TRANSIENT BUBBLE GROWTH IN
A HOMOGENEOUS REACTOR

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ABSTRACT

A mechanism for shutting off a power excursion in a homogeneous reactor is the rapid formation of bubbles. This report presents the results of a study of a model of bubble growth in such reactors. Equations are derived which give upper and lower bounds for the radius of a bubble, as a function of time, under conditions present in an actual reactor. The present work should make possible the deduction of the growth rate of bubble nuclei from observations of void volume and pressure.

I. INTRODUCTION

A mechanism for shutting off a power excursion in a homogeneous reactor is the rapid formation of bubbles. This expands the solution and reduces the reactivity. Since frothing sometimes occurs below the boiling temperature, bubble formation may be caused by the diffusion of radiolytic gases formed in the reactor. Section II of the present report is a study of the growth of bubbles of radiolytic gas (oxygen and hydrogen) under the conditions present in the early part of a reactor excursion (i. e., the part prior to peak power). Section III presents a simple extension of this study so that the growth of the bubbles may be followed through peak-power operation of the reactor.

Some preliminary results concerning the growth of bubbles subject to transient inertial pressure and surface tension are given in Section IV. The problem of bubble nucleation is not discussed in this report, since, for bubbles whose diameter is small compared to one micron, effects not treated here must be important to balance the great pressure caused by surface tension near a surface of great curvature. Bubble buoyancy and ambient water flow are also omitted, as our primary interest is in the initial peak in a fast power excursion. However, it is believed that all the main effects following nucleation up to a time slightly beyond peak power are properly treated for a transient whose e-folding time is less than a few tens of milliseconds. Therefore, the present work should make possible the deduction of the rate of formation of bubble nuclei from observations of void volume and pressure.

II. BUBBLE GROWTH PRIOR TO PEAK POWER

For the purposes of this investigation, it was assumed that:

- a) A spherical bubble is formed with initial radius, R_0 , of such magnitude that effects of surface tension are negligible;
- b) The center of the bubble is at rest in an infinite liquid medium containing dissolved gases;
- c) The concentration of dissolved gases is increasing exponentially due to the formation of radiolytic gases by irradiation;
- d) The gases diffuse through the wall of the bubble causing the bubble to grow while the density of gas in the bubble remains constant;
- e) No significant temperature variation occurs;
- f) The velocity of the bubble wall is small in comparison with the velocity of sound, thereby making the inertial effects of the fluid negligible.

Using the above assumptions, the equation satisfied by the concentration of the dissolved gases is:

$$\frac{\partial c(r, t)}{\partial t} = \kappa \nabla^2 c + S(t) \quad [r > R(t)] \quad \dots (1)$$

where

$c(r, t)$ = the concentration of the dissolved gases,

κ = the diffusivity,

r = the distance measured from the center of the bubble,

$R(t)$ = the radius of the bubble at time t ,

$S(t)$ = the source term.

Basically, the present work differs from previous work on the growth of spherical bubbles by diffusion in that a source term is present in the above equation.^{1,2}

The source term is assumed to be proportional to the power in the reactor.

For bubble growth prior to peak power, the power $P(t)$ may be defined as

$$P(t) = P_0 e^{\alpha t} \quad \dots (2)$$

where

α = the inverse period of the reactor,
 P_0 = the initial power of the reactor.

Figure 1. A Typical Power Trace

For the purposes of illustration, a typical trace of the power in KEWB, as a function of time, is shown in Figure 1 by the heavy line. The dotted line indicates the power as given by Equation (2).

The breakaway point is the point on the curve at which the power deviates significantly from the power as given in Equation 2; the breakaway time is the time at which this deviation occurs.

Thus, our model would be expected to be strictly valid only for the period of time before the breakaway.

Thus,

$$S(t) = \gamma P_0 e^{\alpha t} = S_c e^{\alpha t}, \quad \dots (3)$$

where

$$S_c = P_0 \frac{G\mu}{V_c}, \quad \dots (4)$$

and where

G = the volume of gas produced (at STP) per unit-energy release
 (l/kw-sec),
 V_c = the volume of liquid in the core (l),
 μ = the number of grams per liter of gas at STP.

The mass $M(t)$ of the gas in the bubble is given by the equation

$$M(t) = \frac{4\pi}{3} R^3(t) \rho \quad ,$$

where

ρ = the density of gas in the bubble.

The rate of change of the mass of gas is given by

$$\frac{dM}{dt} = 4\pi R^2(t) \rho \frac{dR}{dt} \quad ,$$

and this must be equal to the rate at which gas flows through the wall of the bubble. This rate is equal to

$$4\pi R^2(t) \kappa \left(\frac{\partial c}{\partial r} \right)_{r=R(t)} \quad .$$

Therefore,

$$4\pi R^2(t) \kappa \left(\frac{\partial c}{\partial r} \right)_{r=R(t)} = 4\pi R^2(t) \rho \frac{dR}{dt} \quad . \quad \dots (5)$$

Under the preceding assumptions, the following boundary-value problem may be formulated:

$$\frac{\partial c}{\partial t} = \kappa \nabla^2 c + S_c e^{\alpha t} \quad \left[4 > R(t) \right] , \quad \dots (6)$$

$$c(r, 0) = c_i$$

$$\lim_{r \rightarrow \infty} c(r, t) = c_i + \frac{S_c}{\alpha} (e^{\alpha t} - 1) \quad , \quad \dots (7)$$

$$c(R(t), t) = c_s \quad , \quad \dots (8)$$

$$\kappa \left(\frac{\partial c}{\partial r} \right)_{r=R(t)} = \rho \frac{dR}{dt} , \quad \dots (9)$$

where

c_i = the initial concentration of gas in the liquid,

c_s = the concentration of gas at saturation.

It is assumed that $c_i > c_s$ (i.e., that the solution is supersaturated).

The basic aim of the work to follow is to obtain an analytical expression for $R(t)$, the radius of the bubble. Since the boundary-value problem formulated above is nonlinear, only approximate expressions for $R(t)$ are found, but these expressions are in the form of upper and lower bounds which do not differ significantly over the interval of time in question.

A. THE INTEGRAL EQUATION FOR $R(t)$

If we define $u(r, t)$ by the equation

$$c(r, t) = \frac{-u(r, t)}{r} + c_i + \frac{S}{\alpha} (e^{\alpha t} - 1) , \quad \dots (10)$$

then

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial r^2} \quad \left[r > R(t) \right] , \quad \dots (11)$$

$$u(r, 0) = 0$$

$$\lim_{r \rightarrow \infty} u(r, t) = 0 , \quad \dots (12)$$

$$u(R(t), t) = R(t) \phi(t)$$

$$\dots (13)$$

$$\phi(t) \equiv c_i - c_s + \frac{S}{\alpha} (e^{\alpha t} - 1) ,$$

$$\rho \frac{dR}{dt} = \frac{\kappa}{R(t)} \left[-\left(\frac{\partial u}{\partial r}\right)_{r=R(t)} + \phi(t) \right] \quad \dots(14)$$

Note that, by hypothesis, $\phi(t) \geq 0$.

It follows from Equation (11) that

$$\int_0^t \int_{R(\xi)}^{\infty} \left[\frac{\partial u}{\partial \xi}(r, \xi) - \kappa \frac{\partial^2 u}{\partial r^2}(r, \xi) \right] dr d\xi \equiv 0 \quad \dots(15)$$

Since $u(r, 0) = 0$, we find that

$$\begin{aligned} \int_0^t \int_{R(\xi)}^{\infty} \frac{\partial u}{\partial \xi} dr d\xi &= \int_{R(t)}^{\infty} u(r, t) dr + \int_{R_0}^{R(t)} u(r, T(r)) dr \\ &\equiv Q \end{aligned}$$

where $t = T(r)$ gives the time dependence of the bubble radius R . $T(r)$ exists and is a single-valued function as long as $R(t)$ is monotone.

By using Equation (14)

$$-\kappa \int_0^t \int_{R(\xi)}^{\infty} \frac{\partial^2 u}{\partial r^2} dr d\xi = -\frac{\rho}{2} [R^2(t) - R_0^2] + \kappa \int_0^t \phi(\xi) d\xi \quad .$$

Thus

$$R^2(t) = R_L^2(t) + \frac{2}{\rho} Q \quad , \quad \dots(16)$$

where

$$R_L^2(t) = R_0^2 + \frac{2\kappa}{\rho} \left[\left(c_i - c_s - \frac{S_c}{a} \right) t + \frac{S_c}{a^2} (e^{at} - 1) \right] \quad , \quad \dots(17)$$

and $u(r, t)$ is the solution of the boundary-value problem given in Equations (11) through (13).

B. UPPER AND LOWER BOUNDS FOR $R(t)$

Consider now the boundary-value problem for $u(r, t)$ as given by Equations (11) through (13). Since $\phi(t) \geq 0$ for all values of t , then none of the values assumed by $u(r, t)$ on the boundary are negative. Hence, by the maximum principle for parabolic equations, $u(r, t) \geq 0$, and, therefore, a lower bound for $R(t)$ is given by $R_L(t)$, i.e.,

$$R_L(t) \leq R(t). \quad \dots (18)$$

A simple upper bound for $R(t)$ may be established in a way which will illustrate the manner in which more complicated upper and lower bounds may be established. It is well known that the complementary error function, $\text{erfc}(x)$, has the following properties:

$$\text{erfc}(0) = 1$$

$$\lim_{x \rightarrow \infty} \text{erfc}(x) = 0.$$

Further, in a semi-infinite media, the complementary error function is a solution of the heat equation in plane geometry. Now, $u(r, t) = R(t)\phi(t)$ for $r = R(t)$. Hence, if we consider the function

$$\text{erfc}\left(\frac{r - R(t)}{2\sqrt{\kappa t}}\right)$$

which has the value of 1 if $r = R(t)$, then

$$U(r, t) = R(t)\phi(t)\text{erfc}\left(\frac{r - R(t)}{2\sqrt{\kappa t}}\right)$$

has the value $R(t)\phi(t)$ for $r = R(t)$. Further, assuming $R(t)$ is bounded in the time interval under consideration,

(AI)

$$\lim_{t \rightarrow 0} U(r, t) = 0 ,$$

$$\lim_{r \rightarrow \infty} U(r, t) = 0 .$$

Finally,

$$\frac{\partial U}{\partial t} - \kappa \frac{\partial^2 U}{\partial r^2} = f(r, t) , \quad \dots (19)$$

and $f(r, t) \geq 0$ for all r and t , assuming $R'(t) > 0$. Then, by the maximum principle for parabolic equations,

$$u(r, t) \leq U(r, t) .$$

A physical picture illustrating the last result may be constructed by considering the temperatures in two semi-infinite slabs whose boundary temperatures are identical with one containing a heat source. In the slab with the heat source, the temperature would never be less than the temperature at the corresponding point in space and time in the other slab.

Consider first the integral

$$I_1 = \int_{R_0}^{R(t)} u[r, T(r)] \, dr .$$

Now,

$$\begin{aligned} I_1 &= \int_{R_0}^{R(t)} R[T(r)] \phi[T(r)] \operatorname{erfc}\left(\frac{r - R[T(r)]}{2\sqrt{T(r)}}\right) \, dr \\ &= \int_{R_0}^{R(t)} r \phi[T(r)] \, dr , \end{aligned}$$

since

$$R[T(r)] = r .$$

Further $\phi(t)$ is a monotone function, and $\phi\{T[R(t)]\} = \phi(t)$ and $\phi[T(R_0)] = \phi(0) = c_i - c_s$. Thus, by the generalized mean-value theorem of integral calculus,

$$I_1 = \phi(0) \frac{\xi^2 - R_0^2}{2} + \phi(t) \frac{R^2(t) - \xi^2}{2} , \quad [R_0 < \xi < R(t)] .$$

On the other hand,

$$\begin{aligned} I_2 &= \int_{R(t)}^{\infty} R(t) \phi(t) \operatorname{erfc}\left(\frac{r - R(t)}{2\sqrt{\kappa t}}\right) dr \\ &= 2R(t) \phi(t) \sqrt{\frac{t}{\pi}} . \end{aligned}$$

By consideration of the values of the physical constants of interest, it may be seen that I_1 is very small in comparison to I_2 , and for that reason will be omitted from further discussion.

Thus,

$$R^2(t) \leq R_u^2(t) = R_L^2(t) + \frac{4}{\rho} R_u(t) \phi(t) \sqrt{\frac{\kappa t}{\pi}} . \quad \dots (20)$$

The general behavior of $R_L(t)$ and $R_u(t)$ is illustrated in Figure 2.

Although $R_u(t) - R_L(t)$ is quite small for small values of t , $R_u(t) - R_L(t)$ becomes quite large for larger values of t and hence a more detailed analysis is necessary. However, the above analysis does give a qualitative idea of the behavior of $R(t)$. It may be noticed that there must be an almost linear behavior

Figure 2. Upper and Lower Bounds

of $R(t)$ for small values of t and an exponential behavior for large values of t . Such a behavior of $R(t)$ indicates that a great deal of difficulty might be encountered in obtaining a numerical solution of the boundary-value problem [Equations (11) through (14)] by the usual machine methods.

During the time of quasi-linear growth of R , the growth is governed chiefly by the initial degree of super-

saturation $c_i - c_s$; during this time $c_i - c_s \gg \int_0^t S(\tau) d\tau$. During the exponential growth, the opposite inequality holds true and gas formed near the bubble wall during its growth is chiefly responsible for further growth.

We next will increase the lower bound and decrease the upper bound. Since $R'(t)$ is assumed to be non-negative, $R(t) > R_0$. If $R(t)$ in Equation (13) is replaced by R_0 , then it would be expected that the solution of the new boundary-value problem would be less than or equal to the solution of the original problem, given in Equations (11) through (14). This may also be shown mathematically, using techniques similar to that outlined above. Further, with $R(t)$ replaced by R_0 in Equation (13), we have a boundary-value problem whose solution is known.³ It is

$$\begin{aligned}
 U_1(r, t) = R_0 \left\{ \left(c_i - c_s - \frac{S_c}{a} \right) \operatorname{erfc} \left(\frac{r - R_0}{2\sqrt{\kappa t}} \right) \right. \\
 + \frac{S_c}{2a} e^{at} \exp \left[(r - R_0) \sqrt{a/\kappa} \right] \operatorname{erfc} \left(\frac{r - R_0}{2\sqrt{\kappa t}} - \sqrt{at} \right) \dots (21) \\
 \left. + \exp \left[(r - R_0) \sqrt{a/\kappa} \right] \operatorname{erfc} \left(\frac{r - R_0}{2\sqrt{\kappa t}} + \sqrt{at} \right) \right\} ,
 \end{aligned}$$

and it may be shown that for all values of r and t that

$$U_1(r, t) \leq u(r, t) \quad \dots (22)$$

We need to evaluate the expression

$$\int_{R(t)}^{\infty} U_1(r, t) \, dr \quad \dots (23)$$

The evaluation of this integral involves the evaluation of a definite integral which is not given in commonly used tables. Therefore it will be given here. If $a \neq 0$, $b \neq 0$, then

$$\int_0^{\infty} e^{ax} \operatorname{erfc}(bx + c) \, dx = \frac{-1}{a} \operatorname{erfc}(c) + \left(\frac{1}{a}\right) \left(\frac{2}{\sqrt{\pi}}\right) \exp\left[\frac{(2bc - a)^2}{4b^2} - c^2\right] \operatorname{erfc}\left(\frac{2bc - a}{2b}\right).$$

... (24)

Letting

$$R_1^2(t) = R_L^2(t) + \frac{2}{\rho} \int_{R(t)}^{\infty} U_1(r, t) \, dr,$$

then it may be seen that

$$R_1^2(t) = R_L^2(t) + 2R_0 \lambda(t), \quad \dots (25)$$

where

$$\lambda(t) = \frac{1}{\rho} \left[\left(c_i - c_s - \frac{S_c}{a} \right) \frac{2\sqrt{\kappa t}}{\sqrt{\pi}} + \frac{S_c}{a} e^{at} \sqrt{\frac{\kappa}{a}} \operatorname{erf} \sqrt{at} \right]. \quad \dots (26)$$

Then $R_1(t)$ is a lower bound for $R(t)$.

Finally, if we let

$$U_2(r, t) = \frac{R(t)}{R_0} U_1(r, t) , \quad \dots (27)$$

then it may be shown that for all values of r and t ,

$$u(r, t) \leq U_2(r, t); \quad \dots (28)$$

hence, $U_2(r, t)$ is an upper bound for $u(r, t)$. Letting

$$R_2^2(t) = R_L^2(t) + \frac{2}{\rho} \int_{R(t)}^{\infty} U_2(r, t) dr, \quad \dots (29)$$

then it may be shown that

$$R_2^2(t) = R_L^2(t) + 2R_2(t)\lambda(t) .^* \quad \dots (30)$$

Collecting these results, we have the following ordering:

$$R_L(t) \leq R_1(t) \leq R(t) \leq R_2(t) \leq R_u(t) . \quad \dots (31)$$

C. NUMERICAL CALCULATIONS

Note first of all that if we set $S_c = 0$ in Equations (25), (26), and (30), then we are in a position to check the results of Plesset and Epstein,² for, in their work, no source term was present in the diffusion equation. Let $\bar{R}_1(t)$ denote the lower bound for the bubble radius if no source term is present, and $\bar{R}_2(t)$ the corresponding upper bound. We see that

$$\bar{R}_1^2(t) = \bar{R}_L^2(t) + 2R_0\bar{\lambda}(t) \quad \dots (32)$$

* Again, the integral corresponding to the integral I_1 of the preceding section may be neglected in comparison to the integral corresponding to the integral I_2 .

(AI)

and

$$\bar{R}_2^2(t) = R_L^2(t) + 2\bar{R}_2(t)\bar{\lambda}(t) \quad , \quad \dots(33)$$

where

$$\bar{R}_L^2(t) = R_0^2 + \frac{2\kappa}{\rho}(c_i - c_s)t \quad \dots(34)$$

and

$$\bar{\lambda}(t) = \left(\frac{c_i - c_s}{\rho} \right) \left(\frac{2\sqrt{\kappa t}}{\sqrt{\pi}} \right) \quad \dots(35)$$

In their work, Plesset and Epstein introduced the dimensionless variables

$$\epsilon = R/R_0 \quad \text{and} \quad x^2 = \left(\frac{2\kappa}{\rho} \right) \left(\frac{c_i - c_s}{R_0^2} \right) t \quad .$$

Letting

$$\epsilon_1(t) = \frac{R_1(t)}{R_0} \quad , \quad \epsilon_2(t) = \frac{R_2(t)}{R_0} \quad ,$$

we find that

$$\epsilon_1(x) = \sqrt{1 + 2\sigma x + x^2} \quad \dots(36)$$

and

$$\epsilon_2(x) = x + \sqrt{1 + (1 + \sigma^2)x^2} \quad , \quad \dots(37)$$

where

$$\sigma = \sqrt{\frac{2(c_i - c_s)}{\pi\rho}} \quad .$$

Figure 3. Comparison of Methods

In Figure 3, for a value of $\sigma = 5.64 \times 10^{-2}$, a plot taken from the paper of Plesset and Epstein for $f = 1.25$ is compared to the bubble radius as given by the methods used in this paper. The exact answer lies between the curves labeled $\epsilon_2(x)$ and $\epsilon_2(x)$, while the answer obtained by the method of Plesset and Epstein lies above the curve $\epsilon_2(x)$. Thus, the method used in this paper, at least in this case, leads to a slightly more accurate approximation to the solution of the boundary-value problem.

To show graphically the nature of the growth of the bubble, a transient with a period of 1.2 sec was taken. The temperature was taken to be 23°C and the pressure 14 cm Hg. The values of the other constants were:*

$$\begin{aligned} P_0 &= 1 \text{ w}, \\ R_0 &= 10^{-3} \text{ cm}, \\ \kappa &= 3.9 \times 10^{-5} \text{ cm}^2/\text{sec}, \\ \rho &= 1.51 \times 10^{-5} \text{ gm/cm}^3, \\ c_i - c_s &= 6.65 \times 10^{-8} \text{ gm/cm}^3. \end{aligned}$$

Figure 4 shows the results of computations for the lower bound of the radius of a bubble as a function of time. The upper curve gives the radius for a bubble whose growth from 10^{-3} cm began at $t = 0$, the middle curve for a bubble whose growth from 10^{-3} cm began 5 sec later at $t = 5$, and the lower curve for a bubble whose growth from 10^{-3} cm began 8 sec later at $t = 8$. In Figure 5 the corresponding results of computations for the upper bounds are shown. It may be noticed that, to plotting accuracy, the upper and lower bounds corresponding to $t = 8$ are almost indistinguishable. However, the upper and lower bounds for $t = 0$ and $t = 5$ do differ significantly, the maximum difference amounting to about 33% in the first case and to about 2% in the second case. A difference of

* These constants were obtained through the cooperation of Dean Gamble, Reactor Kinetics Group, Atomics International.

Figure 4. Lower Bounds ($\tau = 1.2$ sec)

Figure 5. Upper Bounds ($\tau = 1.2$ sec)

2% could easily be tolerated, but a difference of 33% is too large. On the other hand, if a significant amount of bubbles are formed only when the power level is high, then this error is not significant. There is evidence for the belief that this is the case, and this will be discussed later. Further, it may be noticed that at first the bubble radius grows almost linearly, and then grows exponentially. Finally, for larger values of t the radius of the bubble is almost independent of the time at which the bubble was formed.

The time scale here is relative. More relevant is the power level at the particular time. For the 1.2-sec transient studied here, peak power was 125 kw, and breakaway power was about 50 kw which corresponds to a time of 13 sec on the graphs. Thus, the graphs give the bubble radius for times larger than those for which the model is strictly valid.

It might be remarked that although the bubble radius is growing exponentially, the bubble-wall velocity is quite small. At $t = 14$ the bubble-wall velocity is less than or equal to 7.5×10^{-4} cm/sec, which is quite small in comparison to the velocity of sound in water.

Since experimental data on the growth of individual bubbles is not yet available from KEWB, it is impossible to compare the results of these calculations with such data even though a KEWB power trace was used. Hence, we have no practical check on the theory.

III. BUBBLE GROWTH THROUGH PEAK POWER

Corben⁴ has shown that data on power excursions in SPERT may be fitted by using an expression of the form

$$P(t) = \phi_1 e^{at} - \phi_2 e^{2at}, \quad \dots (38)$$

where ϕ_1 and ϕ_2 are constants determined by the data. Using data on the KEWB furnished by the Reactor Kinetics Group at Atomics International, it has been found that, although application of Equation (38) will yield a good approximation for some power excursions through peak power, it will not, in general, yield a good approximation for all of the power excursions in KEWB.

It has been found by experimentation that an expression of the type

$$P(t) = \sum_{i=1}^M \phi_i e^{a_i at}, \quad (M \leq 3) \quad \dots (39)$$

will fit a larger number of the power excursions in KEWB than Equation (1), but will not fit all of the data presently available. The numbers ϕ_i and a_i are constants adjusted to fit the data. At the present time, we will consider only those power excursions which may be adequately represented by an expression of the form of Equation (39). If one considers only the problem of representing the power excursion for a short time past peak power, Equation (39) will be adequate for most of the data available at the present time. Such a representation is used for illustrative purposes only, since a different one may be required for other homogenous reactors.

We thus consider a more general source term

$$S(t) = \gamma P(t) = \sum_{i=1}^M \phi_i e^{a_i at}.$$

If we put this source term into Equation (1), then, by following the procedures outlined in Section II, we can follow the growth of a bubble through peak power under the assumptions used in Section II.

It is convenient for computing purposes to have an expression for the growth of a bubble whose radius was R_0 at a time other than $t = 0$. We will speak of "a bubble formed at time $t = x$." By this, we will mean a bubble which had a radius equal to R_0 at time $t = x$.

We shall denote by $R(t;x)$ the radius at time t of a bubble formed at time $t = x$. Then, corresponding to the previous formulae, we obtain

$$R_1(t;x) = \lambda(t;x) + \sqrt{R_L^2(t;x) + \lambda^2(t;x)}, \quad \dots (40a)$$

$$R_2(t;x) = \sqrt{R_L^2(t;x) + 2R_0 \lambda(t;x)}, \quad \dots (40b)$$

$$\begin{aligned} \lambda(t;x) = \frac{1}{\rho} \left\{ \left[c_x - c_s - \gamma \left(\sum_{i=1}^M \frac{\phi_i}{a_i a} e^{a_i a x} \right) \right] \frac{2}{\sqrt{\pi}} \sqrt{\kappa(t-x)} \right. \\ \left. + \gamma \left[\sum_{i=1}^M \frac{\phi_i}{a_i a} \sqrt{\frac{\kappa}{a_i a}} e^{a_i a(t-x)} \operatorname{erf} \left(\sqrt{a_i a(t-x)} \right) \right] \right\}, \end{aligned} \quad \dots (40c)$$

$$\begin{aligned} R_L^2(t;x) = R_0^2 + \frac{2\kappa}{\rho} \left\{ \left[c_x - c_s - \gamma \left(\sum_{i=1}^M \frac{\phi_i}{a_i a} e^{a_i a x} \right) \right] (t-x) \right. \\ \left. + \gamma \sum_{i=1}^M \frac{\phi_i}{(a_i a)^2} \left(e^{a_i a t} - e^{a_i a x} \right) \right\}, \end{aligned} \quad \dots (40d)$$

$$c_x = c_i + \gamma \sum_{i=1}^M \frac{\phi_i}{(a_i a)^2} \left(e^{a_i a t} - e^{a_i a x} \right) . \quad \dots (40e)$$

All symbols in the equations above have the same meaning as in Section II.

The results of a computation for a power excursion with a period of 1.62 sec are shown in Figures 6 and 7. In these figures, the power as a function of the time is depicted as well as the radii of bubbles formed at varying times. Note that:

- The general shape of the curve for the radius of the bubble is unchanged although the more general source term is used.
- Figure 4, shows that the time lag before exponential growth begins decreased as the power level increased. In Figures 6 and 7 the same result is even more striking.

Figure 6. Lower Bounds
($\tau = 1.62$ sec)

Figure 7. Upper Bounds
($\tau = 1.62$ sec)

In these calculations the value for R_0 was taken to be 10^{-3} cm, the other constants were taken as before. Note also that, if a significant number of bubbles are formed in the early part of the excursion, it would be quite plausible to speak of an average bubble size in the reactor at peak power, for at peak power, there is very little variance in the size of the bubbles formed between $t = 0$ and $t = 12$. However, calculations by Hetrick⁵ on the total void volume in the reactor indicate that there is probably no significant number of bubbles formed (in the sense previously indicated) at an early time in the reactor excursion.

Since, as before, the differences between the upper and lower bounds gets smaller as the time of formation gets larger, this result further indicates that the method of calculation is adequate for the purposes of this calculation. As before, the lower bound on $R(t)$ is strict, while the upper bound is understood to omit an extremely small term.

IV. BUBBLE GROWTH INCLUDING PRESSURE AND SURFACE-TENSION EFFECTS

For KEWB transients obtained in the range for τ from about 2 msec to about 10 msec, it is believed that there may be a significant buildup in the core pressure before peak power is reached. An increase of the pressure in the fluid would tend to make the average size of the bubble smaller than those previously considered and should also slow, if not possibly prevent, the growth of the bubbles. For sufficiently small bubbles, the effect of surface tension becomes important. This section will take into account both of these effects. These effects greatly complicate the equations developed in previous sections, and approximations will be made to simplify certain parts of the computation.

We take the equation

$$p(t) + \frac{2\sigma}{R(t)} = K_2 \rho[R(t)] \quad \dots (41)$$

as the equation of state for the bubble,

where

$p(t)$ = the external pressure,

σ = the surface tension constant,

K_2 = a constant involving the molecular weight of the gas, the gas constant, and the absolute temperature,

$\rho[R(t)]$ = the density of the gas in the bubble of radius $R(t)$.

The boundary-value problem now becomes:

$$\frac{\partial c}{\partial t} = \kappa \nabla^2 c + \gamma P(t), \quad [r > R(t)] ; \quad \dots (42)$$

$$c(r, 0) = c_i, \quad \dots (43)$$

$$\lim_{r \rightarrow \infty} c(r, t) = c_i + \gamma \int_0^t P(\tau) d\tau ;$$

$$c(R(t), t) = K_1 \left[p(t) + \frac{2\sigma}{R(t)} \right] ; \quad \dots (44)$$

and

$$\kappa \left(\frac{\partial c}{\partial r} \right)_{r=R(t)} = \frac{1}{3K_2} \left\{ \frac{d}{dt} [p(t)R(t)] + \left[2p(t) + \frac{4\sigma}{R(t)} \right] \frac{dR}{dt} \right\}. \quad \dots (45)$$

The only new symbol is K_1 which represents the reciprocal of the constant in Henry's law.

As before, we make the transformation

$$c(r, t) = \frac{-u(r, t)}{r} + c_i + \gamma \int_0^t P(\tau) d\tau . \quad \dots (46)$$

This transformation leads to the following boundary-value problem:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial r^2} , \quad [r > R(t)] ; \quad \dots (47)$$

$$u(r, 0) = 0 ,$$

$$\lim_{r \rightarrow \infty} u(r, t) = 0 ; \quad \dots (48)$$

$$u(R(t), t) = R(t) \left\{ \phi(t) - K_1 \left[p(t) + \frac{2\sigma}{R(t)} \right] \right\} ; \quad \dots (49)$$

where

$$\phi(t) = c_i + \gamma \int_0^t P(\tau) d\tau$$

and

$$\begin{aligned} \kappa \left(\frac{\partial u}{\partial r} \right)_{r=R(t)} = & \kappa \left\{ \phi(t) - K_1 \left[p(t) + \frac{2\sigma}{R(t)} \right] \right\} - \frac{R(t)}{3K_2} \left\{ \frac{d}{dt} \left[p(t)R(t) \right] \right. \\ & \left. + \left[2p(t) + \frac{4\sigma}{R(t)} \right] \frac{dR}{dt} \right\} . \end{aligned} \quad \dots (50)$$

As before, we find that

$$\int_0^t \int_{R(\xi)}^\infty \frac{\partial u}{\partial t} dr d\xi = \int_{R(t)}^\infty u(r, t) dr + \int_{R_0}^{R(t)} u(r, T(r)) dr , \quad \dots (51)$$

where, again, $t = T(r)$ is the inverse function of $r = R(t)$. Further,

$$\begin{aligned} - \kappa \int_0^t \int_{R(\xi)}^\infty \frac{\partial^2 u}{\partial r^2} dr d\xi = & \kappa \int_0^t \phi(\tau) d\tau - \kappa K_1 \int_0^t p(\tau) d\tau - 2\kappa K_1 \sigma \int_0^t \frac{d\tau}{R(\tau)} \\ & - \frac{1}{3K_2} R^2(\tau) p(\tau) \Big|_0^t - \frac{1}{3K_2} \int_0^t p(\tau) R(\tau) \frac{dR}{d\tau} d\tau \\ & - \frac{4\sigma}{3K_2} R(\tau) \Big|_0^t \end{aligned}$$

or

$$\begin{aligned} - \int_0^t \int_{R(\xi)}^\infty \frac{\partial^2 u}{\partial r^2} dr d\xi = & \kappa \int_0^t \phi(\tau) d\tau - \kappa K_1 \int_0^t p(\tau) d\tau - \frac{2\kappa K_1 \sigma t}{R(\xi_1)} \\ & - \frac{1}{3K_2} \left[R^2(t) p(t) - R_0^2 p(0) \right] - \frac{1}{3K_2} p(\xi_2) \left[\frac{R^2(t) - R_0^2}{2} \right] \\ & - \frac{4\sigma}{3K_2} \left[R(t) - R_0 \right] , \end{aligned} \quad \dots (52)$$

(A)

where

$$0 < \xi_1, \xi_2 < t \quad .$$

Thus, we may write

$$A(t)R^2(t) + B(t)R(t) + C(t) = 0, \quad \dots (53)$$

where

$$A(t) = \frac{1}{3K_2} \left[p(t) + \frac{p(\xi_2)}{2} \right], \quad \dots (54a)$$

$$B(t) = \frac{4\sigma}{3K_2}, \quad \dots (54b)$$

$$\begin{aligned} -C(t) = & \kappa \int_0^t \phi(\tau) d\tau - \kappa K_1 \int_0^t p(\tau) d\tau - \frac{2\kappa K_1 \sigma t}{R(\xi_1)} \\ & + \frac{R_0^2 p(0)}{3K_2} + \frac{p(\xi_2)R_0^2}{6K_2} + \frac{4\sigma R_0}{3K_2} \end{aligned} \quad \dots (54c)$$

$$+ \int_{R(t)}^{\infty} u(r, t) dr + \int_{R_0}^{R(t)} u(r, T(r)) dr \quad .$$

We shall assume that the pressure and power may be adequately represented by

$$p(t) = \sum_{k=0}^M a_k t^k \quad \dots (55)$$

and

$$P(t) = \sum_{k=0}^N b_k \exp(a\lambda_k t) \quad \dots (56)$$

for some integers M and N and constants a_k ($k = 1, 2, \dots, M$), b_k , and λ_k ($k = 1, 2, \dots, N$).

We shall proceed as previously, and assume that $R(t) = R_0$. Let $u(r, t)$ denote the solution of the boundary value problem [Equations (47) through (49)]. Using Equations (55) and (56), we find that,

$$\begin{aligned} u_1(r, t) = R_0 & \left\{ \left(c_i - \frac{2\sigma K_1}{R_0} - \gamma \sum_{k=0}^N \frac{b_k}{a\lambda_k} \right) \operatorname{erfc} \left(\frac{r - R_0}{2\sqrt{\kappa t}} \right) \right. \\ & + \gamma \sum_{k=0}^N \frac{b_k}{a\lambda_k} \left(\frac{1}{2} e^{a\lambda_k t} \left[\exp \left(-(r - R_0) \sqrt{a\lambda_k / \kappa} \right) \operatorname{erfc} \left(\frac{r - R_0}{2\sqrt{\kappa t}} - \sqrt{a\lambda_k t} \right) \right. \right. \\ & \left. \left. + \exp \left((r - R_0) \sqrt{a\lambda_k / \kappa} \right) \operatorname{erfc} \left(\frac{r - R_0}{2\sqrt{\kappa t}} + \sqrt{a\lambda_k t} \right) \right] \right) \right\} \quad \dots (57) \\ & - K_1 \sum_{k=0}^M a_k \Gamma(k+1) (4t)^k \left[i^{2k} \operatorname{erfc} \left(\frac{r - R_0}{2\sqrt{\kappa t}} \right) \right] \Bigg\} , \end{aligned}$$

where

$$i^n \operatorname{erfc}(x) = \int_x^\infty i^{n-1} \operatorname{erfc} \tau d\tau , \quad \dots (58)$$

(A)

and

$$i^0 \operatorname{erfc}(x) = \operatorname{erfc}(x) \quad \dots (59)$$

We further find that

$$\begin{aligned} \int_{R_0}^{\infty} u(r, t) dr = R_0 & \left[\left(c_i - \frac{2\sigma K_1}{R_0} - \gamma \sum_{k=0}^N \frac{b_k}{a\lambda_k} \right) \frac{2\sqrt{\kappa t}}{\sqrt{\pi}} \right. \\ & + \frac{\gamma}{a} \sum_{k=0}^N \frac{b_k}{\lambda_k} \sqrt{\frac{\kappa}{b_k a}} e^{\lambda_k a t} \operatorname{erf} \left(\sqrt{b_k a t} \right) \\ & \left. - K_1 \sum_{k=0}^N a_k \frac{\Gamma(k+1)}{\Gamma(k+3/2)} t^k \sqrt{\kappa t} \right] \quad \dots (60) \end{aligned}$$

As before, we replace R_0 by $R(t)$. Denote the corresponding values of the bubble radius by $R_1(t)$ and $R_2(t)$, respectively, where $R_1(t)$ and $R_2(t)$ are given by

$$A(t)R_1^2(t) + B(t)R_1(t) + C_1(t) = 0, \quad \dots (61)$$

and

$$A(t)R_2^2(t) + B_2(t)R_2(t) + C_2(t) = 0. \quad \dots (62)$$

The coefficients $A(t)$ and $B(t)$ are given in Equation 54, and the coefficients $C_1(t)$, $B_2(t)$, and $C_2(t)$ are given by

$$\begin{aligned}
 -C_1(t) = & \kappa \int_0^t \phi(\tau) d\tau - \kappa K_1 \int_0^t p(\tau) d\tau - \frac{2\kappa K_1 \sigma t}{R(\xi_1)} \\
 & + \frac{R_0^2}{3K_2} \left[p(0) + \frac{p(\xi_2)}{2} \right] + \frac{4\sigma R_0}{3K_2} \dots (63a)
 \end{aligned}$$

$$+ \int_{R_0}^{\infty} u_1(r, t) dr + \int_{R_0}^{R(t)} u(r, T(r)) dr ,$$

$$\begin{aligned}
 B_2(t) = & \frac{4\sigma}{3K_2} - \left(c_i - \gamma \sum_{i=0}^N \frac{b_k}{a_k} \right) \frac{2\sqrt{\kappa t}}{\sqrt{\pi}} - \frac{\gamma}{a} \sum_{k=0}^N \frac{b_k}{\lambda_k} \sqrt{\frac{\kappa}{b_k a}} e^{\lambda_k a t} \operatorname{erf}(\sqrt{b_k a t}) \\
 & + K_1 \sum_{k=0}^N a_k \frac{\Gamma(k+1)}{\Gamma(k+3/2)} t^k \sqrt{\kappa t} , \dots (63b)
 \end{aligned}$$

$$\begin{aligned}
 -C_2(t) = & + \kappa \int_0^t \phi(\tau) d\tau - \kappa K_1 \int_0^t p(\tau) d\tau - \frac{2\kappa K_1 \sigma t}{R(\xi_1)} - \frac{4\sigma K_1 \sqrt{\kappa t}}{\sqrt{\pi}} \\
 & + \frac{R_0^2}{3K_2} \left[p(0) + \frac{p(\xi_2)}{2} \right] + \frac{4\sigma R_0}{3K_2} + \int_{R_0}^{R(t)} u_2(r, \tau(r)) dr , \dots (63c)
 \end{aligned}$$

where $u_2(r, t)$ is given by the right-hand side of Equation (57) with R_0 replaced by $R(t)$.

A KEWB transient with a period of 10.1 msec was taken for the purpose of giving an illustrative calculation. In order to perform these calculations, a digital computer code was written using Equation (61) and (62) as the basis for the calculation. Beginning at time $t = t_0$, the code computes $R_1(t)$ and $R_2(t)$ for the points $t_n = t_0 + n\Delta t$, $n = 1, 2, \dots$. The values at $t = t_n$ chosen for ξ_1 and

ξ_2 were $(t_0 + t_n)/2$ and t_{n-1} respectively. Figure 8 shows the result of these calculations for a bubble with an initial radius of 1 micron whose growth begins 4 msec before peak power is reached. Note that the comparatively sharp increase in the pressure about 2 msec before peak power is reached is reflected distinctly in the upper bound, while only the rate of growth of the lower bound is slowed. The adjectives upper and lower are used here only in the sense that they are analogous to the upper and lower bounds treated in previous sections. As yet, there is no analytical proof that they are really upper and lower bounds although it is quite plausible.

Figure 8. Growth of a Bubble Including Surface-Tension and Pressure Effects

It will be noted that in several places in this report, KEWB has been used as an example of a homogeneous reactor. On the basis of this example, the specific form for the power trace has been chosen and transient pressure effects have been included in the mathematical model. It should not be inferred from this, however, that direct comparisons between the macroscopic behavior of the void volume observed in KEWB and the microscopic behavior of bubbles as predicted by this theory may be made. In order to bridge this gap, a theory of bubble nucleation needs to be formulated. Further, bubble growth on plates should be considered since it is quite possible that such bubbles may make a significant contribution to the total void volume. On the other hand, knowledge of the growth of bubbles of the type considered in this report is a necessary step in understanding the growth of void volume.

REFERENCES

1. G. Birkhoff, R. S. Margulies, and W. A. Horning, "Spherical Bubble Growth," Physics of Fluids I, (May-June 1958), pp 201-204
2. P. S. Epstein and M. S. Plesset, "On the Stability of Gas Bubbles in Liquid-Gas Solutions," J. Chem. Phys. 18, (1950), pp 1505-1509
3. H. S. Carslaw and J. C. Jaeger, Conditons of Heat in Solids (1st ed., London, Oxford University Press, 1947)
4. H. S. Corben, "Power Bursts in Nuclear Reactors," Nuclear Sci. and Eng. I, (1958), p 105
5. D. L. Hetrick, System Development Corporation, in a conversation with H. P. Flatt, Atomics International, October 1958

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4. H. S. Corben, "Power Bursts in Nuclear Reactors," Nuclear Sci. and Eng. I, (1958), p 105
5. D. L. Hetrick, System Development Corporation, in a conversation with H. P. Flatt, Atomics International, October 1958