Testing Gravity Against Early Time Integrated Sachs-Wolfe Effect

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A generic prediction of general relativity is that the cosmological linear density growth factor \( D \) is scale independent. But in general, modified gravities do not preserve this signature. A scale dependent \( D \) can cause time variation in gravitational potential at high redshifts and provides a new cosmological test of gravity, through early time integrated Sachs-Wolfe (ISW) effect-large scale structure (LSS) cross correlation. We demonstrate the power of this test for a class of \( f(R) \) gravity, with the form \( f(R) = -\lambda_1 H_0^2 \exp(-R/\lambda_2 H_0^2) \). Such \( f(R) \) gravity, even with degenerate expansion history to \( \Lambda CDM \), can produce detectable ISW effect at \( z \gtrsim 3 \) and \( l \gtrsim 20 \). Null detection of such effect would constrain \( \lambda_1 \) to be \( \lambda_2 > 1000 \) at \( > 95\% \) confidence level. On the other hand, robust detection of ISW-LSS cross correlation at high \( z \) will severely challenge general relativity.

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Introduction.— Cosmological observations provide unique tools to study gravity at \( \gtrsim \) Mpc scales. General relativity, with the aid of the cosmological constant, or dark energy with equation of state \( w \sim -1 \), successfully reproduces the accelerated expansion of the Universe, indicated by SN Ia observations\(^1\), along with the flatness of the Universe measured by the cosmic microwave background (CMB)\(^2\) and distance measured by the baryon oscillations\(^3\). However, these observational evidences mainly constrain the mean expansion history of the Universe and can be reproduced by modified gravity such as brane world DGP theory\(^4\) and generalized \( f(R) \) gravity\(^5\). Essentially, the large scale structure (LSS) of the universe, such as weak gravitational lensing\(^6\), is required to break this degeneracy.

General relativity imprints a unique signature in the LSS, which is scale independent linear density growth factor \( D \) at sub-horizon scale after matter-radiation equality epoch\(^7\). Modifications to general relativity not only changes the amplitude of \( D \), but in general, causes \( D \) to be scale dependent. This unique feature of modified gravity has already been noticed in phenomenological theory of modified Newtonian potential\(^8\) and in DGP\(^9\). It can be detected by weak gravitational lensing, galaxy clustering\(^8\) and late time integrated Sachs-Wolfe (ISW) effect\(^8\). Counter-intuitively, in this paper, we show that modified gravity can produce a detectable early time ISW effect.

We investigate a class of \( f(R) \) gravity with action

\[
L = \int (R + f(R))\sqrt{-g}d^4x + L_{\text{matter}} ,
\]

and field equation

\[
(1 + f_R)R_{uv} - \frac{g_{uv}}{2}(R + f - 2\Box f_R) - f_{Ruv} = 8\pi G T_{uv} \ ,
\]

where \( f_R \equiv df/dR \). We design \( f(R) = -\lambda_1 H_0^2 \exp(-R/\lambda_2 H_0^2) \), where \( \lambda_{1,2} \) are two positive dimensionless constants and \( H_0 \) is the Hubble constant at present. If we choose \( \lambda_2 \ll R_{\text{sol}}/H_0^2 \sim \rho_{\text{sol}}/\rho_c \gtrsim 10^{10} \) (\( \rho_c \) is the critical density of the Universe), the exponential damping in \( f(R) \) guarantees that there is no effect of \( f(R) \) to our solar system and the exponential \( f(R) \) can pass all solar system tests. For the \( f(R) \) gravity, the application of Birkhoff theorem to perturbations of a spherically symmetric region leads to scale independent \( D \)\(^10\). But, as pointed out by\(^11\), this approach may be problematic. We clarify this issue by solving the structure evolution of the fully covariant \( f(R) \) gravity to linear order. We find that \( D \) shows nontrivial scale dependence\(^12\).

The \( H-z \) relation of the \( f(R) \) gravity.— Cosmological observations prohibit strong deviation of \( f(R) \) from a cosmological constant. At the limit that \( R(a) \equiv 1/(1 + z) = 1 \ll \lambda_2 H_0^2 \), the \( H-z \) relation of \( f(R) \) gravity can have the same asymptotic behavior as that of \( \Lambda CDM \). At low redshift where \( R(a) \ll \lambda_2 H_0^2 \), \( f(R) \) behaves as a cosmological constant and the \( H-z \) relation resembles that of \( \Lambda CDM \). At high redshifts where \( R \gg \lambda_2 H_0^2 \), \( f(R) \rightarrow 0 \) and \( H(z) \rightarrow \Omega_0^{1/2}(1+z)^{3/2} \). Deviation from \( \Lambda CDM \) happens at some intermediate redshifts where \( R(a) \sim \lambda_2 H_0^2 \) and vanishes toward both higher and lower \( z \). We quantify their difference by solving Eq.\(^2\) of a flat universe to zero order

\[
H^2 + \frac{f}{6} - \frac{\ddot{a}}{a} f_R + H \dot{f}_R = H_0^2 \Omega_0 a^{-3} .
\]

This equation can be rewritten as \( y = \Omega_0 - C(y(a)) \), where \( y \equiv a^3 H^2 \), \( C(y(a)) \equiv [f/6 - \ddot{a}f_R/a + Hf_R]a^3 \) and \( \Omega_0 \) is the dimensionless matter density at present.
Since $C(y(a))$ is completely determined once $y$ as a function of $a$ is given, Eq. (3) can be solved iteratively by the iteration relation $y^{(i+1)} = \Omega_0 - C(y^{(i)})$. To mimic a $ΛCDM$ universe, we fix $\lambda_1$ by requiring $f(R(a = 1)) = -6H_0^2(1 - \Omega_0)$. The iteration converges quickly by taking the initial guess $y^{(0)} = \Omega_0 + (1 - \Omega_0)a_0^3$. For $\lambda_2 \geq 100$, $y^{(1)}$ is accurate to $\sim 1\%$. As expected, for $\lambda_2 \geq 100$, the $H(z)$-relation is almost identical to the corresponding $ΛCDM$ cosmology (Fig. 1). Such $f(R)$ gravity can not be distinguished from $ΛCDM$ by inflation, big bang nucleosynthesis (BBN), primary CMB, SN Ia’s and other measures of $ΛCDM$ relation.

The large scale structure of the $f(R)$ gravity.—We will show that, even with this degeneracy in $H(z)$-relation and solar system behavior, the LSS of the $f(R)$ gravity could be significantly different to that of $ΛCDM$. We choose the Newtonian gauge

$$ds^2 = -(1 + 2\phi)dt^2 + a^2(1 + 2\phi)dx^{i,2}.$$  

There are four perturbation variables $\phi$, $\psi$, the matter over-density $δ$ and the (comoving) peculiar velocity convergence $θ$.

In general relativity, $\phi = -\psi$, as long as there is no anisotropic stress. But in modified gravity, this relation breaks in general. $ij$ ($i \neq j$) component of Eq. (2) provides the relation between $φ$ and $ψ$. For $f(R)$ gravity, due to non-vanishing $f_{R,ij}$ ($i \neq j$), $φ$-$ψ$ relation becomes scale dependent. Throughout this paper, we neglect time derivative terms with respect to spatial derivative terms of corresponding variables. This simplification holds at scales $k \gtrsim aH \lesssim 10^{-3}h/$Mpc. Since we will focus on the ISW effect at $l \gtrsim 20$ and $z \gtrsim 3$ where the relevant $k \gtrsim 5 \times 10^{-3}h/$Mpc, this simplification is sufficiently accurate. We then obtain

$$φ + ψ = \frac{f_{RR}c^2}{1 + f_{R}a^2} (\nabla^2ψ + 2\nabla^2φ).$$  

(5)

In Fourier space, this reads $ψ = -φ(1 - 2Ω)/1 - Q$, where $Q(k,a) \equiv -2f_{RR}c^2k^2/(1 + f_{R})a^2$ and $f_{RR} \equiv d^2f/dR^2$. For clarity, we explicitly show the speed of light $c$. We will see that this scale dependent $ψ$-$φ$ relation has profound effect on the LSS. Combining Eq. (5) and the $tt$ component of Eq. (2) we obtain the new Poisson equation

$$\nabla^2(φ - ψ) = -3H_0^2Ω_0 \frac{\delta}{1 + f_{R}a^2}.$$  

(6)

The energy-momentum tensor is still conserved and provides the remaining two equations:

$$\ddot{\delta} + θ = 0, \dot{θ} + 2Hθ + \frac{1}{a^2}κ\nabla^2ψ = 0.$$  

(7)

Combining all 4 equations, we obtain the main equation of this paper:

$$\delta'' + \delta'(\frac{3}{a} + \frac{H'}{H}) - \frac{δ}{a^2} \frac{1 - 2Ω}{2 - 3Ω} \frac{3H_0^2Ω_0}{a^2H^2(1 + f_{R})} = 0,$$  

(8)

where $' \equiv d/da$. In general relativity, $Q = 0$, so $D$ is scale independent at scales $k \gtrsim aH/c$, no matter what the form of dark energy is. But in $f(R)$, the scale dependent $Q(k,a)$ induces nontrivial scale dependence to $D$. This behavior can not be obtained by a simple change in the effective Newton’s constant. Furthermore, the correction $Q$ has a nontrivial dependence on $a$. This is hard to realize by simply changing the form of the Newtonian potential (e.g. to Yukawa potential).

Since $f_{RR} < 0$, there exist one apparent singularity $Q = 2/3$ in Eq. (5) where only $δ = 0$ solution is accepted and two at $Q = 1/2, 1$ in the $ψ$-$φ$ relation, where only $ψ = 0$ solution is accepted. We leave this issue alone until the discussion section. For the moment, we take a modest goal by only using regions where $Q < 1/2$ to constrain $f(R)$. For $λ_2 = 1000$, this constrains us to region where $k \lesssim 0.012h/$Mpc.

Hereafter, we fix $λ_2 = 1000$. At $z \gg 1$, $H \propto a^{-3/2}$, $D \propto a^{-n}$ when $η = 3Q/(2 - 3Q) \ll 1$. Thus gravitational potential decays at high redshifts with rate $∝ a^{-n}$.

FIG. 1: The $H(z)$-$z$ relation and structure growth in the exponential $f(R)$ gravity. Top left panel: $H(z)$. $λ_2 \rightarrow \infty$ corresponds to $ΛCDM$ cosmology. Top right panel: $Q(k,a) \propto k^2$, which describes the main effect of $f(R)$ gravity to structure formation. We plot the result of $k = 0.01h/$Mpc. Bottom left panel: $f_{R}(a)$, which determines the effective Newton’s constant $G_{eff} = G/(1 + f_{R})$. For $λ_2 \gtrsim 100$, its effect to structure formation can be neglected. Bottom right panel: $D(k, a)/a$ ($λ_2 = 1000$), where the linear density growth factor $D$ is normalized such that $D \rightarrow a$ when $a \rightarrow 0$. 
We solve Eq. 8 numerically. Initial condition is set to
This provides us a unique way to test this form of
$\Lambda$CDM cosmology or dark energy models with
$z$ exponential $f$ the evolution of $\psi$
(with power spectrum distribution, there exists an ISW-LSS cross correlation,
both $\psi$
Here, $\chi$
and $\Delta$
Here,

\[ \frac{\Delta T}{T_{\text{CMB}}} = \int (\psi - \phi) a^2 d\chi. \]  
(9)

Here, $\chi$ is the comoving angular diameter distance. Since both $\psi - \phi$ and the LSS trace the underlying matter distribution, there exists an ISW-LSS cross correlation, with power spectrum

\[ \frac{1}{2\pi} C_{l}^{\text{ISW-LSS}} = \frac{\pi}{l} \int \Delta_{l}^{2}(\psi - \phi) \delta_{\text{LSS}}(\chi) W_{\text{LSS}}(\chi)a^2 d\chi. \]  
(10)

Here, $\delta_{\text{LSS}}$ is the density fluctuation of the LSS tracers, $W_{\text{LSS}}$ is the corresponding weighting function and $\Delta_{l}^{2}$ is the corresponding 3D power spectrum(variance). The above formula adopts the Limber’s approximation, which is sufficiently accurate to serve for our interest at $l \geq 20$. The amplitude and sign of the ISW effect is determined by $A_{\text{ISW}} = D/a - dD/da$. Positive $A_{\text{ISW}}$ means positive correlation between ISW and LSS. For $k \gtrsim 0.007h/\text{Mpc}$, $A_{\text{ISW}}$ has a bump at $z \sim 6$, whose amplitude increases towards small scales (large $k$). This boosts early time small scale ISW signal (Fig. 2).

Here we estimate the S/N of the ISW-LSS cross correlation measurements. Since the exponential $f(R)$ does not affect physics at $z \gtrsim 100$, we adopt the same primordial power spectrum with power index $n = 1$, the same transfer function BBKS and the same amplitude at $a_{i} = 0.01$, as that of the $\Lambda$CDM cosmology. The LSS tracers we choose are 21cm emitting galaxies at $3 < z < 5$, which will be measured by proposed 21cm experiments such as Square Kilometer Array. Singularities presented in the perturbation equations limit us to $l < 60$, where one can neglect shot noises of both the CMB and LSS. At this limit, the S/N of each $l$ is

\[ \left( \frac{S}{N} \right)^2 \equiv \frac{(2l+1) f_{\text{sky}} r^2}{1 + C_{l}^{\text{CMB}}/C_{l}^{\text{ISW}}} \]  
(11)

where $C_{l}^{\text{CMB}}$, $C_{l}^{\text{ISW}}$ and $r$ are the primary CMB power spectrum, ISW power spectrum and the cross correlation coefficient between ISW and LSS, respectively. Since $r$ has very weak dependence on galaxy bias, the estimation presented here is weakly model dependent. We disregard signals from $l < 20$, to reduce confusions of $\Lambda$CDM cosmology or dark energy models. For sparse galaxy sampling which is sufficient for our purpose, SKA is able to cover the whole sky. So we assume that $f_{\text{sky}} = 1$. The cumulative $\sum_{20}^{l_{\text{max}}} (S/N)^2$ is shown in Fig. 2.

The ISW signal peaks at $z \gtrsim 3$ and increases toward high $l$. This is hard to mimic by $\Lambda$CDM, dark energy or many forms of modified gravity. (1) For $\Lambda$CDM or dark energy models with $w \lesssim -1$, the ISW effect vanishes at high $l$. This is hard to mimic by $\Lambda$CDM, dark energy or many forms of modified gravity. (2) For dark energy models with $w \gtrsim -1$, $A_{\text{ISW}}$ does not decrease as fast as that of $\Lambda$CDM. But the ISW signal (including contributions from dark energy fluctuations) decreases toward high $l$ and one does not expect a detectable ISW effect. (3) For DGP, a negative ISW-LSS cross correlation may be expected and one does not expect a detectable ISW effect. (4) For generalized $f \propto (\alpha R^2 + \beta R_{ab}R^{ab} + \gamma R_{abcd}R^{abcd})^{-n}$ ($n > 0$), the ISW effect vanishes at high $z$ because the $f$ correction decreases much faster than the exponential $f(R)$. So we expect that null detection of ISW-LSS cross correlation at $l \geq 20$ and $z \gtrsim 3$ would constrain $\lambda_2$ to $\lambda_2 > 1000$ at $> 2\sigma$ confidence level. On the other hand, a detection of such cross correlation would present as a severe challenge to general relativity.

Discussion.— The scale dependence of $D$, as an unambiguous signature of modified gravity, can in principle be
measured from weak gravitational lensing by the mean of lensing tomography. Since $\phi$ is no longer equal to $-\psi$, we provide the general form of the lensing transformation matrix $A_{ij}$

$$A_{ij} - \delta_{ij} = \int_0^x d\chi (\phi - \psi)_{,ij} W(\chi, \chi_*) , \quad (12)$$

where $W(\chi, \chi_*) = \chi(1 - \chi/\chi_*)$ is the usual lensing kernel. All basic lensing theorems remain unchanged. For example, lensing shear field is still curl free ($f = 1$). For $f(R)$ gravity, relation between the lensing convergence $\kappa = 1 - (A_{11} + A_{22})/2$ and the matter over-density resembles that of the general relativity, with

$$\kappa = \frac{3}{2} H_0^2 \Omega_0 \int \delta a^{-1} W(\chi, \chi_*)(1 + f_R)^{-1} d\chi . \quad (13)$$

It is interesting to see how well weak lensing alone can constrain modified gravity. For the exponential $f(R)$, one complexity is that lensing mainly probes LSS at $z \lesssim 1$, where $Q$ is small and the deviation from a scale independent $D$ is small, so the constraints may be weak. This can be significantly improved by gravitational potential reconstructed from primary CMB. Combining lensing and CMB measurements, it is very promising to measure the evolution of the gravitational potential between $z = 1100$ and $z \sim 0$ robustly. This will put strong constraints on the nature of gravity. Unfortunately, due to singularities in the perturbation equations, we are limited to scales $k \lesssim 0.012 h/$Mpc or $l \lesssim 20$ at $z \lesssim 1$ (for $\lambda_2 = 1000$). Information contained in this region is very limited and could be contaminated by other physics such as dark energy fluctuations. Solving the field equation crossing those singularities consistently is nontrivial. We leave this work for future study.

The $Q = 1/2, 2/3, 1$ singularities may be caused by awkward gauge choice, the neglecting of time derivative terms with respect to corresponding spatial derivative terms, or the failure of the perturbation approach. For example, for $Q = 2/3$, the only solution $\delta = 0$ does not depend on initial conditions. This could be caused by neglecting time derivative times, which erases some degree of freedom. These issues require detailed study. But if these singularities do exist, the LSS can rule out most $f(R)$ gravities as alternatives to dark energy or general relativity, because the existence of singularities in the perturbation equations is generic in many, if not all, $f(R)$ gravities which can reproduce the expansion history of the Universe. To produce a similar expansion history, (1) $R$ should increase when $a$ decreases and (2) $f(R(a = 1)) \leq 0$ in order to have an accelerated expansion at present and $f(R(a \to 0)) = 0$ in order not to affect inflation, BBN and primary CMB. This results in $f_R(a < a_+) \geq f_R(a_+) > 0$. On the other hand, when $a \to 0$, $R \propto a^{-3}$ and increases very quickly. So, $f$ increases toward high redshift and crosses over zero at some epoch and then increases faster than $a^{-3}$. This contradicts our expectation.

To demonstrate the power of LSS to constrain gravity, we adopt a conservative requirement to avoid singularities at $k < k_s$. At the limit that $\lambda_2 \gg 1$, $Q$ peaks at $a = (2 \lambda_2/9 \Omega_0)^{-1/3}$ and the peak amplitude is $\approx 12(1 - \Omega_0)/(2/9 \Omega_{de})^{2/3} \lambda_2^{2/3}(ck/H_0)^2$, where we show the speed of light $c$ explicitly. To avoid singularities at $k < k_s$,

$$\lambda_2 \geq 2.5 \times 10^5 \left( \frac{k_s}{h \text{Mpc}} \right)^{3/2} \quad (14)$$

should be satisfied.

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[7] Neutrino free streaming causes $D$ to be scale dependent, but this effect is small. Fluctuations in dark energy cause $D$ to be scale dependent at $\sim$ horizon scale. But this effect vanishes toward smaller scales.
[12] Xuelei Chen and Tomi Koivisto for helpful discussions.