirements also eliminate the vector terms $\underline{s} \cdot \underline{r}$ and $\underline{s} \cdot \underline{p}$). So we have three possible coordinate tensors of the same form as (62),⁽²⁵⁾

$$R_{2q}(\underline{a}, \underline{a}) = \sum_{\mu} a_{\mu} a_{q-\mu} (11 \mu q-\mu | 2q), \quad (63)$$

with the a_{μ} being spherical components of one of the vectors $\underline{r}$, $\underline{p}$, or $\underline{L}$.

Taking $\underline{a}$ as the unit vector along $\underline{r}$, R_{2q} becomes essentially the spherical harmonic Y_2^q of the polar angles of $\underline{r}$,

$$R_{2q}(\underline{r}, \underline{r}) = \sqrt{\frac{8\pi}{15}} Y_2^q(\theta\phi). \quad (64)$$

The alternative forms in (6), (7), (8) and footnote (25) quoted for the tensor coupling may be derived from the relation

$$\underline{S}_2 \cdot \underline{R}_2(\underline{a}, \underline{b}) = (\underline{s} \cdot \underline{a})(\underline{s} \cdot \underline{b}) - \frac{1}{2} i \underline{s} \cdot (\underline{a} \times \underline{b}) - \frac{1}{3} \underline{s}^2 (\underline{a} \cdot \underline{b}) \quad (65)$$

remembering that $\underline{L} \times \underline{L} = i \underline{L}$, and $\underline{r} \times \underline{r} = \underline{p} \times \underline{p} = 0$.

The spin tensor may be cast into an alternative form for deuterons. The deuteron spin $\underline{s}$ is the vector sum of the neutron and proton spins (neglecting the internal D-state),

$$\underline{S} = \underline{S}_N + \underline{S}_P = \frac{1}{2} (\underline{\sigma}_N + \underline{\sigma}_P),$$

where the $\underline{\sigma}$ are Pauli spin operators. Since it is not possible to form a second rank tensor from $\underline{s}_N$ or $\underline{s}_P$ by themselves (being spin $\frac{1}{2}$), it turns out S_{2q} can be rewritten

$$S_{2q} = \frac{1}{2} \sum_{\mu} (\sigma_N)_{\mu} (\sigma_P)_{q-\mu} (1 \ 1 \ \mu \ q-\mu | 2 \ q). \quad (66)$$

Then using (65),

$$2r^2 \underline{S}_2 \cdot \underline{R}_2(\underline{r}) = (\underline{\sigma}_N \cdot \underline{r})(\underline{\sigma}_P \cdot \underline{r}) - \frac{1}{3} r^2 (\underline{\sigma}_N \cdot \underline{\sigma}_P), \quad (67)$$

which is the form suggested by Goldfarb.⁽¹¹⁾

The matrix elements of these tensor interactions are readily evaluated by standard techniques.⁽¹⁸⁾ Being scalar products they are, of course, all diagonal in the total angular momentum J , but all except (8) have elements linking orbital angular momenta L and $L \pm 2$, as discussed in Sections II and III. We have for spin s ,

$$\begin{aligned} \langle L J | \underline{S}_2 \cdot \underline{R}_2 | L' J \rangle &= (-)^{J-L-s} \sqrt{(2L+1)(2s+1)} W(L L' s s; 2J) \\ &\times \langle s || \underline{S}_2 || s \rangle \langle L || \underline{R}_2 || L' \rangle \quad (68) \end{aligned}$$

For $s = 1$, $\langle s || \underline{S}_2 || s \rangle = \sqrt{5/3}$. We soon find the diagonal $L = L'$ values.

For the coordinate tensor $\underline{R}_2(\underline{r})$ of (64),

$$\langle L J | \underline{S}_2 \cdot \underline{R}_2(\underline{r}) | L J \rangle = - \frac{L}{(2L+3)\sqrt{6}} \quad \text{if } J=L+1,$$

$$\begin{aligned}
 &= +\sqrt{\frac{1}{6}} \quad \text{if } J=L, \\
 &= -\frac{L+1}{(2L-1)\sqrt{6}} \quad \text{if } J=L-1. \quad (69)
 \end{aligned}$$

The momentum tensor $\underline{\underline{R}}_2(\underline{\underline{p}}, \underline{\underline{p}})$ gives

$$\begin{aligned}
 \langle L J | \underline{\underline{S}}_2 \cdot \underline{\underline{R}}_2(\underline{\underline{p}}, \underline{\underline{p}}) | L J \rangle &= -\frac{2L}{3(2L+3)\sqrt{2L+1}} \nabla^2 \quad \text{if } J=L+1, \\
 &= -\frac{2}{3\sqrt{2L+1}} \nabla^2 \quad \text{if } J=L, \\
 &= -\frac{2(L+1)}{3(2L-1)\sqrt{2L+1}} \nabla^2 \quad \text{if } J=L-1. \quad (70)
 \end{aligned}$$

The orbital tensor $\underline{\underline{R}}_2(L, L)$ gives

$$\begin{aligned}
 \langle L J | \underline{\underline{S}}_2 \cdot \underline{\underline{R}}_2(L, L) | L J \rangle &= \frac{1}{6} L(2L-1) \quad \text{if } J=L+1, \\
 &= -\frac{1}{6} (2L-1)(2L+3) \quad \text{if } J=L, \\
 &= \frac{1}{6} (L+1)(2L+3) \quad \text{if } J=L-1. \quad (71)
 \end{aligned}$$

With spin $s = 1$, so $J = L$, or $L \pm 1$, the off-diagonal elements between L and $L+2$ can only occur for $J = L+1$. These are

$$\begin{aligned}
 \langle L+2, L+1 | \underline{\underline{S}}_2 \cdot \underline{\underline{R}}_2(\underline{\underline{r}}, \underline{\underline{r}}) | L, L+1 \rangle &= \frac{\sqrt{\frac{3}{2}} (L+1)(L+2)}{2L+3}, \\
 \langle L+2, L+1 | \underline{\underline{S}}_2 \cdot \underline{\underline{R}}_2(\underline{\underline{p}}, \underline{\underline{p}}) | L, L+1 \rangle &= -\frac{\sqrt{(L+1)(L+2)}}{2L+3} \left[\frac{d^2}{dr^2} - \frac{2L+1}{r^2} \frac{d}{dr} + \frac{L(L+2)}{r^2} \right]. \quad (72)
 \end{aligned}$$

These matrix elements, of course, correspond to integration over angle and spin coordinates only, not the radial coordinate r . Also, in (70) ∇^2 assumes its value for angular momentum L , namely

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{L(L+1)}{r^2} .$$

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- (20) The $(a \ b \ \alpha \ \beta \mid c \ \gamma)$ are Clebsch-Gordan coefficients, denoted $C(a \ b \ c; \alpha \ \beta)$ in reference 18.
- (21) Similar interchannel transitions occur, with their attendant coupled equations, if the potential V is not spherically symmetric, as in the optical model for scattering from deformed nuclei. See, for example, D. M. Chase, L. Wilets, and A.R. Edmonds, Phys. Rev. 110, 1080 (1958).
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- (24) There is no need to use symmetric combinations like $[S_\mu S_\lambda + S_\lambda S_\mu]$ in (62), and (63) below, since the symmetry of the Clebsch-Gordan coefficients does this automatically even when the S_μ do not commute.
- (25) The use of a complex potential to simulate absorption is equivalent to relaxing the time reversal invariance of V . Thus the absorption potential could contain terms of even parity but odd under time reversal associated with a real radial dependence. The only one which can be formed uses

r and p,

$$R_{2q}(\underline{r}, \underline{p}) = \sum_{\mu} \tau_{\mu} p_{q-\mu} (1 \ 1 \ \mu \ q-\mu / 2 \ q)$$

so that

$$\underline{S}_2 \cdot \underline{R}_2(\underline{r}, \underline{p}) = (\underline{S} \cdot \underline{r})(\underline{S} \cdot \underline{p}) - \frac{1}{2} i \underline{S} \cdot \underline{L} - \frac{1}{3} \underline{S}^2 (\underline{r} \cdot \underline{p}).$$

(Parity still excludes the vector terms $\underline{S} \cdot \underline{p}$ and $\underline{S} \cdot \underline{r}$.)

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