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THE APPLICATION OF THE COLLOCATION METHOD USING HERMITE CUBIC SPLINES TO NONLINEAR TRANSIENT ONE-DIMENSIONAL HEAT-CONDUCTION PROBLEMS

by

T. C. Chawla, G. Leaf,
W. L. Chen, and M. A. Grolmes

BASE TECHNOLOGY

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T. C. Chawla, G. Leaf,* W. L. Chen, and M. A. Grolmes

Reactor Analysis and Safety Division

July 1975

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NOMENCLATURE

A = Coefficient matrix

a = Inner boundary

b = Outer boundary

$C_{i,k}$ = Defined by Eqs. 20

C_p = Specific heat

$F_{i,k}$ = Defined by Eqs. 18

h = Coolant heat-transfer coefficient

h_g = Gap heat-transfer coefficient

h_j = Step length in general and in fuel region

H_j = Step length in clad region

$K(T)$ = Thermal conductivity

K_0 = Thermal conductivity at reference temperature T_0

N = Number of intervals in clad region

n = Number of intervals in fuel region

$\dot{q}$ = Rate of power generation per unit volume in fuel pin

r = Radial coordinate

r_j = Knot points in fuel

R_C = External radius of clad

R_F = External radius of fuel

R_j = Knot points in clad

R_o = Internal radius of fuel

T = Temperature

T_{Na} = Coolant temperature

t = Time

x_j = Knot points in general

$S_j(x)$ = Basis function defined by Eq. 6b

$V_j(x)$ = Basis function defined by Eq. 6a

Greek Symbols

α = Thermal diffusivity

β,γ = Constants defined by Eqs. 15a

δ_{ij} = Kronecker delta function

$\eta_{i,k}$ = Collocation points in general and also in fuel region

Θ = Kirchoff's transformed temperature defined by Eq. 3

$\xi_{i,k}$ = Collocation points in clad region

ρ = Density

$\Phi_j(r) = V_j''(r) + \frac{1}{r} V_j'(r)$

$\Psi_j(r) = S_j''(r) + \frac{1}{r} S_j'(r)$

Subscripts

C = Clad region

F = Fuel region

i,j,k = Indices

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ABSTRACT

A collocation method for the solution of one-dimensional parabolic partial differential equations using Hermite splines as approximating functions and Gaussian quadrature points as collocation points is described. The method consists of expanding dependent variables in terms of piecewise cubic Hermite splines in the space variable at each time step. The unknown coefficients in the expansion are obtained at every time step by requiring that the resultant differential equation be satisfied at a number of points (in particular, at the Gaussian quadrature points) in the field equal to the number of unknown coefficients. This collocation procedure reduces the partial differential equation to a system of ordinary differential equations which is solved as an initial-value problem using the steady-state solution as the initial condition. The method thus developed is applied to a two-region nonlinear transient heat-conduction problem and compared with a finite-difference method. It is demonstrated that because of high-order accuracy only a small number of equations are needed to produce desirable accuracy. The method has the desirable characteristic of an analytical method in that it produced point values as against nodal values in the finite-difference scheme.

I. INTRODUCTION

Various numerical methods have been devised for solutions to the problems of transient nonlinear heat conduction. The most common of these are finite-difference methods and finite-element methods using Galerkin or weighted residual procedures. Finite-difference methods are usually of low order and therefore require the solution of large systems of equations in order to achieve a satisfactory truncation error. A large system of equations implies large storage requirements; consequently, time differencing must be done with single-time-step methods whether explicit or

implicit. In an explicit method, the largest time step that can be taken is limited due to stability conditions. On the other hand, an implicit method will allow larger time steps, but only at a higher cost per time step. In either case, substantial machine times are encountered in solving such large systems of equations.

In contrast to low-order finite-difference methods, finite-element methods based on the use of a Galerkin or a weighted residual procedure can achieve high-order accuracy, thereby substantially reducing the size of the numerical system. Hence, the data-handling requirements are less severe and multistep time-differencing methods can be used in order to increase the size of the time steps. On the other hand, for nonlinear problems, these methods usually require the computation of integrals at each time step. This, in turn, implies considerably more arithmetic at each time step when compared to low-order finite-difference methods. Thus, in spite of their greater accuracy, these higher-order Galerkin procedures when applied to nonlinear conduction problems may not be substantially faster than finite-difference methods.

A third method is based on collocation combined with the use of suitable approximating subspaces. For nonlinear problems, this approach has an advantage over a Galerkin-type procedure in that there is much less arithmetic at each time step. On the other hand, it has the disadvantage that, unless the collocation points are suitably chosen, this method produces low-order accuracy regardless of the nature of the approximating subspaces. This implies large systems of equations in order to obtain suitable accuracy, just as in the case of low-order finite-difference methods. However, de Boor and Swartz[1] and Douglas and Depont[2] have shown that if Gaussian quadrature points are selected as collocation points, high-order accuracy can be achieved with suitable approximating subspaces. Hence, in this case, the advantages of a high-order Galerkin-type procedure are achieved, namely, a small system of equations and the use of multistep time differencing. In addition, like finite-difference methods, it requires only a small amount of arithmetic at each time step. In this report, we demonstrate an application of this method to the nonlinear heat-conduction equation; nonlinearity is introduced by requiring the transport properties to be a function of temperature.

II. DESCRIPTION OF THE PROBLEM

For this application, consider a single fuel pin for a nuclear reactor. This pin consists of a hollow cylindrical solid of mixed-oxide fuel encased in a cylindrical sheath of stainless-steel cladding. There is a finite contact resistance between the fuel and the clad, and the outer surface of the clad is cooled by convection. The fuel is heated by a uniform volume heat source which can vary with time. A schematic cross-sectional view of such a system is shown in Fig. 1. The figure contains

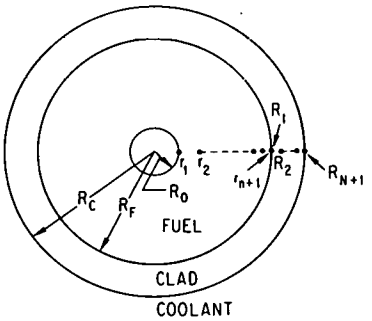

Fig. 1.
Cross-sectional View of a Fuel Pin Showing the Knot Sequences.
ANL Neg. No. 900-4880.

dimensional nomenclature. If the volume heat source in the fuel is denoted by $\dot{q}(t)$, and if axial and circumferential conduction terms are neglected, then the heat-conduction equation can be written in cylindrical coordinates as

$$\rho(T)\ C_p(T)\ \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(rK(T)\ \frac{\partial T}{\partial r}\right) + \dot{q}(t) \tag{1}$$

When the above equation is applied to the clad, the heat-source term is set equal to zero. The boundary conditions which must be satisfied are

$$\frac{\partial T_F}{\partial r} = 0 \quad \text{at} \quad r = R_o\ ; \tag{2a}$$

$$-K_F\ \frac{\partial T_F}{\partial r} = h_g\left[T_F(R_F^-,t) - T_C(R_F^+,t)\right] \quad \text{at } r = R_F\ ; \tag{2b}$$

$$K_F\ \frac{\partial T_F}{\partial r} = K_C\ \frac{\partial T_C}{\partial r} \quad \text{at } r = R_F\ ; \tag{2c}$$

$$- K_C \frac{\partial T_C}{\partial r} = h(t) \left[T_C(R_C,t) - T_{Na}(t) \right] . \qquad (2d)$$

Here, $T_F = T_F(r,t)$ refers to the temperature distribution in the fuel and $T_C = T_C(r,t)$ to the temperature distribution in the clad. The temperature of the external medium (sodium) surrounding the clad is denoted by $T_{Na}(t)$. The definition of the other symbols used is given in the Nomenclature.

The above equations can be simplified somewhat by application of Kirchoff's transformation

$$\Theta = \frac{1}{K_O} \int_{T_O}^{T} K(T)dT . \qquad (3)$$

Upon the introduction of the variable Θ into Eqs. 1 and 2, we obtain

$$\frac{\partial \Theta}{\partial t} = \alpha \left(\frac{\partial^2 \Theta}{\partial r^2} + \frac{1}{r} \frac{\partial \Theta}{r} \right) + \frac{\alpha \dot{q}}{K_O} \quad ; \qquad (4)$$

$$\frac{\partial \Theta_F}{\partial r} = 0 \quad \text{at } r = R_o \; ; \qquad (5a)$$

$$- K_{OF} \frac{\partial \Theta_F}{\partial r} = h_g \left[T_F \left(\Theta_F(R_F^-,t) \right) - T_C \left(\Theta_C(R_F^+,t) \right) \right] \quad \text{at } r = R_F \; ; \qquad (5b)$$

$$K_{OF} \frac{\partial \Theta_F}{\partial r} = K_{OC} \frac{\partial \Theta_C}{\partial r} \quad \text{at } r = R_F \; ; \qquad (5c)$$

$$- K_{OC} \frac{\partial \Theta_C}{\partial r} = h(t) \left[T_C \left(\Theta_C(R_C,t) \right) - T_{Na}(t) \right] . \qquad (5d)$$

Here the diffusivity $\alpha = K(T)/\rho(T)C_p(T)$ must be evaluated by means of the transformation $T = T(\Theta)$ which is the inverse of the transformation given in Eq. 3.

III. A METHOD OF COLLOCATION

We shall seek an approximate solution of Eqs. 4 and 5 using the method of lines in conjunction with collocation. To this end, we assume that at each fixed time the transformed temperature $\Theta(r,t)$ can be approximated by a function which is a piecewise cubic polynomial in r and that the function together with its first derivative is continuous. More specifically, let [a,b] denote either the fuel region $[R_0,R_F]$ or the clad region $[R_F,R_C]$. Let this interval be subdivided by the set of points

$$\pi: a = x_1 < x_2 <\cdots< x_{n+1} = b,\ h_j = x_j - x_{j-1} .$$

Relative to this partition π, the approximating subspace $H_3(\pi)$ will consist of all functions f(x) such that

(1) f(x) is equal to a cubic polynomial in each subinterval $[x_i,x_{i+1}]$ for $1 \leq i \leq n$,

(2) f(x) and f'(x) are continuous at the points x_i for $2 \leq i \leq n$, and

(3) f(x) satisfies the appropriate boundary conditions in Eqs. 5, depending on whether the interval [a,b] is the fuel or the clad.

Since this problem has two regions, we will generate two approximating sets of functions: $H_3^{(F)}(\pi_F)$ and $H_3^{(C)}(\pi_C)$, one relative to a partition π_F of the fuel, the other relative to a partition π_C of the clad. These two sets of functions are required to satisfy the common interface conditions (5b) and (5c).

A convenient basis for generating either set of approximating functions is the set $\{V_j(x), S_j(x)\}_{j=1}^{n+1}$, where

$$V_j(x) = \begin{cases} \left(1 + \dfrac{x - x_j}{h_j}\right)^2 \left[1 - 2\left(\dfrac{x - x_j}{h_j}\right)\right] & \text{for } x_{j-1} \le x \le x_j \\ \left(1 - \dfrac{x - x_j}{h_{j+1}}\right)^2 \left[1 + 2\left(\dfrac{x - x_j}{h_{j+1}}\right)\right] & \text{for } x_j \le x \le x_{j+1} \\ 0 \text{ elsewhere} \end{cases} \tag{6a}$$

$$S_j(x) = \begin{cases} h_j \left(\dfrac{x - x_j}{h_j}\right) \left[1 + \dfrac{x - x_j}{h_j}\right]^2 & \text{for } x_{j-1} \le x \le x_j \\ h_{j+1} \left(\dfrac{x - x_j}{h_{j+1}}\right) \left(1 - \dfrac{x - x_j}{h_{j+1}}\right)^2 & \text{for } x_j \le x \le x_{j+1} \\ 0 \text{ elsewhere} \end{cases} \tag{6b}$$

The functions $V_{j-1}(x)$, $V_j(x)$, $S_{j-1}(x)$, and $S_j(x)$ together with their first derivatives are shown in Figs. 2 and 3. It is assumed that $V_1(x)$ and

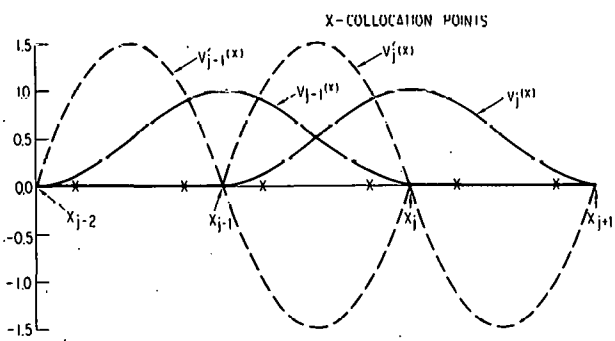

Fig. 2.
Graphs of $V_{j-1}(x)$, $V'_{j-1}(x)$, $V_j(x)$, and $V'_j(x)$.
ANL Neg. No. 900-4453 Rev. 1.

$S_1(x)$ vanish to the left of x_1, whereas $V_{n+1}(x)$, and $S_{n+1}(x)$ vanish to the right of x_{n+1}. From the defining equations, we observe that each of the functions has the following properties:

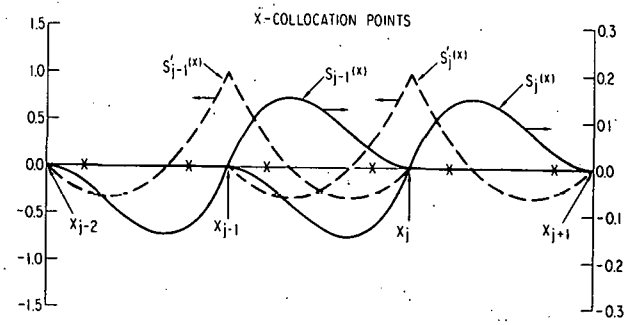

Fig. 3.
Graphs of $S_{j-1}(x)$, $S'_{j-1}(x)$, $S_j(x)$, and $S'_j(x)$.
ANL Neg. No. 900-4452 Rev. 1.

(1) Each $V_j(x)$ and $S_j(x)$ is continuous together with its first derivative in the interval [a,b].

(2) Each $V_j(x)$ and $S_j(x)$ is a cubic polynomial in each subinterval, and they vanish outside the subinterval $[x_{j-1}, x_{j+1}]$.

(3)
$$\left.\begin{aligned} &V_i(x_j) = \delta_{ij};\ V'_i(x_j) \equiv 0 \\ &S_i(x_j) \equiv 0;\ S'_i(x_j) = \delta_{ij} \end{aligned}\right\} \quad 1 \le i,j \le n+1 \ .$$

The set $\{V_i(x), S_i(x)\}_{i=1}^{n+1}$ will form a basis for the set of functions $H_3(\pi)$. Thus, we assume that the transformed temperature field has the form

$$\Theta(x,t) = \sum_{j=1}^{n+1} \left[\Theta(x_j,t)V_j(x) + \Theta'(x_j,t)S_j(x)\right] , \tag{7}$$

where $\{\Theta(x_j,t), \Theta'(x_j,t)\}_{j=1}^{n+1}$ are the unknown coefficients. In view of property (3), these coefficients represent, respectively, the unknown temperature and spatial derivative of temperature at each of the knot points $(x_j,\ 1 \le j \le n+1)$.

Let

$$\pi_F: R_o = r_1 < r_2 < \cdots < r_{n+1} = R_F$$

be a partition of the fuel and

$$\pi_C: R_F = R_1 < R_2 < \cdots < R_{N+1} = R_C$$

a partition of the clad. We seek an approximation to the temperature field $\Theta(r,t)$, of the form shown in Eq. 7, in the fuel and the clad. Substituting Eq. 7 into Eqs. 4 and 5, we obtain the following relation in the fuel:

$$\frac{\partial \Theta_F}{\partial t} = \sum_{j=1}^{n+1} \left[V_j(r)\dot{\Theta}_F(r_j,t) + S_j(r)\dot{\Theta}'_F(r_j,t) \right]$$

$$= \alpha_F \left(\Theta_F(r,t) \right) \left\{ \sum_{j=1}^{n+1} \left[\Theta_F(r_j,t)\ \Phi_j(r) + \Theta'_F(r_j,t)\ \Psi_j(r) \right] \right\}$$

$$+ \alpha_F \left(\Theta_F(r,t) \right) \dot{q}(t)/K_{OF} \qquad \text{for } r_1 \leq r \leq r_{n+1} , \tag{8}$$

where

$$\Phi_j(r) = V''_j(r) + \frac{1}{r} V'_j(r), \quad \Psi_j(r) = S''_j(r) + \frac{1}{r} S'_j(r) .$$

Now

$$\frac{\partial \Theta}{\partial r} = \sum_{j=1}^{n+1} \left[V'_j(r)\Theta_F(r_j,t) + S'_j(r)\Theta'_F(r_j,t) \right] .$$

Since $V'_j(r_1) \equiv 0$, $S'_1(r_1) = 1$, and $S'_j(r_1) \equiv 0$ for $j > 1$, we find that the boundary condition (5a) implies

$$\Theta'_F(r_1,t) \equiv 0 . \tag{9a}$$

Differentiating this expression with respect to time, we find also

$$\dot{\Theta}'_F(r_1,t) \equiv 0 . \tag{9b}$$

Similarly, we find the following relations at $r = r_{n+1}$ from the boundary condition (5b):

$$\Theta_F'(r_{n+1},t) = -\frac{h_g}{K_{OF}}\left[T_F\left(\Theta_F(r_{n+1},t)\right) - T_C\left(\Theta_C(R_1,t)\right)\right] = F_f \qquad (9c)$$

and

$$\dot{\Theta}_F'(r_{n+1},t) = -\frac{\dot{h}_g}{K_{OF}}\left[T_F\left(\Theta_F(r_{n+1},t)\right) - T_C\left(\Theta_C(R_1,t)\right)\right]$$

$$-\frac{h_g}{K_{OF}}\left[\left.\frac{dT_F}{d\Theta_F}\right|_{\Theta_F(r_{n+1},t)}\dot{\Theta}_F(r_{n+1},t) - \left.\frac{dT_C}{d\Theta_C}\right|_{\Theta_C(R_1,t)}\right.$$

$$\left.\times\,\dot{\Theta}_C(R_1,t)\right]$$

$$= f_F + A_F\dot{\Theta}_F(r_{n+1},t) + A_C\dot{\Theta}_C(R_1,t)\ . \qquad (9d)$$

For the clad region (see Fig. 1), we find

$$\frac{\partial\Theta_C(r,t)}{\partial t} = \sum_{j=1}^{N+1}\left[V_j(r)\dot{\Theta}_C(R_j,t) + S_j(r)\dot{\Theta}_C'(R_j,t)\right]$$

$$= \alpha_C\left(\Theta_C(r,t)\right)\sum_{j=1}^{N+1}\left[\Theta_C(R_j,t)\ \Phi_j(r)\right.$$

$$\left.+\ \Theta_C'(R_j,t)\ \Psi_j(r)\right] \quad \text{for } R_1 \le r \le R_{N+1}\ ; \qquad (10)$$

$$\Theta_C'(R_1,t) = \frac{K_{OF}}{K_{OC}}\Theta_F'(r_{n+1},t) = \frac{K_{OF}}{K_{OC}}F_f\ ; \qquad (11a)$$

$$\dot{\Theta}_C'(R_1,t) = \frac{K_{OF}}{K_{OC}}\,\dot{\Theta}_F'(r_{n+1},t)$$

$$= \frac{K_{OF}}{K_{OC}}\left[f_F + A_F\dot{\Theta}_F(r_{n+1},t) + A_C\dot{\Theta}_C(R_1,t)\right]\ ; \qquad \text{(11b)}$$

$$\Theta_C'(R_{N+1},t) = -\frac{h}{K_{OC}}\left[T_C\left(\Theta_C(R_{N+1},t)\right) - T_{Na}(t)\right] = F_C\ ;$$

$$\dot{\Theta}_C'(R_{N+1},t) = -\frac{\dot{h}}{K_{OC}}\left[T_C\left(\Theta_C(R_{N+1},t)\right) - T_{Na}(t)\right]$$

$$-\frac{h}{K_{OC}}\left[\left.\frac{dT_C}{d\Theta_C}\right|_{\Theta_C(R_{N+1},t)}\dot{\Theta}_C(R_{N+1},t) - \dot{T}_{Na}(t)\right] \qquad \text{(11c)}$$

or

$$\dot{\Theta}_C'(R_{N+1},t) = f_C + B_C\dot{\Theta}_C(R_{N+1},t)\ . \qquad \text{(11d)}$$

Here, in Eqs. 9d and 11d, we have made the following substitutions:

$$f_F = -\frac{\dot{h}_g}{K_{OF}}\left[T_F\left(\Theta_F(r_{n+1},t)\right) - T_C\left(\Theta_C(R_1,t)\right)\right]\ ;\ A_F = -\frac{h_g}{K_{OF}}\left.\frac{dT_F}{d\Theta_F}\right|_{\Theta_F(r_{n+1},t)}\ ;$$

$$A_C = \frac{h_g}{K_{OF}}\left.\frac{dT_C}{d\Theta_C}\right|_{\Theta_C(R_1,t)}\ ;\ f_C = -\frac{\dot{h}}{K_{OC}}\left[T_C\left(\Theta_C(R_{N+1},t)\right) - T_{Na}(t)\right] + \frac{h}{K_{OC}}\dot{T}_{Na}(t)\ ;$$

$$B_C = -\frac{h}{K_{OC}}\left.\frac{dT_C}{d\Theta_C}\right|_{\Theta_C(R_{N+1},t)}\ .$$

Substituting Eqs. 9 into Eq. 8, we obtain

$$\sum_{j=2}^{n} \left(V_j(r)\dot{\theta}_F(r_j,t) + S_j(r)\dot{\theta}'_F(r_j,t) \right) + V_1(r)\dot{\theta}_F(r_1,t)$$

$$+ \left[V_{n+1}(r) + S_{n+1}(r)\, A_F \right] \dot{\theta}_F(r_{n+1},t) + S_{n+1}(r)\, A_C \dot{\theta}_C(R_1,t)$$

$$= - S_{n+1}(r) f_F + \frac{\alpha_F(\theta_F(r,t))\dot{q}(t)}{K_{OF}} + \alpha_F \left(\theta_F(r,t) \right)$$

$$\times \left(\sum_{j=1}^{n+1} \theta_F(r_j,t)\, \Phi_j(r) + \sum_{j=2}^{n} \theta'_F(r_j,t)\, \Psi_j(r) \right.$$

$$\left. + F_f\, \Psi_{n+1}(r) \right) \quad \text{for } r_1 \leq r \leq r_{n+1} \,. \tag{12}$$

Substituting Eqs. 11b, 11c, and 11d into Eq. 10, we obtain

$$\sum_{j=2}^{N} \left[V_j(r)\dot{\theta}_C(R_j,t) + S_j(r)\dot{\theta}'_C(R_j,t) \right] + \left(S_1(r)\, A_C \frac{K_{OF}}{K_{OC}} + V_1(r) \right) \dot{\theta}_C(R_1,t)$$

$$+ \left(B_C S_{N+1}(r) + V_{N+1}(r) \right) \dot{\theta}_C(R_{N+1},t) + S_1(r)\, A_F \frac{K_{OF}}{K_{OC}} \dot{\theta}_F(r_{n+1},t)$$

$$= - f_F S_1(r) \frac{K_{OF}}{K_{OC}} - f_C S_{N+1}(r) + \alpha_C \left(\theta_C(r,t) \right)$$

$$\times \left(\sum_{j=1}^{N+1} \theta_C(R_j,t)\, \Phi_j(r) + \sum_{j=2}^{N} \theta'_C(R_j,t)\, \Psi_j(r) + \right.$$

$$+\sum_{j=2}^{N}\Theta_C'(R_j,t)\ \Psi_j(r)+\Psi_1(r)\ F_f\ \frac{K_{OF}}{K_{OC}}+\Psi_{N+1}(r)\ F_C\Bigg)$$

$$\text{for } R_1 \le r \le R_{N+1} \ . \tag{13}$$

The unknown coefficients contained in Eqs. 12 and 13 are:

$$\left\{\Theta_F(r_j,t)\right\}_{j=1}^{n+1}\ ,\quad \left\{\Theta_F'(r_j,t)\right\}_{j=2}^{n}\ ,\quad \left\{\Theta_C(R_j,t)\right\}_{j=1}^{N+1}\ ,\quad \text{and}$$

$$\left\{\Theta_C'(R_j,t)\right\}_{j=2}^{N}\ ,$$

which add up to 2n + 2N unknowns. In order to obtain 2n + 2N equations, we can require that Eq. 12 be satisfied at 2n points in the fuel region and Eq. 13 be satisfied at 2N points in the clad region. Since there are n intervals in the fuel region and N intervals in the clad region, it therefore seems natural to locate two points in each interval. In accordance with the approximation theory as given by Douglas and Dupont[2] and by de Boor and Swartz,[1] the Gauss-Legendre quadrature points of order 2 are chosen as the collocation points in each subinterval $[x_i, x_{i+1}]$. Thus the collocation points are given by

$$\eta_{i,k} = \frac{1}{2}\ (x_{i-1} + x_i) + (-1)^k\ \frac{h_i}{2\sqrt{3}} \qquad 2 \le i \le n+1,\ 1 \le k \le 2. \tag{14}$$

Douglas and Dupont[2] have shown that for a parabolic equation the use of the above collocation points will result in accuracy up to $O(h^4)$ provided the thermal capacity and conductivity have bounded third-order derivatives and the solution $\Theta(x,t)$ has bounded sixth-order spatial derivatives over a fixed time interval.

If we let

$$\beta = \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right) \quad \text{and} \quad \gamma = \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right) \tag{15a}$$

in Eq. 14, we obtain

$$\left.\begin{aligned} \frac{\eta_{j,1} - x_j}{h_j} &= -\beta; \quad \frac{\eta_{j,2} - x_j}{h_j} = -\gamma \\ \frac{\eta_{j,1} - x_{j-1}}{h_j} &= \beta; \quad \frac{\eta_{j,2} - x_{j-1}}{h_j} = \gamma \end{aligned}\right\}. \tag{15b}$$

Using these relations in Eqs. 6 give the following expressions for the values of $V_j(x)$, $S_j(x)$, and their derivatives at the collocation points $\eta_{i,k}$:

$$\left.\begin{aligned} V_{j-1}(\eta_{j,1}) &= V_j(\eta_{j,2}) = (1-\beta)^2(1+2\beta) = V_\beta \\ V_{j-1}(\eta_{j,2}) &= V_j(\eta_{j,1}) = (1-\gamma)^2(1+2\gamma) = V_\gamma \\ S_{j-1}(\eta_{j,1}) &= -S_j(\eta_{j,2}) = h_j\beta(1-\beta)^2 = h_j S_\beta \\ S_{j-1}(\eta_{j,2}) &= -S_j(\eta_{j,1}) = h_j(1-\gamma)^2 = h_j S_\gamma \end{aligned}\right\}; \tag{16a}$$

$$\left.\begin{aligned} V'_{j-1}(\eta_{j,1}) &= -V'_j(\eta_{j,2}) = -6\beta(1-\beta)/h_j = -V'_\beta/h_j \\ V'_{j-1}(\eta_{j,2}) &= -V'_j(\eta_{j,1}) = -6\gamma(1-\gamma)/h_j = -V'_\gamma/h_j \\ S'_{j-1}(\eta_{j,1}) &= S'_j(\eta_{j,2}) = 1 - 4\beta + 3\beta^2 = S'_\beta \\ S'_{j-1}(\eta_{j,2}) &= S'_j(\eta_{j,1}) = 1 - 4\gamma + 3\gamma^2 = S'_\gamma \end{aligned}\right\}; \qquad (16b)$$

$$\left.\begin{aligned} V''_{j-1}(\eta_{j,1}) &= V''_j(\eta_{j,2}) = -6(1-2\beta)/h_j^2 = -V''_\gamma/h_j^2 \\ V''_{j-1}(\eta_{j,2}) &= V''_j(\eta_{j,1}) = -6(1-2\gamma)/h_j^2 = -V''_\gamma/h_j^2 \\ S''_{j-1}(\eta_{j,1}) &= -S''_j(\eta_{j,2}) = -(4-6\beta)/h_j = -S''_\beta/h_j \\ S''_{j-1}(\eta_{j,2}) &= -S''_j(\eta_{j,1}) = -(4-6\gamma)/h_j = -S''_\gamma/h_j \end{aligned}\right\}. \qquad (16c)$$

Evaluating Eq. 12 at the collocation points $\eta_{i,k}$ in the fuel, and noting that each $V_j(x)$ and $S_j(x)$ has its support in $[x_{j-1}, x_{j+1}]$ and, in particular,

$$V_1(r) = 0 \quad \text{for} \quad r \notin [r_1, r_2); \; V_{n+1}(r) = S_{n+1}(r) = 0 \quad \text{for} \quad r \notin (r_n, r_{n+1}],$$

we obtain the following system of equations:

For $i = 2$, $k = 1$ (i.e., $\eta_{2,1}$)

$$V_1(\eta_{2,1})\dot{\theta}_F(r_1,t) + V_2(\eta_{2,1})\dot{\theta}_F(r_2,t) + S_2(\eta_{2,1})\dot{\theta}'_F(r_2,t) = F_{2,1} \,. \qquad (17a)$$

For $i = 2$, $k = 2$ (i.e., $\eta_{2,2}$)

$$V_1(\eta_{2,2})\dot{\Theta}_F(r_1,t) + V_2(\eta_{2,2})\dot{\Theta}_F(r_2,t) + S_2(\eta_{2,2})\dot{\Theta}'_F(r_2,t) = F_{2,2} \ . \tag{17b}$$

In general for $3 \leq i \leq n$, $1 \leq k \leq 2$,

$$V_{i-1}(\eta_{i,k})\dot{\Theta}_F(r_{i-1},t) + V_i(\eta_{i,k})\dot{\Theta}_F(r_i,t) + S_{i-1}(\eta_{i,k})\dot{\Theta}'_F(r_{i-1},t) + S_i(\eta_{i,k})\dot{\Theta}'_F(r_i,t) = F_{i,k} \ . \tag{17c}$$

For $i = n+1$, $1 \leq k \leq 2$ (i.e., $\eta_{n+1,k}$)

$$V_n(\eta_{n+1,k})\dot{\Theta}_F(r_n,t) + \left[V_{n+1}(\eta_{n+1,k}) + S_{n+1}(\eta_{n+1,k})A_F\right]\dot{\Theta}_F(r_{n+1},t) + S_n(\eta_{n+1,k})\dot{\Theta}'_F(r_n,t) + S_{n+1}(\eta_{n+1,k})A_C\dot{\Theta}_C(R_1,t) = F_{n+1,k} \ , \tag{17d}$$

where

$$F_{2,k} = \alpha_F\left(\Theta_F(\eta_{2,k},t)\right)\left(\Theta_F(r_1,t)\ \Phi_1(\eta_{2,k}) + \Theta_F(r_2,t)\ \Phi(\eta_{2,k}) + \Theta'_F(r_2,t)\ \Psi_2(\eta_{2,k}) + \dot{q}(t)/K_{OF}\right) \quad \text{for } 1 \leq k \leq 2 \ ; \tag{18a}$$

$$F_{i,k} = \alpha_F\left(\Theta_F(\eta_{i,k},t)\right)\left(\Theta_F(r_{i-1},t)\ \Phi_{i-1}(\eta_{i,k}) + \Theta_F(r_i,t)\ \Phi_i(\eta_{i,k}) + \Theta'_F(r_{i-1},t)\ \Psi_{i-1}(\eta_{i,k}) + \Theta'_F(r_i,t)\ \Psi_i(\eta_{i,k}) + \dot{q}(t)/K_{OF}\right)$$

$$\text{for } 2 \leq i \leq n,\ 1 \leq k \leq 2 \ ; \tag{18b}$$

$$F_{n+1,k} = - S_{n+1}(\eta_{n+1,k}) f_F + \alpha_F\left(\Theta_F(\eta_{n+1,k},t)\right)\left(F_f\ \Psi_{n+1}(\eta_{n+1,k})\right.$$

$$+ \Theta_F(r_n,t)\ \Phi_n(\eta_{n+1,k}) + \Theta_F(r_{n+1},t)\ \Phi_{n+1}(\eta_{n+1,k})$$

$$\left. + \Theta_F'(r_n,t)\ \Psi_n(\eta_{n+1,k}) + \dot{q}(t)/K_{OF}\right) . \qquad (18c)$$

Similarly, evaluating Eq. 13 at the collocations points which are denoted by $\xi_{i,k}$ and given by Eq. 14 for the clad region, we obtain the following system:

$$\left[S_1(\xi_{2,k})A_C \frac{K_{OF}}{K_{OC}} + V_1(\xi_{2,k})\right]\ \dot{\Theta}_C(R_1,t) + V_2(\zeta_{2,k})\dot{\Theta}_C(R_2,t)$$

$$+ S_2(\xi_{2,k})\dot{\Theta}_C'(R_2,t) + S_1(\xi_{2,k})A_F \frac{K_{OF}}{K_{OC}}\ \dot{\Theta}_F(r_{n+1},t) = C_{2,k}$$

$$\text{for } 1 \le k \le 2 \ ; \qquad (19a)$$

$$V_{i-1}(\xi_{i,k})\dot{\Theta}_C(R_{i-1},t) + V_i(\xi_{i,k})\dot{\Theta}_C(R_i,t) + S_{i-1}(\xi_{i,k})\dot{\Theta}_C'(R_{i-1},t)$$

$$+ S_i(\xi_{i,k})\dot{\Theta}_C'(R_i,t) = C_{i,k}$$

$$\text{for } 2 \le i \le N,\ 1 \le k \le 2 \ ; \qquad (19b)$$

$$V_N(\xi_{N+1,k})\dot{\Theta}_C(R_N,t) + \left(B_C S_{N+1}(\xi_{N+1,k}) + V_{N+1}(\xi_{N+1,k})\right)\ \dot{\Theta}_C(R_{N+1},t)$$

$$+ S_N(\xi_{N+1,k})\dot{\Theta}_C'(R_N,t) = C_{N+1,k}$$

$$\text{for } 1 \le k \le 2 \ , \qquad (19c)$$

where

$$C_{2,k} = - S_1(\xi_{2,k}) f_F \frac{K_{OF}}{K_{OC}} + \alpha_C\left(\Theta_C(\xi_{2,k},t)\right) \left[\Theta_C(R_1,t)\ \Phi_1(\xi_{2,k}) + \Theta_C(R_2,t)\ \Phi_2(\xi_{2,k}) + \Theta_C'(R_2,t)\ \Psi_2(\xi_{2,k}) + \Psi_1(\xi_{2,k}) F_f \frac{K_{OF}}{K_{OC}}\right]$$

$$\text{for } 1 \leq k \leq 2 \ ; \tag{20a}$$

$$C_{i,k} = \alpha_C\left(\Theta_C(\xi_{i,k},t)\right) \left(\Theta_C(R_{i-1},t)\ \Phi_{i-1}(\xi_{i,k}) + \Theta_C(R_i,t)\ \Phi_i(\xi_{i,k}) + \Theta_C'(R_{i-1},t)\ \Psi_{i-1}(\xi_{i,k}) + \Theta_C'(R_i,t)\ \Psi_i(\xi_{i,k})\right)$$

$$\text{for } 2 \leq i \leq N,\ 1 \leq k \leq 2 \ ; \tag{20b}$$

$$C_{N+1,k} = - f_C S_{N+1}(\xi_{N+1,k}) + \alpha_C\left(\Theta_C(\xi_{N+1,k})\right) \left(\Theta_C(R_N,t)\ \Phi_N(\xi_{N+1,k}) + \Theta_C(R_{N+1},t)\ \Phi_{N+1}(\xi_{N+1,k}) + \Theta_C'(R_N,t)\ \Psi_N(\xi_{N+1,k}) + \Psi_{N+1}(\xi_{N+1,k})\ F_C\right)$$

$$\text{for } 1 \leq k \leq 2 \ . \tag{20c}$$

We note that the transformed temperature $\Theta(r,t)$ must be evaluated at the collocation points in order to evaluate the property functions, such as the diffusivity α. For this purpose, we use Eq. 7. For example, in the fuel at the collocation point $\eta_{i,k}\varepsilon(x_{i-1}, x_i)$ we have

$$\Theta_F(\eta_{i,k},t) = \Theta_F(r_{i-1},t)V_{i-1}(\eta_{i,k}) + \Theta_F(r_i,t)V_i(\eta_{i,k})$$

$$+ \Theta'_F(r_{i-1},t)S_{i-1}(\eta_{i,k}) + \Theta'_F(r_i,t)S_i(\eta_{i,k}) \ . \qquad (21)$$

Substituting for the values of the basis functions and their derivatives from Eqs. 16 into Eqs. 17 through 21 and rewriting the resulting equations in matrix form, we obtain

$$\left[A\right]\left\{\dot{U}\right\} = G(U,t) \ , \qquad (22)$$

where U(t) is the 2n + 2N dimensional vector

$$\left\{\Theta_F(r_j,t)\right\}_{j=1}^{n+1} \ U \left\{\Theta'_F(r_j,t)\right\}_{j=2}^{n} \ U \left\{\Theta_C(R_j,t)\right\}_{j=1}^{N+1} \ U \left\{\Theta'_C(R_j,t)\right\}_{j=2}^{N}$$

and G(U,t) is the vector of the entire right-hand side. The details of the above matrix equation are shown in Fig. 4. From an examination of Fig. 4 it is clear that the coefficient matrix [A] is a band matrix. This property is a consequence of the basis functions $V_j(x)$ and $S_j(x)$ having local support. It should be noted that, in general, the coefficient matrix [A] will depend on the temperature because of the convective interface and boundary conditions. This dependence is explicitly shown in Eqs. 9d and 11d. Thus, the matrix [A] has to be inverted at each iteration of every time step. The band structure is utilized in this inversion process.

With given initial values {U(0)}, the system (22) represents a system of nonlinear ordinary differential equations. This system is solved with the ordinary-differential-equation subroutine GEAR.[3]

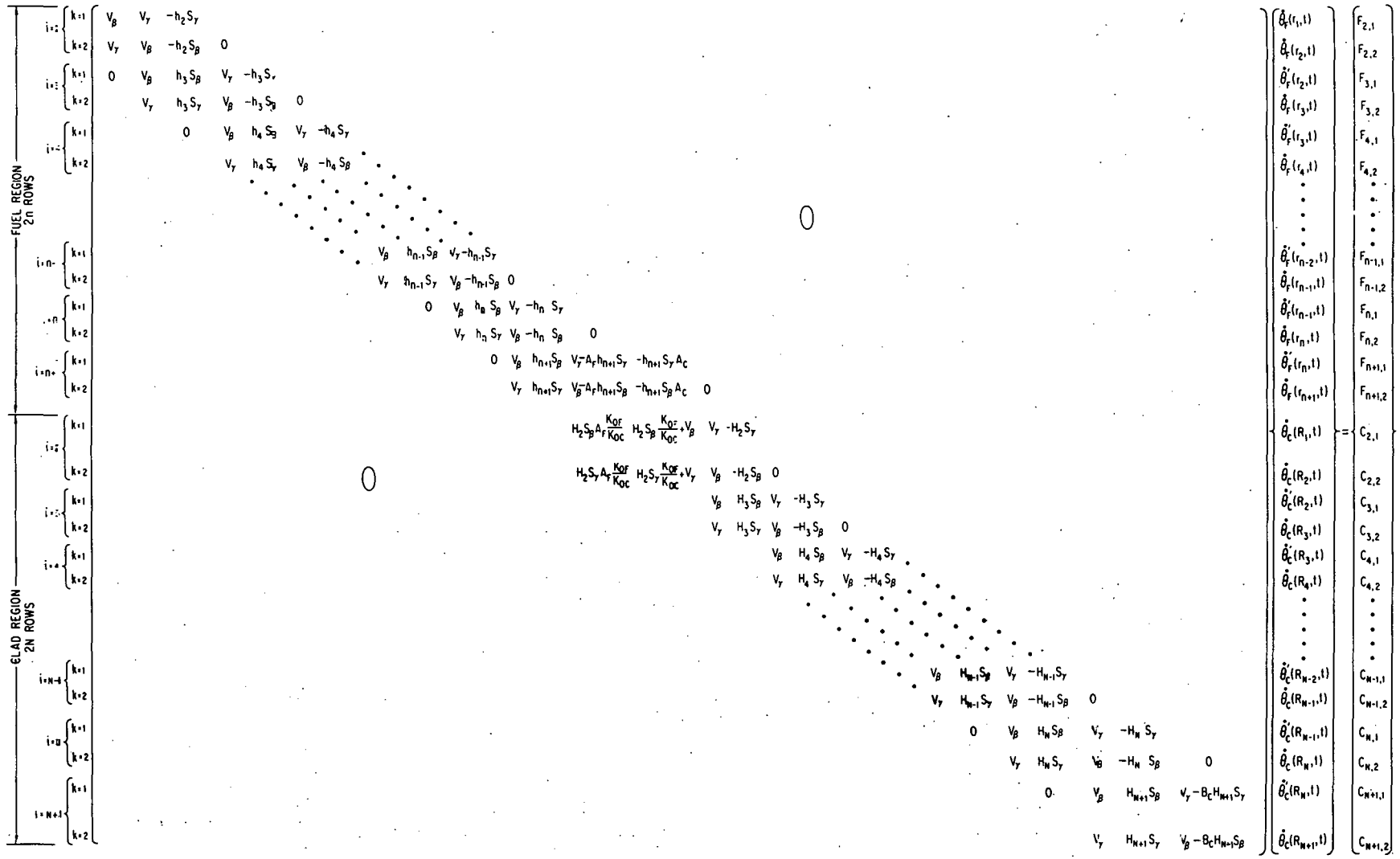

Fig. 4. Display of Matrix Equation (22).
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IV. NUMERICAL RESULTS

The method of collocation as described previously was used for the solution of the transient conduction problem for the cylindrical fuel-clad pin configuration shown in Fig. 1. The transport properties of the materials and the dimensions of the pin are given in Appendix A. The transient was initiated by permitting the volumetric heat source in the fuel to increase exponentially with time as specified in Appendix A. The initial temperature distribution was obtained by a steady-state solution of Eq. 4 with boundary conditions (5) and an initial uniform heat source $\dot{q}(t = 0)$. The effect of step size or the number of knots placed in fuel and clad regions on the accuracy of the solution was investigated. A set of five calculations were made. In the first calculation, two knots were used in the fuel and two in the clad. Since two knots correspond to one subinterval, the temperature distribution in this calculation was approximated with one cubic polynomial in the fuel and another in the clad. The second calculation used three knots in the fuel, while the fourth calculation used four knots in the fuel, with both calculations using two knots in the clad. The fourth calculation used five knots in the fuel and three knots in the clad. The last calculation was made with 20 knots in the fuel and five knots in the clad. The positions of knots chosen both in fuel and cladding for the above five examples are given in Appendix A. In the choice of these positions, no attempt was made to adjust these positions in order to improve the accuracy. It should be noted that it is not the selection of the knot positions which provides high-order accuracy for this collocation procedure, but rather the choice of Gaussian collocation points relative to each subinterval determined by these knots.

For purposes of comparison, the solution to this problem was also approximated by a finite-difference type procedure as used in the THTB program,[4] which is a general-purpose transient-heat-conduction program. In this method, the spatial interval is subdivided into subintervals (called nodes). For one-dimensional problems, the spatial approximation to the conduction equation is essentially the same as the usual three-point approximation to the second derivative and is derived by means of an integral heat balance over each node. The time derivative is approximated by an implicit single-time-step difference procedure. The implicit time

Table I. Comparison of a Finite-difference Method with the Collocation Method at Time = 4.0 sec.

	TEMPERATURE °F																	
	FINITE DIFFERENCE METHOD (THTB CODE)												COLLOCATION METHOD					
Position	80 Nodes in Fuel 15 Nodes in Clad No. of Eqs. = 95			35 Nodes in Fuel 6 Nodes in Clad No. of Eqs. = 41			6 Nodes in Fuel 2 Nodes in Clad No. of Eqs. = 8			4 Nodes in Fuel 2 Nodes in Clad No. of Eqs. = 6			2 Knots in Fuel 2 Knots in Clad No. of Eqs. = 4	3 Knots in Fuel 2 Knots in Clad No. of Eqs. = 6	4 Knots in Fuel 2 Knots in Clad No. of Eqs. = 8	5 Knots in Fuel 3 Knots in Clad No. of Eqs. = 12	20 Knots in Fuel 5 Knots in Clad No. of Eqs. = 46
(Inches)	Time Step (sec) .001 .005 .01 Tolerance (°F) 0.1×10^{-2} CPU Time (sec)			Time Step (sec) .001 .005 .01 Tolerance (°F) 0.1×10^{-2} CPU Time (sec)			Time Step (sec) .001 .005 .01 Tolerance (°F) 0.1×10^{-2} CPU Time (sec)			Time Step (sec) .001 .005 .01 Tolerance (°F) 0.1×10^{-2} CPU Time (sec)			Time Step Variable Relative Error 0.1×10^{-2} CPU Time (sec)	Time Step Variable Relative Error 0.1×10^{-2} CPU Time (sec)	Time Step Variable Relative Error 0.1×10^{-2} CPU Time (sec)	Time Step Variable Relative Error 0.1×10^{-2} CPU Time (sec)	Time Step Variable Relative Error 0.1×10^{-2} CPU Time (sec)
	1318	310	158	501	130	67	95	25	16	65	18	12	2	2	2	3	95
0.03410	3490.1 —	3491.8 +.0487%	3491.4 +.0372%	3491.2 +.0315%	3491.3 +.0344%	3490.9 +.0229%	3488.3 -.0516%	3488.0 -.0602%	3487.6 -.0716%	3517.1 +.7736%	3516.8 +.7650%	3516.5 +.7564%	3497.9 +.2235%	3493.8 +.1360%	3494.3 +.1203%	3492.9 +.0802%	3492.1 +.0573%
0.05841	3039.9 —	3041.2 +.0428%	3040 8 +.0296%	3041.9 +.0658%	3041.8 +.0625%	3041.5 +.0526%	3086.4 +1.5297%	3086.1 +1.8487%	3085.7 +1.5066%	3084.9 +1.4803%	3084.6 +1.4704%	3084.2 +1.4513%	3064.8 +.8191%	3043.2 +.1086%	3042.0 +.0691%	3042.1 +.0724%	3041.6 +.0559%
0.08042	2331.0 —	2331.7 +.0300%	2331.4 +.0172%	2331.8 +.0343%	2331.7 +.0300%	2331.4 +.0172%	2381.2 +2.1536%	2381.0 +2.1450%	2380.7 +2.1321%	2380.5 +2.1236%	2380.3 +2.1150%	2380.0 +2.1021%	2343.3 +.5277%	2331.0 —	2331.8 +.0343%	2332.2 +.0515%	2331.9 +.0386%
0.09509	1772.2 —	1772.6 +.0226%	1772 4 +.0113%	1773.3 +.0621%	1773.3 +.0621%	1773.1 +.0508%	1732.8 +.5981%	1782.7 +.5925%	1782.6 +.5868%	1782.4 +.5756%	1782.3 +.5643%	1782.2 +.5643%	1774.6 +.1354%	1772.2 —	1772.7 +.0282%	1772.8 +.0339%	1772.6 +.0226%
0.10380	740.8 —	740.8 —	740 8 —	740.8 —	740.8 —	740.8 —	740.1 -.0944%	740.1 -.0944%	740.1 -.0944%	740.1 -.0944%	740.1 -.0944%	740.1 -.0944%	740.9 +.0135%	740.8 —	740.8 —	740.8 —	740.8 —
0.11129	698.5 —	698.5 —	698 5 —	698.5 —	698.5 —	698.5 —	698.0 -.0716%	698.0 -.0716%	698.0 -.0716%	698.0 -.0716%	698.0 -.0716%	697.9 -.0859%	698.6 +.0143%	698.5 —	598.5 —	698.5 —	698.5 —

Table II. Comparison of a Finite-difference Method with the Collccation Method at Time = 8.0 sec.

	TEMPERATURE °F																	
	FINITE DIFFERENCE METHOD (THTB CODE)												COLLOCATION METHOD					
Position	80 Nodes in Fuel 15 Nodes in Clad No. of Eqs. = 95			35 Nodes in Fuel 6 Nodes in Clad No. of Eqs. = 41			6 Nodes in Fuel 2 Nodes in Clad No. of Eqs. = 8			4 Nodes in Fuel 2 Nodes in Clad No. of Eqs. = 6			2 Knots in Fuel 2 Knots in Clad No. of Eqs. = 4	3 Knots in Fuel 2 Knots in Clad No. of Eqs. = 6	4 Knots in Fuel 2 Knots in Clad No. of Eqs. = 8	5 Knots in Fuel 3 Knots in Clad No. of Eqs. = 12	20 Knots in Fuel 5 Knots in Clad No. of Eqs. = 46
(Inches)	Time Step (sec) .001 .005 .01 Tolerance (°F) 0.1×10^{-2} CPU Time (sec)			Time Step (sec) .001 .005 .01 Tolerance (°F) 0.1×10^{-2} CPU Time (sec)			Time Step (sec) .001 .005 .01 Tolerance (°F) 0.1×10^{-2} CPU Time (sec)			Time Step (sec) .001 .005 .01 Tolerance (°F) 0.1×10^{-2} CPU Time (sec)			Time Step Variable Relative Error 0.1×10^{-2} CPU Time (sec)	Time Step Variable Relative Error 0.1×10^{-2} CPU Time (sec)	Time Step Variable Relative Error 0.1×10^{-2} CPU Time (sec)	Time Step Variable Relative Error 0.1×10^{-2} CPU Time (sec)	Time Step Variable Relative Error 0.1×10^{-2} CPU Time (sec)
	1318	310	158	501	130	67	95	25	16	55	18	12	2	2	2	3	95
0.03410	4772.8 —	4772.6 -.0042%	4772.0 -.0168%	4772.2 -.0126%	4771.7 -.0230%	4771.1 -.0356%	4762.0 -.2263%	4761.5 -.2368%	476C.9 -.2493%	4793.9 +.4421%	4793.5 +.4337%	4792.8 +.4190%	4767.3 -.1152%	4777.7 +.1027%	4775.1 +.0482%	4773.3 +.0105%	4772.7 -.0021%
0.05841	4178.8 —	4178.5 -.0072%	4177.9 -.0215%	4179.8 +.0239%	4179.4 +.0144%	4178.7 -.0024%	4233.3 +1.3042%	4232.8 +1.2922%	4232.2 +1.2779	4227.8 +1.1726%	4227.3 +1.1606%	4226.7 +1.1463%	4195.3 +.3949%	4183.2 +.1053%	4178.6 -.0048%	4178.7 -.0024%	4178.5 -.0072%
0.08042	3194.9 —	3194.7 -.0063%	3194.1 -.0250%	3155.0 +.0031%	3194.7 -.0063%	3194.1 -.0250%	3241.3 +1.4523%	3240.9 +1.4398%	324C.4 +1.4243%	3238.4 +1.3615%	3238.0 +1.3490%	3237.5 +1.3334%	3202.7 +.2441%	3195.8 +.0281%	3195.2 +.0094%	3194.9 —	3194.8 -.0031%
0.09509	2292.1 —	2291.9 -.0087%	2291.6 -.0218%	2253.6 +.0854%	2293.3 +.0524%	2293.0 +.0393%	2318.7 +1.1605%	2318.4 +1.1474%	2318.1 +1.1343%	2317.1 +1.0907%	2316.8 +1.0776%	2316.5 +1.0645%	2292.3 +.0087%	2292.5 +.0174%	2292.4 +.0131%	2292.1 —	2292.0 -.0044%
0.10380	778.6 —	778.6 —	778.5 -.0128%	778.6 —	778.5 -.0128%	778.5 -.0128%	777.8 -.1027%	777.8 -.1027%	777.7 -.1156%	777.7 -.1156%	777.7 -.1156%	777.6 -.1284%	778.6 —	778.6 —	778.6 —	778.6 —	778.6 —
0.11129	719.0 —	719.0 —	719.0 —	719.0 —	718.9 -.0139%	718.9 -.0139%	718.3 -.0974%	718.3 -.0974%	718.2 -.1113%	718.2 -.1113%	718.2 -.1113%	718.2 -.1113%	719.0 —	719.0 —	719.0 —	719.0 —	719.0 —

differencing leads to a linear system which must be solved at every time step. This system is solved iteratively by means of the point Gauss-Seidel iterative method. With this program, four calculations were made. The first calculation used $N_F = 4$ and $N_C = 2$, where N_F denotes the number of nodes in fuel and N_C denotes the number of nodes in clad; the second calculation used $N_F = 6$ and $N_C = 2$; the third calculation was made with $N_F = 35$ and $N_C = 6$; the fourth calculation was made with $N_F = 80$ and $N_C = 15$. The choice of nodal boundaries as used in the above four calculations is also given in Appendix A. In addition to the effect of spatial mesh size, the effect of the size of the time steps was also investigated. For this purpose three time steps, namely, 0.01, 0.005, and 0.001 sec, were used with this finite difference program.

In Tables I and II we have compared temperature distributions as calculated by the two methods. In Table I, the time is 4 sec and in Table II, 8 sec after the initiation of the power transient. The six positions at which the temperature is shown in these tables are the node positions corresponding to the case of $N_F = 4$ and $N_C = 2$ for the THTB calculation. In the case of the other three calculations, the subintervals were selected such that the original nodal positions corresponding to the case of $N_F = 4$ and $N_C = 2$ were included among these nodes. Thus, the temperatures for the THTB calculations shown in Tables I and II did not have to be interpolated. For the method of collocation, the temperature at these six positions is calculated by means of Eq. 7 which provides an interpolation procedure entirely consistent with the approximation procedure. We have taken the THTB calculation with $N_F = 80$ and $N_C = 15$ corresponding to time step 0.001 sec as the benchmark calculation for these tables. The relative per cent deviation from this benchmark is shown under the corresponding temperatures in the tables. The number of equations solved for each case both with the collocation method and the finite-difference method (note that in case of THTB the number of equations is equal to the total number of nodes) are also shown in these tables. From these tables, we note that for the same number of equations solved, the method of collocation is substantially more accurate than the finite-difference method. In fact, we see that the accuracy obtained with only 12 equations in the collocation method is comparable with the accuracy obtained with 41 equations in the finite-difference method corresponding to

time step of 0.01 sec. These tables also show the corresponding CPU times on IBM 195/360 computer for the various time steps displayed in the tables. It may be seen that CPU time for the collocation method with 12 equations is less by one order of magnitude than for the case of the THTB calculation employing 41 equations yielding the same order of accuracy as the collocation method.

We may also note from these tables that the collocation method with four equations provides a relative accuracy in the temperature at these times of less than 1% which is more than adequate for many engineering calculations.

Figure 5 shows the temperature distribution at time = 8 sec,

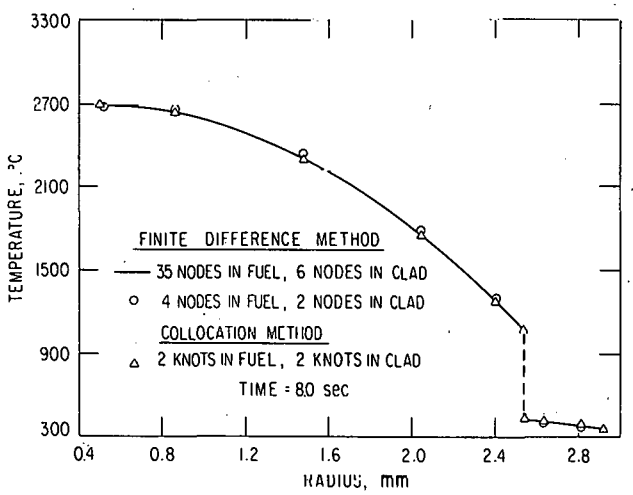

Fig. 5. Illustrative Comparison of a Finite-difference Solution with the Collocation Method at Time = 8.0 sec. ANL Neg. No. 900-4882.

illustrating the comparative accuracies for the two methods. In addition, the figure shows the jump in the temperature at the fuel-clad interface.

V. CONCLUSIONS

The application of this collocation method for this type of nonlinear parabolic equation shows that this method can provide a very accurate numerical solution with a very small number of equations. The method is far more accurate and faster than the usual finite-difference methods.

APPENDIX A

Transport Properties of Fuel and Cladding

Properties of fuel

$$K_F = \begin{cases} 0.0160528/(14.17943 + 0.01183T_F) \text{ Btu/(ft sec °F)} \\ \qquad \text{for } 932 \leq T_F(°F) < 2552 \\ 0.0003612 \text{ Btu/(ft sec °F)} \quad \text{for } T_F \geq 2552°F \end{cases}$$

$$\rho_F = 651.2 \text{ lb}_m/\text{ft}^3$$

$$C_{PF} = [12.54 + 0.0170T_F(°K) - 0.117 \times 10^{-4} T_F^2(°K) + 0.307 \times 10^{-8} T_F^3(°K)]/269.7664 \text{ Btu/(lb}_m \text{ °F)}$$

Properties of cladding

$$K_C = 2.22 \times 10^{-3} + 1.25 \times 10^{-6} T_C(°F) \text{ Btu/(ft sec °F)}$$

$$\rho_C = 485.26 \text{ lb}_m/\text{ft}^3$$

$$C_{PC} = 0.1105 + 2.632 \times 10^{-5} T_C(°F) \text{ Btu/lb}_m \text{ °F}$$

Heat-transfer coefficients

$$h_g = 0.2778 \text{ Btu/(ft}^2 \text{ sec °F)}$$

$$h = 6.944 \text{ Btu/(ft}^2 \text{ sec °F)}$$

Power transient

$$\dot{q}(t) = \dot{q}_o \exp(0.1\ t)\ ,$$

where q_o is the initial steady-state power set at 45266.32 Btu/(ft^3 sec) and t is the time in seconds.

Dimensions of fuel pin

$$R_o = r_1 = 0.02 \text{ in.}$$

$$R_F = r_{n+1} = 0.1 \text{ in.}$$

$$R_C = R_{N+1} = 0.115 \text{ in.}$$

Position of knots (in inches)

Let $(r_1 \rightarrow r_{n+1})$ denote the set of knots in the fuel region and $(R_1 \rightarrow R_{N+1})$ denote the set of knots in the clad region.

For n = 1, N = 1,

$$(r_1 \rightarrow r_2) = 0.02,\ 0.1;\ (R_1 \rightarrow R_2) = 0.1,\ 0.115;$$

for n = 2, N = 1,

$$(r_1 \rightarrow r_3) = 0.02,\ 0.07,\ 0.1;\ (R_1 \rightarrow R_2) = 0.1,\ 0.115;$$

for n = 3, N = 1,

$$(r_1 \rightarrow r_4) = 0.02,\ 0.05,\ 0.08,\ 0.1;$$

$$(R_1 \rightarrow R_2) = 0.1,\ 0.115;$$

for n = 4, N = 2,

$(r_1 \rightarrow r_5)$ = 0.02, 0.04, 0.065, 0.085, 0.1;

$(R_1 \rightarrow R_3)$ = 0.1, 0.1075, 0.115;

for n = 19, N = 4,

$(r_1 \rightarrow r_{20})$ = 0.02, 0.025, 0.031, 0.035, 0.039, 0.043, 0.047, 0.051,
0.055, 0.059, 0.063, 0.067, 0.071, 0.075, 0.079,
0.083, 0.089, 0.093, 0.097, 0.1;

$(R_1 \rightarrow R_5)$ = 0.1, 0.10625, 0.11125, 0.11375, 0.115.

Subinterval boundaries for the THTB conduction code (in inches)

Let N_F denote the number of nodes in fuel and N_C denote the number of nodes in clad, and let r_{Fi} $(0 \leq i \leq N_F)$ and R_{Ci} $(0 \leq i \leq N_C)$ denote the position of node boundaries in fuel and clad, respectively.

For N_F = 4, N_C = 2,

$(r_{FO} \rightarrow r_{F4})$ = 0.02, 0.045, 0.07, 0.09, 0.1;

$(R_{CO} \rightarrow R_{C2})$ = 0.1, 0.1075, 0.115;

for N_F = 6, N_C = 2,

$(r_{FO} \rightarrow r_{F6})$ = 0.02, 0.03, 0.0377574, 0.045, 0.07, 0.090, 0.1;

$(R_{CO} \rightarrow R_{C2})$ = 0.1, 0.1075, 0.115;

for $N_F = 35$, $N_C = 6$,

$(r_{F0} \rightarrow r_{F35})$ = 0.02, 0.022, 0.025, 0.028, 0.031, 0.034, 0.0341997, 0.04, 0.043, 0.045, 0.048, 0.051, 0.054, 0.057, 0.0597868, 0.062, 0.064, 0.066, 0.068, 0.07, 0.072, 0.074, 0.076, 0.078, 0.08, 0.0808378, 0.084, 0.086, 0.088, 0.09, 0.092, 0.094, 0.0961617, 0.098, 0.099, 0.1;

$(R_{C0} \rightarrow R_{C6})$ = 0.1, 0.1025, 0.105074, 0.1075, 0.11, 0.112569, 0.115;

for $N_F = 80$, $N_C = 15$,

$(r_{F0} \rightarrow r_{F14})$ = 0.02 → 0.034 with $\Delta r_F = 0.01$, $r_{F15} = 0.0341997$;

$(r_{F16} \rightarrow r_{F60})$ = 0.036 → 0.08 with $\Delta r_F = 0.01$, $r_{F61} = 0.0808378$;

$(r_{F62} \rightarrow r_{F75})$ = 0.082 → 0.095 with $\Delta r_F = 0.01$, $r_{F76} = 0.0951739$;

$(r_{F77} \rightarrow r_{F80})$ = 0.097 → 0.1 with $\Delta r_F = 0.01$;

$(R_{C0} \rightarrow R_{C3})$ = 0.1 → 0.103 with $\Delta R_C = 0.001$, $R_{C4} = 0.104584$;

$(R_{C5} \rightarrow R_{C11})$ = 0.105 → 0.111 with $\Delta R_C = 0.001$, $R_{C12} = 0.111583$;

$(R_{C13} \rightarrow R_{C15})$ = 0.113 → 0.115 with $\Delta R_C = 0.001$

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