LM-05K174
January 9, 2006

Multipole Analysis of Circular Cylindrical Magnetic Systems

J Selvaggi

NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States, nor the United States Department of Energy, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

MULTIPOLE ANALYSIS OF CIRCULAR CYLINDRICAL MAGNETIC SYSTEMS

By

Jerry P. Selvaggi

A Thesis Submitted to the Graduate
Faculty of Rensselaer Polytechnic Institute
in Partial Fulfillment of the
Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Electrical, Computer, and Systems Engineering

Approved by the
Examining Committee:

Professor S. J. Salon, Thesis Adviser

Professor J. K. Nelson, Member

Professor R. C. Degeneff, Member

Professor K. A. Connor, Member

Dr. M.V.K. Chari, Member

Dr. M. J. Debortoli, Member

Rensselaer Polytechnic Institute
Troy, New York

December 2005
(For Graduation May 2006)

MULTIPOLE ANALYSIS OF CIRCULAR CYLINDRICAL MAGNETIC SYSTEMS

By

Jerry P. Selvaggi

An Abstract of a Thesis Submitted to the Graduate

Faculty of Rensselaer Polytechnic Institute

in Partial Fulfillment of the

Requirements for the Degree of

DOCTOR OF PHILOSOPHY

Major Subject: Electrical, Computer, and Systems Engineering

The original of the complete thesis is on file
in the Rensselaer Polytechnic Institute Library

Examining Committee:

Professor S. J. Salon, Thesis Adviser

Professor J. K. Nelson, Member

Professor R. C. Degeneff, Member

Professor K. A. Connor, Member

Dr. M.V.K. Chari, Member

Dr. M. J. Debortoli, Member

Rensselaer Polytechnic Institute
Troy, New York

December 2005
(For Graduation May 2006)

© Copyright 2003
by
Jerry P. Selvaggi
All Rights Reserved

CONTENTS

LIST OF TABLES	viii
LIST OF FIGURES	ix
ACKNOWLEDGMENT	xiv
ABSTRACT	xv
1. INTRODUCTION	1
1.1 General introduction	1
1.1.1 Statement of Problem	1
1.1.2 Principal advantages	1
1.2 Multipole theory	2
1.2.1 Spherically symmetric systems	4
1.2.2 Circular cylindrical systems	6
1.3 Summary	8
2. CYLINDRICAL GREEN'S FUNCTION EXPANSION	10
2.1 Introduction	10
2.2 Free-space cylindrical Green's function	11
2.2.1 Toroidal functions or Q-functions	13
2.3 Gradient of the free-space cylindrical Green's function and higher order derivatives	15
2.4 Summary	16
3. COULOMB'S LAW FOR MAGNETIC CHARGE	18
3.1 Introduction	18
3.2 Coulomb's law for magnetic charge	20
3.2.1 Toroidal expansion of the magnetic form of Coulomb's law	20
3.2.2 Toroidal expansion of the magnetic field intensity	24
3.3 Summary	26
4. INTEGRAL FORMULATION FOR MAGNETOSTATIC PROBLEMS IN CYLINDRICAL COORDINATES	28
4.1 Introduction	28

4.2	Maxwell's magnetostatic equations	29
4.3	Toroidal expansion of the magnetic scalar potential for a finite cylindrical magnet given a magnetization forcing function	31
4.4	Toroidal expansion of the magnetic vector potential for a finite cylindrical conductor given a current density forcing function	34
4.5	Toroidal expansion of the magnetic vector potential for cylindrical coils with a rectangular cross-section given a current density forcing function	36
4.5.1	Helmholtz coils	38
4.6	Various geometries for which Q-functions are applicable	40
4.6.1	A frustum of a cone	40
4.6.2	A spheroid and a sphere	41
4.7	Summary	42
5.	VALIDATION OF THE TOROIDAL EXPANSION—SIMPLE EXAMPLES	43
5.1	Introduction	43
5.2	Two electric point charges	43
5.3	Circular current loop with applied uniform current source	47
5.4	Mutual inductance between two non-coplanar and parallel current loops	53
5.5	The “electrified disk” problem	56
5.6	The simple but nonlinear pendulum	62
5.7	Summary	67
6.	SPHERICAL MULTIPOLES FROM A TOROIDAL EXPANSION	68
6.1	Introduction	68
6.2	Derivation of a spherical multipole expansion from a given toroidal expansion	68
6.3	Application of the multipole reference table	73
6.3.1	Two electric point charges	73
6.3.2	Circular conducting ring	74
6.3.3	Hollow circular current disk	77
6.4	Spherical harmonics versus toroidal harmonics	84
6.5	Summary	85

7. CHARGE SIMULATION METHOD	87
7.1 Introduction	87
7.2 Charge simulation and the magnetic scalar potential	89
7.3 Charge simulation and the normal component of the magnetic flux density	90
7.4 Summary	91
8. APPLICATION OF THE TOROIDAL EXPANSION FOR COMPUTING THE MAGNETIC FIELD FROM A PERMANENT MAGNET MOTOR .	92
8.1 Introduction	92
8.2 Permanent magnets used in permanent-magnet motors	94
8.2.1 A single permanent magnet	94
8.2.1.1 The magnetic scalar potential method	94
8.2.1.2 The magnetic flux density method	98
8.2.2 Four permanent magnets	100
8.2.2.1 Magnetic scalar potential method	100
8.2.2.2 Magnetic flux density method	107
8.3 Real six-pole motor	107
8.3.1 Balanced motor	109
8.3.2 Unbalanced motor	110
8.4 Summary	111
9. DISCUSSION AND CONCLUSION	117
9.1 Introduction	117
9.2 Solution methods for circular cylindrical electromagnetic systems . . .	117
9.3 Toroidal expansion for cylindrically symmetric problems in magneto- statics	118
9.3.1 Non-cylindrical sources	119
9.4 Future research and general conclusions	119
LITERATURE CITED	122
APPENDICES	
A. VECTOR IDENTITIES AND THEOREMS	134
A.1 Introduction	134
A.2 Vector identities	134

A.3	Vector Theorems	135
A.3.1	Fundamental theorems of vector fields	135
A.3.1.1	Gauss's theorem	135
A.3.1.2	Stokes's theorem	136
A.3.1.3	Helmholtz's theorem	136
A.3.1.4	Poisson's theorem	136
B.	MULTIPOLE EXPANSIONS REPRESENTED BY CARTESIAN TENSORS	138
B.1	Introduction	138
B.2	Multipole analysis in Cartesian tensor notation	138
C.	INVERSE DISTANCE IN CYLINDRICAL COORDINATES	147
C.1	Introduction	147
C.2	Inverse distance function	147
D.	ALTERNATE FORMULATION OF THE FREE-SPACE GREEN'S FUNCTION IN CYLINDRICAL COORDINATES	149
D.1	Introduction	149
D.2	Alternate form for the cylindrical Green's function	149
E.	VALIDATION OF THE CYLINDRICAL GREEN'S FUNCTION	151
E.1	Introduction	151
E.2	Validation of the cylindrical Green's function	151
F.	SOME TABULATED VALUES FOR GAMMA FUNCTIONS, AND THEIR PRODUCTS	154
F.1	Introduction	154
F.2	Gamma functions	154
G.	RECURRENCE RELATIONSHIPS FOR THE Q-FUNCTIONS	156
G.1	Introduction	156
G.2	Recurrence relationship	156
H.	ELLIPTIC INTEGRALS IN TERMS OF Q-FUNCTIONS	157
H.1	Introduction	157
H.2	Elliptic integrals in terms of Q-functions	157

I. DERIVATIVE PROPERTIES OF Q-FUNCTIONS	159
I.1 Introduction	159
I.2 Derivative properties of the Q-function	159
J. INTEGRALS WHICH OCCUR FOR CYLINDRICALLY SYMMETRIC MAGNETIC SYSTEMS	161
J.1 Introduction	161
J.2 Integral relationships	161
J.2.1 Thin disks with constant surface charge density	161
J.2.2 Thin disks with azimuthal current distribution	163
K. RELATIONSHIP BETWEEN THE CYLINDRICAL GREEN'S FUNC- TION AND THE SPHERICAL GREEN'S FUNCTION	164
K.1 Introduction	164
K.2 Cylindrical and spherical Green's functions	164
L. A NUMERICAL COMPARISON BETWEEN A SPHERICAL HARMONIC EXPANSION AND A TOROIDAL EXPANSION	167
L.1 Introduction	167
L.2 Numerical study	167
M. ROTATED CYLINDRICAL COORDINATE SYSTEM	177
M.1 Introduction	177
M.2 Rotation and translation of an electromagnetic source	177
N. BRONZAN'S METHOD	196
N.1 Introduction	196
N.2 General derivation of Bronzan's method	196
N.3 Bronzan's method in the presence of magnetized media	202
N.4 Scalar potential of filamentary circuits	204

LIST OF TABLES

1.1	Names for a few multipoles	2
3.1	Components of the magnetic field intensity	27
5.1	Numerical values for the period of a nonlinear pendulum	65
6.1	Multipole reference table	72
8.1	Motor specifications	109
F.1	Gamma functions	155
G.1	Recurrence relations	156
I.1	The "sign" of the derivatives of the Q-functions	160
L.1	Numerical comparison for EXAMPLE 1	173
L.2	Numerical comparison for EXAMPLE 2	175

LIST OF FIGURES

1.1	Circular loop with linear mass density	3
1.2	Circular loop with linear charge density	5
1.3	Circular loop with a surface charge density	7
1.4	Circular loop with a constant current density	8
2.1	Inverse distance in cylindrical coordinates	11
3.1	Cylindrical permanent magnet with a given magnetization	19
3.2	Four magnetic charges located at arbitrary points in space	21
3.3	Plot of a few Q-functions	22
4.1	Cylindrical permanent magnet with a given magnetization	32
4.2	Circular cylindrical conductor carrying a current density	34
4.3	Incomplete cylindrical conductor	36
4.4	Cylindrical coils	37
4.5	Helmholtz coils	39
4.6	A hollow frustum	41
4.7	Oblate spheroid	42
5.1	Two electric point charges	44
5.2	Electric scalar potential from two point charges	47
5.3	The $m=1$ contribution to the electric scalar potential from two point charges	48
5.4	The $m=3$ contribution to the electric scalar potential from two point charges	48
5.5	The $m=5$ contribution to the electric scalar potential from two point charges	49
5.6	Uniform current Loop	50
5.7	Magnetic vector potential from a uniform current Loop	51

5.8	Two coaxial and noncoplanar uniform current loops	54
5.9	Mutual inductance of two current loops	55
5.10	Mutual inductance of two current loops for $a=1.5\text{m}$ and $b=3\text{m}$	56
5.11	Electrified Disk	57
5.12	The electric scalar potential due to an electrified disk of radius 0.5m at a potential of 100V	60
5.13	Total electric scalar potential computed on the observation cylinder due to an electrified disk	61
5.14	The $m=0$ contribution to the electric scalar potential computed on the observation cylinder due to an electrified disk	62
5.15	Nonlinear pendulum	63
5.16	Period of a nonlinear pendulum	64
6.1	Relationship between spherical and cylindrical coordinates	69
6.2	Infinitely thin current loop	70
6.3	Circular Conducting Ring	74
6.4	Thin hollow circular disk	78
6.5	Comparison between hollow disk and circular loop	82
7.1	Potential cylinder	87
7.2	Charge simulation on a hypothetical cylinder	88
8.1	Charge simulation model	93
8.2	A permanent magnet inside a hypothetical cylinder	95
8.3	Total magnetic scalar potential	95
8.4	The $m=1$ contribution to the total magnetic scalar potential	96
8.5	The $m=3$ contribution to the total magnetic scalar potential	96
8.6	The $m=5$ contribution to the total magnetic scalar potential	97
8.7	Total radial component of the magnetic field intensity	98
8.8	Total axial component of the magnetic field intensity	98
8.9	Total azimuthal component of the magnetic field intensity	99

8.10	The m=1 contribution of the radial component of magnetic field intensity	100
8.11	The m=1 contribution of the axial component of magnetic field intensity	101
8.12	The m=1 contribution of the azimuthal component of magnetic field intensity	101
8.13	The m=3, and m=5 contributions of the magnetic field intensity vector	102
8.14	The radial component of the magnetic field intensity computed from simulated data	103
8.15	The axial component of the magnetic field intensity computed from simulated data	103
8.16	The azimuthal component of the magnetic field intensity computed from simulated data	104
8.17	A 4-pole permanent magnet system inside a hypothetical cylinder . . .	104
8.18	Two-dimensional view of the 4-pole magnetic system	105
8.19	Total magnetic scalar potential	105
8.20	The m=2 contribution to the total magnetic scalar potential	106
8.21	The m=4 contribution to the total magnetic scalar potential	106
8.22	The m=6 contribution to the total magnetic scalar potential	107
8.23	Full meshed model of a 6-pole BLDC motor with endcaps and stator coils	108
8.24	6-pole permanent magnet pole arrangement	108
8.25	The total magnetic scalar potential	109
8.26	The m=1 contribution to the total magnetic scalar potential	110
8.27	The m=2 contribution to the total magnetic scalar potential	110
8.28	The m=3 contribution to the total magnetic scalar potential	111
8.29	The m=4 contribution to the total magnetic scalar potential	111
8.30	The m=5 contribution to the total magnetic scalar potential	112
8.31	The total magnetic scalar potential for 6.5 percent demagnetization of magnet 4	112
8.32	The m=1 contribution for a 6.5 percent demagnetization of magnet 4 .	113

8.33	The $m=2$ contribution for a 6.5 percent demagnetization of magnet 4	114
8.34	The $m=3$ contribution for a 6.5 percent demagnetization of magnet 4	114
8.35	The total magnetic scalar potential for 10 axial off-set	115
8.36	The $m=1$ contribution for a 10 percent axial off-set	115
8.37	The $m=2$ contribution for a 10 percent axial off-set	116
8.38	The $m=3$ contribution for a 10 percent axial off-set	116
9.1	Dead zone surrounding a real cylindrical magnetic source	119
B.1	An electromagnetic source	139
B.2	Origin dependence of a multipole expansion	145
C.1	A cylindrical source of length $2h$	147
H.1	Elliptic integrals and Q-functions	158
K.1	Angular relationships in spherical coordinates	165
L.1	Circular conducting ring	168
L.2	Cylinder inscribed in a sphere	171
L.3	Sphere inscribed in a cylinder	172
M.1	Coordinate system rotations	178
M.2	A rotated and translated current loop	181
M.3	Non-rotated current loop	182
M.4	Rotated current loop	184
M.5	Non-rotated charged line segment	190
M.6	Rotated charged line segment	191
M.7	Comparison between a rotated charged line segment and a non-rotated charged line segment	194
M.8	Comparison between a rotated charged line segment and a non-rotated charged line segment	195
N.1	A general magnetic source	197
N.2	A magnetic source enclosed in a hypothetical sphere	198

N.3	Prohibitive zones in Bronzan's method	200
N.4	Magnetic scalar potential of a current loop valid at all points on the z axis	208
N.5	Magnetic scalar potential of a current loop valid at all points in space .	210
N.6	Magnetic scalar potential of a rectangular current loop valid at all points on the z axis	214
N.7	Magnetic flux density of a rectangular current loop valid at all points on the z axis by using the Biot-Savart law	215
N.8	Magnetic scalar potential of a rectangular current loop valid at all points in space	217

ACKNOWLEDGMENT

I would like to thank my adviser, Professor S. J. Salon, for his assistance, guidance, and most importantly, his patience during my time spent working on this thesis. I would also like to thank the other members of the Doctoral Committee: Professor J. K. Nelson, Professor R. C. Degeneff, Professor K. A. Connor, Dr. M. V. K. Chari, and Dr. M. Debortoli for taking time from their busy schedules in order to grill me. In particular, I would like to thank Dr. Chari for his unwavering desire to question everything that I did which made me check and recheck my results in order to be absolutely thorough. His commitment and guidance is most appreciated.

I am also indebted to Dr. O-Mun Kwon for doing the necessary finite element computations used in chapter 8 of this thesis. I would also like to give thanks to Dr. M. Kupferschmid whose help in Fortran made my life easier. I am also grateful for the long and useful discussions with Dr. R. P. Radlinski of BBN Technologies. Also, I would like to thank Mrs. Rose Carignan for allowing me to annoy her during those times when I didn't feel like doing any work.

I am also grateful to father, Jerry A. Selvaggi, for teaching me how to be an engineer. A special thanks goes to my sister, Dr. Suzanne Selvaggi, whose financial support made my life as a student more bearable. Most importantly, I would like to thank my mom for giving me life, and I wish that she were here to see her son get his Ph.D.

ABSTRACT

This thesis deals with an alternate method for computing the external magnetic field from a circular cylindrical magnetic source. The primary objective is to characterize the magnetic source in terms of its equivalent multipole distribution. This multipole distribution must be valid at points close to the cylindrical source and a spherical multipole expansion is ill-equipped to handle this problem; therefore a new method must be introduced. This method, based upon the free-space Green's function in cylindrical coordinates, is developed as an alternative to the more familiar spherical harmonic expansion.

A family of special functions, called the toroidal functions or Q-functions, are found to exhibit the necessary properties for analyzing circular cylindrical geometries. In particular, the toroidal function of zeroth order, which comes from the integral formulation of the free-space Green's function in cylindrical coordinates, is employed to handle magnetic sources which exhibit circular cylindrical symmetry.

The toroidal functions, also called Q-functions, are the weighting coefficients in a "Fourier series-like" expansion which represents the free-space Green's function. It is also called a toroidal expansion. This expansion can be directly employed in electrostatic, magnetostatic, and electrodynamic problems which exhibit cylindrical symmetry. Also, it is shown that they can be used as an alternative to the Elliptic integral formulation. In fact, anywhere that an Elliptic integral appears, one can replace it with its corresponding Q-function representation.

A number of problems, using the toroidal expansion formulation, are analyzed and compared to existing known methods in order to validate the results. Also, the equivalent multipole distribution is found for most of the solved problems along with its corresponding physical interpretation. The main application is to characterize the external magnetic field due to a six-pole permanent magnet motor in terms of its equivalent multipole distribution.

CHAPTER 1

INTRODUCTION

1.1 General introduction

1.1.1 Statement of Problem

In order to compute the magnetic field from a finite cylindrical magnetic source, a method must be developed which enables one to handle intractable integrals which arise when using an integral formulation. Whether one develops a scalar potential, a vector potential, or whether one computes the field directly from the Biot-Savart formulation, the integral which results is usually analytically intractable in closed form.

If one also wishes to characterize the magnetic source in terms of its dipole moment, its quadrupole moment, etc., then it would be advantageous to solve the integral analytically. In order to compute the magnetic field close to the cylindrical source, a cylindrical coordinate system is the most useful since this exactly maps the source.

This thesis describes a method which allows one to compute the magnetic field from a cylindrical source in terms of its equivalent multipole distribution. The technique which is employed is valid for both static and dynamic problems. It is also valid for general forcing functions such as current densities, magnetization, and others. Although not a requirement, all problems in this thesis are formulated in terms of integrals.

1.1.2 Principal advantages

There are a number of principal advantages of this alternate method. One advantage is that it reduces an integral, whose integrand is a function of Bessel functions or modified Bessel functions, to a summation which can easily be evaluated. It also has a distinct advantage over the more familiar Elliptic integral solution because it doesn't rely on any mathematical transformation. In other words, it is not always immediately apparent that one is dealing with an Elliptic integral. A

n	<i>Naming Convention</i>
0	<i>monopole</i>
1	<i>dipole</i>
2	<i>quadrupole</i>
3	<i>octupole</i>
4	<i>Hexadecapole</i>
5	<i>Tricontadipole</i>
6	<i>Hexacontatetrapole</i>
$\vdots$	$\vdots$
n	$2^n - pole$

Table 1.1: Names for a few multipoles

number of transformations are needed to put the integral in standard form. This alternate formulation, using Q-functions, eliminates all the algebraic steps that are usually common if an Elliptic integral formulation is chosen. Also, this method can be greatly extended with the aid of charge simulation because it enables one to use the magnetic form of Coulomb's law. This is important because this alternate formulation is directly applicable to $|r - r'|^{2n-1}$ type systems. This is not the case with a Elliptic integral formulation. Another very important issue that has been addressed is that a spherical harmonic expansion of the magnetic field external to a circular cylindrical magnetic source does not include those field points which are close to the motor. There are regions of space, called dead zones, in which the spherical harmonic expansion is not valid. With the method proposed in this thesis, the problem of dead zones disappears.

1.2 Multipole theory

The term *multipole* is the name given to certain point charge systems or “poles” that comprise an electromagnetic system. It can also be used as a general description for any other system which allows for a potential function representation. Most notable of these is the Newtonian potential found in gravitational theory [1, 2, 3]. The number of poles in a multipole distribution is always 2^n ; n is called the order of the multipole and can range from 0, 1, 2, 3, to all higher positive integers. Table 1.1 lists the names for a few of the multipoles.

Figure 1.1: Circular loop with linear mass density

Multipole analysis is a mathematical method by which one can convert a complex source into its elementary parts. The source is usually written in terms of an infinite series of elementary sources. The method is usually employed when the solution to a physical problem is written in terms of an integral. This integral is usually not tractable, and a binomial expansion is performed on the integrand in order to evaluate the integral term-by-term. Various physical quantities in gravitation [1, 2, 3], electrochemical analysis [4], nuclear physics [5], electromagnetism [6], [7], and others can be formulated in terms of an integral. The resulting integrals are quite complicated. In other words, one must either choose an alternate formulation of the physical problem or find ways to approximate the integral. Multipole analysis deals with the latter.

Consider the following simple problem in gravitation. A thin circular ring with a linear mass density, $\lambda(\phi)$, is centered at the origin of the x - y plane as shown in Figure 1.1. The gravitational potential at the field point due to an arbitrary source is given by [1]

$$\Phi_P(\mathbf{r}) = G \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r' \quad (1.1)$$

where $\rho(\mathbf{r})$ is the mass density of the source and G is the gravitational constant. Applying Equation (1.1) to the ring yields:

$$\Phi_P(\rho, \phi, z) = Ga \int_0^{2\pi} \frac{\lambda(\phi')}{\sqrt{\rho^2 + a^2 + z^2 - 2a\rho \cos(\phi - \phi')}} d\phi' \quad (1.2)$$

Equation (1.2) is not integrable in terms of elementary functions. In fact, if $\lambda(\phi) = \lambda_o$ is a constant, then Equation (1.2) yields a solution in terms of an Elliptic integral of the first kind. This shows that even the simplest problems may require a specialized technique in order to solve the integral.

Converting Equation (1.2) to spherical coordinates and then expanding its denominator in terms of a binomial expansion, valid for $\mathbf{r} > \mathbf{a}$ or for $\mathbf{r} < \mathbf{a}$, allows one to integrate Equation (1.2) term-by-term. More will be said later about the mathematical details of this process. The two infinite series for $\Phi_P(\rho, \phi, z)$, valid for $\mathbf{r} > \mathbf{a}$ or for $\mathbf{r} < \mathbf{a}$, are called the multipole expansions of the potential. Each term in the expansion has a physical meaning. If $\lambda(\phi) = \lambda_o = \frac{M}{2\pi a}$, where M is the total mass of the ring and $\mathbf{r} > \mathbf{a}$, then the first term in the expansion represents the potential of the ring as if all of the mass were concentrated at its center. The second term is proportional to the center of gravity of the ring. However, the center of gravity is at the origin and therefore this term is zero. The third term is proportional to the moment of inertia of the ring. This term is nonzero. This process can be continued for all higher moments. The series of integrals obtained from this expansion are called the inertia integrals. Expanding an intractable integral in terms of a binomial distribution, and then characterizing the source in terms of its elementary or primitive sources is called multipole analysis. See Appendix B for a more detailed discussion on multipole analysis.

1.2.1 Spherically symmetric systems

If a problem exhibits spherical symmetry, and if one can formulate the solution to the problem in terms of an integral, tractable or not, then one can perform a spherical multipole analysis [7, 8, 9]. The power of a spherical multipole expansion lies in the fact that one can separate, from the integral, the contribution due to the

Figure 1.2: Circular loop with linear charge density

source variables and the contribution due to the field variables. This is only possible in a spherical and a Cartesian coordinate system. This is one of the reasons why a multipole analysis has meaning only in one or both of these coordinate systems.

Other orthogonal coordinate systems do not, in general, allow for the separation of the source variables and the field variables. For example, if Equation (1.2) is written in terms of spherical coordinates then one can directly apply a spherical multipole analysis. However, if no conversion is made then one will find that one can not, in general, separate the field variables from the source variables in the integrand, and an alternate method must be employed in order glean some meaning from the binomial expansion that is used to evaluate the integral term-by-term.

Consider the following simple problem in electrostatics. A thin circular ring with a linear charge density, $\lambda(\phi)$, is centered at the origin of the x - y plane as shown in Figure 1.2. The electrostatic potential at the field point due to an arbitrary source is given by [10, 11]

$$\Phi_P(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r' \quad (1.3)$$

where $\rho(\mathbf{r})$ is the charge density of the source. Applying Equation (1.3) to the

conducting ring yields:

$$\Phi_P(r, \theta, \phi) = \frac{a}{4\pi\epsilon_0} \int_0^{2\pi} \frac{\lambda(\phi')}{\sqrt{r^2 + a^2 - 2ar \cos(\gamma)}} d\phi' \quad (1.4)$$

where $\cos(\gamma) = \sin(\theta) \cos(\phi - \phi')$. Equation (1.4) can be written in terms of a multipole expansion given by

$$\Phi_P(r, \theta, \phi) = \frac{1}{4\pi\epsilon_0} \frac{a}{r} \int_0^{2\pi} \lambda(\phi') \sum_{l=0}^{\infty} \left(\frac{a}{r}\right)^l P[\cos(\gamma)] d\phi' \quad (1.5)$$

valid for $r > a$. The $P[\cos(\gamma)]$ is a function of both the field variables, (θ, ϕ) and source variable, ϕ' . However, $P[\cos(\gamma)]$ can be expanded using a well known summation formula as follows [10]:

$$P[\cos(\gamma)] = 4\pi \sum_{m=-l}^l \frac{1}{2l+1} Y_{lm}^*\left(\frac{\pi}{2}, \phi'\right) Y_{lm}(\theta, \phi) \quad (1.6)$$

where $Y_{lm}(\theta, \phi)$ are called spherical harmonics and $Y_{lm}^*(\theta, \phi)$ are their complex conjugates. Using Equation (1.6), Equation (1.5) can be written as

$$\Phi_P(r, \theta, \phi) = \frac{1}{\epsilon_0} \frac{a}{r} \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{Y_{lm}(\theta, \phi)}{2l+1} \left(\frac{a}{r}\right)^l \int_0^{2\pi} \lambda(\phi') Y_{lm}^*\left(\frac{\pi}{2}, \phi'\right) d\phi' \quad (1.7)$$

Notice that in Equation (1.7) there is a complete separation between the source variables and the field variables. This is, of course, the desired characteristic of the spherical multipole expansion. The integrand of Equation (1.7) is only a function of the source variables.

1.2.2 Circular cylindrical systems

The integral formulation for a circular cylindrical magnetic source expressed in a non-spherical coordinate system can not be directly expanded in a multipole distribution. However, one can expand the integrand in terms of an appropriately converging binomial expansion and then integrate term-by-term. This will not directly yield a multipole distribution. A method must be found which allows one to

Figure 1.3: Circular loop with a surface charge density

recover the spherical multipole distribution.

Also, it is desirable to compute the magnetic field close to the finite cylindrical magnetic source for which a spherical harmonic expansion may not be applicable. The example of an infinitely thin charged ring, as previously discussed, can be solved by a spherical harmonic expansion only because it is infinitely thin. What if one wants to solve the same problem except that the ring has a thickness in the radial direction, and is infinitely thin in the z direction. Also, assume that a constant charge density of σ_0 exists on the ring as shown in Figure 1.3. If one attempts a spherical harmonic expansion of the potential function, one will find that a spherical multipole analysis would generate two mathematical expressions; one expression valid for $r < a$ and another expression valid for $r > b$. What about the region $a < r < b$? This region can not readily be incorporated into a spherical harmonic expansion. What if, instead of constant charge density, σ_0 , a constant current density of $\mathbf{K} = \frac{I}{b-a} \hat{\phi}$ flows in the ring as shown in Figure 1.4? If one wants to know what the magnetic field is at points in the region $a < r < b$ using a magnetic scalar potential formulation then Bronzan's method [12] (see *Appendix N*) or Gray's method [13, 14] needs to be employed and even then only certain regions of space will represent a valid spherical harmonic expansion. A region in space which can

Figure 1.4: Circular loop with a constant current density

not be represented by a spherical harmonic expansion will be called the *dead zone*. In fact, even if a vector potential formulation was chosen, one could not easily find a spherical harmonic expansion valid for $a < r < b$.

In this research, a method is developed which is based on the cylindrical free-space Green's function. This enables one to compute the magnetic field from a circular cylindrical magnetic source. Also, this method can be used to eliminate all *dead zones* which arise when a spherical harmonic formulation is employed. The problem illustrated in Figure 1.3 can be solved by the method developed in this thesis. Also, the problem illustrated in Figure 1.4 can be solved by this same method starting with the vector potential formulation.

1.3 Summary

This thesis makes a new contribution to the solution of circular cylindrical electromagnetic systems. Its primary focus is on developing a multipole description of this system without resorting to a spherical multipole expansion. This method is quite general and can be applied to electrostatic, magnetostatic and electrodynamic problems.

Maxwell's equations, in their most general form, can be formulated in terms of

partial differential equations or in terms of integral equations. This thesis will deal with the latter. This is, in general, not a limitation because one can always recast a partial differential equation in terms of an integral equation. Of course, one needs to take due note of all the boundary conditions. For electrostatic problems, Eyges [11] states it best, “*These equations for the unknown charge distributions have the advantage of being valid for any surface, not only those for which separated solutions of Laplace’s equation exist. Moreover, they reduce by one the dimensionality of the unknown function, since a three-dimensional problem involves an unknown charge distribution on a surface, that is, a function of two variables, and a two-dimensional problem reduces to an unknown function of one variable.*” Of course, the disadvantage of an integral formulation is that they are not easily solved analytically. When all else fails, a numerical solution can always be found.

CHAPTER 2

CYLINDRICAL GREEN'S FUNCTION EXPANSION

2.1 Introduction

There are many techniques used to solve linear differential equations or linear partial differential equations and among these the Green's function approach [15] is one of the most useful. The Green's function is an integral kernel that can be used to solve an inhomogeneous linear differential or an inhomogeneous linear partial differential equation. The Green's function has a simple physical significance. It represents the solution for a unit point source $f(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}')$. In other words, it represents the impulse response to a system [16, 17].

The Green's function may not be unique and finding one for a particular coordinate system which satisfies certain boundary conditions can be a daunting task. In fact, one may not even exist. The nonuniqueness of the Green's function also poses a number of difficulties. For example, one form of the Green's function for a particular geometry may not be suited for a numerical study whereas another form may. One form may lead to analytical complications and another may not. The key is to find the proper form, if one exists, and to make sure that all boundary conditions are satisfied.

A Green's function which is applicable for a boundary value problem is usually found by modifying its free-space Green's function. The free-space Green's function, whose boundary surface is at infinity, is derived by assuming that the forcing function to the linear differential equation or to the linear partial differential equation is a unit point source. Once the free-space Green's function is found, one needs to modify it in some way so that it satisfies certain finite-boundary conditions. This is usually more difficult, but it can be done for certain geometries. Once a Green's function is found which satisfies all boundary conditions then one can solve the same differential or partial differential equation for different forcing functions without having to re-solve the complete problem for each new forcing function. It is very much like a convolution process. This is what makes the Green's function approach very

Figure 2.1: Inverse distance in cylindrical coordinates

attractive.

2.2 Free-space cylindrical Green's function

The reciprocal distance, in cylindrical coordinates, between the source point, (ρ', ϕ', z') , and the observation point, (ρ, ϕ, z) , is given by

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\sqrt{\rho^2 + \rho'^2 + (z - z')^2 - 2\rho\rho' \cos(\phi - \phi')}} \quad (2.1)$$

where $\mathbf{r}'$ represents the position vector of the source point and $\mathbf{r}$ is the position vector of the observation point (see *Appendix C for details*) as shown in Figure 2.1. There are a number of ways in which the reciprocal distance can be represented, and one way utilizes the expansion of the free-space Green's function in cylindrical coordinates. This expansion leads to the expression [10]

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi - \phi')} \left(\int_0^{\infty} K_m(\rho u) I_m(\rho' u) \cos[u(z - z')] du \right) \quad (2.2)$$

where I_m and K_m are modified Bessel functions of the first and second kind respectively. Equation (2.2) is valid if $\rho > \rho'$. Alternatively, Equation (2.2) must be written as

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \left(\int_0^{\infty} K_m(\rho' u) I_m(\rho u) \cos[u(z - z')] du \right) \quad (2.3)$$

when $\rho' > \rho$. Let

$$\Theta_m(\rho, \phi, z) = \int_0^{\infty} K_m(\rho u) I_m(\rho' u) \cos[u(z - z')] du \quad (2.4)$$

$$\Lambda_m(\rho, \phi, z) = \int_0^{\infty} K_m(\rho' u) I_m(\rho u) \cos[u(z - z')] du \quad (2.5)$$

Writing Equation (2.2) and Equation (2.3) in terms of real quantities give

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{2}{\pi} \sum_{m=0}^{\infty} \varepsilon_m \Theta_m(\rho, \phi, z) \cos[m(\phi - \phi')] \quad (2.6)$$

and

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{2}{\pi} \sum_{m=0}^{\infty} \varepsilon_m \Lambda_m(\rho, \phi, z) \cos[m(\phi - \phi')] \quad (2.7)$$

The coefficient, ε_m , is called the Neumann factor [18]. The Neumann factor can be expressed in terms of the Kronecker Delta [19]. This is represented by

$$\varepsilon_m = 2 - \delta_m^0 \quad (2.8)$$

where $\varepsilon_m = 1$ if $m = 0$ and $\varepsilon_m = 2$ for $m \geq 1$.

The numerical solution of Equation (2.4) or of Equation (2.5) requires the accurate evaluation of the infinite integral over a product of modified Bessel functions for all m . This has proven to be a rather difficult problem and it is one reason why Equation (2.2) or Equation (2.3) have not been extensively utilized (*see Appendix D for an alternate formulation of the inverse distance function*). However, one can simplify Equation (2.4) or Equation (2.5) and eliminate the need for numerical integration.

The infinite integrals in Equations (2.4) and (2.5) have been evaluated [20, 21,

22] and are given by

$$\Theta_m(\rho, \phi, z) = \frac{1}{2\sqrt{\rho\rho'}} Q_{m-\frac{1}{2}} \left(\frac{\rho^2 + \rho'^2 + (z - z')^2}{2\rho\rho'} \right) \quad (2.9)$$

$$\Lambda_m(\rho, \phi, z) = \frac{1}{2\sqrt{\rho\rho'}} Q_{m-\frac{1}{2}} \left(\frac{\rho^2 + \rho'^2 + (z - z')^2}{2\rho\rho'} \right) \quad (2.10)$$

See Appendix E for a simple proof of Equations (2.9) or (2.10). Notice that $\Theta_m(\rho, \phi, z) = \Lambda_m(\rho, \phi, z)$. This shows that the free-space Green's function in cylindrical coordinates yields, unlike a spherical harmonic expansion, one expansion. This is a very useful property. In solving cylindrically symmetric magnetic systems, only one expansion is needed. If the physical problem allows for a solution inside the cylindrical magnetic source then the same form of the expansion used for the external solution is also valid for the internal solution.

Utilizing Equation (2.9) or Equation (2.10), one can rewrite Equation (2.6) and Equation (2.7) as [23, 24, 25, 26, 27]

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\pi\sqrt{\rho\rho'}} \sum_{m=0}^{\infty} \varepsilon_m Q_{m-\frac{1}{2}}(\beta) \cos[m(\phi - \phi')] \quad (2.11)$$

where $\beta = \frac{\rho^2 + \rho'^2 + (z - z')^2}{2\rho\rho'} > 1$ and $Q_{m-\frac{1}{2}}(\beta)$ is called a Legendre function of the second kind and of half-integral degree or a toroidal function of zeroth order. They are also referred to as Q-functions. Equation (2.11) represents a Fourier series expansion of the inverse distance function in cylindrical coordinates whose weighting coefficients are Q-functions. This can also be viewed as the binomial expansion of the inverse distance function. However, it is not a spherical multipole expansion. More will be said about this later.

2.2.1 Toroidal functions or Q-functions

An expression for the Legendre function, $Q_\nu(\beta)$, in terms of gamma functions and the hypergeometric function is [28, 29]

$$Q_\nu(\beta) = \frac{\sqrt{\pi}\Gamma(\nu+1)}{\Gamma(\nu+\frac{3}{2})(2\beta)^{\nu+1}} F\left(\frac{\nu+2}{2}, \frac{\nu+1}{2}; \frac{2\nu+3}{2}; \frac{1}{\beta^2}\right) \quad (2.12)$$

Substituting $\nu = m - \frac{1}{2}$ into Equation (2.12) yields:

$$Q_{m-\frac{1}{2}}(\beta) = \frac{\sqrt{\pi}\Gamma\left(m + \frac{1}{2}\right)}{\Gamma(m+1)(2\beta)^{m+\frac{1}{2}}} F\left(\frac{2m+3}{4}, \frac{2m+1}{4}; m+1; \frac{1}{\beta^2}\right) \quad (2.13)$$

The Hypergeometric function $F(a, b; c; \lambda)$ is defined as [30]

$$F(a, b; c; \lambda) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!} \lambda^n \quad (2.14)$$

Substituting $a = \frac{2m+3}{4}$, $b = \frac{2m+1}{4}$, $c = m+1$, and $\lambda = \frac{1}{\beta^2} < 1$ into Equation (2.14) gives

$$F(a, b; c; \lambda) = \frac{\Gamma(m+1)}{\Gamma\left(\frac{2m+3}{4}\right)\Gamma\left(\frac{2m+1}{4}\right)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{4n+2m+3}{4}\right)\Gamma\left(\frac{4n+2m+1}{4}\right)}{(n+m)!n!} \frac{1}{\beta^{2n}} \quad (2.15)$$

Substituting Equation (2.15) into Equation (2.13) yields:

$$Q_{m-\frac{1}{2}}(\beta) = \frac{\sqrt{\pi}\Gamma\left(m + \frac{1}{2}\right)}{\Gamma\left(\frac{2m+3}{4}\right)\Gamma\left(\frac{2m+1}{4}\right)(2\beta)^{m+\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{4n+2m+3}{4}\right)\Gamma\left(\frac{4n+2m+1}{4}\right)}{(n+m)!n!} \frac{1}{\beta^{2n}} \quad (2.16)$$

The following expressions are used to further simplify Equation (2.16):

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad (2.17)$$

$$\Gamma\left(m + \frac{1}{2}\right) = \frac{(2m-1)!!}{2^m} \sqrt{\pi} \quad (2.18)$$

$$\Gamma(\xi)\Gamma\left(\xi + \frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma(2\xi)}{2^{2\xi-1}} \quad (2.19)$$

where $(2m-1)!! = 1 \cdot 3 \cdot 5 \cdot 7 \cdots (2m-1)$. Notice that $(2m-1)!!$ is defined to be 1 for all values for which $2m-1 < 0$.

Equation (2.19) is called the Legendre's duplication formula. Setting $\xi = \frac{4n+2m+1}{4}$ in Legendre's duplication formula gives, for all $m, n \geq 0$,

$$\Gamma\left(\frac{4n+2m+3}{4}\right)\Gamma\left(\frac{4n+2m+1}{4}\right) = \frac{\sqrt{2\pi}(4n+2m-1)!!}{2^{4n+2m}} \quad (2.20)$$

See Appendix F for useful relationships involving the Gamma function. Substituting Equation (2.20) into Equation (2.16), for all $m \geq 0$, yields [31]:

$$Q_{m-\frac{1}{2}}(\beta) = \frac{\pi}{(2\beta)^{m+\frac{1}{2}} 2^m} \sum_{n=0}^{\infty} \frac{(4n+2m-1)!!}{2^{2n} (n+m)!n!} \frac{1}{(2\beta)^{2n}} \quad (2.21)$$

where $(4n+2m-1)!! = 1 \cdot 3 \cdot 5 \cdot 7 \cdots (4n+2m-1)$ for all $m, n \geq 0$.

2.3 Gradient of the free-space cylindrical Green's function and higher order derivatives

It is often necessary in electromagnetic field problems to know not only the inverse distance function but its gradient. The inverse distance is useful when formulating the electric scalar potential, the magnetic scalar potential, or the magnetic vector potential in terms of an integral. However, in magnetostatic problems, it is necessary to compute the magnetic field intensity, $\mathbf{H}$, or the magnetic flux density, $\mathbf{B}$. This requires a knowledge of the gradient of the inverse distance function (see *Appendix I for details on the derivatives of Q -functions*). In cylindrical coordinates, one must find an expression for the gradient of Equation (2.11).

The is given by

$$\begin{aligned} \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) &= \frac{1}{\pi} \sum_{m=0}^{\infty} \varepsilon_m \left(\frac{\partial}{\partial \rho} \left(\frac{Q_{m-\frac{1}{2}}(\beta)}{\sqrt{\rho\rho'}} \right) \cos[m(\phi - \phi')] \hat{\rho} + \right. \\ &\quad \left. \frac{\partial}{\partial \phi} \left(\cos[m(\phi - \phi')] \right) \frac{Q_{m-\frac{1}{2}}(\beta)}{\sqrt{\rho\rho'}} \frac{\hat{\phi}}{\rho} + \right. \\ &\quad \left. \frac{\partial}{\partial z} \left(Q_{m-\frac{1}{2}}(\beta) \right) \frac{\cos[m(\phi - \phi')]}{\sqrt{\rho\rho'}} \hat{z} \right) \end{aligned} \quad (2.22)$$

For magnetic field problems which exhibit cylindrical symmetry, one may also need to know the higher order derivatives of $|\mathbf{r} - \mathbf{r}'|^{-1}$. More specifically, the ex-

pansion for $|\mathbf{r} - \mathbf{r}'|^{-2l-1}$ for $l \geq 0$ will be required. This expansion is given by [24]

$$\begin{aligned} \frac{1}{|\mathbf{r} - \mathbf{r}'|^{2l+1}} &= \frac{1}{(2\rho\rho')^{l+\frac{1}{2}} [\beta - \cos(\phi - \phi')]^{l+\frac{1}{2}}} \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{(2\rho\rho')^{l+\frac{1}{2}} \Gamma(l + \frac{1}{2}) (\beta^2 - 1)^{\frac{l}{2}}} \cdot \\ &\quad \sum_{m=0}^{\infty} \varepsilon_m Q_{m-\frac{1}{2}}^l(\beta) \cos[m(\phi - \phi')] \end{aligned} \quad (2.23)$$

where $Q_{m-\frac{1}{2}}^l(\beta)$ are the associated toroidal functions. Notice, that Equation (2.23) reduces to Equation (2.11) for $l = 0$. Also, one can write [24]

$$\begin{aligned} |\mathbf{r} - \mathbf{r}'|^{2l-1} &= (2\rho\rho')^{l-\frac{1}{2}} [\beta - \cos(\phi - \phi')]^{l-\frac{1}{2}} \\ &= \frac{1}{2} \left(\frac{2}{\pi}\right)^{\frac{3}{2}} \Gamma(l + \frac{1}{2}) (2\rho\rho')^{l-\frac{1}{2}} (\beta^2 - 1)^{\frac{l}{2}} \cdot \\ &\quad \sum_{m=0}^{\infty} \varepsilon_m \frac{\Gamma(m - l + \frac{1}{2})}{\Gamma(m + l + \frac{1}{2})} Q_{m-\frac{1}{2}}^l(\beta) \cos[m(\phi - \phi')] \end{aligned} \quad (2.24)$$

for $l \geq 0$. For example, if $l = 1$, Equation (2.24) reduces to

$$|\mathbf{r} - \mathbf{r}'| = \frac{1}{\pi} \sqrt{\rho\rho'} \sqrt{\beta^2 - 1} \sum_{m=0}^{\infty} \frac{\varepsilon_m}{m^2 - \frac{1}{4}} Q_{m-\frac{1}{2}}^1(\beta) \cos[m(\phi - \phi')] \quad (2.25)$$

Equations (2.23) and (2.24) are quite useful for solving electromagnetic radiation problems in circular cylindrical coordinates.

2.4 Summary

In this chapter, a free-space Green's function for a cylindrical coordinate system is introduced. It is shown that the cylindrical free-space Green's function given by Jackson [10] and by Smythe [19] is written in terms of an integral. It turns out that this integral can be evaluated in terms of non-elementary functions. The analytical solution opened the door to an alternate description of a cylindrically symmetric magnetic system. This mathematical description employs the use of Q-functions, and these functions are predominantly used for problems which exhibit toroidal sym-

metry. However, they have not been extensively applied to cylindrical geometries in the engineering world [32, 33], and in fact, recent literature only sparsely mentions the restricted class of toroidal functions which are used for cylindrical geometries [34, 35, 36]. Other formulations, specifically those due to Kildishev [37, 38, 39], have made contributions to this problem by employing a spheroidal harmonic analysis.

Unlike an Elliptic integral approach, where a fair amount of algebraic manipulation is necessary in order to recognize that the integral formulation actually fits the proper Elliptic integral form, the Q -function allows one to quickly formulate the problem. The inverse distance in cylindrical coordinates can be written down immediately in terms of a toroidal expansion using Equation (2.11). This expansion, along with its higher order derivatives, can greatly simplify the mathematics used for cylindrical geometries.

CHAPTER 3

COULOMB'S LAW FOR MAGNETIC CHARGE

3.1 Introduction

It is well known that time-independent electric fields can always be attributed to electric charges which in turn can be considered sources of those fields. However, time-independent magnetic fields have no such physical counterpart. All experiments, to date, have given no positive indication for the existence of magnetic charges. P.A.M. Dirac [40] showed that the existence of magnetic monopoles could explain the quantization of electric charge, and showed that if they exist, they must carry a “magnetic charge”. In fact, in high energy physics a number of theories rely on the existence of magnetic charge to make theoretical predictions. This aside, the question is whether one can use the idea of magnetic charge, fictitious or not, to make magnetic field computations. This, of course, is exactly what can be done in order to solve permanent magnet systems whose magnetization is independent of the applied external magnetic fields [10].

For example, it is desired to compute the magnetic field from a cylindrical permanent magnet as shown in Figure 3.1. The magnetic scalar potential at points external to the cylindrical permanent magnet can be written as [10, 41, 42]

$$\Phi_P(\rho, \phi, z) = -\frac{1}{4\pi} \int_V \frac{\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r' + \frac{1}{4\pi} \int_S \frac{\mathbf{n}' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dS' \quad (3.1)$$

where P is the location of the field point. This can be rewritten as

$$\Phi_P(\rho, \phi, z) = \frac{1}{4\pi} \int_V \frac{\rho_M(r')}{|\mathbf{r} - \mathbf{r}'|} d^3r' + \frac{1}{4\pi} \int_S \frac{\sigma_M(r')}{|\mathbf{r} - \mathbf{r}'|} dS' \quad (3.2)$$

where $\rho_M = -\nabla' \cdot \mathbf{M}(\mathbf{r}')$ is an equivalent magnetic volume charge density and $\sigma_M = \mathbf{n}' \cdot \mathbf{M}(\mathbf{r}')$ is an equivalent surface charge density. This method is sometimes referred to as the *equivalent pole method* for computing the external magnetic field from a permanent magnet. It relies on the fact that the magnetic scalar potential

Figure 3.1: Cylindrical permanent magnet with a given magnetization

or the magnetic field can be computed from an equivalent distribution of fictitious magnetic volume or surface charge densities. The magnetic field intensity external to the magnet is written as

$$\mathbf{H}_P(\rho, \phi, z) = \frac{1}{4\pi} \int_V \frac{\rho_M(r')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3r' + \frac{1}{4\pi} \int_S \frac{\sigma_M(r')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dS' \quad (3.3)$$

Equation (3.3) is completely analogous to the equation used to compute the external electric field from a polarized dielectric [42].

It is quite reasonable to extend the idea of fictitious charge densities to fictitious point charges. In other words, can one compute the magnetic field from a point charge distribution? The answer is yes, and this idea will be an important part of this thesis.

3.2 Coulomb's law for magnetic charge

Coulomb's law for the electric potential in the *mks* system of units, for N point charges, is well known, and is given by

$$\Phi_P(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^N \frac{q_k}{|\mathbf{r} - \mathbf{r}_k|} \quad (3.4)$$

where P is the location of the field point and $\mathbf{r}$ is the distance vector measured from the origin of some coordinate system to the observation point, and $\mathbf{r}_k$ is the distance vector measured from the origin of the coordinate system to the location of each point charge. One can write, by direct analogy, the magnetic form of Coulomb's law [43] as

$$\Phi_P(\mathbf{r}) = \frac{1}{4\pi} \sum_{k=1}^N \frac{\Omega_k}{|\mathbf{r} - \mathbf{r}_k|} \quad (3.5)$$

where P is the location of the field point and Ω_k are the magnetic charges whose dimensions are *amperes · meters*. Unlike electrical charge, all magnetic charges must sum to zero. This is the requirement that must be met in order to maintain $\nabla \cdot \mathbf{B} = 0$. In other words, no monopoles are allowed to exist.

3.2.1 Toroidal expansion of the magnetic form of Coulomb's law

Using Equation (3.5), one can compute the magnetic scalar potential in cylindrical coordinates from a point charge distribution. This can be written as

$$\Phi_P(\rho, \phi, z) = \frac{1}{4\pi} \sum_{k=1}^N \frac{\Omega_k}{\sqrt{\rho^2 + \rho_k^2 + (z - z_k)^2 - 2\rho\rho_k \cos[m(\phi - \phi_k)]}} \quad (3.6)$$

where ρ'_k , ϕ'_k , and z'_k represent the location of the k^{th} point charge. Employing Equation (2.11), one can rewrite Equation (3.6) as

$$\Phi_P(\rho, \phi, z) = \frac{1}{4\pi^2} \sum_{k=1}^N \sum_{m=0}^{\infty} \frac{\varepsilon_m \Omega_k}{\sqrt{\rho\rho_k}} Q_{m-\frac{1}{2}}(\beta_k) \cos[m(\phi - \phi_k)] \quad (3.7)$$

where $\beta_k = \frac{\rho^2 + \rho_k^2 + (z - z_k)^2}{2\rho\rho_k} > 1$ and $Q_{m-\frac{1}{2}}(\beta_k)$ is given by Equation (2.21). Equation (3.7) represents the toroidal expansion for the magnetic scalar potential computed at

Figure 3.2: Four magnetic charges located at arbitrary points in space

some arbitrary point in space due to an arbitrary magnetic point charge distribution in a cylindrical coordinate system. For example, consider the four magnetic point charges in Figure 3.2. Equation (3.7), for $N = 4$, represents the toroidal expansion of the magnetic scalar potential at the field point, (ρ, ϕ, z) , due to four magnetic point charges whose coordinates are given in terms of their cylindrical coordinates relative to some fixed origin in space. The equation which represents the magnetic scalar potential at the field point is given by

$$\begin{aligned} \Phi_P(\rho, \phi, z) = & \frac{1}{4\pi^2\sqrt{\rho}} \sum_{m=0}^{\infty} \varepsilon_m \left\{ \frac{\Omega_1}{\sqrt{\rho_1}} Q_{m-\frac{1}{2}}(\beta_1) \cos[m(\phi - \phi_1)] + \right. \\ & \frac{\Omega_2}{\sqrt{\rho_2}} Q_{m-\frac{1}{2}}(\beta_2) \cos[m(\phi - \phi_2)] + \frac{\Omega_3}{\sqrt{\rho_3}} Q_{m-\frac{1}{2}}(\beta_3) \cos[m(\phi - \phi_3)] + \\ & \left. \frac{\Omega_4}{\sqrt{\rho_4}} Q_{m-\frac{1}{2}}(\beta_4) \cos[m(\phi - \phi_4)] \right\} \end{aligned} \quad (3.8)$$

In general, Equation (3.7) is valid at any point in space not coincident with the source. Although this is obvious physically, the condition, $\beta_k = \frac{\rho^2 + \rho_k^2 + (z - z_k)^2}{2\rho\rho_k} > 1$, mathematically ensures the convergence of the toroidal expansion at all field points

Figure 3.3: Plot of a few Q-functions

not coincident with any source point.

As the field point moves closer to the source point, more terms in the expansion will be required to accurately compute the magnetic scalar potential. However, it will be shown that the toroidal expansion is a highly convergent expansion and in most cases only a few terms in the expansion will be necessary to accurately compute the magnetic scalar potential or the magnetic field. In fact, the Q-functions are monotonically decreasing functions and their convergence is assured [44]. Figure 3.3 shows a few representative Q-functions. One can see from Figure 3.3 that the Q-functions look very much like decaying exponentials. This very important property enables the Q-function to rapidly converge.

Equation (3.7) can be rewritten in a way which shows the contribution from each m . This expansion is given by

$$\Phi_P^{(m)}(\rho, \phi, z) = \frac{1}{4\pi^2} \sum_{k=1}^N \frac{\varepsilon_m \Omega_k}{\sqrt{\rho \rho_k}} Q_{m-\frac{1}{2}}(\beta_k) \cos[m(\phi - \phi_k)] \quad (3.9)$$

where P is the field point. One can view this expansion as the cylindrical counterpart to the more familiar spherical multipole expansion of a potential in a spherical

coordinate system. However, one must recognize that it is not a spherical multipole expansion. Unlike a spherical multipole expansion for a magnetic system, the dipole term in a cylindrical coordinate representation, for example, is built into the entire $Q_{\frac{1}{2}}(\beta)$ term. In other words, the $l = 1$ term in Equation (1.7), if it is nonzero, is exactly the dipole term of the spherical multipole expansion. However, for a cylindrical magnetic system, the $m = 1$ term in Equation (3.9) incorporates not only the dipole term but an infinite series of higher harmonics. A more detailed discussion of this will be given in chapter 6.

A few representative terms in the expansion of Equation (3.9) are given as follows:

$$\begin{aligned} \Phi_P^{(0)}(\rho, \phi, z) = & \frac{1}{4\pi^2} \left[\frac{\Omega_1}{\sqrt{\rho\rho_1}} Q_{-\frac{1}{2}}(\beta_1) + \frac{\Omega_2}{\sqrt{\rho\rho_2}} Q_{-\frac{1}{2}}(\beta_2) + \right. \\ & \left. \dots \frac{\Omega_N}{\sqrt{\rho\rho_N}} Q_{-\frac{1}{2}}(\beta_N) \right] \end{aligned} \quad (3.10)$$

$$\begin{aligned} \Phi_P^{(1)}(\rho, \phi, z) = & \frac{1}{2\pi^2} \left[\frac{\Omega_1}{\sqrt{\rho\rho_1}} Q_{\frac{1}{2}}(\beta_1) \cos[(\phi - \phi_1)] + \right. \\ & \frac{\Omega_2}{\sqrt{\rho\rho_2}} Q_{\frac{1}{2}}(\beta_2) \cos(\phi - \phi_2) + \\ & \left. \dots \frac{\Omega_N}{\sqrt{\rho\rho_N}} Q_{\frac{1}{2}}(\beta_N) \cos(\phi - \phi_N) \right] \end{aligned} \quad (3.11)$$

$$\begin{aligned} \Phi_P^{(2)}(\rho, \phi, z) = & \frac{1}{2\pi^2} \left[\frac{\Omega_1}{\sqrt{\rho\rho_1}} Q_{\frac{3}{2}}(\beta_1) \cos[2(\phi - \phi_1)] + \right. \\ & \frac{\Omega_2}{\sqrt{\rho\rho_2}} Q_{\frac{3}{2}}(\beta_2) \cos[2(\phi - \phi_2)] + \\ & \left. \dots \frac{\Omega_N}{\sqrt{\rho\rho_N}} Q_{\frac{3}{2}}(\beta_N) \cos[2(\phi - \phi_N)] \right] \end{aligned} \quad (3.12)$$

$$\begin{aligned}
& \vdots \\
\Phi_P^{(m)}(\rho, \phi, z) &= \frac{1}{2\pi_o^2} \left[\frac{\Omega_1}{\sqrt{\rho\rho_1}} Q_{m-\frac{1}{2}}(\beta_1) \cos[m(\phi - \phi_1)] + \right. \\
& \quad \frac{\Omega_2}{\sqrt{\rho\rho_2}} Q_{m-\frac{1}{2}}(\beta_2) \cos[m(\phi - \phi_2)] + \\
& \quad \left. \dots \frac{\Omega_N}{\sqrt{\rho\rho_N}} Q_{m-\frac{1}{2}}(\beta_N) \cos[m(\phi - \phi_N)] \right] \quad (3.13)
\end{aligned}$$

One can consider the total potential at the observation point as being derived from the contributions of each potential at a given m . This can be expressed as

$$\begin{aligned}
\Phi_P(\rho, \phi, z) &= \sum_{m=0}^{\infty} \Phi_P^{(m)}(\rho, \phi, z) \\
&= \Phi_P^{(0)}(\rho, \phi, z) + \Phi_P^{(1)}(\rho, \phi, z) + \Phi_P^{(2)}(\rho, \phi, z) + \dots \\
& \quad \Phi_P^{(M-2)}(\rho, \phi, z) + \Phi_P^{(M-1)}(\rho, \phi, z) + \Phi_P^{(M)}(\rho, \phi, z) + \dots \quad (3.14)
\end{aligned}$$

An important feature of Equation (3.14), which will be discussed later, is that for a cylindrically symmetric magnetic system it is highly convergent and only a few terms may be needed to accurately reproduce the total scalar potential.

3.2.2 Toroidal expansion of the magnetic field intensity

Once the magnetic scalar potential is known, it is necessary to compute its gradient in order to find the magnetic field intensity. The magnetic field intensity is given by

$$\mathbf{H}_P(\rho, \phi, z) = -\nabla \Phi_P \quad (3.15)$$

By employing Equations (2.21), (3.7), and (3.15), the $\mathbf{H}$ field at the observation point is written as

$$\begin{aligned}
\mathbf{H}_P(\rho, \phi, z) &= -\frac{1}{4\pi^2} \sum_{k=1}^N \sum_{m=0}^{\infty} \varepsilon_m \frac{\Omega_k}{\sqrt{\rho\rho_k}} \left\{ \sqrt{\rho\rho_k} \frac{\partial}{\partial \rho} \left(\frac{Q_{m-\frac{1}{2}}(\beta_k)}{\sqrt{\rho\rho_k}} \right) \right. \\
& \quad \cos m(\phi - \phi_k) \hat{\rho} + Q_{m-\frac{1}{2}}(\beta_k) \frac{\partial}{\partial \phi} (\cos[m(\phi - \phi_k)]) \frac{\hat{\phi}}{\rho} + \\
& \quad \left. \cos[m(\phi - \phi_k)] \frac{\partial}{\partial z} (Q_{m-\frac{1}{2}}(\beta_k)) \hat{z} \right\} \quad (3.16)
\end{aligned}$$

For $m = 0$, the magnetic field intensity, $\mathbf{H}_P^{(0)}(\rho, \phi, z)$, is given by

$$\begin{aligned} \mathbf{H}_P^{(0)}(\rho, \phi, z) = & -\frac{1}{4\pi^2} \sum_{k=1}^N \frac{\Omega_k}{\sqrt{\rho\rho_k}} \left\{ \left[\sqrt{\rho\rho_k} \frac{\partial}{\partial \rho} \left(\frac{Q_{m-\frac{1}{2}}(\beta_k)}{\sqrt{\rho\rho_k}} \right) \right] \hat{\rho} + \right. \\ & \left. \left[\frac{\partial}{\partial z} \left(Q_{m-\frac{1}{2}}(\beta_k) \right) \right] \hat{z} \right\} \end{aligned} \quad (3.17)$$

Computing the necessary derivatives yields:

$$\begin{aligned} \mathbf{H}_P^{(0)}(\rho, \phi, z) = & \frac{1}{4\pi^2} \sum_{k=1}^N \frac{\Omega_k}{\sqrt{\rho\rho_k}} \left\{ \frac{1}{2\rho_k\beta_k} \left[Q_{-\frac{1}{2}}(\beta_k) + \right. \right. \\ & \left. \frac{4\pi}{\sqrt{2\beta_k}} \left(1 - \frac{\rho_k\beta_k}{\rho} \right) \sum_{n=1}^{\infty} \frac{nA_n}{2^{2n}(2\beta_k)^{2n}} \right] \hat{\rho} + \\ & \frac{1}{2\beta_k} \left(\frac{z - z_k}{\rho\rho_k} \right) \left[Q_{-\frac{1}{2}}(\beta_k) + \right. \\ & \left. \left. \frac{4\pi}{\sqrt{2\beta_k}} \sum_{n=1}^{\infty} \frac{nA_n}{2^{2n}(2\beta_k)^{2n}} \right] \hat{z} \right\} \end{aligned} \quad (3.18)$$

where $A_n = \frac{(4n-1)!!}{[n!]^2}$. For $m \geq 1$, the magnetic field intensity, $\mathbf{H}^{(m)}(\rho, \phi, z)$, is given by

$$\begin{aligned} \mathbf{H}_P^{(m)}(\rho, \phi, z) = & -\frac{1}{2\pi^2} \sum_{k=1}^N \sum_{m=1}^{\infty} \frac{\Omega_k}{\sqrt{\rho\rho_k}} \left\{ \right. \\ & \left[\cos[m(\phi - \phi_k)] \left\{ \sqrt{\rho\rho_k} \frac{\partial}{\partial \rho} \left(\frac{Q_{m-\frac{1}{2}}(\beta_k)}{\sqrt{\rho\rho_k}} \right) \right\} \right] \hat{\rho} + \\ & \left[Q_{m-\frac{1}{2}}(\beta_k) \frac{\partial}{\partial \phi} (\cos[m(\phi - \phi_k)]) \right] \frac{\hat{\phi}}{\rho} + \\ & \left. \left[\cos[m(\phi - \phi_k)] \frac{\partial}{\partial z} \left(Q_{m-\frac{1}{2}}(\beta_k) \right) \right] \hat{z} \right\} \end{aligned} \quad (3.19)$$

Computing the necessary derivatives yields:

$$\begin{aligned}
\mathbf{H}_P^{(m)}(\rho, \phi, z) = & \frac{1}{2\pi^2} \sum_{k=1}^N \sum_{m=1}^{\infty} \frac{q_k}{\sqrt{\rho\rho_k}} \left(\pi \left[\frac{1}{\rho_k\beta_k} - \frac{1}{\rho} \right] \cos[m(\phi - \phi_k)] \cdot \right. \\
& \left[\frac{1}{(2\beta_k)^{m+\frac{1}{2}}} \frac{1}{2^m} \sum_{n=0}^{\infty} \frac{(2n+m) A_{nm}}{2^{2n} (2\beta_k)^{2n}} + \right. \\
& \left. \left. \frac{1}{2\pi \left(1 - \frac{\rho_k\beta_k}{\rho}\right)} Q_{m-\frac{1}{2}}(\beta_k) \right] \hat{\rho} + \right. \\
& \left. \left[m \sin(\phi - \phi_k) \right] Q_{m-\frac{1}{2}}(\beta_k) \right] \frac{\hat{\phi}}{\rho} + \\
& \frac{\pi(z - z_k)}{\rho\rho_k\beta_k} \cos[m(\phi - \phi_k)] \cdot \\
& \left[\frac{1}{(2\beta_k)^{m+\frac{1}{2}}} \frac{1}{2^m} \sum_{n=0}^{\infty} \frac{(2n+m) A_{nm}}{2^{2n} (2\beta_k)^{2n}} \right. \\
& \left. \left. + \frac{1}{2\pi} Q_{m-\frac{1}{2}}(\beta_k) \right] \hat{z} \right) \tag{3.20}
\end{aligned}$$

where $A_{nm} = \frac{(4n+2m-1)!!}{n!(n+m)!}$. An expansion for the magnetic field intensity, analogous to Equation (3.14), is given by

$$\begin{aligned}
\mathbf{H}_P(\rho, \phi, z) = & \sum_{m=0}^{\infty} \mathbf{H}_P^{(m)}(\rho, \phi, z) \\
= & \mathbf{H}_P^{(0)}(\rho, \phi, z) + \mathbf{H}_P^{(1)}(\rho, \phi, z) + \mathbf{H}_P^{(2)}(\rho, \phi, z) + \\
& \mathbf{H}_P^{(3)}(\rho, \phi, z) + \dots \mathbf{H}_P^{(M-2)}(\rho, \phi, z) + \mathbf{H}_P^{(M-1)}(\rho, \phi, z) + \\
& \mathbf{H}_P^{(M)}(\rho, \phi, z) + \dots \tag{3.21}
\end{aligned}$$

Equation (3.21) is a highly convergent series. The magnetic components of $\mathbf{H}_P(\rho, \phi, z)$ can be tabulated for quick reference as shown in Table 3.1 . See Appendix I for details on how to compute the derivatives of $Q_{m-\frac{1}{2}}(\beta_k)$.

3.3 Summary

This chapter introduces the idea of the magnetic point charge and the application of Coulomb's law for a magnetic point charge distribution. In particular,

$m = 0$		
$H_\rho^{(0)}(\rho, \phi, z) = -\frac{1}{4\pi^2} \sum_{k=1}^N \frac{q_k}{\sqrt{\rho\rho_k}}$	$\left \frac{\partial}{\partial \rho} \left(Q_{-\frac{1}{2}}(\beta_k) \right) - \frac{1}{2\rho} Q_{-\frac{1}{2}}(\beta_k) \right $	
$H_\phi^{(0)}(\rho, \phi, z) = 0$		
$H_z^{(0)}(\rho, \phi, z) = -\frac{1}{4\pi^2} \sum_{k=1}^N \frac{q_k}{\sqrt{\rho\rho_k}}$	$\left \frac{\partial}{\partial z} Q_{-\frac{1}{2}}(\beta_k) \right $	
$m \geq 1$		
$H_\rho^{(m)}(\rho, \phi, z) = -\frac{1}{2\pi^2} \sum_{k=1}^N \frac{q_k}{\sqrt{\rho\rho_k}}$	$\left\{ \frac{\left[\frac{\partial}{\partial \rho} \left(Q_{m-\frac{1}{2}}(\beta_k) \right) - \frac{1}{2\rho} Q_{m-\frac{1}{2}}(\beta_k) \right]}{\cos[m(\phi - \phi_k)]} \right\}$	
$H_\phi^{(m)}(\rho, \phi, z) = -\frac{1}{2\pi^2} \sum_{k=1}^N \frac{q_k}{\sqrt{\rho\rho_k}}$	$\left\{ \frac{1}{\rho} Q_{m-\frac{1}{2}}(\beta_k) \frac{\partial}{\partial \phi} (\cos[m(\phi - \phi_k)]) \right\}$	
$H_z^{(m)}(\rho, \phi, z) = -\frac{1}{2\pi^2} \sum_{k=1}^N \frac{q_k}{\sqrt{\rho\rho_k}}$	$\left\{ \left \frac{\partial}{\partial z} \left(Q_{m-\frac{1}{2}}(\beta_k) \right) \right \cos[m(\phi - \phi_k)] \right\}$	

Table 3.1: Components of the magnetic field intensity

the application of Coulomb's law for a point charge distribution, in a cylindrical coordinate system, is used to find the magnetic scalar potential and the magnetic field intensity. The scalar potential and the magnetic field intensity are written in terms of a highly convergent toroidal expansion which will be used in conjunction with the charge simulation method, discussed in chapter 7, to compute the external magnetic field from any circular cylindrical magnetic source.

CHAPTER 4

INTEGRAL FORMULATION FOR MAGNETOSTATIC PROBLEMS IN CYLINDRICAL COORDINATES

4.1 Introduction

A magnetostatic field can be described by the vectors, $\mathbf{H}$, $\mathbf{B}$, $\mathbf{J}$, or $\mathbf{A}$ where $\mathbf{H}$ is the magnetic field intensity, $\mathbf{B}$ is magnetic induction or magnetic flux density, $\mathbf{J}$ is the volume current density, and $\mathbf{A}$ is the magnetic vector potential. In a current-free region of space, $\mathbf{J}=0$, the magnetostatic field can be computed from a magnetic scalar potential function, Φ_M . It is generally easier to compute the potential functions $\mathbf{A}$ or Φ_M instead of the vectors $\mathbf{H}$ or $\mathbf{B}$, and this chapter will discuss various integral formulations of the former.

In free space there exists a simple constitutive relationship between $\mathbf{B}$ and $\mathbf{H}$, and this is given by

$$\mathbf{B} = \mu_0 \mathbf{H} \quad (4.1)$$

where μ_0 is the permeability of free space and its value is $4\pi \cdot 10^{-7} \text{V} \cdot \text{s}/\text{A} \cdot \text{m}$.

In contrast to the magnetostatic field in free space, there is no general law which relates $\mathbf{B}$ and $\mathbf{H}$ for an arbitrary medium. However, for the majority of common materials the correlation between $\mathbf{B}$ and $\mathbf{H}$ is given by

$$\mathbf{B} = \mu \mathbf{H} \quad (4.2)$$

where μ , a constant, is the permeability of the material medium. The material medium for which μ is not a function of $\mathbf{H}$ is called a magnetically linear medium, and in turn, the medium for which the correlation between $\mathbf{B}$ and $\mathbf{H}$ is not dependent on the direction of $\mathbf{H}$ is called a magnetically isotropic medium. In the general case, the medium may be neither isotropic nor linear—that is, the correlation between $\mathbf{H}$ and $\mathbf{B}$ depends on the direction of $\mathbf{H}$ relative to various characteristic directions in the medium. Crystals, for example, exhibit both non-isotropic as well as non-linear

behavior, and therefore Equation (4.2) would, in general, have to be modified. This chapter will only consider magnetically linear and magnetically isotropic mediums.

4.2 Maxwell's magnetostatic equations

Maxwell's magnetostatic equations in differential form are

$$\nabla \times \mathbf{H} = \mathbf{J} \quad (4.3)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (4.4)$$

and in integral form are

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} \quad (4.5)$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \quad (4.6)$$

Equations (4.5) or (4.6), and more generally, Poisson's theorem for vector fields(*see Appendix A*), implies that the magnetic vector potential can be written as

$$\mathbf{A} = \frac{1}{4\pi} \int_{All\ Space} \frac{\nabla \times \mathbf{B}}{|\mathbf{r} - \mathbf{r}'|} d^3r' + \mathbf{A}_0 \quad (4.7)$$

Let $\mathbf{A}_0 = 0$ for simplicity. Taking the curl of Equation (4.1) and employing Equation (4.3) allows one to rewrite Equation (4.7) as

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{All\ Space} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r' \quad (4.8)$$

This is the magnetic vector potential at some point, P , in free space due to a volume current density. Two important properties of the magnetic vector potential in free-space given by Equation (4.8) are

$$\nabla \cdot \mathbf{A} = 0 \quad (4.9)$$

and

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} \quad (4.10)$$

where Equation (4.10) is Poisson's equation. Equations (4.9) and (4.10) can easily be shown to be true [41].

In a current-free region of space, Equation (4.3) can be written as

$$\nabla \times \mathbf{H} = 0 \quad (4.11)$$

where the magnetic field intensity, $\mathbf{H}$, can be expressed as the gradient of some potential function, namely:

$$\mathbf{H} = -\nabla \Phi_P \quad (4.12)$$

where Φ_P is called the magnetic scalar potential. Equation (4.4) can be rewritten as

$$\nabla \cdot \mu \nabla \Phi_P = 0 \quad (4.13)$$

Equation (4.13) can be expanded to give

$$\nabla^2 \Phi_P + \frac{1}{\mu} \nabla \mu \cdot \nabla \Phi_P = 0 \quad (4.14)$$

If μ is constant, then Equation (4.14) reduces to

$$\nabla^2 \Phi_P = 0 \quad (4.15)$$

which is Laplace's equation for a magnetostatic potential in a current-free medium of constant μ . This simplifies the analysis somewhat because it reduces the problem to one in which all the known mathematical techniques for handling electrostatic problems can be used.

In material media, Maxwell's equations can be augmented by considering a *magnetization vector*. This vector, $\mathbf{M}$, is frequently written as

$$\mathbf{M} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{H} \quad (4.16)$$

and is used when considering magnetized sources.

Maxwell's magnetostatic equations, whether in differential or integral form, seem simple in theory, but in practice their solution can be quite complicated. The remaining sections in this chapter will deal with various ways one can formulate these equations, in integral form, for magnetic sources which exhibit cylindrical symmetry. The toroidal expansion of the magnetic form of Coulomb's law has already been considered in chapter 3. However, one can generalize this to handle arbitrary forcing functions.

4.3 Toroidal expansion of the magnetic scalar potential for a finite cylindrical magnet given a magnetization forcing function

In analyzing a permanent-magnet system [45, 46, 47, 48, 49], one can often find a scalar potential function. This function is then used to compute the magnetic field intensity by using Equation (4.12). In a current-free region of space with a given magnetization, one can employ Poisson's integral formula to derive the magnetic scalar potential due to a magnetized source. Using Equation (A.25), the magnetic scalar potential due to a magnetized source at some arbitrary point in space external to the source can be written as

$$\Phi_P(\mathbf{r}) = \frac{1}{4\pi} \int_{All\ Space} \frac{\nabla' \cdot \mathbf{H}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r' \quad (4.17)$$

where P is the location of the field point. Substituting Equation (4.16) in (4.17) yields:

$$\begin{aligned} \Phi_P(\mathbf{r}) &= -\frac{1}{4\pi} \int_{All\ Space} \frac{\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r' \\ &= -\frac{1}{4\pi} \int_V \frac{\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r' + \frac{1}{4\pi} \oint_S \frac{\mathbf{n}' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dS \\ &= \frac{1}{4\pi} \int_V \frac{\rho_M(r')}{|\mathbf{r} - \mathbf{r}'|} d^3r' + \frac{1}{4\pi} \int_S \frac{\sigma_M(r')}{|\mathbf{r} - \mathbf{r}'|} dS' \end{aligned} \quad (4.18)$$

Figure 4.1: Cylindrical permanent magnet with a given magnetization

Jackson [10] and Jefimenko [41] give clear expositions of Equation (4.18). This is the *equivalent pole method* discussed in chapter 3.

The application of Equation (4.18) for a cylindrically symmetric magnetized system can best be understood by solving a simple example. Consider a cylindrical magnet as shown in Figure 4.1. Let the magnetization vector be $\mathbf{M}(\rho', \phi', z')$. Employing Equation (2.11), Equation (4.18) becomes

$$\begin{aligned}
 \Phi_P(\rho, \phi, z) = & \frac{1}{4\pi^2\sqrt{\rho}} \sum_{m=0}^{\infty} \varepsilon_m \left\{ - \int_{-h}^h \int_0^{2\pi} \int_0^R \sqrt{\rho'} \left(\nabla' \cdot \mathbf{M}(\rho', \phi', z') \right) Q_{m-\frac{1}{2}}(\beta) \cdot \right. \\
 & \cos[m(\phi - \phi')] d\rho' d\phi' dz' + \\
 & \sqrt{R} \int_{-h}^h \int_0^{2\pi} \left(\hat{\rho} \cdot \mathbf{M}(R, \phi', z') \right) Q_{m-\frac{1}{2}}(\beta_{\rho'=R}) \cos[m(\phi - \phi')] d\phi' dz' + \\
 & \int_0^{2\pi} \int_0^R \sqrt{\rho'} \left(\hat{z} \cdot \mathbf{M}(\rho', \phi', h) \right) Q_{m-\frac{1}{2}}(\beta_{z'=h}) \cos[m(\phi - \phi')] d\rho' d\phi' - \\
 & \int_0^{2\pi} \int_0^R \sqrt{\rho'} \left(\hat{z} \cdot \mathbf{M}(\rho', \phi', -h) \right) Q_{m-\frac{1}{2}}(\beta_{z'=-h}) \cdot \\
 & \left. \cos[m(\phi - \phi')] d\rho' d\phi' \right\} \quad (4.19)
 \end{aligned}$$

Equation (4.19) represents a general expression for the magnetic scalar potential at some external field point due to a solid cylindrical permanent magnet with a given

general magnetization vector. Hollow cylindrical magnets can be handled in the same way. If, for example, the magnetization vector is given by $\mathbf{M} = M_0 \hat{z}$, Equation (4.19) reduces to

$$\begin{aligned}\Phi_P(\rho, z) &= \frac{M_0}{4\pi^2\sqrt{\rho}} \sum_{m=0}^{\infty} \varepsilon_m \int_0^{2\pi} \int_0^R \sqrt{\rho'} \left\{ Q_{m-\frac{1}{2}}(\beta_{z'=h}) - Q_{m-\frac{1}{2}}(\beta_{z'=-h}) \right\} \\ &\quad \cos[m(\phi - \phi')] d\rho' d\phi' \\ &= \frac{M_0}{2\pi\sqrt{\rho}} \int_0^R \sqrt{\rho'} \left\{ Q_{-\frac{1}{2}}(\beta_{z'=h}) - Q_{-\frac{1}{2}}(\beta_{z'=-h}) \right\} d\rho'\end{aligned}\quad (4.20)$$

Equation (4.20) represents the magnetic scalar potential at some field point external to the magnetized cylinder which has a constant magnetization in the z -direction. Employing Equation (2.21) enables one to rewrite Equation (4.20) as

$$\begin{aligned}\Phi_P(\rho, z) &= \frac{M_0}{2\pi} \sum_{n=0}^{\infty} A_n \rho^{2n} \int_0^R \left\{ \frac{\rho'^{2n+1}}{[\rho^2 + \rho'^2 + (z-h)^2]^{2n+\frac{1}{2}}} - \right. \\ &\quad \left. \frac{\rho'^{2n+1}}{[\rho^2 + \rho'^2 + (z+h)^2]^{2n+\frac{1}{2}}} \right\} d\rho'\end{aligned}\quad (4.21)$$

where $A_n = \frac{(4n-1)!!}{2^{2n}(n!)^2}$. See Appendix J for the evaluation of all integrals. Consider the case where the field point is located on the z axis at a point $|z| > h$. The magnetic scalar potential becomes

$$\begin{aligned}\Phi_P(0, z) &= \lim_{\rho \rightarrow 0} \frac{M_0}{2\pi} \sum_{n=0}^{\infty} A_n \rho^{2n} \int_0^R \left\{ \frac{\rho'^{2n+1}}{[\rho^2 + \rho'^2 + (z-h)^2]^{2n+\frac{1}{2}}} - \right. \\ &\quad \left. \frac{\rho'^{2n+1}}{[\rho^2 + \rho'^2 + (z+h)^2]^{2n+\frac{1}{2}}} \right\} d\rho' \\ &= \frac{M_0}{2\pi} \int_0^R \left(\frac{\rho'}{\sqrt{\rho'^2 + (z-h)^2}} - \frac{\rho'}{\sqrt{\rho'^2 + (z+h)^2}} \right) d\rho' \\ &= \frac{M_0}{2} \left(\sqrt{R^2 + (z-h)^2} - \sqrt{R^2 + (z+h)^2} + 2h \right)\end{aligned}\quad (4.22)$$

The magnetic field intensity is found by using Equation (4.22) and Equation (4.12).

Figure 4.2: Circular cylindrical conductor carrying a current density

This yields:

$$\mathbf{H}_P(0, z) = \frac{M_0}{2} \left(\frac{|z + h|}{\sqrt{R^2 + (z + h)^2}} - \frac{|z - h|}{\sqrt{R^2 + (z - h)^2}} \right) \hat{z} \quad (4.23)$$

Equation (4.23) is a well known expression found in many text books on electro-magnetism [41, 42]

4.4 Toroidal expansion of the magnetic vector potential for a finite cylindrical conductor given a current density forcing function

In developing the toroidal expansion for a circular cylindrical geometry, one can use a finite cylinder as a “building block” for more complex magnetic systems. However, one must realize that there will be no return current for this hypothetical model and the results must be interpreted with care. Consider a finite circular cylindrical conductor carrying a general current density, $\mathbf{J}(\rho, \phi, z)$, as shown in Figure 4.2 . Using Equation (4.8) and Equation (2.11), the magnetic vector potential at

some arbitrary field point external to the conductor can be written as

$$\mathbf{A}_P(\rho, \phi, z) = \frac{\mu_0}{4\pi^2\sqrt{\rho}} \sum_{m=0}^{\infty} \varepsilon_m \int_{-h}^h \int_0^{2\pi} \int_0^R \mathbf{J}(\rho', \phi', z') Q_{m-\frac{1}{2}}(\beta) \cdot \cos[m(\phi - \phi')] \sqrt{\rho'} d\rho' d\phi' dz' \quad (4.24)$$

where P is the location of the field point. Employing Equation (2.21) reduces Equation (4.24) to

$$\mathbf{A}_P(\rho, \phi, z) = \frac{\mu_0}{4\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\varepsilon_m A_{nm}}{2^m} \rho^{2n+m} \int_{-h}^h \int_0^{2\pi} \int_0^R \frac{\rho'^{2n+m+1} \mathbf{J}(\rho', \phi', z')}{[\rho^2 + \rho'^2 + (z-h)^2]^{\frac{4n+2m+1}{2}}} \cdot \cos[m(\phi - \phi')] d\rho' d\phi' dz' \quad (4.25)$$

Equation (4.25) can be used as a basic “building block” for solid cylindrical conductors. For hollow conductors, one can quickly write

$$\mathbf{A}_P(\rho, \phi, z) = \frac{\mu_0}{4\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\varepsilon_m A_{nm}}{2^m} \rho^{2n+m} \int_{-h}^h \int_0^{2\pi} \int_a^b \frac{\rho'^{2n+m+1} \mathbf{J}(\rho', \phi', z')}{[\rho^2 + \rho'^2 + (z-h)^2]^{\frac{4n+2m+1}{2}}} \cdot \cos[m(\phi - \phi')] d\rho' d\phi' dz' \quad (4.26)$$

where the integration on ρ' is from the inner radius, a , to the outer radius, b . For a solid circular cylindrical conductor with a volume-sector removed [50, 51, 52, 53, 54], one can write

$$\mathbf{A}_P(\rho, \phi, z) = \frac{\mu_0}{4\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\varepsilon_m A_{nm}}{2^m} \rho^{2n+m} \int_{-h}^h \int_{\alpha_2}^{2\pi+\alpha_1-\alpha_2} \int_0^R \frac{\rho'^{2n+m+1} \mathbf{J}(\rho', \phi', z')}{[\rho^2 + \rho'^2 + (z-h)^2]^{\frac{4n+2m+1}{2}}} \cdot \cos[m(\phi - \phi')] d\rho' d\phi' dz' \quad (4.27)$$

where α_1 and α_2 are shown in Figure 4.3 . A number of cylindrical geometries can be handled by the Q-function in the same manner as was done in this section. One of the most important parts of the toroidal expansion is that the integration on the ϕ' variable can, for most realistic current density forcing functions, be integrated in closed form in terms of elementary functions. This is generally not true with an Elliptic integral formulation. Also, the Q-function has built within it the spherical

Figure 4.3: Incomplete cylindrical conductor

multipole expansion. This enables one to filter, from the toroidal expansion, a multipole distribution close to the cylindrical structure. This can not be done directly with a spherical harmonic expansion. More will be said about this in chapter 6.

4.5 Toroidal expansion of the magnetic vector potential for cylindrical coils with a rectangular cross-section given a current density forcing function

Coils with a rectangular cross-section find considerable application in many engineering disciplines [55, 56, 57, 58]. Transformers, electrical machines, magnetic resonance imaging(MRI) devices, Helmholtz coils, and solenoids are just a few of the many possible applications. The basic structure that is considered is shown in Figure 4.4 . For simplicity, two circular coils which carry current densities whose magnitudes vary in the ρ and the z directions, but whose direction is purely azimuthal, will be considered. However, any number of coils with any orientation in space(see *Appendix M*) and with more complex forcing functions can be solved using a toroidal expansion. The differential magnetic vector potential for the configuration

Figure 4.4: Cylindrical coils

in Figure 4.4 can be written as

$$\begin{aligned}
 d\mathbf{A}_P(\rho, \phi, z) &= d\mathbf{A}_1(\rho, \phi, z) + d\mathbf{A}_2(\rho, \phi, z) \\
 &= \frac{\mu_0}{4\pi} \left[\frac{J_1(\rho'_1, z'_1) \hat{\phi}'_1 \rho'_1 d\rho'_1 d\phi'_1 dz'_1}{\sqrt{\rho^2 + \rho'^2_1 + (z - z'_1)^2 - 2\rho\rho'_1 \cos(\phi - \phi'_1)}} + \right. \\
 &\quad \left. \frac{J_2(\rho'_2, z'_2) \hat{\phi}'_2 \rho'_2 d\rho'_2 d\phi'_2 dz'_2}{\sqrt{\rho^2 + \rho'^2_2 + (z - z'_2)^2 - 2\rho\rho'_2 \cos(\phi - \phi'_2)}} \right] \\
 &= \frac{\mu_0}{4\pi^2 \sqrt{\rho}} \sum_{m=0}^{\infty} \varepsilon_m \left\{ J_1(\rho'_1, z'_1) \sqrt{\rho'_1} Q_{m-\frac{1}{2}}(\beta_1) \cdot \right. \\
 &\quad \cos[m(\phi - \phi'_1)] \hat{\phi}'_1 d\rho'_1 d\phi'_1 dz'_1 + \\
 &\quad J_2(\rho'_2, z'_2) \sqrt{\rho'_2} Q_{m-\frac{1}{2}}(\beta_2) \cdot \\
 &\quad \left. \cos[m(\phi - \phi'_2)] \hat{\phi}'_2 d\rho'_2 d\phi'_2 dz'_2 \right\} \quad (4.28)
 \end{aligned}$$

where

$$\hat{\phi}'_1 = -\sin(\phi'_1)\hat{x} + \cos(\phi'_1)\hat{y} \quad (4.29)$$

$$\hat{\phi}'_2 = -\sin(\phi'_2)\hat{x} + \cos(\phi'_2)\hat{y} \quad (4.30)$$

$$\beta_1 = \frac{\rho^2 + \rho_1'^2 + (z - z_1')^2}{2\rho\rho_1'} \quad (4.31)$$

$$\beta_2 = \frac{\rho^2 + \rho_2'^2 + (z - z_2')^2}{2\rho\rho_2'} \quad (4.32)$$

The vector potential can be written as

$$\begin{aligned} \mathbf{A}(\rho, \phi, z) = & \frac{\mu_0}{4\pi^2\sqrt{\rho}} \sum_{m=0}^{\infty} \varepsilon_m \left\{ \int_{h_1}^{h_2} \int_0^{2\pi} \int_{R_1}^{R_2} J_1(\rho'_1, z'_1) \sqrt{\rho'_1} Q_{m-\frac{1}{2}}(\beta_1) \cdot \right. \\ & \left[-\sin(\phi'_1)\hat{x} + \cos(\phi'_1)\hat{y} \right] \cos[m(\phi - \phi'_1)] \hat{\phi}'_1 d\rho'_1 d\phi'_1 dz'_1 + \\ & \int_{h_3}^{h_4} \int_0^{2\pi} \int_{R_3}^{R_4} J_2(\rho'_2, z'_2) \sqrt{\rho'_2} Q_{m-\frac{1}{2}}(\beta_2) \cdot \\ & \left. \left[-\sin(\phi'_2)\hat{x} + \cos(\phi'_2)\hat{y} \right] \cos[m(\phi - \phi'_2)] \hat{\phi}'_2 d\rho'_2 d\phi'_2 dz'_2 \right\} \quad (4.33) \end{aligned}$$

Only the $m = 1$ term survives the ϕ'_1 and ϕ'_2 integration. Equation (4.33) reduces to

$$\begin{aligned} \mathbf{A}(\rho, z) = & \frac{\mu_0 \hat{\phi}}{2\pi\sqrt{\rho}} \left\{ \int_{h_1}^{h_2} \int_{R_1}^{R_2} J_1(\rho'_1, z'_1) \sqrt{\rho'_1} Q_{m-\frac{1}{2}}(\beta_1) d\rho'_1 dz'_1 + \right. \\ & \left. \int_{h_3}^{h_4} \int_{R_3}^{R_4} J_2(\rho'_2, z'_2) \sqrt{\rho'_2} Q_{m-\frac{1}{2}}(\beta_2) d\rho'_2 dz'_2 \right\} \quad (4.34) \end{aligned}$$

4.5.1 Helmholtz coils

Consider Equation (4.34) when $R_2 \rightarrow R_1 = a$, $R_3 \rightarrow R_4 = b$, $h_2 \rightarrow h_1 = -h$, and $h_4 \rightarrow h_3 = h$. This describes infinitely thin Helmholtz coils [59, 60, 61, 62, 63] as shown in Figure 4.5 . Using Equation (4.34), the magnetic vector potential can be written as

$$\mathbf{A}(\rho, z) = \frac{\mu_0}{2\pi} \left\{ I_1 \sqrt{\frac{a}{\rho}} Q_{\frac{1}{2}}(\xi_1) + I_2 \sqrt{\frac{b}{\rho}} Q_{\frac{1}{2}}(\xi_2) \right\} \hat{\phi} \quad (4.35)$$

Figure 4.5: Helmholtz coils

where

$$\xi_1 = \frac{\rho^2 + a^2 + (z + h)^2}{2a\rho} \quad (4.36)$$

$$\xi_2 = \frac{\rho^2 + b^2 + (z - h)^2}{2b\rho} \quad (4.37)$$

Employing Equation (2.21), with $m = 1$, allows one to rewrite Equation (4.35) as

$$\begin{aligned} \mathbf{A}(\rho, z) = & \frac{\mu_0}{4} \sum_{n=0}^{\infty} \frac{(4n+1)!!}{2^{2n}(n+1)!n!} \left\{ I_1 \frac{\rho^{2n+1} a^{2n+2}}{[\rho^2 + a^2 + (z+h)^2]^{2n+\frac{3}{2}}} + \right. \\ & \left. I_2 \frac{\rho^{2n+1} b^{2n+2}}{[\rho^2 + b^2 + (z-h)^2]^{2n+\frac{3}{2}}} \right\} \hat{\phi} \end{aligned} \quad (4.38)$$

If $a = b$ and $I_1 = I_2 = I$ in Equation (4.38), it reduces to

$$\begin{aligned} \mathbf{A}(\rho, z) = & \frac{\mu_0 I \rho a^2}{4} \sum_{n=0}^{\infty} \frac{(4n+1)!! (\rho a)^{2n}}{2^{2n}(n+1)!n!} \left\{ \frac{1}{[\rho^2 + a^2 + (z+h)^2]^{2n+\frac{3}{2}}} + \right. \\ & \left. \frac{1}{[\rho^2 + a^2 + (z-h)^2]^{2n+\frac{3}{2}}} \right\} \hat{\phi} \end{aligned} \quad (4.39)$$

Likewise, if $a = b$ and $I_1 = -I_2 = I$ in Equation (4.38) then it reduces to

$$\mathbf{A}(\rho, z) = \frac{\mu_0 I \rho a^2}{4} \sum_{n=0}^{\infty} \frac{(4n+1)!! (\rho a)^{2n}}{2^{2n} (n+1)! n!} \left\{ \frac{1}{[\rho^2 + a^2 + (z+h)^2]^{2n+\frac{3}{2}}} - \frac{1}{[\rho^2 + a^2 + (z-h)^2]^{2n+\frac{3}{2}}} \right\} \hat{\phi} \quad (4.40)$$

Notice, that for points on the z axis, $\mathbf{A}(0, z) = 0$. This is exactly what should occur. The magnetic flux density can easily be computed by taking the *curl* of Equation (4.39) or Equation (4.40). The magnetic flux density for a Helmholtz coil is computed in chapter 5. One will find that the B_ρ component on the z axis is zero, but the B_z component is nonzero.

4.6 Various geometries for which Q-functions are applicable

This section introduces a few geometrical configurations for which the Q-function is directly applicable. The mathematics which describes each particular problem will not be formulated, but a basic description of how to apply the toroidal expansion will be given. Some of the configurations involve geometries that are mathematically more suited for a non-cylindrical coordinate system, but can still be formulated using a toroidal expansion. The reason for using a Q-function representation for these types of problems becomes clear when one considers its numerical properties. The Q-function is a monotonically decreasing function, and it is therefore extremely well-suited to numerical analysis.

4.6.1 A frustum of a cone

Consider a conductor in the shape of a frustum of a cone as shown in Figure 4.6 . This geometry is handled quite easily by the toroidal expansion. If a current density, $\mathbf{J}$, is flowing in the conductor and one wishes to compute the magnetic flux density external to the conductor, then a toroidal expansion can easily be applied. For this geometry, unlike that of a circular cylinder, there is a geometric variation in the ρ -direction.

Figure 4.6: A hollow frustum

4.6.2 A spheroid and a sphere

Consider, for example, an oblate spheroidal surface. This is a surface of revolution obtained by rotating an ellipse about its minor axis as shown in Figure 4.7 where $a > c$. For an oblate spheroid with the z axis as the symmetry axis, its equation is

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1$$

If Laplace's equation, or its integral counterpart, needed to be solved for an oblate spheroidal geometry, then oblate spheroidal coordinates would be the most appropriate. However, one could actually use a toroidal expansion to solve the problem. Since the toroidal expansion is highly convergent, it may be advantageous to use the Q-function approach. For example, if one wanted to know the potential everywhere external to a hollow oblate spheroidal conductor raised to some potential V_0 , then one could consider the oblate spheroid built from a series of thin circular rings. These rings, for an oblate spheroid, would be centered about the z -axis as shown in Figure 4.7. Of course, a sphere is just a degenerate form of a spheroid and can be solved using the same reasoning. In fact, solving Laplace's equation on a spherical surface in this way is one method for checking the validity of the toroidal

Figure 4.7: Oblate spheroid

expansion.

The charge simulation method, discussed in chapter 7, allows one to solve many problems which do not exhibit circular cylindrical symmetry. This extends the usefulness of the toroidal expansion.

4.7 Summary

In this chapter, a number of useful cylindrical geometries were considered. These geometries form the basic “building blocks” for many types of electromagnetic systems which exhibit circular cylindrical symmetry. Whether one is interested in computing the magnetic field from a cylindrical conductor, hollow or solid, or whether one is interested in designing and optimizing circular coil arrangements with rectangular cross-sections, this chapter gives the basic geometries which can be used to fit the model of a specific problem. This is, by no means, an exhaustive treatment of all the possible geometries that can be tackled with Q-functions. However, examples in this chapter can be used to help model geometries, which upon first glance, may not seem applicable to a Q-function representation.

CHAPTER 5

VALIDATION OF THE TOROIDAL EXPANSION—SIMPLE EXAMPLES

5.1 Introduction

This chapter deals with the application of Q-functions and the corresponding toroidal expansion. The goal will be to solve some familiar problems in electromagnetics and then to compare the solutions with those that were found by using more familiar methods. It is meant only to be an introduction and not an exhaustive treatment in the application of the Q-function. Specifically chosen problems will illustrate the various ways one can quickly formulate electrostatic and magnetostatic problems. Also, it will give a brief treatment of the simple, but nonlinear pendulum problem in order to illustrate its usefulness in numerical analysis.

5.2 Two electric point charges

A simple application of Q-functions involves the calculation of the electric potential for a discrete point charge distribution. One can consider any number of point charges, but for simplicity only two point charges will be used. These charges will be located on the x -axis as shown in Figure 5.1. The charges are equal in magnitude, but opposite in sign. This is a nice example for illustrating that the Q-functions allow for an alternative description to the solution of cylindrically symmetric systems. If one wishes to find the potential of this system when $R \gg a$, the dipole field, it will be advantageous to first look at the spherical harmonic solution. The potential at the field point, in spherical coordinates, can be written as

$$\Phi_P(r, \theta, \phi) = \frac{q}{4\pi\epsilon_o} \left[\frac{1}{\sqrt{R^2 + a^2 - 2aR \cos(\gamma_1)}} - \frac{1}{\sqrt{R^2 + a^2 + 2aR \cos(\gamma_2)}} \right] \quad (5.1)$$

Figure 5.1: Two electric point charges

For $R > a$, Equation (5.1) can be written

$$\Phi_P(r, \theta, \phi) = \frac{q}{4\pi\epsilon_o R} \sum_{l=0}^{\infty} \left(\frac{a}{R}\right)^l \left[P_l[\cos(\gamma_1)] - P_l[\cos(\gamma_2)] \right] \quad (5.2)$$

where γ_1 is the angle between vectors $\mathbf{a} = a\hat{i}$ and $\mathbf{R}$, and γ_2 is the angle between vectors $\mathbf{a} = -a\hat{i}$ and $\mathbf{R}$. One can write Equation (5.2), in terms of spherical harmonics, as

$$\Phi_P(r, \theta, \phi) = \frac{q}{4\pi\epsilon_o R} \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \left(\frac{a}{R}\right)^l \sum_{m=-l}^l \left\{ Y_{lm}^*\left(\frac{\pi}{2}, 0\right) - Y_{lm}^*\left(\frac{\pi}{2}, \pi\right) \right\} Y_{lm}(\theta, \phi) \quad (5.3)$$

Equation (5.3) can be expanded to give the following:

$$\begin{aligned} \Phi_P(r, \theta, \phi) = & \frac{q}{4\pi\epsilon_o R} \left\{ 2 \left(\frac{a}{R}\right) \sin(\theta) \cos(\phi) + \frac{1}{4} \left(\frac{a}{R}\right)^3 \sin(\theta) \cos(\phi) \cdot \right. \\ & \left. \left(10 \sin^2(\theta) \cos(2\phi) - 5 \cos(2\theta) - 7 \right) + \dots \right\} \end{aligned} \quad (5.4)$$

The dipole potential is given by

$$\Phi_P(r, \theta, \phi)^{dipole} = \frac{p}{4\pi\epsilon_o R^2} \sin(\theta) \cos(\phi) \quad (5.5)$$

or

$$\Phi_P(r, \theta, \phi)^{dipole} = \frac{\mathbf{p} \cdot \mathbf{R}}{4\pi\epsilon_0 R^3} \quad (5.6)$$

where the dipole moment, $\mathbf{p}$, is

$$\mathbf{p} = 2qa\hat{x} \quad (5.7)$$

Applying Q-functions to this problem, one can write the potential as

$$\Phi_P(\rho, \phi, z) = \frac{q}{4\pi^2\epsilon_0} \frac{1}{\sqrt{\rho a}} \sum_{m=0}^{\infty} \varepsilon_m [1 - (-1)^m] Q_{m-\frac{1}{2}} \left(\frac{\rho^2 + a^2 + z^2}{2a\rho} \right) \cos(m\phi) \quad (5.8)$$

Only odd m 's contribute to the sum in Equation (5.8). The resulting equation for the potential becomes

$$\Phi_P(\rho, \phi, z) = \frac{q}{\pi^2\epsilon_0} \frac{1}{\sqrt{\rho a}} \sum_{l=0}^{\infty} Q_{2l+\frac{1}{2}} \left(\frac{\rho^2 + a^2 + z^2}{2a\rho} \right) \cos[(2l+1)\phi] \quad (5.9)$$

This can be expanded to yield:

$$\begin{aligned} \Phi_P(\rho, \phi, z) = & \frac{q}{\pi^2\epsilon_0} \frac{1}{\sqrt{\rho a}} \left\{ Q_{\frac{1}{2}} \left(\frac{\rho^2 + a^2 + z^2}{2a\rho} \right) \cos(\phi) + \right. \\ & Q_{\frac{5}{2}} \left(\frac{\rho^2 + a^2 + z^2}{2a\rho} \right) \cos(3\phi) + Q_{\frac{9}{2}} \left(\frac{\rho^2 + a^2 + z^2}{2a\rho} \right) \cos(5\phi) + \\ & \left. Q_{\frac{13}{2}} \left(\frac{\rho^2 + a^2 + z^2}{2a\rho} \right) \cos(7\phi) + \dots \right\} \end{aligned} \quad (5.10)$$

Consider just the first term in the expansion, namely

$$\begin{aligned} \Phi_P(\rho, \phi, z)^{(l=0)} &= \frac{q}{\pi^2\epsilon_0} \frac{1}{\sqrt{\rho a}} Q_{\frac{1}{2}} \left(\frac{\rho^2 + a^2 + z^2}{2a\rho} \right) \cos(\phi) \\ &= \frac{q}{2\pi\epsilon_0} \frac{\rho a}{(\rho^2 + a^2 + z^2)^{\frac{3}{2}}} \sum_{n=0}^{\infty} \frac{(4n+1)!!}{2^{2n} (n+1)!n!} \left(\frac{\rho a}{\rho^2 + a^2 + z^2} \right)^{2n} \cos(\phi) \\ &= \frac{q}{2\pi\epsilon_0} \frac{\rho a}{(\rho^2 + a^2 + z^2)^{\frac{3}{2}}} \left\{ 1 + \frac{1 \cdot 3 \cdot 5}{2^2(2!)(1!)} \left(\frac{\rho a}{\rho^2 + a^2 + z^2} \right)^2 + \right. \\ & \quad \left. \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2^3(3!)(2!)} \left(\frac{\rho a}{\rho^2 + a^2 + z^2} \right)^4 + \dots \right\} \cos(\phi) \end{aligned} \quad (5.11)$$

Substitute the following equations

$$\rho = R \sin(\theta) \quad (5.12)$$

$$z = R \cos(\theta) \quad (5.13)$$

$$R^2 = \rho^2 + z^2 \quad (5.14)$$

into Equation (5.11) and then expand for $R \gg a$ to obtain

$$\begin{aligned} \Phi_P(r, \theta, \phi)^{(l=0)} &= \frac{q}{2\pi\epsilon_0} \frac{Ra \sin(\theta)}{(R^2 + a^2)^{\frac{3}{2}}} \left\{ 1 + \frac{1 \cdot 3 \cdot 5}{2^2(2!)(1!)} \frac{R^2 a^2 \sin^2(\theta)}{(R^2 + z^2)^2} + \right. \\ &\quad \left. \frac{R^4 a^4 \sin^4(\theta)}{(R^2 + z^2)^4} + \dots \right\} \cos(\phi) \end{aligned} \quad (5.15)$$

$$\begin{aligned} \Phi_P(r, \theta, \phi)^{(l=0)} &\simeq \frac{q}{2\pi\epsilon_0} \frac{Ra \sin(\theta) \cos(\phi)}{(R^2 + a^2)^{\frac{3}{2}}} \text{ for } R \gg a \\ &= \frac{p \sin(\theta) \cos(\phi)}{4\pi\epsilon_0 R^2} \left[1 + \left(\frac{a}{R} \right)^2 \right]^{-\frac{3}{2}} \\ &\simeq \frac{p \sin(\theta) \cos(\phi)}{4\pi\epsilon_0 R^2} \text{ for } R \gg a \\ \Phi_P(r, \theta, \phi)^{dipole} &= \frac{\mathbf{p} \cdot \mathbf{R}}{4\pi\epsilon_0 R^3} \end{aligned} \quad (5.16)$$

After some algebra, one can see that through the use of the Q-functions the same dipole potential is reached. This is, of course, what should happen. This example was used to illustrate the validity of the toroidal expansion. If one were just interested in the far field then writing the potential in terms of spherical harmonics would yield the dipole potential rather quickly. However, the use of the Q-functions gives an alternative way of approaching the problem and it allows one to find the potential close to the charge distribution using fewer terms than a spherical harmonic expansion (see *Appendix L for a more in depth numerical study*). Also, Equation (5.9) is valid at any point in space not coincident with either point charge. However, the spherical harmonic solution is valid only for $R > a$. One needs to write another expansion valid for $R < a$. In other words, the spherical harmonic solution requires two separate expansions to describe the potential at all points in space not coincident with the charges.

Figure 5.2: Electric scalar potential from two point charges

What does the potential look like on a hypothetical cylinder enclosing the two charges? For example, let $a = 0.4 \text{ m}$, and let the observation cylinder have a radius of 0.6 m and a height of 0.6 m . Also, let $\frac{q}{4\pi\epsilon_0} = 1$ for simplicity. Figure 5.2 represents a plot of the total potential as seen on the observation cylinder enclosing the two point charges. Also, Figure 5.3 represents the $m = 1$ contribution to the total electric scalar potential. Figure 5.4 represents the $m = 3$ contribution to the total electric scalar potential. Figure 5.5 represents the $m = 5$ contribution to the total electric scalar potential. The $m = 1$ contribution is about 80% of the total. One can see that the $\frac{q}{\pi^2\epsilon_0} \frac{1}{\sqrt{\rho a}} Q_{\frac{1}{2}} \left(\frac{\rho^2 + a^2 + z^2}{2a\rho} \right) \cos(\phi)$ contribution, which incorporates the spherical dipole moment, is the dominant contribution to the total potential even at distances which are relatively close to the field points.

5.3 Circular current loop with applied uniform current source

A current loop lying in the $x - y$ plane and centered at the origin is carrying a uniform current. The objective will be to compute the magnetic vector potential at some arbitrary field point not coincident with the current loop. It is advantageous

Figure 5.3: The $m=1$ contribution to the electric scalar potential from two point charges

Figure 5.4: The $m=3$ contribution to the electric scalar potential from two point charges

Figure 5.5: The $m=5$ contribution to the electric scalar potential from two point charges

to derive from first principles the magnetic vector potential from a circular current loop situated on the $x - y$ plane as shown in Figure 5.6.

The differential magnetic vector potential at the field point, P , is given by

$$d\mathbf{A}_P = \frac{\mu_0 I}{4\pi} \frac{a d\phi'}{|\mathbf{R} - \mathbf{a}|} \hat{\phi}' \quad (5.17)$$

where

$$\frac{1}{|\mathbf{R} - \mathbf{a}|} = \frac{1}{\sqrt{\rho^2 + a^2 + z^2 - 2a\rho \cos(\phi - \phi')}} \quad (5.18)$$

The magnetic vector potential at some arbitrary point in space not coincident with the current loop can be written as

$$\mathbf{A}_P(\rho, \phi, z) = \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \frac{[-\sin(\phi')\hat{x} + \cos(\phi')\hat{y}] d\phi'}{\sqrt{\rho^2 + a^2 + z^2 - 2a\rho \cos(\phi - \phi')}} \quad (5.19)$$

where the unit vector, $\hat{\phi}'$, has been replaced by its Cartesian equivalent. Using

Figure 5.6: Uniform current Loop

Equation (2.11), Equation (5.18) can be written as

$$\frac{1}{|\mathbf{R} - \mathbf{a}|} = \frac{1}{\pi\sqrt{\rho a}} \sum_{m=0}^{\infty} \varepsilon_m Q_{m-\frac{1}{2}}(\xi) \cos \left[m \left(\phi - \phi' \right) \right] \quad (5.20)$$

where $\xi = \frac{\rho^2 + a^2 + z^2}{2a\rho}$. Substituting Equation (5.20) into Equation (5.19) yields

$$\begin{aligned} \mathbf{A}_P(\rho, \phi, z) &= \frac{\mu_0 I a}{4\pi^2} \frac{1}{\sqrt{\rho a}} \sum_{m=0}^{\infty} \varepsilon_m Q_{m-\frac{1}{2}}(\xi) \cdot \\ &\int_0^{2\pi} \left(\left[-\sin(\phi') \hat{x} + \cos(\phi') \hat{y} \right] \cos \left[m \left(\phi - \phi' \right) \right] \right) d\phi' \quad (5.21) \end{aligned}$$

Equation (5.21) is integrated to

$$\mathbf{A}_P(\rho, z) = \frac{\mu_0 I}{2\pi} \sqrt{\frac{a}{\rho}} Q_{\frac{1}{2}}(\xi) \hat{\phi} \quad (5.22)$$

Equation (5.22) gives a mathematically compact form for the magnetic vector potential [23]. Figure 5.7 is a plot of the $|\mathbf{A}|$ on the surface of a hypothetical cylinder of radius of 0.80 m and a length of 0.50 m. The radius of the current loop is 0.75 m and the uniform current is 100 A.

Figure 5.7: Magnetic vector potential from a uniform current Loop

Using Equation (2.21) with $m = 1$, and $\rho' = a$, $Q_{\frac{1}{2}}(\xi)$ becomes

$$Q_{\frac{1}{2}}(\xi) = \frac{\pi}{2} \left(\frac{a\rho}{\rho^2 + a^2 + z^2} \right)^{\frac{3}{2}} \sum_{n=0}^{\infty} \frac{(4n+1)!!}{2^{2n}n!(n+1)!} \left(\frac{a\rho}{\rho^2 + a^2 + z^2} \right)^{2n} \quad (5.23)$$

The magnetic vector potential is now given by

$$\mathbf{A}_P(\rho, z) = \frac{\mu_0 I}{4} \frac{a^2 \rho}{(\rho^2 + a^2 + z^2)^{\frac{3}{2}}} \sum_{n=0}^{\infty} \frac{(4n+1)!!}{2^{2n}n!(n+1)!} \left(\frac{a\rho}{\rho^2 + a^2 + z^2} \right)^{2n} \hat{\phi} \quad (5.24)$$

A few terms in the expansion are

$$\begin{aligned} \mathbf{A}_P(\rho, z) = \frac{\mu_0 I}{4} \frac{a^2 \rho}{(\rho^2 + a^2 + z^2)^{\frac{3}{2}}} & \left\{ 1 + \frac{1 \cdot 3 \cdot 5}{2^2 \cdot 1! \cdot 2!} \left(\frac{a\rho}{\rho^2 + a^2 + z^2} \right)^2 + \right. \\ & \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2^4 \cdot 2! \cdot 3!} \left(\frac{a\rho}{\rho^2 + a^2 + z^2} \right)^4 + \\ & \left. \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13}{2^6 \cdot 3! \cdot 4!} \left(\frac{a\rho}{\rho^2 + a^2 + z^2} \right)^6 + \dots \right\} \hat{\phi} \end{aligned} \quad (5.25)$$

If one substitutes Equations (5.12) through (5.14) into Equation (5.25), the resulting expansion becomes

$$\begin{aligned} \mathbf{A}_P(R, \theta) = \frac{\mu_0 I}{4} \frac{a^2 R \sin(\theta)}{(a^2 + R^2)^{\frac{3}{2}}} \left\{ 1 + \frac{1 \cdot 3 \cdot 5}{2^2 \cdot 1! \cdot 2!} \left(\frac{aR \sin(\theta)}{R^2 + a^2} \right)^2 + \right. \\ \left. \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2^4 \cdot 2! \cdot 3!} \left(\frac{aR \sin(\theta)}{R^2 + a^2} \right)^4 + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13}{2^6 \cdot 3! \cdot 4!} \left(\frac{aR \sin(\theta)}{R^2 + a^2} \right)^6 + \dots \right\} \hat{\phi} \end{aligned} \quad (5.26)$$

Equation (5.26) is identical to the Elliptic integral expansion given by Jackson [10] on page 182 of his text. A direct relationship exists among the complete Elliptic integrals of the first and second kind and the Q-functions[28]. It can be shown that

$$Q_{\frac{1}{2}}(\xi) = \xi \sqrt{\frac{2}{\xi+1}} K \left(\sqrt{\frac{2}{\xi+1}} \right) - \sqrt{2(\xi+1)} E \left(\sqrt{\frac{2}{\xi+1}} \right) \quad (5.27)$$

and

$$Q_{-\frac{1}{2}}(\xi) = \sqrt{\frac{2}{\xi+1}} K \left(\sqrt{\frac{2}{\xi+1}} \right) \quad (5.28)$$

where $K \left(\sqrt{\frac{\xi+1}{2}} \right)$ and $E \left(\sqrt{\frac{\xi+1}{2}} \right)$ are complete Elliptic integrals [60, 64] of the first and second kind respectively. One can obtain a relationship between Q-functions and Elliptic integrals by using the recurrence relationship

$$Q_{m+\frac{3}{2}}(\xi) = 4 \left(\frac{M+1}{2M+3} \right) \xi Q_{M+\frac{1}{2}}(\xi) - \left(\frac{2M+1}{2M+3} \right) Q_{M-\frac{1}{2}}(\xi) \quad (5.29)$$

For example, using Equations (??) and (5.28), one can compute $Q_{\frac{3}{2}}(\xi)$, $Q_{\frac{5}{2}}(\xi)$, ..., $Q_{m+\frac{3}{2}}(\xi)$ in terms of Elliptic integrals (see *Appendix H for details*).

One way to derive the magnetic flux density from a single current loop as shown in Figure 5.4 is to simply take the *curl*, in cylindrical coordinates, of the magnetic vector potential. This yields:

$$\mathbf{B}_P = \nabla \times \mathbf{A}_P = -\frac{\hat{\rho}}{\rho} \frac{\partial}{\partial z} [\rho A_\phi] + \frac{\hat{z}}{\rho} \frac{\partial}{\partial \rho} [\rho A_\phi] \quad (5.30)$$

where

$$A_\phi = \frac{\mu_0 I}{2\pi} \sqrt{\frac{a}{\rho}} Q_{\frac{1}{2}}(\xi) \quad (5.31)$$

Employing Equation (5.31) and Equation(5.30) yields:

$$\mathbf{B}_P(\rho, z) = \frac{\mu_0 I}{2\pi} \sqrt{\frac{a}{\rho}} \left\{ -\frac{\partial}{\partial z} \left[Q_{\frac{1}{2}}(\xi) \right] \hat{\rho} + \left(\frac{1}{2\rho} Q_{\frac{1}{2}}(\xi) + \frac{\partial}{\partial \rho} \left[Q_{\frac{1}{2}}(\xi) \right] \right) \hat{z} \right\} \quad (5.32)$$

The magnetic flux density is written as

$$\mathbf{B}_P = B_\rho \hat{\rho} + B_z \hat{z} \quad (5.33)$$

where

$$B_\rho(\rho, z) = \frac{\mu_0 I a^2}{4} \frac{z\rho}{(\rho^2 + a^2 + z^2)^{\frac{5}{2}}} \sum_{n=0}^{\infty} R_n \left(\frac{a\rho}{\rho^2 + a^2 + z^2} \right)^{2n} \quad (5.34)$$

$$B_z(\rho, z) = \frac{\mu_0 I a^2}{4} \sum_{n=0}^{\infty} (a\rho)^{2n} \left[\frac{2S_n}{(\rho^2 + a^2 + z^2)^{2n+\frac{3}{2}}} - \frac{\rho^2 R_n}{(\rho^2 + a^2 + z^2)^{2n+\frac{5}{2}}} \right] \quad (5.35)$$

and

$$R_n = \frac{(4n+3)!!}{2^{2n} n! (n+1)!} \quad (5.36)$$

$$S_n = \frac{(4n+1)!!}{2^{2n} (n!)^2} \quad (5.37)$$

Equations (5.34) and (5.35) give the components of the magnetic flux density at any point in space from a single current loop centered at the origin and lying on the $x - y$ plane

5.4 Mutual inductance between two non-coplanar and parallel current loops

This example considers the calculation of the mutual inductance between two non-coplanar and parallel current loops [63, 66, 67, 68], shown in Figure 5.8 , by employing the use of the Q-function. Consider first the magnetic vector potential

Figure 5.8: Two coaxial and noncoplanar uniform current loops

given by Equation (5.22) with $\rho = b$, $z = h$, and $\xi_1 = \frac{b^2 + a^2 + h^2}{2ab}$. Also, let $I_1 = I_2 = I$. The mutual inductance between the two current loops, shown in Figure 5.8, is given by

$$M_{12} = \frac{1}{I} \int_0^{2\pi} \mathbf{A}_1 \cdot d\mathbf{l}_2 \quad (5.38)$$

$$\begin{aligned} &= \frac{\mu_0}{2\pi} \sqrt{\frac{a}{b}} Q_{\frac{1}{2}} \left(\frac{b^2 + a^2 + h^2}{2ab} \right) \int_0^{2\pi} b d\phi_2 \\ &= \mu_0 \sqrt{ab} Q_{\frac{1}{2}} \left(\frac{b^2 + a^2 + h^2}{2ab} \right) \end{aligned} \quad (5.39)$$

Employing Equation (5.23) yields:

$$\begin{aligned} M_{12} &= \frac{\pi\mu_0}{2} \frac{a^2 b^2}{(a^2 + b^2 + h^2)^{\frac{3}{2}}} \cdot \\ &\quad \sum_{n=0}^{\infty} \frac{(4n+1)!!}{2^{2n} n! (n+1)!} \left(\frac{ab}{a^2 + b^2 + h^2} \right)^{2n} \end{aligned} \quad (5.40)$$

Equation (5.40) represents the mutual inductance between two non-coplanar and parallel uniform current loops. Consider the case when $b = a$ and $h > 2a$. Imposing

Figure 5.9: Mutual inductance of two current loops

this condition yields:

$$\begin{aligned}
 M_{12} &= \frac{\pi\mu_0}{2} \frac{a^4}{(h^2 + 2a^2)^{\frac{3}{2}}} \left[1 + \frac{1 \cdot 3 \cdot 5}{2^2 1! 2!} \frac{a^4}{(h^2 + 2a^2)^2} + \right. \\
 &\quad \left. \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2^3 2! 3!} \frac{a^6}{(h^2 + 2a^2)^4} + \dots \right] \\
 &= \frac{\pi\mu_0 a}{2} \left[\left(\frac{a}{h} \right)^3 - 3 \left(\frac{a}{h} \right)^5 + \frac{75}{8} \left(\frac{a}{h} \right)^7 + \dots \right] \quad (5.41)
 \end{aligned}$$

Equation (5.41) is identical to the solution given by Jackson [10] in problem 5.34 on page 234 of his text. Figure 5.9 shows the plot of Equation (5.39) for $b = 3.0 \text{ m}$, and with a and h varying. Figure 5.10. is a two dimensional plot for $a = 1.5 \text{ m}$ and $b = 3.0 \text{ m}$ with a varying h .

Figure 5.10: Mutual inductance of two current loops for $a=1.5\text{m}$ and $b=3\text{m}$

5.5 The “electrified disk” problem

The electrified disk problem has peaked the interest of a number of physicists and mathematicians for many years [69, 70, 71, 72]. The basic problem is to find the potential at an arbitrary point in space due to an infinitely thin circular disk which has been charged or *electrified*. The assumption is that no other conductors are nearby so that one does not need to consider induction. This simplification, by no means, trivializes the problem.

This seemingly simple problem exhibits enough mathematical complexity to make it an interesting problem to discuss. It was first solved by the famous George Green [71] in 1838, and then by the Wilhelm Weber in 1873. Also Lord Kelvin solved the same problem using a limiting procedure for the solution of Laplace’s equation in ellipsoidal coordinates. Copson [69] solved the same problem directly by using complex variable theory. Copson’s method, which will not be discussed in

Figure 5.11: Electrified Disk

this thesis, requires a more thorough theoretical understanding of integral equations. The method discussed in this thesis gives a completely new way of looking at this problem which relies on the Q -function.

Consider an infinitely thin conducting disk of radius, a , with an impressed voltage, V_0 , as shown in Figure 5.11. How does one compute the potential at an arbitrary point in space? One way to solve this problem using an integral formulation requires a knowledge of the charge density on the disk. The charge density is certainly not constant. In fact, it has been shown that the charge density on the disk is given by

$$\sigma(\rho) = \frac{q}{4\pi a \sqrt{a^2 - \rho^2}} \quad (5.42)$$

where q is the total charge on the disk, and $\sigma(\rho)$ is the charge density on one side of the disk. Some of the more recent sources such as Friedberg [73], Eyges [74], Davis and Reitz [75], and Griffiths and Li [76] have all made contributions to the conducting disk or conducting needle problem. Knowing the charge density allows one to immediately formulate the differential electric scalar potential. This is given by

$$d\Phi_P(\rho, \phi, z) = \frac{2q}{16\pi^2 \epsilon_0 a} \int_0^{2\pi} \int_0^a \frac{\rho' d\rho' d\phi'}{\sqrt{a^2 - \rho'^2} \sqrt{\rho^2 + a^2 + z^2 - 2a\rho \cos(\phi - \phi')}} \quad (5.43)$$

where the factor of 2 in front of the charge, q , implies that one must consider both sides of the disk. Also, employing Equation (2.11), Equation (2.21), and then integrating Equation (5.43), one obtains

$$\begin{aligned}
\Phi_P(\rho, \phi, z) &= \frac{2q}{16\pi^3\epsilon_0 a \sqrt{\rho}} \sum_{m=0}^{\infty} \varepsilon_m \int_0^{2\pi} \int_0^a \frac{Q_{m-\frac{1}{2}}(\beta) \sqrt{\rho'} \cos[m(\phi - \phi')]}{\sqrt{a^2 - \rho^2}} d\rho' d\phi' \\
&= \frac{q}{8\pi^2\epsilon_0 a} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varepsilon_m \frac{(4n+2m-1)!!(\rho)^{2n+m}}{2^{2n+m} (n+m)!n!} \cdot \\
&\quad \int_0^{2\pi} \int_0^a \frac{(\rho')^{2n+m+1} \cos[m(\phi - \phi')]}{\sqrt{a^2 - \rho^2} (\rho^2 + \rho'^2 + z^2)^{2n+m+\frac{1}{2}}} d\rho' d\phi'
\end{aligned} \tag{5.44}$$

where P is the location of the field point. The $m = 0$ term is the only term which survives the integration in Equation (5.44).

$$\Phi_P(\rho, z) = \frac{q}{4\pi\epsilon_0 a} \sum_{n=0}^{\infty} \frac{(4n-1)!!(\rho)^{2n}}{2^{2n} (n!)^2} \int_0^a \frac{(\rho')^{2n+1} d\rho'}{\sqrt{a^2 - \rho^2} (\rho^2 + \rho'^2 + z^2)^{2n+\frac{1}{2}}} \tag{5.45}$$

Evaluating the integral in Equation (5.45) yields:

$$\begin{aligned}
\Phi_P(\rho, z) &= \frac{q}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{(4n-1)!!}{(2n+1)!!2^n n!} \frac{(\rho a)^{2n}}{(\rho^2 + z^2)^{2n+\frac{1}{2}}} \\
&\quad F\left(1+n, \frac{1}{2}+2n; \frac{3}{2}+n; -\frac{a^2}{\rho^2+z^2}\right)
\end{aligned} \tag{5.46}$$

where $F\left(1+n, \frac{1}{2}+2n; \frac{3}{2}+n; -\frac{a^2}{\rho^2+z^2}\right)$ is defined by Equation (2.14). Equation (5.46) simplifies for points on the z axis. In other words as $\rho \rightarrow 0$ in Equation (5.46), the potential reduces to

$$\Phi_P(\rho, z) = \frac{q}{4\pi\epsilon_0} \left[\frac{\pi}{2} + \tan^{-1} \left(\frac{a^2 - z^2}{2a|z|} \right) \right] \tag{5.47}$$

and as $z \rightarrow 0$, Equation (5.47) reduces to

$$\Phi(0, 0) = V_0 = \frac{q}{8\pi\epsilon_0} \tag{5.48}$$

where the capacitance is $C = 8\pi\epsilon_0$. This is a well known result. Using Equation

(5.48), one can write Equation (5.46) as

$$\Phi_P(\rho, z) = \frac{2aV_0}{\pi} \sum_{n=0}^{\infty} \frac{(4n-1)!!}{(2n+1)!!2^n n!} \frac{(\rho a)^{2n}}{(\rho^2 + z^2)^{2n+\frac{1}{2}}} \cdot F\left(1+n, \frac{1}{2}+2n; \frac{3}{2}+n; -\frac{a^2}{\rho^2 + z^2}\right) \quad (5.49)$$

Equation (5.49) gives the electric potential at an arbitrary point in space. In fact, one can find a closed form solution in terms of elementary functions [19, 72]. It is given by

$$\Phi_P(\rho, z)_{closed\ form} = \frac{2V_0}{\pi} \tan^{-1} [\Theta(\rho, z, a)] \quad (5.50)$$

where

$$\Theta(\rho, z, a) = a\sqrt{2} \left(\rho^2 + z^2 - a^2 + \sqrt{(\rho^2 + z^2 - a^2)^2 + 4a^2 z^2} \right)^{-\frac{1}{2}} \quad (5.51)$$

One can now deduce that

$$\tan^{-1} [\Theta(\rho, z, a)] = a \sum_{n=0}^{\infty} \frac{(4n-1)!!}{(2n+1)!!2^n n!} \frac{(\rho a)^{2n}}{(\rho^2 + z^2)^{2n+\frac{1}{2}}} \cdot F\left(1+n, \frac{1}{2}+2n; \frac{3}{2}+n; -\frac{a^2}{\rho^2 + z^2}\right) \quad (5.52)$$

The author believes that Equation (5.46) or Equation (5.49) represents a new and alternate formulation of the potential from an electrified disk, and that Equation (5.52) gives the series expansion for the inverse tangent function for argument $\Theta(\rho, z, a)$. Figure 5.12. shows a plot of either Equation (5.49) or Equation (5.50) for $V_0 = 100$ V and $a = 0.5$ m. Figure 5.13 represents a plot of the total electric scalar potential, due to the electrified disk, computed on an observation cylinder with a radius of 0.6 m and a length of 1.2 m. Figure 5.14 represents the $m = 0$ contribution to the total electric scalar potential. One can see that the $m = 0$ contribution looks identical to the total scalar potential. This is what should be expected since the electrified disk has a non-zero total charge and therefore a monopole distribution must dominate the potential.

The importance of this section does not lie in the development of a new for-

Figure 5.12: The electric scalar potential due to an electrified disk of radius 0.5m at a potential of 100V

mulation for the electrified disk, but in the fact that the Q-function technique is applicable to three-dimensional problems. The electrified disk is a two dimensional problem, and therefore one can not, in general, use the results of an infinitely thin disk to make predictions about its three dimensional finite cylindrical counterpart. The author's interest ultimately lies, for this particular problem, in finding the charge density of a finite cylinder by using the Q-function formulation. Unfortunately, to date, he has not had success. The method requires one to solve a complicated integral equation, with a Q-function kernel, for the unknown charge density. However, the fact that this formulation has, to the best of his knowledge, never been attempted allows for a new avenue of discovery.

Smythe wrote a number of excellent papers [77, 78, 79, 80, 81] attempting to solve the three dimensional cylindrical problem and much more, and there has been renewed interest in computing charge densities on various geometries [73, 76, 80, 83,

Figure 5.13: Total electric scalar potential computed on the observation cylinder due to an electrified disk

84]. Ferguson and Duncan [85] solve for the charge density on a finite and hollow cylindrical tube without end caps. Also, C. Chakraborty, Poddar, A. Chakraborty and Das [86] compute the charge density and capacitance of isolated cylinders and cones in free space using a moment method formulation. Their technique is strictly a numerical one. All the techniques, to date, have not succeeded in finding a closed form expression for the charge density on a cylinder. Smythe's work appears to be the standard by which all other methods are measured, unfortunately, in "real world" electrostatic problems, conductors are usually not isolated from one another which further complicates the problem.

The author is constantly reminded of the words that James Clerk Maxwell wrote for a series of lectures [87] that he gave at the Cavendish Laboratory from around 1874 to 1879. *"The calculation of the distribution of electrification on the surface of a conductor when electrified bodies are placed near it is in general an*

Figure 5.14: The $m=0$ contribution to the electric scalar potential computed on the observation cylinder due to an electrified disk

operation beyond the powers of existing mathematical methods.” This statement can be quite humbling, and it is as true today as it was in Maxwell’s day.

The use of the integral formulation with a Q-function kernel, for electrostatic problems, suffers from the same mathematical difficulties as other methods; one needs to know a charge density and finding this may not be possible except by numerical means. Fortunately, magnetostatic problems do not suffer from this requirement and the full power of the Q-function and the corresponding toroidal expansion can be brought to bear.

5.6 The simple but nonlinear pendulum

The simple, but nonlinear pendulum is a nice example for comparing the Elliptic integral formulation to the Q-function formulation. Figure 5.15 shows a

Figure 5.15: Nonlinear pendulum

nonlinear pendulum. It is assumed that the mass of the bob is much greater than the mass of the inflexible chord which connects it to the point O . Damping is assumed to be negligible. The differential equation which describes the motion of the pendulum for all angles $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ is given by

$$I \frac{d^2 \theta}{dt^2} + mgL \sin(\theta) = 0 \quad (5.53)$$

where the moment of inertia, I , is given by

$$I = mL^2 \quad (5.54)$$

One can now write the period of motion of this pendulum as

$$T(\theta_0) = 4 \sqrt{\frac{L}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{2 [\cos(\theta) - \cos(\theta_0)]}} \quad (5.55)$$

Figure 5.16: Period of a nonlinear pendulum

Making the substitutions, $\cos(\theta) = 1 - 2\sin^2(\frac{\theta}{2})$ and $\sin(\frac{\theta}{2}) = \sin(\frac{\theta_0}{2})\sin(\phi)$, reduces Equation (5.55) to

$$\begin{aligned} T(\theta_0) &= 4\sqrt{\frac{L}{g}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \sin^2(\frac{\theta_0}{2})\sin^2(\frac{\phi}{2})}} \\ &= 4\sqrt{\frac{L}{g}} K(\zeta) \end{aligned} \quad (5.56)$$

where $\zeta = \sin(\frac{\theta_0}{2})$. Equation (5.56) is the equation given in most texts on dynamics. One can also write Equation (5.56) as

$$T(\zeta) = 2\pi\sqrt{\frac{L}{g}} \sum_{n=0}^{\infty} \frac{[(2n-1)!!]^2}{2^n n! (2n)!!} \zeta^{2n} \quad (5.57)$$

Figure 5.16 gives the plot of the period of the pendulum for $\theta_0 = \frac{\pi}{4}$ and $L = 0.5$ m in terms of an Elliptic integral of the first kind. Using the relationships found in

n	$T_{Q-function}$	$T_{Elliptic\ Integral}$
0	1.4742267269	1.4192268951
1	1.4759522231	1.4711871368
2	1.4759581136	1.4754674250
3	1.4759581389	1.4759027262
4	1.4759581390	1.4759515336
5	1.4759581390	1.4759573232
6	1.4759581390	1.4759580356
7	1.4759581390	1.4759581256
8	1.4759581390	1.4759581372
9	1.4759581390	1.4759581387
10	1.4759581390	1.4759581389

Table 5.1: Numerical values for the period of a nonlinear pendulum

Appendix H, one can rewrite Equation (5.56) in terms of Q-functions. This is given by

$$T(\zeta) = 4\sqrt{\frac{L}{g}} \frac{1}{\zeta} Q_{-\frac{1}{2}}\left(\frac{2-\zeta^2}{\zeta^2}\right) \quad (5.58)$$

or

$$T(\zeta) = 4\pi\sqrt{\frac{L}{2g(2-\zeta^2)}} \sum_{n=0}^{\infty} \frac{(4n-1)!!}{2^{4n}(n!)^2} \left(\frac{\zeta^2}{2-\zeta^2}\right)^{2n} \quad (5.59)$$

The plot of the period is the same as Figure 5.16. However, it takes around three times as many terms using the Elliptic integral formulation as it does in using the Q-function formulation in order to arrive at the same period with an eight decimal point accuracy. This is shown in Table 5.1. One can easily compute the $n = 0$ term using the Q-function formulation. It is about 99.9% of the actual period, and the $n = 0$ term using the Elliptic integral formulation is about 96.2% of the actual period. One may also recognize that the $n = 0$ term for the Elliptic integral formulation yields the period of a linear pendulum give by

$$T = 2\pi\sqrt{\frac{L}{g}} \quad (5.60)$$

where as the $n = 0$ term for the Q-function formulation yields:

$$T = 4\pi \sqrt{\frac{L}{2g(2 - \zeta^2)}} \quad (5.61)$$

How does one reconcile this? Remember, the $n = 0$ term represents an approximation to the true period of a nonlinear pendulum. For the elliptic integral formulation, the $n = 0$ term exactly represents the solution to the simple linear pendulum. However, the $n = 0$ term for the Q-function formulation represents the first term of the nonlinear pendulum. In other words, to recover the solution to the simple linear pendulum, one must take the limit as θ_0 goes to zero in Equation (5.61). That is

$$\begin{aligned} T &= \lim_{\theta_0 \rightarrow 0} 4\pi \sqrt{\frac{L}{2g[2 - \sin^2(\frac{\theta_0}{2})]}} \\ &= 4\pi \sqrt{\frac{L}{4g}} \\ &= 2\pi \sqrt{\frac{L}{g}} \end{aligned} \quad (5.62)$$

What does this say about the two formulations? If one wants to approximate the period of a real pendulum, employing the Q-function formulation will guarantee that you compute a number which more closely resembles its true value. In other words, one can use Equation (5.61) to approximate the true period of a real pendulum. If, for example, $\theta_0 = \frac{\pi}{3}$, then Equation (5.61) yields a value for the period which is 99.6% of the true value. The first term in the Elliptic integral formulation yields a period which is 93.2% of the true value.

Although the relationship among the Q-functions and the Elliptic integrals of the first and the second kind have been known for over a century, one will find very few references which take advantage of the Q-function's analytical and numerical properties.

5.7 Summary

This chapter shows how the Q-function can be used to solve problems that have been previously solved by methods that are more familiar. A series of examples were carefully chosen in order to clearly illustrate the mechanics of applying the Q-function to geometries which exhibit cylindrical symmetry. Although, the example problems chosen are in no way exhaustive in scope, it is hoped that the solutions were given in enough detail to allow the reader to apply the Q-function to a much wider range of applications. This approach to solving cylindrically symmetric systems may yield new results, as well as a new way to look at old problems. For example, the solution to the well-known electrified disk problem yielded previously unknown formulas, and the pendulum problem gives a glimpse of the excellent numerical properties of the Q-function.

This chapter does not give detailed mathematical proofs on properties of the Q-function, but only attempts to justify its use by comparing it to other known methods. However, Snow [24] gives a thorough mathematical analysis of the Q-function. His text gives a detailed analysis of a number of special functions using complex variable theory, and he derives a plethora of useful addition formulas that can not be found in other texts.

CHAPTER 6

SPHERICAL MULTIPOLES FROM A TOROIDAL EXPANSION

6.1 Introduction

The toroidal expansion has been shown to be a practical alternative to the more familiar spherical harmonic expansion and the Elliptic integral formulation. It represents an alternative method for computing the electrostatic and magnetostatic field from a system which exhibits cylindrical symmetry. It also exhibits certain properties which allow one to quickly find the equivalent spherical multipoles. Knowing the analytical form of the free-space Green's function allows for a direct comparison between a spherical multipole expansion and a toroidal function expansion. It enables one to know which toroidal function yields a monopole term, a dipole term, a quadrupole term, and the remaining 2^n -pole terms.

The physics of the problem will dictate which Q-functions may or may not contribute to the solution. The toroidal functions or the Q-functions, which appear in the solution, will have built within it a series of spherical multipoles each of which incorporates an infinite number of specific spherical multipoles, but only the first two Q-functions allow for a monopole and a dipole, respectively. A reference table is developed which emphasizes the correlation between a specific spherical multipole and a specific toroidal function. This table can be used as a qualitative tool for making predictions about the spherical multipole moments.

6.2 Derivation of a spherical multipole expansion from a given toroidal expansion

Chapter 2 introduced the reader to the toroidal expansion, and in particular, the Q-function. Equation (2.21) represented the Q-functions which are the weighting coefficients in the toroidal expansion. In order to derive the spherical multipole expansion from a given toroidal expansion, it is advantageous to rewrite Equation

Figure 6.1: Relationship between spherical and cylindrical coordinates

(2.21) as

$$\frac{1}{\pi \sqrt{\rho \rho'}} Q_{m-\frac{1}{2}}(\rho, z) = \frac{(\rho \rho')^m}{2^m (\rho^2 + \rho'^2 + z^2)^{m+\frac{1}{2}}} \cdot \sum_{n=0}^{\infty} \frac{(4n+2m-1)!!}{2^{2n} (n+m)! n!} \frac{(\rho \rho')^{2n}}{(\rho^2 + \rho'^2 + z^2)^{2n}} \quad (6.1)$$

where the field variables are (ρ, ϕ, z) and source variables are (ρ', ϕ', z') .

Figure 6.1 illustrates the coordinate relationship between spherical coordinates and cylindrical coordinates. The toroidal expansion written in terms of spherical coordinates allows for a direct comparison between the toroidal expansion and the spherical harmonic expansion (see *Appendix K for details*). However, without loss of generality, one can derive the pertinent features of this relationship by choosing an appropriate example.

Consider, for example, an infinitely thin circular current loop of radius, a , lying on the $x - y$ plane and carrying a uniform current, I_ϕ , as shown in Figure 6.2.

This represents the case in which $\rho' = a$ in Equation (6.1). Substituting the

Figure 6.2: Infinitely thin current loop

following coordinate system transformation equations:

$$\rho = R \sin(\theta) \quad (6.2)$$

$$z = R \cos(\theta) \quad (6.3)$$

$$R^2 = \rho^2 + z^2 \quad (6.4)$$

into Equation (6.1) yields, for all $m \geq 0$, the following series expansions:

$$\begin{aligned} \frac{1}{\pi \sqrt{\rho a}} Q_{-\frac{1}{2}}(\rho, z) &= \frac{1}{\sqrt{R^2 + a^2}} \left[1 + \frac{3}{4} \frac{R^2 a^2 \sin^2(\theta)}{(R^2 + a^2)^2} + \right. \\ &\quad \left. \frac{105}{64} \frac{R^4 a^4 \sin^4(\theta)}{(R^2 + a^2)^4} + \dots \right] \end{aligned} \quad (6.5)$$

$$\begin{aligned} \frac{1}{\pi \sqrt{\rho a}} Q_{\frac{1}{2}}(\rho, z) &= \frac{R a \sin(\theta)}{2 (R^2 + a^2)^{\frac{3}{2}}} \left[1 + \frac{15}{8} \frac{R^2 a^2 \sin^2(\theta)}{(R^2 + a^2)^2} + \right. \\ &\quad \left. \frac{945}{192} \frac{R^4 a^4 \sin^4(\theta)}{(R^2 + a^2)^4} + \dots \right] \end{aligned} \quad (6.6)$$

$$\frac{1}{\pi\sqrt{\rho a}}Q_{\frac{3}{2}}(\rho, z) = \frac{R^2 a^2 \sin^2(\theta)}{4(R^2 + a^2)^{\frac{5}{2}}} \left[\frac{3}{2} + \frac{105}{24} \frac{R^2 a^2 \sin^2(\theta)}{(R^2 + a^2)^2} + \frac{3465}{256} \frac{R^4 a^4 \sin^4(\theta)}{(R^2 + a^2)^4} + \dots \right] \quad (6.7)$$

$$\begin{aligned} & \vdots \\ \frac{1}{\pi\sqrt{\rho a}}Q_{M-\frac{1}{2}}(\rho, z) &= \frac{(Ra \sin(\theta))^M}{2^M (R^2 + a^2)^{M+\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(4n + 2M - 1)!!}{2^{2n+M} (n + M)! n!} \cdot \left(\frac{Ra \sin(\theta)}{R^2 + a^2} \right)^{2n} \end{aligned} \quad (6.8)$$

For large R , Equations (6.5) through (6.8) yield:

$$\frac{1}{\pi\sqrt{\rho a}}Q_{-\frac{1}{2}}(\rho, z) \implies \frac{1}{R}, \frac{1}{R^3}, \frac{1}{R^5} \cdots \frac{1}{R^{odd}} \quad (6.9)$$

$$\frac{1}{\pi\sqrt{\rho a}}Q_{\frac{1}{2}}(\rho, z) \implies \frac{1}{R^2}, \frac{1}{R^4}, \frac{1}{R^6} \cdots \frac{1}{R^{even}} \quad (6.10)$$

$$\frac{1}{\pi\sqrt{\rho a}}Q_{\frac{3}{2}}(\rho, z) \implies \frac{1}{R^3}, \frac{1}{R^5}, \frac{1}{R^7} \cdots \frac{1}{R^{odd}} \quad (6.11)$$

$$\begin{aligned} & \vdots \\ \frac{1}{\pi\sqrt{\rho a}}Q_{M-\frac{1}{2}}(\rho, z) &\implies \frac{1}{R^{M+1}}, \frac{1}{R^{M+3}}, \frac{1}{R^{M+5}}, \dots \end{aligned} \quad (6.12)$$

The right hand sides of relationships (6.9) through (6.12) represent which spherical multipoles reside within a given toroidal function shown on the left hand side. For example, $\frac{1}{\pi\sqrt{\rho a}}Q_{-\frac{1}{2}}(\rho, z)$ has built within it a spherical monopole, octupole, hexadecapole, etc. This can best be summarized by Table 6.1 which illustrates which spherical multipole moment contributes to a specific toroidal function.

The $m = 0$ term includes the monopole(2^0), quadrupole(2^2), the hexadecapole(2^4), etc. The $m = 1$ term includes the dipole(2^1), the octupole(2^3), etc. The $m=2$ term includes the quadrupole(2^2), the hexadecapole(2^4), etc. The table can be extended to include any number of Q-functions. An important feature to recognize is that if the $\frac{1}{\pi\sqrt{\rho a}}Q_{-\frac{1}{2}}(\xi)$ function does not contribute then there can be no monopole contribution. Also, if the $\frac{1}{\pi\sqrt{\rho a}}Q_{\frac{1}{2}}(\xi)$ function does not contribute then there can be no

m	$Q_{m-\frac{1}{2}}(\rho, z)$	<i>spherical monopole</i> 2^0	<i>spherical dipole</i> 2^1	<i>spherical quadrupole</i> 2^2	<i>spherical octupole</i> 2^3	<i>spherical hexadecapole</i> 2^4	$\dots$
0	$\frac{1}{\pi\sqrt{\rho a}}Q_{-\frac{1}{2}}(\rho, z)$	$\times$		$\times$		$\times$	
1	$\frac{1}{\pi\sqrt{\rho a}}Q_{\frac{1}{2}}(\rho, z)$		$\times$		$\times$		
2	$\frac{1}{\pi\sqrt{\rho a}}Q_{\frac{3}{2}}(\rho, z)$			$\times$		$\times$	
3	$\frac{1}{\pi\sqrt{\rho a}}Q_{\frac{5}{2}}(\rho, z)$				$\times$		
4	$\frac{1}{\pi\sqrt{\rho a}}Q_{\frac{7}{2}}(\rho, z)$					$\times$	
$\vdots$	$\vdots$						$\ddots$

Table 6.1: Multipole reference table

dipole contribution. If, for example, the $\frac{1}{\pi\sqrt{\rho a}}Q_{\frac{3}{2}}(\xi)$ function does not contribute, but the $\frac{1}{\pi\sqrt{\rho a}}Q_{-\frac{1}{2}}(\xi)$ function does, then there still exists a quadrupole contribution since it appears in the $\frac{1}{\pi\sqrt{\rho a}}Q_{-\frac{1}{2}}(\xi)$ function; therefore, a monopole contribution must also exist. In other words, if a quadrupole contribution is present, but no $\frac{1}{\pi\sqrt{\rho a}}Q_{\frac{3}{2}}(\xi)$ function appears in the solution then one must expect a monopole contribution. However, it is possible to have a $\frac{1}{\pi\sqrt{\rho a}}Q_{\frac{3}{2}}(\xi)$ contribution, but no $\frac{1}{\pi\sqrt{\rho a}}Q_{-\frac{1}{2}}(\xi)$ contribution. This would also yield a quadrupole contribution but no monopole contribution. Finally, both could contribute. These are just a few of the many scenarios, some of which will be discussed in this section, which one could imagine. One could use the multipole reference table to develop a whole family of solution types in terms of toroidal or Q-functions.

As was shown in Equation (5.22), the magnetic vector potential for the current loop at the field point can be written as

$$\mathbf{A}(\rho, z) = \frac{\mu_0 I_\phi}{2\pi} \sqrt{\frac{a}{\rho}} Q_{\frac{1}{2}}(\xi) \hat{\phi} \quad (6.13)$$

One should recognize, with the aid of Table 6.1, that a dipole(2^1), an octupole(2^3), a tricontadipole(2^5), and all $2^{n_{\text{odd}}}$ -poles, beginning with $n = 1$, exist. The far field looks like a dipole. This is, in fact, what one would expect. Although a specific example of a current loop was used to develop the multipole reference table, the table is quite general.

6.3 Application of the multipole reference table

This section includes some simple, but illustrative examples which will highlight the salient features of the Q-functions. An attempt will be made to clarify the results as well as give some physical insight into the meaning of a toroidal expansion.

6.3.1 Two electric point charges

The example of two point charges considered in chapter 5 gives a nice application of the multipole reference table. This example will illustrate the application of Q-functions and how they compare to spherical harmonics. The potential at the field point in spherical coordinates was written as

$$\begin{aligned} \Phi_P(R, \theta, \phi) = \frac{q}{4\pi\epsilon_o R} \left\{ 2 \left(\frac{a}{R} \right) \sin(\theta) \cos(\phi) + \right. \\ \left. \frac{1}{4} \left(\frac{a}{R} \right)^3 \cos(\phi) \sin(\theta) \left[10 \sin^2(\theta) \cos(2\phi) - \right. \right. \\ \left. \left. 5 \cos(2\theta) - 7 \right] + \dots \right\} \end{aligned} \quad (6.14)$$

where P is the location of the field point. The dipole potential was given by

$$\Phi_P(R, \theta, \phi)^{dipole} = \frac{p}{4\pi\epsilon_0 R^2} \sin(\theta) \cos(\phi) \quad (6.15)$$

The electric scalar potential in terms of a toroidal expansion was written as

$$\begin{aligned} \Phi_P(\rho, \phi, z) = \frac{q}{\pi^2\epsilon_0} \frac{1}{\sqrt{\rho a}} \sum_{l=0}^{\infty} Q_{2l+\frac{1}{2}} \left(\frac{\rho^2 + a^2 + z^2}{2a\rho} \right) \cdot \\ \cos[(2l+1)\phi] \end{aligned} \quad (6.16)$$

and this was expanded to yield:

$$\begin{aligned} \Phi_P(\rho, \phi, z) = \frac{q}{\pi^2\epsilon_0} \frac{1}{\sqrt{\rho a}} \left[Q_{\frac{1}{2}} \left(\frac{\rho^2 + a^2 + z^2}{2a\rho} \right) \cos(\phi) + \right. \\ \left. Q_{\frac{5}{2}} \left(\frac{\rho^2 + a^2 + z^2}{2a\rho} \right) \cos(3\phi) + \dots \right] \end{aligned} \quad (6.17)$$

Figure 6.3: Circular Conducting Ring

Also, for $R \gg a$ one obtained

$$\Phi_P(R, \theta, \phi)^{dipole} = \frac{p \sin(\theta) \cos(\phi)}{4\pi\epsilon_0 R^2} \quad (6.18)$$

One can see that the $m = 1, 3, 5, \dots (2m + 1)$ terms contribute. One could have predicted, using the multipole reference table, which terms contributed without actually solving the problem. The table functions as a qualitative tool to help predict which toroidal function dominates the potential distribution and it can also help to predict which higher order terms appear in the expansion.

6.3.2 Circular conducting ring

Consider, as a second example, an infinitely thin circular conducting ring of radius, a , lying in the x - y plane and centered at the origin as shown in Figure 6.3. A linear charge density, for example, of the form

$$\lambda(\phi) = \frac{q}{2\pi a} \cos(p\phi) \quad (6.19)$$

will be considered. The cases $p = 0$ and $p = 1$ will be analyzed. When $p = 0$ a uniform linear charge density, $\frac{q}{2\pi a}$, is imposed on the circular conducting ring. In other words, a total charge, q , resides on the conducting ring. This should indicate

the presence of a monopole contribution.

One can write the differential scalar potential at the field point as

$$d\Phi_P = \frac{1}{4\pi\epsilon_0} \frac{\lambda(\phi') ad\phi'}{|\mathbf{R} - \mathbf{a}|} \quad (6.20)$$

Substitute Equation (6.19) and Equation (2.11) for $p = 0$ into Equation (6.20) to obtain

$$d\Phi_P = \frac{q}{8\pi^3\epsilon_0} \frac{1}{\sqrt{\rho a}} \sum_{m=0}^{\infty} \varepsilon_m Q_{m-\frac{1}{2}}(\xi) \cdot \cos \left[m \left(\phi - \phi' \right) \right] d\phi' \quad (6.21)$$

where

$$\xi = \frac{\rho^2 + a^2 + z^2}{2\rho a} > 1 \quad (6.22)$$

Integrating both sides of Equation (6.21) yields:

$$\Phi_P(\rho, \phi, z) = \frac{q}{8\pi^3\epsilon_0} \frac{1}{\sqrt{\rho a}} \sum_{m=0}^{\infty} \varepsilon_m Q_{m-\frac{1}{2}}(\xi) \cdot \int_0^{2\pi} \cos \left[m \left(\phi - \phi' \right) \right] d\phi' \quad (6.23)$$

Only the $m = 0$ term survives the integration in Equation (6.23). The electric scalar potential at any point in space, not coincident with the conducting loop, is given by

$$\Phi_P(\rho, z) = \frac{q}{4\pi^2\epsilon_0\sqrt{\rho a}} Q_{-\frac{1}{2}}(\xi) \quad (6.24)$$

where from Equation (2.21), for $m = 0$, one obtains

$$Q_{-\frac{1}{2}}(\xi) = \frac{\pi\sqrt{a\rho}}{\sqrt{\rho^2 + a^2 + z^2}} \sum_{n=0}^{\infty} \frac{(4n-1)!!}{2^{2n}(n!)^2} \cdot \frac{(\rho a)^{2n}}{(\rho^2 + a^2 + z^2)^{2n}} \quad (6.25)$$

One can see from Table 6.1 that the far field looks like a monopole field because of the presence of $\frac{1}{\pi\sqrt{\rho a}} Q_{-\frac{1}{2}}(\xi)$. Using Equation (6.25) allows one to rewrite Equation

(6.24) as

$$\Phi_P(\rho, z) = \frac{q}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{(4n-1)!!}{2^{2n}(n!)^2} \frac{(\rho a)^{2n}}{(\rho^2 + a^2 + z^2)^{2n+\frac{1}{2}}} \quad (6.26)$$

Equation (6.26) represents the electric scalar potential at an arbitrary field point in space not coincident with the source.

The electric field in cylindrical coordinates can be written as

$$\mathbf{E}_P(\rho, \phi, z) = - \left[\frac{\partial \Phi_P}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial \Phi_P}{\partial \phi} \hat{\phi} + \frac{\partial \Phi_P}{\partial z} \hat{z} \right] \quad (6.27)$$

The components of the $\mathbf{E}$ -field are given by

$$E_\rho(\rho, z) = \frac{q}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{(4n-1)!! (\rho a)^{2n}}{2^{2n}(n!)^2 \rho} \left[\frac{2n(\rho^2 - z^2 - a^2) + \rho^2}{(\rho^2 + a^2 + z^2)^{2n+\frac{3}{2}}} \right] \quad (6.28)$$

$$E_\phi = 0 \quad (6.29)$$

$$E_z(\rho, z) = \frac{q}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{(4n+1)!!}{2^{2n}(n!)^2} \frac{z (\rho a)^{2n}}{(\rho^2 + a^2 + z^2)^{2n+\frac{3}{2}}} \quad (6.30)$$

Expanding Equations (6.28) and (6.30) yields:

$$E_\rho(\rho, z) = \frac{q}{4\pi\epsilon_0} \left[\frac{\rho}{(\rho^2 + a^2 + z^2)^{\frac{3}{2}}} + \frac{1 \cdot 3}{2^2(1!)^2} \frac{(\rho a)^2 (3\rho^2 - 2z^2 - 2a^2)}{\rho (\rho^2 + a^2 + z^2)^{\frac{7}{2}}} + \dots \right] \quad (6.31)$$

$$E_z(\rho, z) = \frac{q}{4\pi\epsilon_0} \left[\frac{z}{(\rho^2 + a^2 + z^2)^{\frac{3}{2}}} + \frac{1 \cdot 3 \cdot 5}{2^2(1!)^2} \frac{z (\rho a)^2}{(\rho^2 + a^2 + z^2)^{\frac{7}{2}}} + \dots \right] \quad (6.32)$$

The electric field at some point $(0, 0, z)$ on the positive z axis is given by the well known result [41]

$$\mathbf{E}_P(z) = \frac{q}{4\pi\epsilon_0} \frac{z}{(a^2 + z^2)^{\frac{3}{2}}} \hat{z} \quad (6.33)$$

which is easily confirmed by letting $\rho \rightarrow 0$ in Equations (6.31) and (6.32).

Now consider the linear charge density in Equation (6.19) for $p = 1$. This is

given by

$$\lambda(\phi) = \frac{q}{2\pi a} \cos(\phi) \quad (6.34)$$

The total charge on the ring for $p = 1$ is now zero, since

$$\int_0^{2\pi} \lambda(\phi') a d\phi' = 0 \quad (6.35)$$

This should indicate that a monopole contribution is nonexistent. The electric scalar potential can now be written as

$$\begin{aligned} \Phi_P(\rho, \phi, z) = & \frac{q}{8\pi^3 \epsilon_0} \frac{1}{\sqrt{\rho a}} \sum_{m=0}^{\infty} \varepsilon_m Q_{m-\frac{1}{2}}(\xi) \cdot \\ & \int_0^{2\pi} \cos \left[m \left(\phi - \phi' \right) \right] \cos(\phi') d\phi' \end{aligned} \quad (6.36)$$

Only the $m = 1$ term survives the integration in Equation (6.36). The electric scalar potential at any point in space not coincident with the conducting loop is given by

$$\Phi_P(\rho, \phi, z) = \frac{q}{4\pi^2 \epsilon_0 \sqrt{\rho a}} Q_{\frac{1}{2}}(\xi) \cos(\phi) \quad (6.37)$$

One must remember that the Neumann factor, ε_1 , is equal to two. Equation (6.37) yields, for the far field, a dipole potential because of the presence of $\frac{1}{\pi\sqrt{\rho a}} Q_{\frac{1}{2}}(\xi)$.

Equation (6.19) acts much like a filter; it filters out a specific dominant multipole moment. If $p = 2$, then a quadrupole contribution would exist because of the presence of $\frac{1}{\pi\sqrt{\rho a}} Q_{\frac{3}{2}}(\xi)$ and no monopole or dipole contribution is allowed. This is true because $\frac{1}{\pi\sqrt{\rho a}} Q_{-\frac{1}{2}}(\xi)$ and $\frac{1}{\pi\sqrt{\rho a}} Q_{\frac{1}{2}}(\xi)$ do not appear in the solution. One can continue this process for any p .

6.3.3 Hollow circular current disk

Consider a thin and hollow circular current disk with a azimuthal current distribution given by

$$\mathbf{K}(\phi) = \frac{I}{2(b-a)} [1 + \cos(p\phi)] \hat{\phi} \quad (6.38)$$

Figure 6.4: Thin hollow circular disk

where b is the outer radius of the disk and a is the inner radius of the disk as shown in Figure 6.4. The unit vector $\hat{\phi}$ can be rewritten, using the Cartesian unit vectors, as

$$\hat{\phi} = -\sin(\phi)\hat{x} + \cos(\phi)\hat{y} \quad (6.39)$$

The goal is to compute the magnetic vector potential at any point in space not coincident with the disk. This problem can be solved by an Elliptic integral formulation [33, 65, 91].

An alternative to the more familiar Elliptic integral formulation is to employ a toroidal function expansion. This results in a solution which is more easily formulated than the Elliptic integral solution.

The differential vector potential at some field point, P , is written as

$$d\mathbf{A}_P = \frac{\mu_0}{4\pi} \frac{\mathbf{K}(\phi') dS}{\sqrt{\rho^2 + \rho'^2 + z^2 - 2\rho\rho' \cos(\phi - \phi')}} \quad (6.40)$$

Replacing the denominator in Equation (6.40) with Equation (2.11) yields:

$$d\mathbf{A}_P = \frac{\mu_0 I}{8\pi^2 (b-a) \sqrt{\rho\rho'}} \sum_{m=0}^{\infty} \varepsilon_m Q_{m-\frac{1}{2}}(\xi) \cdot \frac{F_{pm}(\phi') \rho' d\rho' d\phi'}{F_{pm}(\phi') \rho' d\rho' d\phi'} \quad (6.41)$$

where

$$F_{pm}(\phi') = \frac{[-\sin(\phi')\hat{x} + \cos(\phi')\hat{y}][1 + \cos(p\phi)]}{\cos[m(\phi - \phi')]} \quad (6.42)$$

The magnetic vector potential can now be written as

$$\mathbf{A}_P(\rho, \phi, z) = \frac{\mu_0 I}{8\pi^2 (b-a) \sqrt{\rho}} \sum_{m=0}^{\infty} \varepsilon_m \int_a^b \int_0^{2\pi} \sqrt{\rho'} Q_{m-\frac{1}{2}}(\xi) \cdot \frac{F_{pm}(\phi') d\rho' d\phi'}{F_{pm}(\phi') d\rho' d\phi'} \quad (6.43)$$

where

$$\xi = \frac{\rho^2 + \rho'^2 + z^2}{2\rho\rho'} \quad (6.44)$$

The azimuthal integration in Equation (6.43) is an integration on the product of simple sinusoidal functions. In general, the azimuthal integration is the more difficult one to handle when an Elliptic integral approach is used, but this is not the case when a toroidal expansion is employed. The azimuthal integration can usually be evaluated in closed form. However, consider the case when $p = 0$. This represents a thin hollow disk carrying a uniform and linear current density, $K = \frac{I}{b-a}$. For $p = 0$, the $m = 1$ term is the only term which survives the ϕ' integration. This is given by

$$\int_0^{2\pi} F_{p1}(\phi') d\phi' = 2\pi \hat{\phi} \quad (6.45)$$

Employing Equation (6.45), reduces the magnetic vector potential to a very compact form given by

$$\mathbf{A}_P(\rho, z) = \frac{\mu_0 I \hat{\phi}}{2\pi (b-a) \sqrt{\rho}} \int_a^b \sqrt{\rho'} Q_{\frac{1}{2}}(\xi) d\rho' \quad (6.46)$$

One can see that the vector potential involves an integration of $Q_{\frac{1}{2}}(\xi)$. This precludes one from directly using the multipole reference table. However, Equation (6.46) can be put into a form in which a multipole analysis can be analyzed. It has already been shown that

$$Q_{\frac{1}{2}}(\xi) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(4n+1)!!}{2^{2n}(n+1)!n!} \frac{(\rho\rho')^{2n+\frac{3}{2}}}{(\rho^2 + \rho'^2 + z^2)^{2n+\frac{3}{2}}} \quad (6.47)$$

Substituting Equation (6.47) into Equation (6.46) yields the magnetic vector potential.

$$\begin{aligned} \mathbf{A}_P(\rho, z) &= \frac{\mu_0 I \hat{\phi}}{4(b-a)} \sum_{n=0}^{\infty} \frac{(4n+1)!! \rho^{2n+1}}{2^{2n}(n+1)!n!} \cdot \\ &\quad \int_a^b \frac{\rho'^{2n+2}}{(\rho'^2 + \rho^2 + z^2)^{2n+\frac{3}{2}}} d\rho' \end{aligned} \quad (6.48)$$

Equation (6.48) is integrable on ρ' for any value of n .

If one were strictly interested in an analytical solution then the integral in Equation (6.48) must be evaluated, and this can be done in terms of hypergeometric functions. Namely,

$$\begin{aligned} I_n(\rho, z, a, b) &= \int_a^b \frac{\rho'^{2n+2}}{(\rho'^2 + \rho^2 + z^2)^{2n+\frac{3}{2}}} d\rho' \\ &= f_n(\rho, z, b) - f_n(\rho, z, a) \end{aligned} \quad (6.49)$$

where

$$f_n(\rho, z, b) = b^{2n+3} \frac{\mathbf{F}\left(n + \frac{3}{2}, 2n + \frac{3}{2}; n + \frac{5}{2}; -\frac{b^2}{\rho^2 + z^2}\right)}{(2n+3)(\rho^2 + z^2)^{2n+\frac{3}{2}}} \quad (6.50)$$

and

$$f_n(\rho, z, a) = a^{2n+3} \frac{\mathbf{F}\left(n + \frac{3}{2}, 2n + \frac{3}{2}; n + \frac{5}{2}; -\frac{a^2}{\rho^2 + z^2}\right)}{(2n+3)(\rho^2 + z^2)^{2n+\frac{3}{2}}} \quad (6.51)$$

The analytical expression for the magnetic vector potential can now be written as

$$\mathbf{A}_P(\rho, z) = \frac{\mu_0 I \hat{\phi}}{4(b-a)} \sum_{n=0}^{\infty} \frac{(4n+1)!! \rho^{2n+1}}{2^{2n}(n+1)!n!} I_n(\rho, z, a, b) \quad (6.52)$$

The hypergeometric functions [30] expressed in Equation (6.50) and in Equation (6.51) all have closed form solutions for each n .

Consider what happens to Equation (6.52) as the thickness of the hollow disk in the radial direction approaches zero. In other words, what happens as b becomes smaller and smaller until $b = a$? This question is easily answered by applying L'Hopital's rule. Mathematically, it is written as

$$\tilde{\mathbf{A}}_P(\rho, z) = \lim_{b \rightarrow a} \frac{\frac{\mu_0 \hat{I} \phi}{4} \sum_{n=0}^{\infty} \frac{(4n+1)!! \rho^{2n+1}}{2^{2n} (n+1)! n!} \frac{d}{db} [I_n(\rho, z, a, b)]}{\frac{d}{db} [(b-a)]}. \quad (6.53)$$

Performing the differentiations and then letting $b \rightarrow a$ yields:

$$\begin{aligned} \tilde{\mathbf{A}}_P(\rho, z) &= \frac{\mu_0 \hat{I} \phi}{4} \sum_{n=0}^{\infty} \frac{(4n+1)!!}{2^{2n} (n+1)! n!} \cdot \\ &\quad \frac{\rho^{2n+1} a^{2n+2}}{(\rho^2 + z^2 + a^2)^{2n+\frac{3}{2}}} \end{aligned} \quad (6.54)$$

Equation (6.54) can be rewritten as

$$\begin{aligned} \tilde{\mathbf{A}}_P(\rho, z) &= \frac{\mu_0 \hat{I} \phi}{4} \sqrt{\frac{a}{\rho}} \sum_{n=0}^{\infty} \frac{(4n+1)!!}{2^{2n} (n+1)! n!} \cdot \\ &\quad \frac{(\rho a)^{2n+\frac{3}{2}}}{(\rho^2 + z^2 + a^2)^{2n+\frac{3}{2}}} \\ &= \frac{\mu_0 \hat{I} \phi}{2\pi} \sqrt{\frac{a}{\rho}} Q_{\frac{1}{2}}(\xi) \end{aligned} \quad (6.55)$$

Equation (6.55) is identical to Equation (6.13). It is the magnetic vector potential of a thin circular ring of radius a carrying a uniform current as shown in Figure 6.2. Of course, this should be expected. The thin and hollow circular disk carrying a uniform and linear current density as shown in Figure 6.4 should degenerate into the current loop carrying a uniform current. This is graphically illustrated in Figure 6.5 . One can see that, as the outer radius b of the flat hollow circular disk approaches the inner radius a , the vector potential of the disk approaches that of a current loop of radius a . The vertical axis in 6.5 is normalized by multiplying the left and

COMPARISON BETWEEN HOLLOW DISK AND CIRCULAR LOOP AT Z=1

Figure 6.5: Comparison between hollow disk and circular loop

right hand sides of Equation (6.52) by $\frac{4(b-a)}{\mu_0 I}$ and considering only the magnitude of the magnetic vector potential. The horizontal axis measures the change in the ρ variable. All values of magnetic vector potential are computed, for this particular comparison, on the $z = 1$ m plane. The hollow flat disk has an outer radius of $b = 2$ m and an inner radius of $a = 1$ m. The current loop has a radius of $a = 1$ m and is represented by the curve closest to the ρ axis in Figure 6.5.

If a numerical solution is desired, and in particular if $\frac{a^2}{\rho^2+z^2} < 1$ and $\frac{b^2}{\rho^2+z^2} < 1$, then the magnetic vector potential can be put into a more useful form. Using Equations (6.50), (6.51), and Equation (6.56) given below

$$F(\alpha, \beta; \gamma; \lambda) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{l=0}^{\infty} \frac{\Gamma(\alpha+l)\Gamma(\beta+l)}{\Gamma(\gamma+l)l!} \lambda^l \quad (6.56)$$

where $\alpha = n + \frac{3}{2}$, $\beta = 2n + \frac{3}{2}$, $\gamma = n + \frac{5}{2}$, $|\lambda(\rho, z, b)| = \frac{b^2}{\rho^2+z^2} < 1$ and $|\lambda(\rho, z, a)| =$

$\frac{a^2}{\rho^2+z^2} < 1$, one can write the magnetic vector potential as

$$\mathbf{A}_P(\rho, z) = \frac{\mu_0 I \hat{\phi}}{4(b-a)} \sum_{n,l=0}^{\infty} R_{nl} \frac{\rho^{2n+1}}{(\rho^2+z^2)^{2n+l+\frac{3}{2}}} \cdot \left[b^{2l+2n+3} - a^{2l+2n+3} \right] \hat{\phi} \quad (6.57)$$

where

$$R_{nl} = \frac{(-1)^l (4n+1)!! (4n+2l+1)!!}{2^{2n+l} (2l+2n+3)l!(n+1)!n!} \quad (6.58)$$

Equation (6.57) is most useful for numerical computation and is directly applicable for doing a multipole analysis. For example, consider the first term in the double sum of Equation (6.57). This is the dominant term. One can express the magnetic vector potential of the first term in Equation (6.57) as

$$\mathbf{A}_P(\rho, z) = \frac{\mu_0 I \hat{\phi}}{4(b-a)} R_{00} \frac{\rho(b^3 - a^3)}{(\rho^2+z^2)^{\frac{3}{2}}} \hat{\phi} \quad (6.59)$$

where $R_{00} = \frac{1}{3}$. Substituting Equation (6.4) into Equation (6.59) yields:

$$\mathbf{A}_P(R, \theta) = \frac{\mu_0 I \sin(\theta)}{12 R^2} (b^2 + ab + a^2) \hat{\phi} \quad (6.60)$$

As $b \rightarrow a$ in Equation (6.60), one obtains

$$\mathbf{A}_P(R, \theta) = \frac{\mu_0 I \hat{\phi} a^2 \sin(\theta)}{4 R^2} \quad (6.61)$$

Equation (6.61) is the same result which one obtains by taking the first term in the expansion of Equation (6.13) for $R \gg a$. Equation (6.60) shows that a dipole field is the dominant contribution as R becomes large. This should have been predicted from the geometry of the problem. One can continue the process to obtain all higher order multipoles. Although Equation (6.57) is in a mathematical form which can help to predict the multipole distribution, one can use the integral form given in Equation (6.48) to perform the same task. In fact, if one were interested in only a few terms in the multipole expansion then Equation (6.48) would be preferable.

6.4 Spherical harmonics versus toroidal harmonics

The power of a spherical harmonic representation lies in its ability to represent a complex electromagnetic source in terms of some primitive quantity. In particular, a complex magnetostatic source can be represented in terms of its dipole, its quadrupole, its octupole, etc. The magnetic field far from the source may appear to the observer as if it were generated by a simple uniform current loop. The observer sees only the dipole. As the observer moves closer to the source, he begins to see the effects of other multipole components adding vectorially to the dipole field. The source no longer appears as if it were generated from a simple current loop. In fact, the components of the dipole field in spherical coordinates as seen by the observer are given by [10]

$$B_r = \frac{\mu_0 \cos(\theta)}{2\pi r^3} |\mathbf{m}| \quad (6.62)$$

$$B_\theta = \frac{\mu_0 \sin(\theta)}{4\pi r^3} |\mathbf{m}| \quad (6.63)$$

where $|\mathbf{m}| = \pi a^2 I$ is the dipole moment. As the observer moves closer to the actual source, Equations (6.62) and (6.63) are no longer true. The actual magnetostatic source may be quite complex. However, if a spherical harmonic representation of that source can be found then one can choose as many primitive moments as desired to reproduce the actual field within a desired accuracy.

The Q-function representation, unlike a spherical harmonic representation, is not able to directly yield the primitive moments from the corresponding toroidal expansion and one should not expect it to. However, using the method developed in this chapter, one can regain the spherical multipole moments without much difficulty. For example, it was shown that if the field, as seen by an observer far from the source, looks like a dipole field then the toroidal expansion must contain a $Q_{\frac{1}{2}}(\xi)$ as its first weighting coefficient. If the observer sees, far from the source, a quadrupole effect then there exists a $Q_{\frac{3}{2}}(\xi)$ term and it must be the first weighting coefficient. If the observer sees, far from the source, an octupole effect then the first term in the toroidal expansion must be a $Q_{\frac{5}{2}}(\xi)$ term, etc. In other words, the first term in the series expansion of the first weighting coefficient has built within it the necessary

dipole, or quadrupole, or octupole, etc. One must employ Equations (6.2) through (6.4) in order to regain the spherical harmonic distribution. This is a mandatory step in the process of transforming one expansion to the other.

The multipole reference table was initially developed to simplify this process. However, it has evolved into a useful tool for determining which weighting coefficients are present and which can be ignored. In other words, one does not need to compute all the toroidal harmonics starting with $Q_{\frac{1}{2}}(\xi)$. This is better illustrated in chapter 8.

6.5 Summary

It has been shown that, for certain electromagnetic systems which exhibit cylindrical symmetry, the toroidal function expansion gives an alternate mathematical description and that this description allows for a direct comparison to a spherical multipole expansion. Table 6.1 gives a quick way to determine which spherical multipoles exist given a toroidal function expansion. The argument, ξ , in $Q_{m-\frac{1}{2}}(\xi)$ is a function of two field variables, ρ and z . It is for this reason that one can directly apply a multipole analysis. The argument of the Q-function has been built within it the radius of a hypothetical sphere surrounding a cylinder. A simple coordinate transformation from cylindrical coordinates to spherical coordinates allows one to obtain an equivalent spherical multipole decomposition of a cylindrically symmetric system.

The main focus of the toroidal expansion has been in its application to various cylindrical systems in electrostatics and magnetostatics, but it is certainly not limited to these areas of interest. In studying electrodynamic problems involving cylindrical geometries where the free-space Green's function can be employed allows for a direct formulation of the mathematical problem. Also, the free-space Green's function in cylindrical coordinates, as with any known free-space Green's function in any particular orthogonal coordinate system, can be modified to handle finite boundaries such as cylinders within cylinders or cylinders which have a prescribed boundary condition. Just as the free-space Green's function for a spherically symmetric system must be modified to handle spherical geometries with prescribed

boundary conditions; so too must the free-space Green's function in cylindrical coordinates be modified to handle prescribed boundary conditions on a cylindrical surface. This allows one to tackle problems in cylindrical coordinates which are usually solved, for example, using a Bessel function description. Also, the Q-function, results in a more direct mathematical formulation than does the equivalent Elliptic integral approach.

If one compares, for example, a Q-function formulation to the equivalent Elliptic integral formulation for a cylindrically symmetric system then one will note that the latter needs a number of transformations in order to put the integral into a proper mathematical form. This is not needed if a toroidal function formulation is used. In other words, no mathematical transformation need be made on the inverse distance function in cylindrical coordinates given by Equation (2.1). The right hand side of Equation (2.1) can be directly written in terms of its Fourier series representation given by Equation (2.11). In fact, integrals of the form

$$\int_V \frac{F(\rho', \phi', z') \rho' d\rho' d\phi' dz'}{(\rho^2 + \rho'^2 + (z - z')^2 - 2\rho\rho' \cos(\phi - \phi'))^{l+\frac{1}{2}}} \quad (6.64)$$

for $l = 0, 1, 2, 3$, etc., and for a general forcing function, $F(\rho', \phi', z')$, can be more quickly formulated using Equation (2.23).

CHAPTER 7

CHARGE SIMULATION METHOD

7.1 Introduction

Charge simulation is a method [88, 89, 90] by which a configuration of simulation charges is determined, and whose potential function or field function approximates the true potential or field of the actual electric or magnetic source. The first type of charge simulation which will be considered will be that derived from a known magnetic scalar potential function. The second type of charge simulation which will be considered will be that derived from experimental data such as the normal component of the magnetic flux density measured on a closed hypothetical cylinder surrounding a real magnetic source.

Assume that a magnetic scalar potential can be computed close to the magnetic source. Also, consider enclosing the real source in a hypothetical surface that is close to the actual magnetic source as shown in Figure 7.1. For this thesis, a

Figure 7.1: Potential cylinder

Figure 7.2: Charge simulation on a hypothetical cylinder

hypothetical cylindrical surface will be chosen. Compute the magnetic scalar potential on a chosen grid of points lying on the hypothetical cylinder. The hypothetical cylinder will be called the *potential cylinder*. The basic idea of charge simulation for a magnetic system is to replace the actual magnetic source with a new source made up entirely of fictitious magnetic charges which yields the same computed potentials on the potential cylinder as the original source did. This is shown in Figure 7.2. The *charge cylinder* completely encloses the actual source, but lies inside the potential cylinder. Alternatively, Figure 7.2 could represent the charges computed from experimental data. This, in fact, is more probable, since it is much easier to measure the magnetic flux density than it is to measure the magnetic scalar potential. Although the mathematical model is different in both cases, the basic procedure for charge simulation remains the same.

Once the charge cylinder is found which reproduces the correct potentials on the potential cylinder or the correct normal component of the field on the *field cylinder*, then one can use the charge cylinder and the magnetic form of Coulomb's law to compute the magnetic scalar potential or the magnetic field anywhere external to this new source. In contrast to finite-difference and finite-element methods, the

charge simulation method is well suited for unbounded electromagnetic problems, but can be modified to handle bounded problems. The hypothetical surface used in charge simulation could be any closed surface surrounding the real source. However, choosing the appropriate geometry that fits the particular problem may simplify the analysis.

7.2 Charge simulation and the magnetic scalar potential

In order to perform the charge simulation method, one must formulate the correct mathematical steps. One can write the magnetic form of Coulomb's Law, as given by Equation (3.4), in matrix form as

$$\begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_n \end{bmatrix} = \frac{1}{4\pi} \begin{bmatrix} \frac{1}{r_{11}} & \frac{1}{r_{12}} & \cdots & \frac{1}{r_{1n}} \\ \frac{1}{r_{21}} & \frac{1}{r_{22}} & \cdots & \frac{1}{r_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{r_{n1}} & \frac{1}{r_{n2}} & \cdots & \frac{1}{r_{nn}} \end{bmatrix} \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \vdots \\ \Omega_n \end{bmatrix} \quad (7.1)$$

where Φ is the magnetic scalar potential, r is the distance between the source point and the field point, and Ω is the fictitious charge. It is necessary to choose a grid with enough points so that an accurate reproduction of the original computed potentials can be established. This involves a bit of trial-and-error. There appears to be no easy way to predict how many charges would be necessary to produce an accurate potential function a' priori. This is not based on any known mathematical law, but on the author's research and is subject to change. In order to compute the charges, one needs to compute the inverse of the $\frac{1}{r_{ij}}$ matrix as shown in Equation (7.2). The inversion process could take some time because the number of potential points could

be large in number.

$$\begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \vdots \\ \Omega_n \end{bmatrix} = 4\pi \begin{bmatrix} \frac{1}{r_{11}} & \frac{1}{r_{12}} & \cdots & \frac{1}{r_{1n}} \\ \frac{1}{r_{21}} & \frac{1}{r_{22}} & \cdots & \frac{1}{r_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{r_{n1}} & \frac{1}{r_{n2}} & \cdots & \frac{1}{r_{nn}} \end{bmatrix}^{-1} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_n \end{bmatrix} \quad (7.2)$$

7.3 Charge simulation and the normal component of the magnetic flux density

In general, it is not feasible to measure the magnetic scalar potential, and therefore some other method must be used to employ the charge simulation technique for a real magnetic source. This is done by measuring the normal component of the magnetic flux density. The matrix equation which expresses the charge simulation method, given measured normal components of the magnetic flux density, is

$$\begin{bmatrix} H_\rho \\ H_z \end{bmatrix} = \frac{1}{4\pi} \begin{bmatrix} \frac{\partial}{\partial \rho} \left(\frac{1}{r_{cylinder-wall}} \right) & \frac{\partial}{\partial \rho} \left(\frac{1}{r_{cylinder-caps}} \right) \\ \frac{\partial}{\partial z} \left(\frac{1}{r_{cylinder-wall}} \right) & \frac{\partial}{\partial z} \left(\frac{1}{r_{cylinder-caps}} \right) \end{bmatrix} \begin{bmatrix} \Omega_{cylinder-wall} \\ \Omega_{cylinder-caps} \end{bmatrix} \quad (7.3)$$

and the charges are given by

$$\begin{bmatrix} \Omega_{cylinder-wall} \\ \Omega_{cylinder-caps} \end{bmatrix} = 4\pi \begin{bmatrix} \frac{\partial}{\partial \rho} \left(\frac{1}{r_{cylinder-wall}} \right) & \frac{\partial}{\partial \rho} \left(\frac{1}{r_{cylinder-caps}} \right) \\ \frac{\partial}{\partial z} \left(\frac{1}{r_{cylinder-wall}} \right) & \frac{\partial}{\partial z} \left(\frac{1}{r_{cylinder-caps}} \right) \end{bmatrix}^{-1} \begin{bmatrix} H_\rho \\ H_z \end{bmatrix} \quad (7.4)$$

7.4 Summary

The charge simulation method is a general method which can be applied to any magnetic source. If a real magnetic source is enclosed in a fictitious closed surface, then charge simulation can be used for computing the external magnetic field from the newly generated charged surface. However, for real magnetic sources, one must have a method for computing the potentials on the potential surface before using the charge simulation method. This can be done, for complex geometries, by employing a finite element method or an integral equation formulation [89].

A combination of charge simulation and an appropriate numerical method allows one to compute an equivalent multipole distribution valid for all points outside the charged surface. For a spherical boundary surface, a spherical multipole distribution is direct, but for a non-spherical surface it can be much more difficult to regain the spherical multipole distribution. Chapter 7 will show how this is accomplished when a cylindrical surface is chosen as the closed boundary surface.

CHAPTER 8

APPLICATION OF THE TOROIDAL EXPANSION FOR COMPUTING THE MAGNETIC FIELD FROM A PERMANENT MAGNET MOTOR

8.1 Introduction

Up to now, the main focus has been on the development of the necessary mathematical machinery needed to understand the toroidal expansion. Once this was done, a series of examples were used to validate the method. Then, in chapter 6, a method was introduced for computing the spherical multipoles from a knowledge of the toroidal expansion.

This chapter deals with the application of the Q-function and its corresponding toroidal expansion to compute the external magnetic field from a permanent magnet motor. In order to do this, a circular cylindrical boundary is chosen as the hypothetical surface which encloses the motor. Since the motor is cylindrical in shape, a cylindrical surface is the most practical surface to choose. This hypothetical surface will be the *potential cylinder* discussed in chapter 7. From a knowledge of the potentials, one uses the charge simulation method to compute the corresponding charges on the *charge cylinder*. The charge cylinder acts as the new magnetic source for computing the external magnetic field from the motor. The details on how to compute the potentials on some hypothetical potential cylinder can be found in the Kwon [89]. A combination of the finite element method and an integral equation formulation is used to compute the potentials on the potential cylinder. Once this is done, the charge simulation method is employed to compute the necessary charges which act as the new magnetic source. The subsequent computations for the potential, or the magnetic field intensity are only valid in a region external to the charged cylinder.

In order to apply the charge simulation technique and the toroidal expansion to a cylindrical magnetic source, one needs to build an appropriate cylindrical grid

Figure 8.1: Charge simulation model

as shown in Figure 7.2 of chapter 7. This, at first glance, seems simple. However, one of the difficulties encountered with the cylindrical grid is the ill-conditioned matrix which results due to the 90° angle between the cylindrical caps and the cylindrical shell. This type of ill-conditioned matrix does not occur when charge simulation is applied to a spherical surface. A number of attempts were made to rectify this problem, and one way which worked is illustrated in Figure 8.1. Figure 8.1 represents a model of the potential and a charged surface as seen from the x axis. One will notice that the corners of the cylindrical model were eliminated from the cylindrical grid in order to resolve the ill-conditioned matrix problem which occurred when the caps of the cylinders were brought into contact with the cylindrical shell. This may seem to void the application of the Q-function method which was derived from the free-space Green's function in cylindrical coordinates. However, the gap that exists between the caps and the cylindrical shell is very small compared to the radius of the cylinder and this, for the most part, corrects the ill-conditioned matrix. The author still feels that a more sophisticated grid would yield better results and

this is a topic for future research.

8.2 Permanent magnets used in permanent-magnet motors

In order to compute the magnetic field from a permanent-magnet motor, it is first necessary to model the permanent magnets alone in order to test the ideas developed in this thesis. A single parallelepiped permanent magnet and four parallelepiped permanent magnets will be used as test cases.

The first method, which will be called *The Magnetic Scalar Potential Method*, allows one to compute the external magnetic field from a known magnetic scalar potential and the second method, which will be called *The Magnetic Flux Density Method*, allows one to compute the external magnetic field from a knowledge of the normal component of the magnetic flux density measured on a hypothetical external cylinder close to the actual motor. This hypothetical external cylinder is a “measurement surface” in which measurements are made of the three components of the magnetic flux density. A knowledge of the normal component of the magnetic flux density enables one to apply Equation (6.4) in order to compute the charges on the charge cylinder.

8.2.1 A single permanent magnet

Figure 8.2 is a model of a permanent magnet surrounded by a hypothetical cylinder. The potential cylinder, not shown in Figure 8.2, lies outside but very close to the charge cylinder.

8.2.1.1 The magnetic scalar potential method

The magnetic scalar potential due to the permanent magnet is computed on a cylindrical grid which lies on the potential cylinder. A knowledge of these potentials allows one to compute the charges by using Equation (6.2). The potential values on the potential cylinder can be computed, for this problem, analytically or numerically. Figure 8.3 represents a three-dimensional plot of the total magnetic scalar potential on an observation cylinder which is coaxial, but external to the charge cylinder. The magnetic scalar potentials were computed using Equation (3.7). The charge

Figure 8.2: A permanent magnet inside a hypothetical cylinder

This is the total $\Phi_P(\rho, \phi, z)$

Figure 8.3: Total magnetic scalar potential

cylinder, of course, acts as the new magnetic source. For this problem, the charge cylinder has a radius of 1.0 m and a length of 3.0 m . The observation cylinder has a radius of 1.2 m and a length of 3.2 m . The magnet is cubic in shape with a side of length 0.1 m centered at the origin as shown in Figure 8.2. Since the goal is to compute the equivalent spherical multipoles, one needs to know which spherical multipole moments due to a single permanent magnet are present at some field point external to the charge cylinder. One can apply the ideas discussed in chapter 7 to this problem. A few plots of the components that make up the toroidal expansion will

Figure 8.4: The $m=1$ contribution to the total magnetic scalar potential

Figure 8.5: The $m=3$ contribution to the total magnetic scalar potential

be shown in order to verify which Q-functions contribute to the toroidal expansion. Figure 8.4 is a plot of the $m = 1$ contribution to the total magnetic scalar potential. Figure 8.5 is a plot of the $m = 3$ contribution to the total magnetic scalar potential, and Figure 8.6 is a plot of the $m = 5$ contribution to the total magnetic scalar potential. The even values of m make no contribution to the toroidal expansion and the odd values of m contribute less as m increases from 1 to higher values. In fact,

This is the $\Phi_P^{(5)}(\rho, \phi, z)$ component

Figure 8.6: The $m=5$ contribution to the total magnetic scalar potential

one can see that the $m = 1$ plot of Figure 8.4 looks identical to Figure 8.3 which is a plot of the total magnetic scalar potential. In other words, the $m = 1$ contribution is the dominant term in the toroidal expansion and can be used to represent the total magnetic scalar potential. From the multipole reference table of chapter 7, one can see that this represents an equivalent spherical dipole potential even at a relatively close distance to the source. The $m = 3, 5, 7, \dots (2m + 1)$ contribute less and less as all odd m values increase.

The magnetic field intensity, H , due to the permanent magnet, can be computed from a knowledge of the magnetic scalar potential by using Equation (3.16). Figure 8.7 is a plot of the total H_ρ . Figure 8.8 is a plot of the total H_z , and Figure 8.9 is a plot of the total H_ϕ . Figure 8.10 is a plot of the $m = 1$ component of the radial component of the magnetic field intensity. Figure 8.11 is a plot of the $m = 1$ component of the axial component of the magnetic field intensity, and Figure 8.12 is a plot of the $m = 1$ component of the azimuthal component of the magnetic field intensity. Figure 8.13 shows plots of the $m = 3$ and $m = 5$ components of the magnetic field intensity.

One may have recognized from the knowledge that the $m = 1$ term was the first non-vanishing contribution present and that the octupole, the tricontadipole, etc., are the only other possible terms that could have been present in the toroidal

$H_\rho(\rho, \phi, z)$ component of the vector potential

Figure 8.7: Total radial component of the magnetic field intensity

$H_z(\rho, \phi, z)$ component of the vector potential

Figure 8.8: Total axial component of the magnetic field intensity

expansion. This is illustrated by using the multipole reference table.

8.2.1.2 The magnetic flux density method

In the absence of real measured data, this section simply verifies Equation (6.4). The computed magnetic field intensity found in Section 8.2.1.1 is used as simulated measured data to compute the charges on the same charge cylinder as in Section 8.2.1.1. In other words, the observation cylinder is used as an equivalent

$H_\phi(\rho, \phi, z)$ component of the vector potential

Figure 8.9: Total azimuthal component of the magnetic field intensity

“potential cylinder” except that the normal components of the magnetic field intensity are used instead of the magnetic scalar potential values. Equation (6.4) is then employed to compute the charges on the charge cylinder, which in theory, should be the same as those computed in Section 8.2.1.1. Once this is accomplished, the magnetic field intensity is computed on the same external hypothetical cylinder as in Section 8.2.1.1. This enables one to verify if Equation (6.4) is correct. Figure 8.14 represents a plot of the H_ρ component of the magnetic field intensity vector computed from simulated data. Figure 8.15 represents a plot of the H_z component of the magnetic field intensity vector computed from simulated data. Likewise, Figure 8.16 represents a plot of H_ϕ component of the magnetic field intensity vector computed from simulated data.

The charges that were found using Equation (6.4) were not identical to those found from the magnetic scalar potential method but were very close. In fact, Figures 8.14 through 8.16 look identical to Figures 8.7 through 8.9 shown in Section 8.2.1.1.

The application of Equation (6.4) is used whenever real measured data is available. In the absence of this data, the magnetic scalar potential method is used. Of course, this method requires a detailed model of the actual motor as illustrated by Kwon [89].

The $H_{\rho}^{(1)}(\rho, \phi, z)$ component

Figure 8.10: The $m=1$ contribution of the radial component of magnetic field intensity

8.2.2 Four permanent magnets

Figure 8.17 is a model of a 4-pole permanent-magnet system. Once again, the four permanent magnets are the actual sources of the magnetic field. The four magnets are all cubic in shape with a side length of 0.1 m . The observation cylinder has the same dimensions as before and the center of each magnet is symmetrically located at 0.15 m from the origin as shown in Figure 8.18 .

8.2.2.1 Magnetic scalar potential method

The equivalent charge system is found which acts as the new magnetic source and an equivalent multipole distribution is found external to the charge cylinder. Figure 8.19 is a plot of the total magnetic scalar potential on an observation cylinder due to the four permanent magnets. As before, it is desirable to compute the equivalent spherical multipole distribution. Figure 8.20 gives a plot of the $m = 2$ contribution to the total magnetic scalar potential. The $m = 0$ and $m = 1$ contributions are zero. A symmetric 4-pole permanent magnetic system has the $m = 2$ term as its first non-vanishing Q-function. Comparing Figure 8.19 with Figure 8.20, one can see that the $m = 2$ contribution is the dominant contribution to the total magnetic scalar potential. The multipole reference table of chapter 7

Figure 8.11: The m=1 contribution of the axial component of magnetic field intensity

Figure 8.12: The m=1 contribution of the azimuthal component of magnetic field intensity

Figure 8.13: The m=3, and m=5 contributions of the magnetic field intensity vector

RADIAL COMPONENT OF THE MAGNETIC FIELD INTENSITY FROM SIMULATED DATA

Total $H_\rho(\rho, \phi, z)$ from simulated data

Figure 8.14: The radial component of the magnetic field intensity computed from simulated data

AXIAL COMPONENT OF THE MAGNETIC FIELD INTENSITY FROM SIMULATED DATA

Total $H_z(\rho, \phi, z)$ from simulated data

Figure 8.15: The axial component of the magnetic field intensity computed from simulated data

AZIMUTHAL COMPONENT OF THE MAGNETIC FIELD INTENSITY FROM SIMULATED DATA

Total $H_\phi(\rho, \phi, z)$ from simulated data

Figure 8.16: The azimuthal component of the magnetic field intensity computed from simulated data

Figure 8.17: A 4-pole permanent magnet system inside a hypothetical cylinder

Figure 8.18: Two-dimensional view of the 4-pole magnetic system

This is the total $\Phi_P(\rho, \phi, z)$

Figure 8.19: Total magnetic scalar potential

shows that the only other possible contributions to the magnetic scalar potential can come from the $m = 4, 6, 8, \dots (2m + 2)$ terms. Figure 8.21 gives a plot of the $m = 4$ contribution to the total magnetic scalar potential, and one can see that it is negligible. This says that a cancellation in the magnetic scalar potential is taking place due to that fact that a $\cos[4(\phi - \phi')]$ multiplies each $Q_{\frac{7}{2}}(\xi)$ for each charge and then summed over all charges. Notice, that the multipole reference table allows for two spherical hexadecapole ($2^4 - pole$) contributions; one from the $m = 2$

Figure 8.20: The m=2 contribution to the total magnetic scalar potential

Figure 8.21: The m=4 contribution to the total magnetic scalar potential

term and the other from the $m = 4$ term. The contribution of the hexadecapole from the $m = 2$ term must be negligible. Consider the $m = 6$ term which has built within it a spherical hexacontatetrapole ($2^6 - pole$) contribution. Figure 8.22 is a plot of the $m = 6$ contribution to the total magnetic scalar potential. The magnetic scalar potential for the $m = 6$ contribution is small, but not negligible. This says that, although some cancellation may be taking place because of the $\cos[6(\phi - \phi')]$ multiplying each $Q_{\frac{11}{2}}(\xi)$, the $m = 6$ term does contribute. Also, a

Figure 8.22: The m=6 contribution to the total magnetic scalar potential

hexacontatetrapole contribution can come from the $m = 2$ and $m = 4$ terms, and these will also contribute to the total spherical hexacontatetrapole. These are not included in Figure 8.22. In fact, it can be shown that the $m = 2, 6, 10, \dots, 4m + 2$ are the only terms which contribute. Although, as m increases, the contribution from the higher harmonics rapidly diminishes. This type of analysis can be extended to include all higher order terms.

8.2.2.2 Magnetic flux density method

The magnetic flux density method yields the same results as does the magnetic scalar potential method. In other words, the plots shown in Figures 8.19 through 8.21 look identical to the ones computed using Equation (6.4). Also, higher order harmonics up to $m = 10$ were tested and they look identical to the ones computed using the magnetic scalar potential method.

8.3 Real six-pole motor

A real 6 – *pole* permanent magnet brushless DC motor including permanent magnets, stator iron, winding currents, and endcaps is used to test the toroidal expansion. Figure 8.23 shows the finite element model of the 6 – *pole* BLDC motor. Details are found in Kwon [86]. Figure 8.24 represents the pole arrangement of the

Figure 8.23: Full meshed model of a 6-pole BLDC motor with endcaps and stator coils

Figure 8.24: 6-pole permanent magnet pole arrangement

Poles	6	Type	BLDC
Power [Hp]	30	Turns per Coil	6
Stack Length	12 in.	Stator Slots	36
Coil Type	Lap winding	Rating Current [peak]	$45\sqrt{2}$
Winding	Y-connection	Pitch	5/6
Permanent Magnet	$\frac{R_{in}: 2.289 \text{ in.}}{R_{out}: 2.461 \text{ in.}}$ Angle: 25.71°	Stator	$\frac{R_{in}: 2.5 \text{ in.}}{R_{out}: 4.83 \text{ in.}}$

Table 8.1: Motor specifications

Figure 8.25: The total magnetic scalar potential

6 – *pole* motor. The motor specifications are shown Table 8.1.

In order to perform the charge simulation method, a combination of finite element analysis and integral equation formulation was used to compute the potentials on the potential cylinder. This is also discussed at length in Kwon’s thesis [89].

8.3.1 Balanced motor

The plot of the total scalar potential for a balanced 6 – *pole* motor is shown in Figure 8.25 . Figures 8.26 through 8.30 represent the various components which contribute to the magnetic scalar potential of a real 6 – *pole* motor under full load conditions.

Figure 8.26: The m=1 contribution to the total magnetic scalar potential

Figure 8.27: The m=2 contribution to the total magnetic scalar potential

8.3.2 Unbalanced motor

A few cases will be considered for an unbalanced 6 – *pole* motor under full load conditions. Figures 8.31 through 8.34 represent the various components which contribute to the magnetic scalar potential of a real 6 – *pole* motor with a 6.5 % demagnetization of magnet 4.

Figures 8.35 through 8.38 represent the various components which contribute to the magnetic scalar potential of a real 6 – *pole* motor with a 10 % axial off-set.

Figure 8.28: The m=3 contribution to the total magnetic scalar potential

Figure 8.29: The m=4 contribution to the total magnetic scalar potential

8.4 Summary

The main focus of this chapter is to illustrate how the toroidal expansion can be employed to characterize a permanent-magnet motor in terms of its equivalent multipole distribution. Initially, a simple permanent-magnet system is used as a test case for the application of the Q-function, and then the multipole reference table was used to help interpret the results.

It is shown that for a balanced set of permanent magnets, the Q-function for-

Figure 8.30: The m=5 contribution to the total magnetic scalar potential

Figure 8.31: The total magnetic scalar potential for 6.5 percent demagnetization of magnet 4

mulation predicts that the dominant term in the expansion is $m = \frac{\text{Pole number}}{2}$. For example, a single permanent magnet yields the $m = 1$ term as the dominant contribution to the magnetic scalar potential or the magnetic field intensity. A balanced 4 – pole magnetic system yields the $m = 2$ term as the dominant contribution. A balanced 10 – pole would yield the $m = 5$ term as the dominant contribution, etc.

For a real 6 – pole motor under balanced conditions, the $m = 3$ term is the major contribution. However, under unbalanced conditions, things change. Other

This is the $\Phi_P^{(1)}(\rho, \phi, z)$ component

Figure 8.32: The m=1 contribution for a 6.5 percent demagnetization of magnet 4

terms in the toroidal expansion contribute. This is to be expected. The results given for the 6 – *pole* motor are verified by Kwon [89]. Also, the results of this chapter agree with those computed using a spherical harmonic expansion. However, this comparison is only valid in a region of space in which the sphere completely encloses the motor. No comparison can be made in a region of space close to the motor where the spherical harmonic expansion is not valid. This region is called the “dead zone” and will be discussed more thoroughly in chapter 9.

Figure 8.33: The m=2 contribution for a 6.5 percent demagnetization of magnet 4

Figure 8.34: The m=3 contribution for a 6.5 percent demagnetization of magnet 4

This is the total $\Phi_P(\rho, \phi, z)$

Figure 8.35: The total magnetic scalar potential for 10 axial off-set

This is the $\Phi_P^{(1)}(\rho, \phi, z)$ component

Figure 8.36: The $m=1$ contribution for a 10 percent axial off-set

This is the $\Phi_P^{(2)}(\rho, \phi, z)$ component

Figure 8.37: The m=2 contribution for a 10 percent axial off-set

This is the $\Phi_P^{(3)}(\rho, \phi, z)$ component

Figure 8.38: The m=3 contribution for a 10 percent axial off-set

CHAPTER 9

DISCUSSION AND CONCLUSION

9.1 Introduction

A brief introduction to the more familiar mathematical methods used to solve electrodynamic problems which exhibit circular cylindrical symmetry will be discussed. The Q-function and the corresponding toroidal expansion represent an alternative method for the solution to these same types of problems. This method should not be used to displace the more familiar methods, but to enhance them. A brief summary of the more important aspects of the toroidal expansion will also be mentioned. The author will also discuss topics for future research.

9.2 Solution methods for circular cylindrical electromagnetic systems

Bessel functions and Elliptic integrals are the two types of formulations which are generally used for analyzing circular cylindrical magnetic systems. There are numerous texts and articles which discuss their properties and their applications. Moon, and Spencer's text [92], for example, gives a very readable treatment on the Bessel function approach to the solution of boundary value problems in cylindrical coordinates. Also, Jackson [10], and Smythe [19] are just two of the more advanced electromagnetic texts which discuss an Elliptic integral formulation for the solution of various problems which exhibit cylindrical symmetry.

The Bessel function formulation usually occurs in the context of a boundary value problem in cylindrical coordinates. For example, Laplace's equation, and Helmholtz's equation can usually be solved by *The Method of Separation of Variables*. Eisenhart [93] was the first to give a detailed description of the conditions needed to be satisfied for Laplace's equation and Helmholtz's equation to be separable. Also, Moon and Spencer wrote a series of excellent papers [94, 95, 96, 97, 98] which enhances Eisenhart's work. In particular, Moon and Spencer wrote a paper

[99] detailing the separation of variable method used in electromagnetic theory.

A number of excellent methods based on *Operational Calculus* [100] have been employed to solve problems which exhibit a circular cylindrical geometry. However, these methods require a much higher level of mathematical sophistication, and this has limited their use in engineering applications.

In a thorough literature search, one will find numerous papers dealing with both the Elliptic integral formulation and the Bessel function formulation, but very little mention of the Q-function formulation [32, 34, 35, 36] to the solution of circular cylindrical systems, and virtually no mention of its application to electromagnetic problems [31, 101]. The toroidal expansion is a third method for analyzing electromagnetic problems which exhibit circular cylindrical symmetry. It extends the number of ways one can tackle these problems and that, among other reasons, makes it useful.

9.3 Toroidal expansion for cylindrically symmetric problems in magnetostatics

The solution to cylindrically symmetric problems in magnetostatics was initially motivated by the need to find the external magnetic field from a finite cylindrical source. The method that was previously employed [89, ?] was a spherical harmonic expansion coupled with the charge simulation method. This allowed one to compute the magnetic field on a hypothetical sphere surrounding the actual magnetic source. This technique is appropriate when the far field is desired, but one can not use this method to compute the field close to the real cylindrical source because there exists a dead zone. This was mentioned in chapter 1. The dead zone which surrounds a magnetic cylindrical source is shown in Figure 9.1 . One can see that a toroidal expansion is valid in regions where a spherical harmonic expansion is prohibited. Also, in order to test the toroidal expansion, magnetic field data is measured close to the magnetic source. This data can not be tested if a spherical harmonic expansion is used. However, the Q-function formulation is well-suited to the task.

Figure 9.1: Dead zone surrounding a real cylindrical magnetic source

9.3.1 Non-cylindrical sources

Applying the Q-function to non-cylindrical magnetic sources has been discussed briefly in this thesis. It is important to realize that the toroidal expansion can be used for magnetic sources which do not exhibit cylindrical symmetry. In fact, the finite element method coupled with the charge simulation method [89, 90] and the Q-function can be used to expand a non-cylindrical source in terms of a toroidal expansion. The only requirement is that the magnetic source must be enclosed in a hypothetical cylindrical boundary. This expansion yields an alternative approach to the more familiar spherical harmonic expansion.

9.4 Future research and general conclusions

This thesis has concentrated on a magnetostatic integral formulation [?, 104], but there is a much wider range of topics for which the Q-function is applicable. For example, the Q-function can be applied to time-dependent magnetic field problems. This is one area of interest for future research. Also, boundary value problems have not been studied in this thesis. The Q-function formulation has been applied to

problems whose boundary is at infinity. In fact, the free-space Green's function was directly used to formulate the inverse distance function in terms of a Fourier series expansion. This Fourier series expansion was called a toroidal expansion whose weighting coefficients are the Q-functions. However, all free-space Green's functions can be modified to handle finite boundaries. This would possibly allow one to use a Q-function formulation where only a Bessel function approach existed. This adds another tool to the engineer's arsenal for tackling cylindrically symmetric electromagnetic problems.

The range of engineering applications which involve cylindrical geometries is immense. Areas such as electrodynamics, acoustics, continuum mechanics, dynamics of structures, and others can all benefit from adding another powerful mathematical tool which can be used to help solve problems which exhibit cylindrical symmetry.

The simple, but nonlinear pendulum discussed in Section 5.6 showed that the Q-function could be used to calculate the period of a pendulum using less terms than the more familiar Elliptic integral formulation. In other words, the Q-function could be used for its excellent numerical properties alone. One can employ the Q-function for acoustic scattering off finite cylinders. It can be employed to tackle structural and vibration problems which exhibit cylindrical symmetry [105, 106, 107]. One, in theory, can reformulate many of the problems of the past century, which exhibited circular cylindrical symmetry, in terms of Q-functions. In fact, the author of this thesis has done this for a few selected problems with much success. Solving an old problem in a new way can sometimes lead to new and useful interpretations.

Although a number of problems in areas other than magnetostatics were considered, the basic tenor of this thesis was a magnetostatic formulation of the Q-function and its application. This should be extended to include a dynamic formulation. There is no theoretical reason prohibiting the application of the Q-function to electrodynamic problems such as, radiation [108, 109], eddy currents [110, 111, 112], time varying electric and magnetic fields [113, 114, 115], etc. Also, one can employ the Q-function formulation for computing electromagnetic forces in cylindrical coils [116, 117]. The author has also applied the Q-function approach to superconducting cylindrical disks [118]. These are just a few of the research areas which the author

believes could benefit from a Q-function formulation.

LITERATURE CITED

- [1] Oliver Dimon Kell, *Foundations of Potential Theory*, Dover Publications, Inc., 1953.
- [2] W.J. Strenberg, and T.L. Smith, *The theory of Potential and Spherical Harmonics*, University of Toronto Press, 1964.
- [3] W.D. MacMillan, *The theory of Potential*, Dover Publications Inc., 1958.
- [4] A.D. Buckingham, "Molecular Quadrupole Moments," *Quarterly Reviews*, 13, 1929.
- [5] K Parker, "The Effect of Nuclear Multipole Moments on Electron Scattering," *Proc. Phys. Soc. A* 66, 1953.
- [6] E.B. Graham, J. Pierrus, and R.E. Raab, "Multipole Moments and Maxwell's Equations," *Journal of Physics B: Atomic, Molecular, and Optical Physics*, 25, 1992.
- [7] R.E. Raab, and O.L. De Lange, *Multipole Theory in Electromagnetism-Classical, Quantum, and Symmetry Aspects, with Applications*, Oxford University Press, 2005.
- [8] J. P. Wikswo, and K.R. Swinney, "A comparison of scalar multipole expansions," *Journal of Applied Physics*, 56 (11), December 1, 1984.
- [9] J. P. Wikswo, and K.R. Swinney, "Scalar multipole expansions and their dipole expansions," *Journal of Applied Physics*, 57 (9), May 1, 1984.
- [10] J.D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, 3rd ed., 1999.
- [11] L.Eyges, *The Classical Electromagnetic Field*, Dover Publications Inc., 1972.
- [12] J.B. Bronzan, "The Magnetic Scalar Potential," *American Journal of Physics*, 39, 1971.

- [13] C.G. Gray, "Simplified derivation of the magnetostatic multipole expansion using the scalar potential," *American Journal of Physics*, 46 (5), 1978.
- [14] C.G. Gray, "Magnetic multipole expansions using the scalar potential," *American Journal of Physics*, 47 (5), 1979.
- [15] I. Stakgold, *Green's Functions and Boundary Value Problems*, John Wiley & Sons, 1979.
- [16] J. Mathews, and R.L. Walker, *Mathematical Methods of Physics*, W.A. Benjamin Inc, 1970.
- [17] B. Friedman, *Principals and Techniques of Applied Mathematic*, Dover Publications Inc., 1990.
- [18] P. M. Morse and H.Feshbach, *Methods of Theoretical Physics*, McGraw Hill, 1953.
- [19] W.R. Smythe, *Static and Dynamic Electricity*, McGraw-Hill, 3rd ed., 1968.
- [20] E.W. Hobson, *On Green's Function for a Circular Disk, with applications to Electrostatic Problems*, Transactions of the Cambridge Philosophical Society, Vol. 18, 1900.
- [21] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, 4th ed., 1980.
- [22] Z.X. Wang and D.R. Guo, *Special Functions*, World Scientific, 1989.
- [23] C. Snow, *Formulas for Computing Capacitance and Inductance*, National Bureau of Standards Circular, 544, September 1, 1954.
- [24] C. Snow, *Hypergeometric and Legendre Functions with Applications to Integral Equations of Potential Theory*, National Bureau of Standards Applied Mathematics Series, 19, May 1, 1952.

- [25] C. Snow, "Potential Problems and Capacitance for a Conductor Bounded by Two Intersecting Spheres," *Journal of Research of the National Bureau of Standards*, Vol. 43, October 1949.
- [26] C. Snow, "A Standard of Small Capacitance," *Journal of Research of the National Bureau of Standards*, Vol. 42, March 1949.
- [27] C. Snow, "Magnetic Fields of Cylindrical Coils," *Natl. Bur. Stand., Applied Mathematics*, Ser 38, 1953.
- [28] N.N. Lebedev, *Special Functions and Their Applications*, Prentice-Hall, 1965.
- [29] E.T. Copson, *An Introduction to the Theory of a Complex Variable*, Oxford University Press, 1st ed., 1955.
- [30] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, Cambridge University Press, 4th ed., 1952.
- [31] J.P. Selvaggi, S. Salon, O-Mun Kwon, and M.V.K. Chari, "Calculating the External Magnetic Field From Permanent Magnets in Permanent-Magnet Motors-An Alternative Method," *IEEE Transactions on Magnetics*, Vol. 40, NO. 5, September 2004.
- [32] P.L. Walstrom, "Dipole-Sheet Multipole Magnets for Accelerators," *IEEE Transactions on Magnetics*, Vol. 30, No. 4, July 1994.
- [33] J.T. Conway, "Exact Solutions for the Magnetic Fields of Axisymmetric Solenoids and Current Distributions," *IEEE Transactions on Magnetics*, Vol. 37, No. 4, July 2001.
- [34] H.S. Cohl, A.R.P. Rau, J.E. Tohline, D.S. Browne, J.E. Cazes, and E.I. Barnes, "Useful alternative to the multipole expansion of $1/r$ potentials," *Physical Review A*, Vol. 64, 2001.
- [35] H.S. Cohl, J.E. Tohline, and A.R.P. Rau, "Developments in determining the gravitational potential using toroidal functions," *Astron. Nachr.*, 321, 2000.

- [36] H.S. Cohl and J.E. Tohline, "A Compact Cylindrical Green's Function Expansion for the Solution of Potential Problems," *The Astrophysical Journal*, 527, December 1999.
- [37] Kildishev, A.V. "Application of Spheroidal Functions in Magnetostatics, " *IEEE Transactions of Magnetism*, Vol. 40, No. 2, July 2004.
- [38] Kildishev, A.V. "Multipole Imaging of an Elongated Magnetic Source, " *IEEE Transactions of Magnetism*, Vol. 36, No. 5, July 2000.
- [39] Kildishev, A.V. "Multipole analysis of an elongated magnetic source by a cylindrical sensor array, " *IEEE Transactions of Magnetism*, Vol. 38, No. 5, July 2002.
- [40] J.J. Sakurai, *Modern Quantum Mechanics*, Addison-Wesley Publishing Company Inc., Revised Ed. 1994.
- [41] O.D. Jefimenko, *Electricity and Magnetism*, Electret Scientific Company, 1989.
- [42] D.M. Cook, *The Theory of the Electromagnetic Field*, Dover Publications Inc., 2003.
- [43] J. A Stratton, *Electromagnetic Theory*, McGraw-Hill Book Company, 1941.
- [44] W.L. Ferrar, *A Text-Book of Convergence*, Clarendon Press, 1938.
- [45] E.P. Furlani, "The field analysis and simulation of a permanent-magnet bias-field device for magneto-optic recording," *Journal of Physics D: Applied Physics*, 30 1997.
- [46] E.P. Furlani, "A formula for the Levitation Force between Magnetic Disks," *IEEE Transactions of Magnetism*, Vol. 29, No. 6, November 1993.
- [47] E.P. Furlani, S. Reznik, and W. Janson, "A Three-Dimensional Field Solution for Bipolar Cylinders," *IEEE Transactions of Magnetism*, Vol. 30, No. 5, September 1994.

- [48] E.P. Furlani, "Analytical analysis of magnetically coupled multipole cylinders," *Journal of Physics. D: Applied Physics*, 33 (2000).
- [49] E.P.Furlani, S. Reznik, and A. Kroll, "A Three-Dimensional Field Solution for Radially Polarized Cylinders," *IEEE Transactions of Magnetics*, Vol. 31, No. 1, January 1995.
- [50] L. K. Urankar, "Vector Potential and Magnetic Field of Current-Carrying Finite Arc Segment in Analytical Form, Part I: Filament Approximation," *IEEE Transactions of Magnetics*, Vol. Mag-16, No. 5, September 1980.
- [51] L. K. Urankar, "Vector Potential and Magnetic Field of Current-Carrying Finite Arc Segment in Analytical Form, Part II: Thin Sheet Approximation," *IEEE Transactions of Magnetics*, Vol. Mag-18, No. 3, September 1982.
- [52] L. K. Urankar, "Vector Potential and Magnetic Field of Current-Carrying Finite Arc Segment in Analytical Form, Part III: Exact Computation for Rectangular Cross Section," *IEEE Transactions of Magnetics*, Vol. Mag-18, No. 6, September 1982.
- [53] L. K. Urankar, "Vector Potential and Magnetic Field of Current-Carrying Finite Arc Segment in Analytical Form, Part IV: General Three-Dimensional Current Density," *IEEE Transactions of Magnetics*, Vol. Mag-20, No. 6, September 1984.
- [54] L. K. Urankar, "Vector Potential and Magnetic Field of Current-Carrying Finite Arc Segment in Analytical Form, Part V: Polygon Cross Section," *IEEE Transactions of Magnetics*, Vol. Mag-26, No. 3, September 1990.
- [55] S. Babic and M.M. Gavrilovic, "New Expression for Calculating Magnetic Fields due to Current-Carrying Solid Conductors," *IEEE Transactions on Magnetics*, Vol. 33, No. 5, September 1997.
- [56] C. Akyel, S. Babic, and S. Kincic. "New and Fast Procedures for Calculating the Mutual Inductance of Coaxial Circular Coils (Circular Coil-Disk Coil)," *IEEE Transactions on Magnetics*, Vol. 98, No. 5, September 2002.

- [57] D. Yu and K.S. Han, "Self-Inductance of Air-Core Circular Coils with Rectangular Cross Section," IEEE Transactions on Magnetics, Vol. Mag-23, No. 6, November 1987.
- [58] D.J. Craik, "Magnetostatics of Axially symmetric structures," Journal of Physics D: Applied Physics, Vol. 7, 1974.
- [59] S. R. Trout, "Use of Helmholtz Coils for Magnetic Measurements," IEEE Transactions on Magnetics, Vol. 23, No. 4, November 1988.
- [60] R.A. Schill, "General Relation for the Vector Magnetic Field of a Circular Current Loop: A Closer Look," IEEE Transactions on Magnetics, Vol. 39, No. 2, March 2003.
- [61] J.P. Blewett, "Magnetic Field Configurations Due to Air Core Coils," Journal of Applied Physics, Vol. 18, November 1947.
- [62] E.M. Purcell, "Helmholtz coils revisited," American Journal of Physics, Vol. 57, No. 1, January 1989.
- [63] T. Tominaka, M. Okamura, and T. Katayama, "Contribution of Iron Yoke on Helical Coils," IEEE Transactions on Applied Superconductivity, Vol. 12, No. 1 March 2002.
- [64] R.H. Good, "Elliptic integrals, the forgotten functions," European Journal of Physics, 22, 2001.
- [65] S. Babic, C. Akyel, and J. Salon, "New Procedures for Calculating the Mutual Inductance of the System: Filamentary Circular Coil-Massive Circular Solenoid," IEEE Transactions of Magnetics, Vol. 39, No. 3, May 2003.
- [66] S. Babic and C. Akyel, "Improvement in Calculation of the Self- and Mutual Inductance of Thin-Wall Solenoids and Disk Coils," IEEE Transactions of Magnetics, Vol. 36, No. 4, July 2000.

- [67] Ki-Bong Kim, E. Levi, Z. Zabar, and L. Birenbaum, "Mutual Inductance of Noncoaxial Circular Coils with Constant Current Density," IEEE Transactions of Magnetics, Vol. 33, No. 5, September 1997.
- [68] D.J. Craik and A.J. Harrison, "Comparative magnetostatic analyses of axisymmetric systems," J. Phys. D: Appl. Phys., Vol. 7, 1974.
- [69] E.T. Copson, "On the problem of the electrified disk," Proc. Edin. Math. Soc. 8, 14, 1947.
- [70] J. H. Jeans, *The Mathematical Theory of Electricity and Magnetism*, Cambridge University Press, 1960.
- [71] G. Green, *Mathematical Papers*, Chelsea Publishing Company, New York, 1970.
- [72] E. Weber, *Electromagnetic Theory, Volume I-Mapping of Fields*, John Wiley & Sons Inc. New York, 1950.
- [73] R. Friedberg, "The electrostatics and magnetostatics of a conducting disk," American Journal of Physics, 61 (12), December 1993.
- [74] L. Eyges, "Solution of boundary value problems with Laplace's equation for ellipsoids and elliptic cylinders," J. Math. Phys., 21 (3), March 1980.
- [75] L. C. Davis, and J.R. Reitz "Solution to Potential Problems near a Conducting Semi-Infinite Sheet or Conducting Disk," AJP, Vol. 39, 1971.
- [76] D. J. Griffiths, and Y. Li, "Charge density on a conducting needle," American Journal of Physics, 64 (6), June 1996.
- [77] W.R. Symthe, "Charged Right Circular Cylinder," Journal of Applied Physics, Vol. 27, No. 8, October 1956.
- [78] W.R. Symthe, "Charged Right Circular Cylinder," Journal of Applied Physics, Vol. 33, No. 10, October 1962.

- [79] W.R. Smythe, "Charged Sphere in Cylinder," Journal of Applied Physics, Vol. 31, No. 3, March 1960.
- [80] W.R. Smythe, "Charged Spheroid in Cylinder," Journal of Mathematical Physics, Vol. 4, No. 6, June 1963.
- [81] W.R. Smythe, "Charged Disk in Cylindrical Box," Journal of Applied Physics, 24, 70, 1953.
- [82] P.K. Wang, "Calculation of Electrostatic Fields Surrounding Finite Circular Cylindrical Conductors," IEEE Transactions on Antennas and Propagation, Vol. AP-32, No. 9, September 1984.
- [83] G. Planinsic and T. Prosen, "Conducting rod on the axis of a charge ring: The Kelvin water drop generator," American Journal of Physics, Vol. 68, No. 12, December 2000.
- [84] S. Kato and M. Kobayashi, "Magnetic Charge Densities Around Edges of Circular Cylinders," IEEE Transactions on Magnetics, Vol. 32, No. 3, May 1996.
- [85] T. R. Ferguson, and R.H. Duncan, "Charged Cylindrical Tube," Journal of Applied Physics, Vol. 32, No. 7, July 1961.
- [86] C. Chakraborty, D. R. Poddar, A. Chakraborty, and B. N. Das, "Electrostatic Charge Distribution and Capacitance of Isolated Cylinders and Truncated Cones in Free Space," IEEE Transactions on Electromagnetic Compatibility, Vol. 35, No. 1, February 1993.
- [87] J.C. Maxwell, *An Elementary Treatise on Electricity*, Dover Publications, Inc., 2nd ed., 2005.
- [88] A.J. Schwab, *Field Theory Concepts*, Springer-Verlag, 1988.
- [89] O. Kwon, "The analysis of effects of certain asymmetries in electrical machines," Ph.D. dissertation, Rensselaer Polytechnic Institute, Troy NY, August 2003.

- [90] Chari, M.V.K., Salon, S.J., *Numerical Methods in Electromagnetism*, Academic Press, 2000.
- [91] S. Babic, S. Salon, "The Mutual Inductance of Two Thin Coaxial Disk Coils in Air," IEEE Transactions of Magnetics, Vol. 40, No. 2, March 2004.
- [92] P. Moon and D.E. Spencer, *Field Theory For Engineers*, D. Van Nostrand Company Inc., 1961.
- [93] L.R. Eisenhart, "Separable Systems of Stackel," Annals of Mathematics, Vol. 35, No. 2, April 1934.
- [94] P. Moon and D.E. Spencer, "Separability in a Class of Coordinate Systems," Journal of Franklin Institute, September 1952.
- [95] P. Moon and D.E. Spencer, "Some Coordinate Systems Associated with Elliptic Functions," Journal of Franklin Institute, June 1953.
- [96] P. Moon and D.E. Spencer, "Theorems on Separability in Riemannian n -Space," Journal of Franklin Institute, August 1952.
- [97] P. Moon and D.E. Spencer, "Recent Investigations of the Separation of Laplace's Equation," Journal of Franklin Institute, April 1953.
- [98] P. Moon, and D.E. Spencer, "Separability Conditions for the Laplace and Helmholtz Equations," Journal of Franklin Institute, June 1952.
- [99] D.E. Spencer, "Separation of Variables in Electromagnetic Theory," Journal of Applied Physics, Vol. 22, No. 4, April 1951.
- [100] I. N. Sneddon, *The Use of Integral Transform*, McGraw-Hill Book Company Inc., 1972.
- [101] P. Moon and D.E. Spencer, "Cylindrical and Rotational Coordinate Systems," Journal of Franklin Institute, October 1951.
- [102] O.Kwon, C.Surussavadee, M.V.K Chari, S.Salon, K.Sivasubramaniam, "Analysis of the Far Field Permanent Magnet Motors and study of effects of

geometric asymmetries and unbalance in magnet design,” IEEE Transactions on Magnetics, Vol. 40, March 2004.

- [103] H. Batemanm, *Partial Differential Equations of Mathematical Physics*, Dover Publications, 1944.
- [104] H. Bateman, “Some Integral Equations of Potential Theory,” Journal of Applied Physics, Vol. 17, February 1946.
- [105] I. N. Sneddon, *Fourier Transforms*, McGraw-Hill Book Company Inc., 1st ed., 1951.
- [106] N. W. McLachlan, *Bessel Functions For Engineer*, Clarendon Press, 2nd ed., 1955.
- [107] N. W. McLachlan, *Ordinary Non-Linear Differential Equations In Engineering And Physical Sciences*, 2nd ed., Clarendon Press, 1958.
- [108] P.L.Overfelt, “Near Fields of the Constant Current Thin Circular Loop Antenna of Arbitrary Radius,” IEEE Transactions on Antennas and Propagation, Vol. 44, No. 2, February 1996.
- [109] D.H. Werner, and R. Mittra, *Frontiers in Electromagnetics*, IEEE Press, 2000.
- [110] R. Hagel, L. Gong, and R. Unbehauen, “Evaluation of the Influence of a Finitely Long Circular Cylindrical Shield on the Electromagnetic Field,” IEEE Transactions on Magnetics, Vol. 28, No. 5, September 1992.
- [111] T.H. Fawzi, K.F. Ali, and P. E. Burke, “Eddy Current Losses in Finite Length Conducting Cylinders,” IEEE Transactions of Magnetics, Vol. Mag-19, No. 5, September 1983.
- [112] C.V. Dodd and W.E. Deeds, “Analytical Solutions to Eddy-Current Probe-Coil Problems,” Journal of Applied Physics, Vol. 39, No. 6, May 1968.

- [113] C.J. Bouwkamp and N.G. de Bruijn, "The Electrical Field of a Point Charge Inside a Cylinder, in Connection with Wave Guide Theory," Journal of Applied Physics, Vol. 18, June 1947.
- [114] M.J. Lahart, "Use of electromagnetic scalar potentials in boundary value problems," American Journal of Physics, Vol. 72, No.1, January 2004.
- [115] M.A. Avila, "Magnetic fields of spherical, cylindrical, and ellipsoidal electrical charge superficial distributions at rotation," Revista Mexicana De Fisica, 49, April 2003.
- [116] Ki-Bong Kim, E. Levi, Z. Zabbar, and L. Birenbaum, "Restoring Force Between Two Noncoaxial Circular Coils," IEEE Transactions on Magnetics, Vol. 32, No. 2, March 1996.
- [117] N.J. Groom, "Expanded Equations for Torque and Force on a Cylindrical Permanent Magnet Core in a Large-Gap Magnetic Suspension System," NASA Technical Paper, 3638, February 1997.
- [118] A.Badia and H.C. Freyhardt, "Meissner state properties of a superconducting disk in a non-uniform magnetic field," Journal of Applied Physics, Vol. 83, No. 5, March 1 1998.
- [119] O.D. Jefimenko, *Causality, Electromagnetic Induction, and Gravitation*, Electret Scientific Company, 2000.
- [120] J. Bouwkamp, and H.B.G.Casimir, "On Multipole Expansions In The Theory Of Electromagnetic Radiation," Physica, 20, 539, 1954.
- [121] T. MacRobert, *Spherical Harmonics*, Pergamon Press, 1967.
- [122] H. Gonzalez, S.R. Juarez, P. Kielanowski, and M. Loewe, "Multipole expansion in magnetostatics," American Journal of Physics, Vol. 66, No. 3, March 1998.
- [123] C.G. Gray, "Multipole expansions of electromagnetic fields using Debye potential," American Journal of Physics, Vol. 46, No. 2, February 1978.

- [124] G.A. Estevez, "Solution to a classical problem in electrostatics in Legendre polynomials expansion," American Journal of Physics, Vol. 56, No. 12, December 1988.
- [125] J.C. Maxwell. *A Treatise on Electricity and Magnetism*, Dover Publications Inc., Vol. 1&2, 1954.
- [126] H.D. Block, *Introduction to Tensor Analysis*, Charles E Merrill Books Inc., 1962.
- [127] J.L. Synge, and A. Schild, *Tensor Calculus*, Dover Publications Inc., 1969.
- [128] G. N. Watson, *A Treatise On The Theory Of Bessel Functions*, Cambridge University Press, 1962.
- [129] M.J. Lighthill, *Introduction to Fourier Analysis and Generalized Functions*, Cambridge University Press, 1960.
- [130] A. Papoulis, *The Fourier Integral and its Applications*, McGraw-Hill Book Company Inc., 1962.
- [131] G. Temple, "Theories and Applications of Generalized Functions", J. Lond. Math. Soc. 28, 134-148, 1953.
- [132] M. Abramowitz, and I. Stegun, *Handbook of Mathematical Functions*, Dover Publications Inc., 1972.
- [133] Waterloo Maple Inc., Maple, Version 9.5, Waterloo, Ontario, Canada N2L 6C2, 2004.
- [134] Wolfram Research Inc., Mathematica, Version 5.0, Champaign, IL, 2003.
- [135] The MathWorks Inc., Matlab, Version 7.0.4, Natick, MA 01760-2098, 2005.
- [136] A. J. Pettofrezzo, *Matrices And Transformations*, Dover Publications, Inc., 1978.

APPENDIX A

VECTOR IDENTITIES AND THEOREMS

A.1 Introduction

This Appendix will tabulate some of the more useful vector identities and vector theorems used throughout this thesis. Detailed proofs of all the theorems can be found in most texts of vector calculus. The theorems will not be written in their most general form but will be expressed in the form which most often appears in physics or engineering texts.

Poisson's theorem is introduced because of its importance in expressing magnetic field calculations in terms of an integral formulation. This will be used quite often in this thesis.

A.2 Vector identities

$$\nabla(\Phi U) = \Phi \nabla U + U \nabla \Phi \quad (\text{A.1})$$

$$\begin{aligned} \nabla(\mathbf{F} \cdot \mathbf{G}) &= (\mathbf{F} \cdot \nabla) \mathbf{G} + \mathbf{F} \times (\nabla \times \mathbf{G}) \\ &\quad + (\mathbf{G} \cdot \nabla) \mathbf{F} + \mathbf{G} \times (\nabla \times \mathbf{F}) \end{aligned} \quad (\text{A.2})$$

$$\nabla \cdot (\Phi \mathbf{F}) = \Phi \nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla \Phi \quad (\text{A.3})$$

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}) \quad (\text{A.4})$$

$$\nabla \times (\Phi \mathbf{F}) = \Phi \nabla \times \mathbf{F} + \nabla \Phi \times \mathbf{F} \quad (\text{A.5})$$

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = (\mathbf{G} \cdot \nabla) \mathbf{F} + \mathbf{F} (\nabla \cdot \mathbf{G}) - (\mathbf{F} \cdot \nabla) \mathbf{G} - \mathbf{G} (\nabla \cdot \mathbf{F}) \quad (\text{A.6})$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0 \quad (\text{A.7})$$

$$\nabla \times \nabla U = 0 \quad (\text{A.8})$$

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla (\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F} \quad (\text{A.9})$$

$$\nabla^2 (\nabla \times \mathbf{F}) = \nabla \times (\nabla^2 \mathbf{F}) \quad (\text{A.10})$$

$$\nabla^2 (\nabla \Phi) = \nabla (\nabla^2 \Phi) \quad (\text{A.11})$$

A.3 Vector Theorems

$$\oint_C \mathbf{F} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \text{ (Stokes's theorem)} \quad (\text{A.12})$$

$$\oint_C U d\mathbf{l} = \int_S d\mathbf{S} \times \nabla U \quad (\text{A.13})$$

$$\oint_C \mathbf{F} \times d\mathbf{l} = \int_S (\nabla \cdot \mathbf{F}) d\mathbf{S} - \int_S \nabla (\mathbf{F} \cdot d\mathbf{S}) \quad (\text{A.14})$$

$$\oint_S \mathbf{F} \cdot d\mathbf{S} = \int_V (\nabla \cdot \mathbf{F}) dV \text{ (Gauss's theorem)} \quad (\text{A.15})$$

$$\oint_S U d\mathbf{S} = \int_V (\nabla U) dV \quad (\text{A.16})$$

$$\oint_S \mathbf{F} \times d\mathbf{S} = - \int_V (\nabla \times \mathbf{F}) dV \quad (\text{A.17})$$

$$\oint_S \mathbf{F} (\mathbf{G} \cdot d\mathbf{S}) = \int_V [(\nabla \cdot \mathbf{G}) \mathbf{F} + (\mathbf{G} \cdot \nabla) \mathbf{F}] dV \quad (\text{A.18})$$

Green's theorems : U_1, U_2 are Scalar functions

$$\oint_S U_1 \nabla U_2 \cdot d\mathbf{S} = \int_V (U_1 \nabla^2 U_2 + \nabla U_1 \cdot \nabla U_2) dV \quad (\text{A.19})$$

$$\oint_S (U_1 \nabla U_2 - U_2 \nabla U_1) \cdot d\mathbf{S} = \int_V (U_1 \nabla^2 U_2 - U_2 \nabla^2 U_1) dV \quad (\text{A.20})$$

$$\oint_S \nabla U \cdot d\mathbf{S} = \int_V \nabla^2 U dV \quad (\text{A.21})$$

Poisson's theorem : $\mathbf{G} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F})$

$$\mathbf{F} = -\frac{1}{4\pi} \int_V \frac{\mathbf{G}}{|\mathbf{r} - \mathbf{r}'|} dV' \quad (\text{A.22})$$

A.3.1 Fundamental theorems of vector fields

A.3.1.1 Gauss's theorem

Gauss's theorem states that the flux integral of the outward normal component of a vector function, $\mathbf{F}$, over a closed and bounded surface, S , is equal to the volume integral of the $\nabla \cdot \mathbf{F}$ over the volume enclosed by the surface, S . This is

mathematically expressed as equation (A.15)

A.3.1.2 Stokes's theorem

Unlike Gauss's theorem which applies to the flux integral of a vector function, Stokes's theorem applies to a circulation integral. The circulation integral of a vector function, $\mathbf{F}$, which traces a closed and counterclockwise path is equal to the integral of the normal component of the $\nabla \times \mathbf{F}$ over the surface which bounds the closed path. This is mathematically expressed as Equation (A.12)

A.3.1.3 Helmholtz's theorem

A vector function, $\mathbf{F}$, is said to be uniquely determined throughout all space by its curl and by its divergence, and must be regular at infinity. That is, it must approach zero at least as $\frac{1}{R^2}$.

A.3.1.4 Poisson's theorem

Poisson's theorem is an extremely useful theorem for expressing a vector in terms of an integral. Equation (A.22) represents Poisson's theorem for vector fields [41, 119]. Mathematically, Poisson's theorem is written as

$$\mathbf{F} = -\frac{1}{4\pi} \int_{All\ Space} \frac{\nabla (\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F})}{|\mathbf{r} - \mathbf{r}'|} d^3r' \quad (\text{A.23})$$

Corollary to Poisson's theorem A vector function, $\mathbf{F}$, whose divergence and whose curl are both zero outside some finite region of space can be expressed as

$$\mathbf{F} = -\nabla\Phi + \nabla \times \mathbf{A} \quad (\text{A.24})$$

where

$$\Phi = \frac{1}{4\pi} \int_{All\ Space} \frac{\nabla \cdot \mathbf{F}}{|\mathbf{r} - \mathbf{r}'|} d^3r' + \Phi_0 \quad (\text{A.25})$$

and

$$\mathbf{A} = \frac{1}{4\pi} \int_{All\ Space} \frac{\nabla \times \mathbf{F}}{|\mathbf{r} - \mathbf{r}'|} d^3r' + \mathbf{A}_0 \quad (\text{A.26})$$

where $\mathbf{A}$ is called the vector potential of $\mathbf{F}$ and where Φ_0 is an arbitrary constant. However, the choice of $\mathbf{A}_0$ can be the gradient of any function since its curl will always be zero, and therefore, it has no effect on $\mathbf{F}$ as computed from Equation (A.24). The method or technique for choosing a possible $\mathbf{A}_0$ from a list of possible expressions is called a *gauge transformation*. For example, if $\mathbf{A}_0$ is chosen to be a constant, then it is called the *Coulomb gauge*. The integrals given by Equations (A.23), (A.25), and (A.26), are called the *Poisson integrals*. These integrals form the basis of the integral formulation for all the magnetic field problems discussed in this thesis.

APPENDIX B

MULTIPOLE EXPANSIONS REPRESENTED BY CARTESIAN TENSORS

B.1 Introduction

Multipole theory in electromagnetism [120, 121, 122, 123, 124] has a long and fruitful history. In fact, Clerk Maxwell [125], used the same analysis to evaluate many of the integrals he obtained in solving various electromagnetic problems. This Appendix will discuss some important aspects of multipole theory as it pertains to classical electromagnetic theory, and its application to the various topics discussed in this thesis. A Cartesian tensor notation [126, 127] will be employed in order to formulate a number of useful equations in a more compact form. The analysis in this appendix is done in the spirit of Raab and De Lange [7].

B.2 Multipole analysis in Cartesian tensor notation

Figure B.1 illustrates a general electromagnetic source. A classical multipole distribution can be developed by considering the inverse distance function. It can be written as

$$\begin{aligned} \frac{1}{|\mathbf{R} - \mathbf{r}|} &= \frac{1}{\sqrt{R^2 + r^2 - 2Rr \cos(\gamma)}} \\ &= \frac{1}{\sqrt{R^2 - 2\mathbf{R} \cdot \mathbf{r} + r^2}} \end{aligned} \quad (\text{B.1})$$

For $|\mathbf{R}| > |\mathbf{r}|$, one obtains

$$\frac{1}{|\mathbf{R} - \mathbf{r}|} = \frac{1}{R} \left(1 + \frac{1}{R^2}(r^2 - 2\mathbf{R} \cdot \mathbf{r}) \right)^{-\frac{1}{2}} \quad (\text{B.2})$$

and for $|\mathbf{r}| > |\mathbf{R}|$, one obtains

$$\frac{1}{|\mathbf{R} - \mathbf{r}|} = \frac{1}{r} \left(1 + \frac{1}{r^2}(R^2 - 2\mathbf{R} \cdot \mathbf{r}) \right)^{-\frac{1}{2}} \quad (\text{B.3})$$

Figure B.1: An electromagnetic source

Consider the case when $|\mathbf{R}| > |\mathbf{r}|$. With a little algebra, Equation (B.2) can be expanded in a binomial expansion as follows:

$$\begin{aligned}
 \frac{1}{|\mathbf{R} - \mathbf{r}|} &= \frac{1}{R} - \frac{1}{2R^3} \left(r^2 - 2\mathbf{R} \cdot \mathbf{r} \right) + \frac{3}{8R^5} \left(r^4 - 4r^2(\mathbf{R} \cdot \mathbf{r}) + 4(\mathbf{R} \cdot \mathbf{r})^2 \right) - \\
 &\quad \frac{5}{16R^7} \left(r^6 - 6r^4(\mathbf{R} \cdot \mathbf{r}) + 12r^2(\mathbf{R} \cdot \mathbf{r})^2 - 8(\mathbf{R} \cdot \mathbf{r})^3 \right) + \\
 &\quad \frac{35}{128R^9} \left(r^8 - 8r^6(\mathbf{R} \cdot \mathbf{r}) + 24r^4(\mathbf{R} \cdot \mathbf{r})^2 - 32r^2(\mathbf{R} \cdot \mathbf{r})^3 + \right. \\
 &\quad \left. 16(\mathbf{R} \cdot \mathbf{r})^4 \right) - \frac{63}{256R^{11}} \left(r^{10} - 10r^8(\mathbf{R} \cdot \mathbf{r}) + 40r^6(\mathbf{R} \cdot \mathbf{r})^2 - \right. \\
 &\quad \left. 80r^4(\mathbf{R} \cdot \mathbf{r})^3 + 80r^2(\mathbf{R} \cdot \mathbf{r})^4 - 32(\mathbf{R} \cdot \mathbf{r})^5 \right) + \dots
 \end{aligned} \tag{B.4}$$

If one collects terms of the same order in r^n , then one can obtain the following equations.

$$O(r^0) \Rightarrow \frac{1}{R} \quad (\text{B.5})$$

$$O(r^1) \Rightarrow \frac{\mathbf{R} \cdot \mathbf{r}}{R^3} \quad (\text{B.6})$$

$$O(r^2) \Rightarrow \frac{3(\mathbf{R} \cdot \mathbf{r})^2 - r^2 R^2}{2R^5} \quad (\text{B.7})$$

$$O(r^3) \Rightarrow \frac{5(\mathbf{R} \cdot \mathbf{r})^3 - 3R^2 r^2 (\mathbf{R} \cdot \mathbf{r})}{2R^7} \quad (\text{B.8})$$

$$O(r^4) \Rightarrow \frac{35(\mathbf{R} \cdot \mathbf{r})^4 - 30R^2 r^2 (\mathbf{R} \cdot \mathbf{r})^2 + 3R^4 r^4}{8R^9} \quad (\text{B.9})$$

$$O(r^5) \Rightarrow \frac{63(\mathbf{R} \cdot \mathbf{r})^5 - 70R^2 r^2 (\mathbf{R} \cdot \mathbf{r})^3 + 15R^4 r^4 (\mathbf{R} \cdot \mathbf{r})}{8R^{11}} \quad (\text{B.10})$$

$\vdots$

Using Equations (B.5) through (B.10), Equation (B.4) can be rewritten as follows:

$$\begin{aligned} \frac{1}{|\mathbf{R} - \mathbf{r}|} &= \frac{1}{R} + \frac{\mathbf{R} \cdot \mathbf{r}}{R^3} + \frac{3(\mathbf{R} \cdot \mathbf{r})^2 - r^2 R^2}{2R^5} + \frac{5(\mathbf{R} \cdot \mathbf{r})^3 - 3R^2 r^2 (\mathbf{R} \cdot \mathbf{r})}{2R^7} + \\ &\quad \frac{35(\mathbf{R} \cdot \mathbf{r})^4 - 30R^2 r^2 (\mathbf{R} \cdot \mathbf{r})^2 + 3R^4 r^4}{8R^9} + \\ &\quad \frac{63(\mathbf{R} \cdot \mathbf{r})^5 - 70R^2 r^2 (\mathbf{R} \cdot \mathbf{r})^3 + 15R^4 r^4 (\mathbf{R} \cdot \mathbf{r})}{8R^{11}} + \dots \end{aligned} \quad (\text{B.11})$$

Assume, for the time being, that the source in Figure B.1 is an electrostatic charge density. The electrostatic scalar potential can be written as

$$\begin{aligned} \Phi(\mathbf{R}) &= \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r})}{|\mathbf{R} - \mathbf{r}|} d^3r \\ &= \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}) \left\{ \frac{1}{R} + \frac{\mathbf{R} \cdot \mathbf{r}}{R^3} + \frac{3(\mathbf{R} \cdot \mathbf{r})^2 - r^2 R^2}{2R^5} + \right. \\ &\quad \frac{5(\mathbf{R} \cdot \mathbf{r})^3 - 3R^2 r^2 (\mathbf{R} \cdot \mathbf{r})}{2R^7} + \\ &\quad \frac{35(\mathbf{R} \cdot \mathbf{r})^4 - 30R^2 r^2 (\mathbf{R} \cdot \mathbf{r})^2 + 3R^4 r^4}{8R^9} + \\ &\quad \left. \frac{63(\mathbf{R} \cdot \mathbf{r})^5 - 70R^2 r^2 (\mathbf{R} \cdot \mathbf{r})^3 + 15R^4 r^4 (\mathbf{R} \cdot \mathbf{r})}{8R^{11}} + \dots \right\} d^3r \end{aligned} \quad (\text{B.12})$$

One can now formulate Equation (B.12) in terms of Cartesian tensor notation. This is given by

$$\begin{aligned}\Phi(\mathbf{R}) &= \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r})}{|\mathbf{R} - \mathbf{r}|} d^3r \\ &= \frac{1}{4\pi\epsilon_0} \left\{ \frac{q}{R} + \frac{R_i}{R^3} p_i + \frac{3R_i R_j - R^2 \delta_{ij}}{2R^5} q_{ij} + \right. \\ &\quad \left. \frac{5R_i R_j R_k - R^2 (R_i \delta_{jk} + R_j \delta_{ki} + R_k \delta_{ij})}{2R^7} q_{ijk} + \dots \right\} \quad (\text{B.13})\end{aligned}$$

where δ_{ij} is the Kronecker delta function given by

$$\delta_{ij} = \begin{cases} 1 & \forall i = j \\ 0 & \forall i \neq j \end{cases} \quad (\text{B.14})$$

The multipole moments are defined as

$$q = \int \rho(\mathbf{r}) d^3r \quad (\text{B.15})$$

$$p_i = \int r_i \rho(\mathbf{r}) d^3r \quad (\text{B.16})$$

$$q_{ij} = \int r_i r_j \rho(\mathbf{r}) d^3r \quad (\text{B.17})$$

$$q_{ijk} = \int r_i r_j r_k \rho(\mathbf{r}) d^3r \quad (\text{B.18})$$

$$q_{ijkl} = \int r_i r_j r_k r_l \rho(\mathbf{r}) d^3r \quad (\text{B.19})$$

$$q_{ijklm} = \int r_i r_j r_k r_l r_m \rho(\mathbf{r}) d^3r \quad (\text{B.20})$$

$$\begin{aligned} &\vdots \\ q_{ijklm\dots s} &= \int r_i r_j r_k r_l r_m \dots r_s \rho(\mathbf{r}) d^3r \quad (\text{B.21}) \end{aligned}$$

In Equations (B.15) through (B.21), q represents the electric monopole moment, p_i represents the electric dipole moment, q_{ij} represents the electric quadrupole moment, q_{ijk} represents the electric octupole moment, and q_{ijkl} represents the electric hexadecapole moment, etc. This process can be continued indefinitely. These moment are called the primitive moments as apposed to the traceless moments. The

traceless moment, for an electrostatic system, exists for all moments beyond the dipole. Traceless moments are sometime useful because their description requires a fewer number of independent components. For example, the traceless quadrupole moment can be written as

$$\Lambda_{ij} = \frac{1}{2} (3q_{ij} - q_{kk}\delta_{ij}) \quad (\text{B.22})$$

where $\delta_{ii} = 3$, and $\Lambda_{ii} = 0$. The quadrupole potential, for example, can be written as

$$\Phi(\mathbf{R})_{quadrupole} = \frac{1}{4\pi\epsilon_0} \frac{(3R_i R_j - R^2 \delta_{ij})}{2R^5} \frac{1}{3} (2\Lambda_{ij} + q_{kk}\delta_{ij}) \quad (\text{B.23})$$

The traceless moment description is discussed in-depth in Raab and De Lange [7].

In order to derive the Cartesian tensor formulation given by Equation (B.13), one needs indicial notation or Einstein's summation notation. For example, the dot product of two vectors can be written as

$$\mathbf{R} \cdot \mathbf{r} = R_i r_i \quad (\text{B.24})$$

where a repeated index means a summation on that index. In other words, $R_i r_i$ implies $\sum_{i=1}^3 R_i r_i$. This allows one to rewrite the electric dipole potential as

$$\begin{aligned} \Phi(\mathbf{R})_{dipole} &= \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}) \frac{\mathbf{R} \cdot \mathbf{r}}{R^3} d^3r \\ &= \frac{R_i}{4\pi\epsilon_0 R^3} \int r_i \rho(\mathbf{r}) d^3r \\ &= \frac{R_i p_i}{4\pi\epsilon_0 R^3} \end{aligned} \quad (\text{B.25})$$

In order to write the electric quadrupole moment in tensor notation, one needs to rewrite the vector expression, $3(\mathbf{R} \cdot \mathbf{r})^2 - r^2 R^2$, in indicial notation. This is done by

considering the following

$$\begin{aligned}
3(\mathbf{R} \cdot \mathbf{r})^2 - r^2 R^2 &= 3(\mathbf{R} \cdot \mathbf{r})^2 - (\mathbf{r} \cdot \mathbf{r})^2 R^2 \\
&= 3 \left(\sum_{i=1}^3 R_i r_i \right)^2 - R^2 \sum_{i=1}^3 r_i r_i \\
&= 3(R_1 r_1 + R_2 r_2 + R_3 r_3)^2 - R^2(r_1^2 + r_2^2 + r_3^2) \\
&= 3(R_1^2 r_1^2 + R_1 r_1 R_2 r_2 + R_1 r_1 R_3 r_3 + R_2 r_2 R_3 r_3 + \\
&\quad R_2^2 r_2^2 + R_2 r_2 R_3 r_3 + R_3 r_3 R_1 r_1 + R_3 r_3 R_2 r_2 + R_3^2 r_3^2) - \\
&\quad R^2(r_1^2 + r_2^2 + r_3^2) \\
&= \sum_{i,j=1}^3 R_i R_j r_i r_j - \sum_{i,j=1}^3 r_i r_j \delta_{ij} \\
&\Rightarrow (3R_i R_j - R^2 \delta_{ij}) r_i r_j
\end{aligned} \tag{B.26}$$

The electric quadrupole potential can be written as

$$\begin{aligned}
\Phi(\mathbf{R})_{quadrupole} &= \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}) \frac{3(\mathbf{R} \cdot \mathbf{r})^2 - r^2 R^2}{2R^5} d^3r \\
&= \frac{1}{4\pi\epsilon_0} \frac{(3R_i R_j - R^2 \delta_{ij})}{2R^5} \int r_i r_j \rho(\mathbf{r}) d^3r \\
&= \frac{1}{4\pi\epsilon_0} \frac{(3R_i R_j - R^2 \delta_{ij})}{2R^5} q_{ij}
\end{aligned} \tag{B.27}$$

Consider the octupole potential given by

$$\begin{aligned}
\Phi(\mathbf{R})_{octupole} &= \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}) \frac{5(\mathbf{R} \cdot \mathbf{r})^3 - 3R^2 r^2 (\mathbf{R} \cdot \mathbf{r})}{2R^7} d^3r \\
&= \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}) (5(\mathbf{R} \cdot \mathbf{r})^3 - 3R^2 r^2 (\mathbf{R} \cdot \mathbf{r})) d^3r
\end{aligned} \tag{B.28}$$

Using the same methodology for the octupole contribution as was used for the

quadrupole contribution, one can write the following:

$$\begin{aligned}
5(\mathbf{R} \cdot \mathbf{r})^3 - 3R^2 r^2 (\mathbf{R} \cdot \mathbf{r}) &= 5(\mathbf{R} \cdot \mathbf{r})^3 - 3R^2 (\mathbf{r} \cdot \mathbf{r})^2 (\mathbf{R} \cdot \mathbf{r}) \\
&= 5 \left(\sum_{i=1}^3 R_i r_i \right)^3 - 3R^2 \sum_{i=1}^3 r_i r_i \sum_{k=1}^3 R_k r_k \\
&= 5 \sum_{i,j,k=1}^3 R_i R_j R_k r_i r_j r_k - 3R^2 \sum_{i,j=1}^3 r_i r_j \delta_{ij} \sum_{k=1}^3 R_k r_k \\
&= 5 \sum_{i,j,k=1}^3 R_i R_j R_k r_i r_j r_k - 3R^2 \sum_{i,j,k=1}^3 R_k r_i r_j r_k \delta_{ij} \\
&= \sum_{i,j,k=1}^3 (5R_i R_j R_k - 3R^2 R_k \delta_{ij}) r_i r_j r_k \\
&= \sum_{i,j,k=1}^3 \left(5R_i R_j R_k - R^2 (R_i \delta_{jk} + R_j \delta_{ki} + R_k \delta_{ij}) \right) r_i r_j r_k \\
&\Rightarrow \left(5R_i R_j R_k - R^2 (R_i \delta_{jk} + R_j \delta_{ki} + R_k \delta_{ij}) \right) r_i r_j r_k
\end{aligned} \tag{B.29}$$

The electric octupole potential can be written as

$$\begin{aligned}
\Phi(\mathbf{R})_{octupole} &= \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}) \frac{5(\mathbf{R} \cdot \mathbf{r})^3 - 3R^2 r^2 (\mathbf{R} \cdot \mathbf{r})}{2R^7} d^3r \\
&= \frac{1}{4\pi\epsilon_0} \frac{5R_i R_j R_k - R^2 (R_i \delta_{jk} + R_j \delta_{ki} + R_k \delta_{ij})}{2R^7} \int r_i r_j r_k \rho(\mathbf{r}) d^3r \\
&= \frac{1}{4\pi\epsilon_0} \frac{5R_i R_j R_k - R^2 (R_i \delta_{jk} + R_j \delta_{ki} + R_k \delta_{ij})}{2R^7} q_{ijk}
\end{aligned} \tag{B.30}$$

This methodology can be used to represent as many terms in the multipole expansion as desired by employing Cartesian tensor notation.

An important aspect of the electric multipole is that it is, in general, origin dependent. For example, Figure B.2 illustrates a general shifted electromagnetic source. The position of the source point has been shifted relative to the origin of the original coordinate system. It is given by

$$\mathbf{r}' = \mathbf{r} - \mathbf{u} \tag{B.31}$$

Figure B.2: Origin dependence of a multipole expansion

If the source represents a charge density, then one can write, for example, the quadrupole moment relative to the original coordinate system as

$$\begin{aligned}
 q'_{ij} &= \int (r_i - u_i)(r_j - u_j) \rho(\mathbf{r}) d^3r \\
 &= \int (r_i r_j - r_i u_j - u_i r_j + u_i u_j) \rho(\mathbf{r}) d^3r \\
 &= \int r_i r_j \rho(\mathbf{r}) d^3r - u_j \int r_i \rho(\mathbf{r}) d^3r - u_i \int r_j \rho(\mathbf{r}) d^3r + u_i u_j \int \rho(\mathbf{r}) d^3r \\
 &= q_{ij} - p_i u_j - p_j u_i + q u_i u_j
 \end{aligned} \tag{B.32}$$

Equation (B.32) shows that the quadrupole moment is independent of the origin of the coordinate system as long as no dipole moment, or monopole moment exist. For an electric multipole moment, the first non-vanishing multipole is independent of the shift in the origin of the coordinate system. This same procedure can be applied to a magnetic system and it can be shown that, in general, the magnetic multipole

moments are dependent on the origin of the coordinate system.

This appendix introduced the multipole expansion by discussing the electrostatic potential of a charged system, but the analysis can be used to formulate the multipole expansion for a magnetic system. It can also be used to formulate an induced multipole expansion. This can occur, for example, when an electrified body is in the presence of another electrified body or an externally applied electric field. One can apply multipole analysis to dynamic scalar and dynamic vector potentials or to the Biot-Savart law. It can also be used to develop force and torque equations.

APPENDIX C

INVERSE DISTANCE IN CYLINDRICAL COORDINATES

C.1 Introduction

This appendix gives a simple description of the inverse distance function in cylindrical coordinates. The model used is a general model which has been applied to the charge simulation method, and many of the examples used in this thesis.

C.2 Inverse distance function

The inverse distance in cylindrical coordinates between the source point and the field point as shown in Figure C.1 can easily be derived as follows:

Figure C.1: A cylindrical source of length $2h$

$$\mathbf{r} - \mathbf{r}' = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z} \quad (\text{C.1})$$

$$\begin{aligned} |\mathbf{r} - \mathbf{r}'| &= \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} \\ &= \sqrt{[\rho \cos(\phi) - \rho' \cos(\phi')]^2 + [\rho \sin(\phi) - \rho' \sin(\phi')]^2 + [z - z']^2} \\ &= \sqrt{\rho^2 + \rho'^2 + (z - z')^2 - 2\rho\rho' \cos(\phi - \phi')} \end{aligned} \quad (\text{C.2})$$

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\sqrt{\rho^2 + \rho'^2 + (z - z')^2 - 2\rho\rho' \cos(\phi - \phi')}} \quad (\text{C.3})$$

Equation (C.1) is the closed form expression for the inverse distance function between the source point and the field point in a cylindrical coordinate system. This closed form expression can be represented by various summation formulas. More specifically, it has been written in terms of a Fourier series expansion whose weighting coefficients are the Q-functions. Although there are other ways to represent Equation (C.1), the Q-function representation allows for a very simple and direct application of Coulomb's law. This was most useful in the application of the charge simulation method.

APPENDIX D

ALTERNATE FORMULATION OF THE FREE-SPACE GREEN'S FUNCTION IN CYLINDRICAL COORDINATES

D.1 Introduction

This Appendix contains an alternate formulation for the free-space cylindrical Green's function. Instead of an infinite integral of a product of modified Bessel functions of order m , the infinite integral contains a product of Bessel functions of order m [21, 22, 124]. This formulation is more useful for a numerical study. The integrand of the defining integral has a non-oscillatory nature which makes numerical integration less problematic than it is for Equations (2.4) or (2.5)

D.2 Alternate form for the cylindrical Green's function

The reciprocal distance can also be written as

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{m=-\infty}^{\infty} e^{im(\phi - \phi')} \left(\int_0^{\infty} J_m(\rho u) J_m(\rho' u) e^{-u(z - z')} du \right) \quad (\text{D.1})$$

or

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{m=0}^{\infty} \varepsilon_m \cos \left[m \left(\phi - \phi' \right) \right] \left(\int_0^{\infty} J_m(\rho u) J_m(\rho' u) e^{-u(z - z')} du \right) \quad (\text{D.2})$$

Gradshteyn and Ryzhik [21] give the following result:

$$\int_0^{\infty} J_m(\rho u) J_m(\rho' u) e^{-u(z - z')} du = \frac{1}{\pi \sqrt{\rho \rho'}} Q_{m-\frac{1}{2}} \left(\frac{\rho^2 + \rho'^2 + (z - z')^2}{2\rho \rho'} \right) \quad (\text{D.3})$$

where $z \geq z'$ in Equations (D.1) and (D.2). The quantity $(z - z')$ must always be a positive quantity or zero for the integral in Equations (D.1) and (D.2) to converge. This can be done by replacing $(z - z')$ with $|z - z'|$. One will notice from Equation

(D.3), Equation (2.4) or Equation (2.5) that

$$\int_0^\infty K_m(\rho u) I_m(\rho' u) \cos \left[u (z - z') \right] du = \frac{\pi}{2} \int_0^\infty J_m(\rho u) J_m(\rho' u) \cdot e^{-u|z-z'|} du \quad (\text{D.4})$$

The right hand side of Equation (D.4) is more useful for numerical integration because it involves an integral with *bounded z modes*. The left hand side of Equation (D.4) involves an integral with oscillating *z modes* and integrals of this type are more difficult to handle numerically.

APPENDIX E

VALIDATION OF THE CYLINDRICAL GREEN'S FUNCTION

E.1 Introduction

There are a number of ways to mathematically prove that Equation (2.11) is correct. A Fourier series approach or a complex variable approach [24] are just two of the more useful ways. Another, and quite powerful way, is to apply distribution theory [129, 130, 131]. This approach enables one to take the existing solution and work backwards to see if one regains the inverse distance function given by Equation (2.1)

E.2 Validation of the cylindrical Green's function

The following relationship can be found in Lebedev [28] for the definition of the Q-function in terms of an integral.

$$\begin{aligned}
 Q_{m-\frac{1}{2}}(\cosh(\alpha)) &= \int_0^\pi \frac{\cos(mu')}{\sqrt{2(\cosh(\alpha) - 2\cos(u'))}} du' \\
 &= \frac{1}{2\sqrt{2}} \int_0^{2\pi} \frac{\cos(mu')}{\sqrt{(\cosh(\alpha) - \cos(u'))}} du' \quad (E.1)
 \end{aligned}$$

for $m \geq 0$, where $\cosh(\alpha) = \beta$, $\alpha > 0$, and therefore $\alpha = \cosh^{-1}(\beta) = \ln(\beta + \sqrt{\beta^2 - 1})$. Multiplying Equation (E.1) by e^{imu} gives

$$\begin{aligned}
 Q_{m-\frac{1}{2}}(\cosh(\alpha))e^{imu} &= \frac{e^{imu}}{2\sqrt{2}} \int_0^{2\pi} \frac{\cos(mu')}{\sqrt{\cosh(\alpha) - \cos(u')}} du' \\
 &= \frac{e^{imu}}{4\sqrt{2}} \int_0^{2\pi} \frac{e^{iu'} + e^{-iu'}}{\sqrt{\cosh(\alpha) - \cos(u')}} du' \\
 &= \frac{1}{4\sqrt{2}} \int_0^{2\pi} \frac{e^{im(u'+u)} + e^{-im(u'-u)}}{\sqrt{\cosh(\alpha) - \cos(u')}} du' \quad (E.2)
 \end{aligned}$$

Performing a summation on m from $-\infty$ to $+\infty$ on both sides of Equation (E.2) yields:

$$\sum_{m=-\infty}^{\infty} Q_{m-\frac{1}{2}}(\cosh(\alpha))e^{imu} = \frac{1}{4\sqrt{2}} \sum_{m=-\infty}^{\infty} \left(\int_0^{2\pi} \frac{e^{im(u'+u)} + e^{-im(u'-u)}}{\sqrt{\cosh(\alpha) - \cos(u')}} du' \right) \quad (\text{E.3})$$

Employing the expressions for the Dirac delta function, one can write

$$\delta(u' + u) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(u'+u)} \quad (\text{E.4})$$

$$\delta(u' - u) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(u'-u)} \quad (\text{E.5})$$

These expressions are the summation versions of the familiar integral expression given by

$$\delta(x' - x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(x'-x)} d\omega \quad (\text{E.6})$$

Equation (E.3) becomes

$$\sum_{m=-\infty}^{\infty} Q_{m-\frac{1}{2}}(\cosh(\alpha))e^{imu} = \frac{\pi}{2\sqrt{2}} \int_0^{2\pi} \frac{\delta(u' + u) + \delta(u' - u)}{\sqrt{\cosh(\alpha) - \cos(u')}} du' \quad (\text{E.7})$$

The right side of Equation (E.7) can now be integrated. The result is as follows:

$$\frac{\pi}{2\sqrt{2}} \int_0^{2\pi} \frac{\delta(u' + u) + \delta(u' - u)}{\sqrt{\cosh(\alpha) - \cos(u')}} du' = \frac{\pi}{\sqrt{2}} \frac{1}{\sqrt{\cosh(\alpha) - \cos(u)}} \quad (\text{E.8})$$

Equation (E.3) becomes

$$\sum_{m=0}^{\infty} \varepsilon_m Q_{m-\frac{1}{2}}(\beta) \cos(m\phi) = \frac{\pi}{\sqrt{2}} \frac{1}{\sqrt{\beta - \cos(\phi)}} \quad (\text{E.9})$$

where $u = \phi$ and $\beta = \cosh(\alpha)$. Using Equation (E.9) and Equation (2.11), the following closed form expression is obtained for the reciprocal distance between two

points in cylindrical coordinates.

$$\begin{aligned}
 \frac{1}{|\mathbf{r} - \mathbf{r}'|} &= \frac{1}{\sqrt{2\rho\rho'}} \frac{1}{\sqrt{\beta - \cos(\phi - \phi')}} \\
 &= \frac{1}{\sqrt{\rho^2 + \rho'^2 + (z - z')^2 - 2\rho\rho' \cos(\phi - \phi')}} \quad (\text{E.10})
 \end{aligned}$$

This proves that if one starts with Equations (2.9) or (2.10), then one can regain the reciprocal distance in cylindrical coordinates given by Equation (2.1)

APPENDIX F

SOME TABULATED VALUES FOR GAMMA FUNCTIONS, AND THEIR PRODUCTS

F.1 Introduction

This appendix introduces the Gamma function and a few its more useful relationships that are used in the derivation of the Q-function given by Equation (2.21).

F.2 Gamma functions

One definition of the Gamma function and that which is used in this thesis is given by

$$\Gamma(\kappa) = \int_0^{\infty} e^{-\lambda} \lambda^{\kappa-1} d\lambda \quad (\text{F.1})$$

for real $\kappa > 0$. Important mathematical relations can be developed from Equation (F.1) and a few of the most useful ones are given by

$$\Gamma(\kappa + 1) = \kappa \Gamma(\kappa) \quad (\text{F.2})$$

$$\Gamma(\kappa) \Gamma(1 - \kappa) = \frac{\pi}{\sin(\pi \kappa)} \quad (\text{F.3})$$

$$2^{2\kappa-1} \Gamma(\kappa) \Gamma(\kappa + \frac{1}{2}) = \sqrt{\pi} \Gamma(2\kappa) \quad (\text{F.4})$$

For integral κ , one can show that

$$\Gamma(n + 1) = n! \quad (\text{F.5})$$

$$\Gamma(n + \frac{1}{2}) = \frac{(2n-1)!!}{2^n} \sqrt{\pi} \quad (\text{F.6})$$

$$\Gamma\left(\frac{2n+3}{4}\right) \Gamma\left(\frac{2n+1}{4}\right) = \frac{(2n-1)!!}{2^{2n-\frac{1}{2}}} \pi \quad (\text{F.7})$$

Table F.1 gives a number of values of various Gamma functions and their products. These values are useful in the derivation of the summation formula for the Q-function given by Equation (2.21)

n	$\Gamma(n + \frac{1}{2})$	$\Gamma(\frac{2n+3}{4}) \Gamma(\frac{2n+1}{4})$
0	$\sqrt{\pi}$	$\sqrt{2\pi}$
1	$\frac{1}{2}\sqrt{\pi}$	$\frac{\sqrt{2}}{4}\pi$
2	$\frac{3}{4}\sqrt{\pi}$	$\frac{3\sqrt{2}}{16}\pi$
3	$\frac{15}{8}\sqrt{\pi}$	$\frac{15\sqrt{2}}{64}\pi$
4	$\frac{105}{16}\sqrt{\pi}$	$\frac{105\sqrt{2}}{256}\pi$
5	$\frac{945}{32}\sqrt{\pi}$	$\frac{945\sqrt{2}}{1024}\pi$
6	$\frac{10395}{64}\sqrt{\pi}$	$\frac{10395\sqrt{2}}{4096}\pi$
7	$\frac{135135}{128}\sqrt{\pi}$	$\frac{135135\sqrt{2}}{16384}\pi$
$\vdots$	$\vdots$	$\vdots$
N	$\frac{(2N-1)!!}{2^N}\sqrt{\pi}$	$\frac{\sqrt{2\pi}(2N-1)!!}{2^{2N}}, N \geq 1$

Table F.1: Gamma functions

APPENDIX G

RECURRENCE RELATIONSHIPS FOR THE Q-FUNCTIONS

G.1 Introduction

This appendix introduces a recurrence relationship which is helpful in computing higher order Q-functions. The recurrence relationship is also useful for transforming Elliptic integral formulations into Q-function formulations shown in Appendix G.

G.2 Recurrence relationship

The higher half-integral degree Legendre functions can be found by employing the recurrence relationship [28].

$$Q_{\eta+1}(\beta) = \left(\frac{2\eta+1}{\eta+1} \right) \beta Q_{\eta}(\beta) - \left(\frac{\eta}{\eta+1} \right) Q_{\eta-1}(\beta) \quad (\text{G.1})$$

Letting $\eta = m + \frac{1}{2}$ in Equation (G.1) gives the recurrence relationship

$$Q_{m+\frac{3}{2}}(\beta) = 4 \left(\frac{m+1}{2m+3} \right) \beta Q_{m+\frac{1}{2}}(\beta) - \left(\frac{2m+1}{2m+3} \right) Q_{m-\frac{1}{2}}(\beta) \quad (\text{G.2})$$

Table G.1 lists a few higher order Q-functions computed using Equation (G.2)

m	$Q_{m+\frac{3}{2}}(\beta)$
0	$\frac{4}{3}\beta_k Q_{\frac{1}{2}}(\beta) - \frac{1}{3}Q_{-\frac{1}{2}}(\beta)$
1	$\frac{8}{5}\beta_k Q_{\frac{3}{2}}(\beta) - \frac{3}{5}Q_{\frac{1}{2}}(\beta)$
2	$\frac{12}{7}\beta_k Q_{\frac{5}{2}}(\beta) - \frac{5}{7}Q_{\frac{3}{2}}(\beta)$
3	$\frac{16}{9}\beta_k Q_{\frac{7}{2}}(\beta) - \frac{7}{9}Q_{\frac{5}{2}}(\beta)$
4	$\frac{20}{11}\beta_k Q_{\frac{9}{2}}(\beta) - \frac{9}{11}Q_{\frac{7}{2}}(\beta)$
$\vdots$	$\vdots$
M	$4 \left(\frac{M+1}{2M+3} \right) \beta Q_{M+\frac{1}{2}}(\beta) - \left(\frac{2M+1}{2M+3} \right) Q_{M-\frac{1}{2}}(\beta)$

Table G.1: Recurrence relations

APPENDIX H

ELLIPTIC INTEGRALS IN TERMS OF Q-FUNCTIONS

H.1 Introduction

This appendix illustrates the mathematical relationship among the Q-functions, and the Elliptic integrals of the first and second kind [28, 132]. Also, a number of useful equations that can quickly be applied to convert an Elliptic integral formula in terms of a Q-function are introduced.

H.2 Elliptic integrals in terms of Q-functions

One can obtain a relationship between Q-functions and the complete Elliptic integrals of the first, $K(x)$, and second kind, $E(x)$, respectively by using the recurrence relationship for Legendre functions given in appendix G.

$$Q_{m+\frac{3}{2}}(\xi) = 4 \left(\frac{m+1}{2m+3} \right) \xi Q_{m+\frac{1}{2}}(\xi) - \left(\frac{2m+1}{2m+3} \right) Q_{m-\frac{1}{2}}(\xi) \quad (\text{H.1})$$

It can be shown that [28]

$$Q_{-\frac{1}{2}}(\xi) = \sqrt{\frac{2}{\xi+1}} K \left(\sqrt{\frac{2}{\xi+1}} \right) \quad (\text{H.2})$$

and

$$Q_{\frac{1}{2}}(\xi) = \xi \sqrt{\frac{2}{\xi+1}} K \left(\sqrt{\frac{2}{\xi+1}} \right) - \sqrt{2(\xi+1)} E \left(\sqrt{\frac{2}{\xi+1}} \right) \quad (\text{H.3})$$

Employing Equations (H.1), (H.2), and (H.3) yield all the higher half-integral degree Q-functions in terms of Elliptic integrals. A few of these are

$$Q_{\frac{3}{2}}(\xi) = \frac{1}{3} (4\xi^2 - 1) \sqrt{\frac{2}{\xi+1}} K \left(\sqrt{\frac{2}{\xi+1}} \right) - \frac{4}{3} \xi \sqrt{2(\xi+1)} E \left(\sqrt{\frac{2}{\xi+1}} \right) \quad (\text{H.4})$$

Figure H.1: Elliptic integrals and Q-functions

$$Q_{\frac{5}{2}}(\xi) = \frac{1}{15} (32\xi^3 - 17\xi) \sqrt{\frac{2}{\xi+1}} K\left(\sqrt{\frac{2}{\xi+1}}\right) - \frac{1}{15} (32\xi^2 - 9) \sqrt{2(\xi+1)} E\left(\sqrt{\frac{2}{\xi+1}}\right) \quad (\text{H.5})$$

$$Q_{\frac{7}{2}}(\xi) = \frac{1}{105} (384\xi^4 - 304\xi^2 + 25) \sqrt{\frac{2}{\xi+1}} K\left(\sqrt{\frac{2}{\xi+1}}\right) - \frac{1}{105} (384\xi^3 - 208\xi) \sqrt{2(\xi+1)} E\left(\sqrt{\frac{2}{\xi+1}}\right) \quad (\text{H.6})$$

One will notice that the Q-function representation yields a much more mathematically compact form as compared to the corresponding Elliptic integral representation. A few representative plots of the Elliptic integrals and Q-functions are shown in Figure H.1

APPENDIX I

DERIVATIVE PROPERTIES OF Q-FUNCTIONS

I.1 Introduction

The derivatives of the Q-functions are required when the gradient of the potential function is needed or when a direct formulation of field quantities is desired. This appendix will give some of the more useful derivative properties.

I.2 Derivative properties of the Q-function

In order to compute the derivatives of $Q_{m-\frac{1}{2}}(\beta)$, one can recast Equation (2.21) into a more easily differentiable forms. One can rewrite Equation (2.21) as

$$Q_{m-\frac{1}{2}}(\beta) = \pi\sqrt{2} \sum_{n=0}^{\infty} \frac{(4n+2m-1)!!}{(n+m)!n!} \frac{1}{(4\beta)^{\frac{4n+2m+1}{2}}} \quad (\text{I.1})$$

where $(4n+2m-1)!! = 1 \cdot 3 \cdot 5 \cdot 7 \cdots (4n+2m-1)$. For $m = 0$, Equation (I.1) becomes

$$Q_{-\frac{1}{2}}(\beta) = \pi\sqrt{2} \left[\frac{1}{(4\beta)^{\frac{1}{2}}} + \sum_{n=1}^{\infty} \frac{(4n-1)!!}{[n!]^2} \frac{1}{(4\beta)^{\frac{4n+1}{2}}} \right] \quad (\text{I.2})$$

where $(4n-1)!! = 1 \cdot 3 \cdot 5 \cdot 7 \cdots (4n-1)$. The derivatives of $Q_{m-\frac{1}{2}}(\beta)$ in cylindrical coordinates are given by, for all $m \geq 0$,

$$\frac{\partial}{\partial \rho} [Q_{m-\frac{1}{2}}(\beta)] = A(\rho, z, \rho', z') \sum_{n=0}^{\infty} \frac{(4n+2m+1)(4n+2m-1)!!}{(n+m)!n! (4\beta)^{\frac{4n+2m+1}{2}}} \quad (\text{I.3})$$

and

$$\frac{\partial}{\partial z} [Q_{m-\frac{1}{2}}(\beta)] = B(\rho, z, \rho', z') \sum_{n=0}^{\infty} \frac{(4n+2m+1)(4n+2m-1)!!}{(n+m)!n! (4\beta)^{\frac{4n+2m+1}{2}}} \quad (\text{I.4})$$

In particular, for $m = 0$, the corresponding derivatives are given by

$$\frac{\partial}{\partial \rho} [Q_{-\frac{1}{2}}(\beta)] = A(\rho, z, \rho', z') \left[\frac{1}{(4\beta)^{\frac{1}{2}}} + \sum_{n=1}^{\infty} \frac{(4n+1)(4n-1)!!}{[n!]^2 (4\beta)^{\frac{4n+1}{2}}} \right] \quad (\text{I.5})$$

$\rho > \rho' \forall m \geq 0$			$\rho < \rho' \forall m \geq 0$		
$\frac{\partial}{\partial \rho} [Q_{m-\frac{1}{2}}(\beta)]$	< 0 if $\rho^2 - \rho'^2 > (z - z')^2$ $= 0$ if $\rho^2 - \rho'^2 = (z - z')^2$ > 0 if $\rho^2 - \rho'^2 < (z - z')^2$		$\frac{\partial}{\partial \rho} [Q_{m-\frac{1}{2}}(\beta)]$	> 0	
$\frac{\partial}{\partial z} [Q_{m-\frac{1}{2}}(\beta)]$	< 0 if $z > z'$ $= 0$ if $z = z'$ > 0 if $z < z'$		$\frac{\partial}{\partial z} [Q_{m-\frac{1}{2}}(\beta)]$	< 0 if $z > h$ > 0 if $z < -h$	
$\frac{\partial}{\partial \phi} [Q_{m-\frac{1}{2}}(\beta)]$	$= 0$		$\frac{\partial}{\partial \phi} [Q_{m-\frac{1}{2}}(\beta)]$	$= 0$	

Table I.1: The "sign" of the derivatives of the Q-functions

and

$$\frac{\partial}{\partial z} [Q_{-\frac{1}{2}}(\beta)] = B(\rho, z, \rho', z') \left[\frac{1}{(4\beta)^{\frac{1}{2}}} + \sum_{n=0}^{\infty} \frac{(4n+1)(4n-1)!!}{[n!]^2 (4\beta)^{\frac{4n+1}{2}}} \right] \quad (\text{I.6})$$

where

$$A(\rho, z, \rho_k, z_k) = -\frac{\pi}{\sqrt{2}} \frac{1}{\beta} \left[\frac{\partial}{\partial \rho} (\beta) \right] \quad (\text{I.7})$$

$$B(\rho, z, \rho_k, z_k) = -\frac{\pi}{\sqrt{2}} \frac{1}{\beta} \left[\frac{\partial}{\partial z} (\beta) \right] \quad (\text{I.8})$$

$$\frac{\partial}{\partial \rho} (\beta) = \frac{1}{\rho'} - \frac{\beta}{\rho} \quad (\text{I.9})$$

$$\frac{\partial}{\partial z} (\beta) = \frac{z - z'}{\rho \rho'} \quad (\text{I.10})$$

It is useful to know where the derivatives of Q-functions are positive or negative. In calculating the magnetic field, it would be helpful if one knew before hand the “sign” of the result. The “sign” convention for the Q-functions is defined by the location of the observation point relative to the source point. The only restriction is that the observation point must not coincide with the source point. Table I.1 summarizes the mathematical relations needed to determine the “sign” of the derivative of $Q_{m-\frac{1}{2}}(\beta)$ with respect to the field point (ρ, ϕ, z) . The Q-functions, $Q_{m-\frac{1}{2}}(\beta)$, are positive for all $m \geq 0$.

APPENDIX J

INTEGRALS WHICH OCCUR FOR CYLINDRICALLY SYMMETRIC MAGNETIC SYSTEMS

J.1 Introduction

The solution to circular cylindrical magnetic systems often requires the evaluation of integrals which may or may not be analytically tractable. Although it is not possible to know all the possible integral forms which may arise in modeling a circular cylindrical magnetic system, this appendix will consider some of the more useful ones. All hand computations were checked using Maple [133] and Mathematica [134]. These commercial software packages, along with Matlab [135], were also used to develop all the plots found in this thesis.

J.2 Integral relationships

A number of integrals which are encountered frequently, and which are tractable, occur when considering the magnetic vector potential for circular cylindrical geometries.

J.2.1 Thin disks with constant surface charge density

Integrals which occur for thin-disk problems can often be solved by modeling the problem in terms of a known surface charge density. These are usually given by

$$I_n(\rho, z, l, a, b) = \int_a^b \frac{\rho'^{2n+1}}{\left(\rho^2 + \rho'^2 + (z+l)^2\right)^{n+\frac{1}{2}}} d\rho' \quad (\text{J.1})$$

This type of integral is encountered for cylindrical permanent magnets with a constant magnetic surface charge density. They also occur for thin electrostatic disks

which have an assumed constant surface charge density.

$$I_n(\rho, z, l, a, b) = \frac{1}{2(n+1)(\rho^2 + (z+l)^2)^{n+\frac{1}{2}}} \left\{ b^{2n+2} \cdot F\left(n + \frac{1}{2}, n+1; n+2; \frac{b^2}{\rho^2 + (z+l)^2}\right) - a^{2n+2} \cdot F\left(n + \frac{1}{2}, n+1; n+2; \frac{a^2}{\rho^2 + (z+l)^2}\right) \right\} \quad (\text{J.2})$$

For $n = 0$, the integral evaluates to

$$I_0(\rho, z, l, a, b) = \left\{ \sqrt{\rho^2 + b^2 + (z+l)^2} - \sqrt{\rho^2 + a^2 + (z+l)^2} \right\} \quad (\text{J.3})$$

For $n = 1$, the integral evaluates to

$$I_1(\rho, z, l, a, b) = \left\{ \frac{b^2 + 2[\rho^2 + (z+l)^2]}{\sqrt{\rho^2 + b^2 + (z+l)^2}} - \frac{a^2 + 2[\rho^2 + (z+l)^2]}{\sqrt{\rho^2 + a^2 + (z+l)^2}} \right\} \quad (\text{J.4})$$

For $n = 2$, the integral evaluates to

$$I_2(\rho, z, l, a, b) = \frac{8(\rho^2 + (z+l)^2)^2 + 12\rho'^2(\rho^2 + (z+l)^2)^2 + 3\rho'^4}{(\rho'^2 + \rho^2 + (z+l)^2)^{n+\frac{3}{2}}} \Big|_a^b \quad (\text{J.5})$$

This can be continued for any desired n . In fact, one can write

$$I(\rho, z, l, a, b) = \sum_{n=0}^{\infty} I_n(\rho, z, l, a, b) \quad (\text{J.6})$$

Usually, only a few terms in equation are necessary to get an accurate result.

J.2.2 Thin disks with azimuthal current distribution

Integrals of the form

$$I_n(\rho, z, l, a, b) = \int_a^b \frac{\rho'^{2n+2}}{(\rho^2 + \rho'^2 + (z+l)^2)^{2n+\frac{3}{2}}} d\rho' \quad (\text{J.7})$$

occur for thin disks with an azimuthal current distribution. It is evaluated as follows:

$$\begin{aligned} I_n(\rho, z, a, b) = & b^{2n+3} \frac{\text{F}\left(n + \frac{3}{2}, 2n + \frac{3}{2}; n + \frac{5}{2}; -\frac{b^2}{\rho^2 + z^2}\right)}{(2n+3)(\rho^2 + (z+l)^2)^{2n+\frac{3}{2}}} - \\ & a^{2n+3} \frac{\text{F}\left(n + \frac{3}{2}, 2n + \frac{3}{2}; n + \frac{5}{2}; -\frac{a^2}{\rho^2 + z^2}\right)}{(2n+3)(\rho^2 + (z+l)^2)^{2n+\frac{3}{2}}} \end{aligned} \quad (\text{J.8})$$

For $n = 0$, the integral evaluates to

$$\begin{aligned} I_0(\rho, z, a, b) = & \frac{a}{\sqrt{\rho^2 + a^2 + (z+l)^2}} - \frac{b}{\sqrt{\rho^2 + b^2 + (z+l)^2}} + \\ & \ln \left(\frac{b + \sqrt{\rho^2 + b^2 + (z+l)^2}}{a + \sqrt{\rho^2 + a^2 + (z+l)^2}} \right) \end{aligned} \quad (\text{J.9})$$

For $n = 1$, the integral evaluates to

$$\begin{aligned} I_1(\rho, z, a, b) = & \frac{4a^2 + 3a(\rho^2 + (z+l)^2)}{\sqrt{\rho^2 + a^2 + (z+l)^2}} - \frac{4b^2 + 3b(\rho^2 + (z+l)^2)}{\sqrt{\rho^2 + b^2 + (z+l)^2}} + \\ & \ln \left(\frac{b + \sqrt{\rho^2 + b^2 + (z+l)^2}}{a + \sqrt{\rho^2 + a^2 + (z+l)^2}} \right) \end{aligned} \quad (\text{J.10})$$

This can be continued for any desired n .

Other integrals which are not encountered in this thesis but which occur frequently in the solution of circular cylindrical magnetic systems can also be simply represented in terms of a Q-function representation.

APPENDIX K

RELATIONSHIP BETWEEN THE CYLINDRICAL GREEN'S FUNCTION AND THE SPHERICAL GREEN'S FUNCTION

K.1 Introduction

This appendix introduces a general relationship between the spherical harmonic formulation and the Q-function formulation. It can be used to generate the multipole reference table given in chapter 7. Unlike the derivation in chapter 7, the formulation in this Appendix is more general.

K.2 Cylindrical and spherical Green's functions

The Green's function, in spherical coordinates, is written as

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r} \right)^n P_n[\cos(\gamma)] \quad \text{where } r > r' \quad (\text{K.1})$$

where $\cos(\gamma)$ is given by

$$\cos(\gamma) = \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' = \cos(\theta) \cos(\theta') + \sin(\theta) \sin(\theta') \cos(\phi - \phi') \quad (\text{K.2})$$

where the spherical coordinate angular relationships are shown in Figure K.1

Equation (K.1) can be put in the form

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} \sqrt{\frac{r'}{r}} \sum_{l=0}^{\infty} \sum_{m=-l}^l \left(\frac{r'}{r} \right)^{n-\frac{1}{2}} \frac{(n-m)!}{(n+m)!} P_l^m[\cos(\theta)] P_l^m[\cos(\theta')] \cdot e^{im(\phi-\phi')} \quad (\text{K.3})$$

Figure K.1: Angular relationships in spherical coordinates

where the following addition theorem for spherical harmonics is employed.

$$P_n [\cos(\gamma)] = \sum_{m=-n}^n \frac{(n-m)!}{(n+m)!} P_n^m [\cos(\theta)] P_n^m [\cos(\theta')] e^{im(\phi-\phi')} \quad (\text{K.4})$$

One can rewrite Equation (K.1) as

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} \sqrt{\frac{r'}{r}} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \sum_{n=|m|}^{\infty} \left(\frac{r'}{r}\right)^{n-\frac{1}{2}} \frac{(n-m)!}{(n+m)!} P_n^m [\cos(\theta)] \cdot P_n^m [\cos(\theta')] \quad (\text{K.5})$$

Equations (2.2) and (2.3) yielded the Green's function in cylindrical coordinates.

This was given by

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\pi \sqrt{\rho \rho'}} \sum_{m=-\infty}^{\infty} Q_{m-\frac{1}{2}}(\beta) e^{im(\phi-\phi')} \quad (\text{K.6})$$

One can recast Equation (K.6) in terms of spherical coordinates as

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\pi \sqrt{rr' \sin(\theta) \sin(\theta')}} \sum_{m=-\infty}^{\infty} \left[Q_{m-\frac{1}{2}}(\tilde{\beta}) e^{im(\phi-\phi')} \right] \quad (\text{K.7})$$

Writing Equation (K.7) in terms of real quantities yields:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\pi \sqrt{rr' \sin(\theta) \sin(\theta')}} \sum_{m=0}^{\infty} \varepsilon_m Q_{m-\frac{1}{2}}(\tilde{\beta}) \cos \left[m(\phi - \phi') \right] \quad (\text{K.8})$$

where

$$\tilde{\beta} = \frac{r^2 + r'^2 - 2rr' \cos(\theta) \cos(\theta')}{2rr' \sin(\theta) \sin(\theta')} > 1 \quad (\text{K.9})$$

Equation (K.9) is nothing more than β transformed from cylindrical coordinates to spherical coordinates by employing the coordinate transformation equations given below.

$$\rho = r \sin(\theta) \quad (\text{K.10})$$

$$\rho' = r' \sin(\theta') \quad (\text{K.11})$$

$$z = r \cos(\theta) \quad (\text{K.12})$$

$$z' = r' \cos(\theta') \quad (\text{K.13})$$

They relate the spherical coordinates to the cylindrical coordinates. Given a point in space represented in spherical coordinates, one can transform that point into cylindrical coordinates and *visa-versa*.

Equation (K.1) and Equation (K.8) are two different representations for the same Green's function. This allows one to transform the more familiar multipole expansion for spherical symmetry given by Equation (K.1) into a Fourier expansion given by Equation (K.8).

APPENDIX L

A NUMERICAL COMPARISON BETWEEN A SPHERICAL HARMONIC EXPANSION AND A TOROIDAL EXPANSION

L.1 Introduction

It is always important to compare a new or an alternate formulation of a physical problem with those that are well established. One formulation which has a long history and is used in many scientific field is the spherical harmonic expansion. In electromagnetics, the spherical harmonic expansion has become ubiquitous. It has become the method of choice for solving the difficult integral formulations found in graduate texts and published papers. It has the advantage of being able to represent the solutions of electromagnetic field problems in terms of primitive charge distributions or other primitive quantities. The corresponding multipole expansion that is generated has a physical meaning and can give the scientist important insight into the problem.

This appendix concentrates on the numerical aspects of the spherical harmonic expansion and it will compare them with those of the toroidal expansion. It will be shown, through simple examples, that the toroidal expansion converges more rapidly and that it represents a useful alternative to the more well established spherical harmonic expansion.

L.2 Numerical study

In order to gain further confidence in the Q-Function approach to the solution of problems which exhibit cylindrical symmetry, it would be beneficial to show how it compares to the Legendre polynomial approach. Consider a conducting ring of radius, a , situated on the x - y plane with its origin coincident with the Cartesian axes. Also, consider this conducting ring to be charged with a uniform line charge

Figure L.1: Circular conducting ring

distribution denoted by λ . The total charge is denoted by q and therefore,

$$q = 2\pi a\lambda \quad (\text{L.1})$$

One can apply the differential form of Coulomb's law to find the electric potential at any point in space from the charged ring. The differential form of Coulomb's law is given by

$$d\Phi_P = \frac{1}{4\pi\epsilon_0} \frac{dq}{|\mathbf{R} - \mathbf{a}|} \quad (\text{L.2})$$

where $dq = \lambda a d\phi$. Figure L.1 illustrates the problem. One can integrate the differential form of Coulomb's law to give

$$\begin{aligned} \Phi_P &= \frac{a}{4\pi\epsilon_0} \int_0^{2\pi} \frac{\lambda d\phi'}{|\mathbf{R} - \mathbf{a}|} \\ &= \frac{q}{8\pi^2\epsilon_0} \int_0^{2\pi} \frac{d\phi'}{|\mathbf{R} - \mathbf{a}|} \end{aligned} \quad (\text{L.3})$$

It is now necessary, if one wants to make an accurate comparison between the Legendre polynomial approach and the Q-function approach, to write the reciprocal distance, $|\mathbf{R} - \mathbf{a}|^{-1}$, in spherical and cylindrical coordinates. If one writes, in spherical coordinates, the reciprocal distance between the field point and the source

point, then one obtains

$$\frac{1}{|\mathbf{R} - \mathbf{a}|} = \frac{1}{\sqrt{R^2 + a^2 - 2aR \sin(\theta) \cos(\phi - \phi')}} \quad (\text{L.4})$$

Alternatively, the reciprocal distance in cylindrical coordinates is given by

$$\frac{1}{|\mathbf{R} - \mathbf{a}|} = \frac{1}{\sqrt{\rho^2 + \rho'^2 + (z - z')^2 - 2\rho\rho' \cos(\phi - \phi')}} \quad (\text{L.5})$$

In spherical coordinates, one can expand Equation (L.4) in terms of Legendre polynomials and then substitute the expression into Equation (L.3) and integrate. The result is given in Jackson [10] and in Smythe [19]. The solution is

$$\Phi_P(R, \theta) = \frac{q}{4\pi\epsilon_0 a} \sum_{n=0}^{\infty} \left(\frac{a}{R}\right)^{n+1} P_n[\cos(\theta)] P_n(0) \quad (\text{L.6})$$

for $R > a$ and

$$\Phi_P(R, \theta) = \frac{q}{4\pi\epsilon_0 a} \sum_{n=0}^{\infty} \left(\frac{R}{a}\right)^n P_n[\cos(\theta)] P_n(0) \quad (\text{L.7})$$

for $R < a$. Also, $P_n(0) = 0$ for odd n and $P_n(0) = (-1)^{\frac{n}{2}} \left(\frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdot 8 \cdots n}\right)$ for even n . Let's consider the expansion valid for $R > a$. A few terms in this expansion are given by

$$\begin{aligned} \Phi_P(R, \theta) = & \frac{q}{4\pi\epsilon_0 R} \left[1 - \frac{1}{4} \left(\frac{a}{R}\right)^2 \left(3 \cos^2(\theta) - 1\right) + \right. \\ & \left. \frac{3}{64} \left(\frac{a}{R}\right)^4 \left(35 \cos^4(\theta) - 30 \cos^2(\theta) + 3\right) + \dots \right] \end{aligned} \quad (\text{L.8})$$

and for $a > R$, a few terms in the expansion are given by

$$\begin{aligned} \Phi_P(R, \theta) = & \frac{q}{4\pi\epsilon_0 a} \left[1 - \frac{1}{4} \left(\frac{R}{a}\right)^2 \left(3 \cos^2(\theta) - 1\right) + \right. \\ & \left. \frac{3}{64} \left(\frac{R}{a}\right)^4 \left(35 \cos^4(\theta) - 30 \cos^2(\theta) + 3\right) + \dots \right] \end{aligned} \quad (\text{L.9})$$

In cylindrical coordinates, one can expand Equation (L.5) in terms of Q-functions to obtain

$$\frac{1}{|\mathbf{R} - \mathbf{a}'|} = \frac{1}{\pi\sqrt{\rho a}} \sum_{m=0}^{\infty} \varepsilon_m Q_{m-\frac{1}{2}}(\xi) \cos \left[m \left(\phi - \phi' \right) \right] \quad (\text{L.10})$$

Substituting Equation (L.10) into Equation (L.3) yields

$$\Phi_P(\rho, \phi, z) = \frac{q}{8\pi^2\epsilon_0} \int_0^{2\pi} \left(\frac{1}{\pi\sqrt{\rho a}} \sum_{m=0}^{\infty} \varepsilon_m Q_{m-\frac{1}{2}}(\xi) \cos \left[m \left(\phi - \phi' \right) \right] \right) d\phi' \quad (\text{L.11})$$

Since the series is uniformly convergent, it is permissible to interchange the operation of integration and summation. This gives

$$\Phi_P(\rho, \phi, z) = \frac{q}{8\pi^3\epsilon_0} \frac{1}{\sqrt{\rho a}} \sum_{m=0}^{\infty} \varepsilon_m Q_{m-\frac{1}{2}}(\xi) \int_0^{2\pi} \cos \left[m \left(\phi - \phi' \right) \right] d\phi' \quad (\text{L.12})$$

Equation (L.12) is identically zero for all values of $m \geq 1$. This says that the solution is not ϕ dependent. For $m = 0$, the integral evaluates to 2π . The electric potential at any arbitrary point in space, not coincident with the source point, is

$$\Phi_P(\rho, z) = \frac{q}{4\pi^2\epsilon_0\sqrt{\rho a}} Q_{-\frac{1}{2}}(\xi) \quad (\text{L.13})$$

where

$$\begin{aligned} Q_{-\frac{1}{2}}(\xi) &= \frac{\pi}{(2\xi)^{\frac{1}{2}}} \left[1 + \sum_{n=1}^{\infty} \frac{(4n-1)!!}{2^{2n} [n!]^2} \frac{1}{(2\xi)^{2n}} \right] \\ &= \frac{\pi}{(2\xi)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(4n-1)!!}{2^{2n} [n!]^2} \frac{1}{(2\xi)^{2n}} \end{aligned} \quad (\text{L.14})$$

and

$$\xi = \frac{\rho^2 + a^2 + z^2}{2\rho a} > 1 \quad (\text{L.15})$$

Unlike the Legendre polynomial solution which needs two expansions to represent the potential anywhere in space, the Q-function approach yields a solution which is valid anywhere in space not coincident with the conducting ring.

In order to make a comparison between the electric potential given in terms

Figure L.2: Cylinder inscribed in a sphere

of Legendre polynomials and the electric potential given in terms of Q-functions, one needs to compare Equation (L.6) or Equation (L.7) with Equation (L.13). The comparison is only valid where a hypothetical spherical surface intersects with a hypothetical cylindrical surface. At these points, the total potential must match. It is important to note that the total electric potential given in terms of Legendre polynomials and Q-functions is what must be considered when making a comparison. The individual terms in either expansion will not be equal. If the cylinder is enclosed within the sphere then the intersection of the two surfaces are given by two the circles shown in red in Figure L.2 . Everywhere on these circles the total electric potential given by the two different expansions must equate. If the sphere is enclosed within the cylinder as shown in Figure L.3 then the intersection of the two surfaces form a circle shown in red, lying in the x - y plane, and the two points located at $(0, 0, -R)$ and $(0, 0, +R)$ shown in blue.

Numerical Results

EXAMPLE 1:

The following example compares the Legendre polynomial expansion and the toroidal expansion for the physical representation shown in Figure L.2. Specifically,

Figure L.3: Sphere inscribed in a cylinder

it compares the potential values on the circle above the x - y plane. The circle intersects the sphere at $\cos(\theta) = \frac{3}{\sqrt{13}}$ for $\rho = 2.0 \text{ m}$, $z = 3.0 \text{ m}$, and $R = \sqrt{13} \text{ m}$. The ring of charge has a radius of 1.0 m . The ratio of $\frac{q}{\epsilon_0}$ is set equal to *unity*. With this information, one must employ Equation (L.6). The Legendre polynomial expansion is given by

$$\begin{aligned}
 \Phi_P \left(\sqrt{13}, \cos^{-1} \left[\frac{3}{\sqrt{13}} \right] \right) &= \frac{1}{4\pi} \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{13}} \right)^{n+1} P_n \left(\frac{3}{\sqrt{13}} \right) P_n(0) \\
 &= \frac{1}{4\pi\sqrt{13}} \left[1 - \frac{1}{4} \left(\frac{1}{\sqrt{13}} \right)^2 \left(3 \left[\frac{3}{\sqrt{13}} \right]^2 - 1 \right) + \right. \\
 &\quad \left. \frac{3}{64} \left(\frac{1}{\sqrt{13}} \right)^4 \left(35 \left[\frac{3}{\sqrt{13}} \right]^4 - 30 \left[\frac{3}{\sqrt{13}} \right]^2 + \right. \right. \\
 &\quad \left. \left. 3 \right) + \dots \right] \tag{L.16}
 \end{aligned}$$

$2n$	<i>Legendre Polynomial Expansion</i> $\Phi_P(R, \theta)$	n	<i>Q-function Expansion</i> $\Phi_P(\rho, z)$
0	0.0220708195	0	0.02126797387
2	0.0216137316	1	0.02159350409
4	0.0216076461	2	0.02160803668
6	0.0216089478	3	0.02160885229
8	0.0216089092	4	0.02160890301
10	0.0216089063	5	0.02160890635
12	0.0216089066	6	0.02160890658
14	0.0216089066	7	0.02160890659
16	0.02160890659	8	0.02160890659
18	0.02160890659	9	0.02160890659
20	0.02160890659	10	0.02160890659
22	0.02160890659	11	0.02160890659
24	0.02160890659	12	0.02160890659

Table L.1: Numerical comparison for EXAMPLE 1

and the Q-function expansion is written as

$$\begin{aligned}
\Phi_P(2, 3) &= \frac{1}{4\pi_0\sqrt{14}} \left(1 + \sum_{n=1}^{\infty} \frac{(4n-1)!!}{(14)^{2n} [n!]^2} \right) \\
&= \frac{1}{4\pi_0\sqrt{14}} \left[1 + \frac{1 \cdot 3}{(14)^2 (1!)^2} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{(14)^4 (2!)^2} + \right. \\
&\quad \left. \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11}{(14)^6 (3!)^2} \right]
\end{aligned} \tag{L.17}$$

Table L.1 gives the potential in terms of a Legendre polynomial expansion and in terms of a toroidal expansion.

One can see that, indeed, the two solutions converge to the same potential and that the Q-function approach converges using less terms. One must remember that the Q-functions are monotonically decreasing functions. This is a very useful property for any convergent function to have. The Legendre polynomials, on the other hand, do not exhibit this property. However, both converge quickly when the field point is far from the source. The Legendre polynomials are multiplied by $R^{-(n+1)}$ and, therefore, will converge rapidly due to the presence of a large R . Alternatively, as the field point gets closer to the source, the Legendre polynomial approach should converge more slowly than the Q-function approach. EXAMPLE

2 will illustrate this.

EXAMPLE 2:

The following example is a continuation of EXAMPLE 1 except that the circle intersects the sphere at $\cos(\theta) = \frac{.5}{\sqrt{1.46}}$ for $\rho = 1.1$ m, $z = 0.5$ m, and $R = \sqrt{1.46}$ m. The ring of charge still has a radius, a , of 1.0 m. The ratio of $\frac{q}{\epsilon_0}$ is set equal to *unity*. The Legendre polynomial expansions is given by

$$\begin{aligned} \Phi_P \left(\sqrt{1.46}, \cos^{-1} \left[\frac{.5}{\sqrt{1.46}} \right] \right) &= \frac{1}{4\pi} \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{1.46}} \right)^{n+1} P_n \left(\frac{.5}{\sqrt{1.46}} \right) P_n(0) \\ &= \frac{1}{4\pi\sqrt{1.46}} \left[1 - \frac{1}{4} \left(\frac{1}{\sqrt{1.46}} \right)^2 \left(3 \left[\frac{.5}{\sqrt{1.46}} \right]^2 - 1 \right) + \right. \\ &\quad \left. \frac{3}{64} \left(\frac{1}{\sqrt{1.46}} \right)^4 \left(35 \left[\frac{.5}{\sqrt{1.46}} \right]^4 - 30 \left[\frac{.5}{\sqrt{1.46}} \right]^2 + 3 \right) + \dots \right] \end{aligned} \quad (\text{L.18})$$

and the toroidal expansion is written as

$$\begin{aligned} \Phi_P(1.1, .5) &= \frac{1}{4\pi_0\sqrt{2.46}} \left[1 + \sum_{n=1}^{\infty} \frac{(4n-1)!! (1.1)^{2n}}{(4.92)^{2n} [n!]^2} \right] \\ &= \frac{1}{4\pi_0\sqrt{2.46}} \left[1 + \frac{(1 \cdot 3) (1.1)^2}{(4.92)^2 (1!)^2} + \frac{(1 \cdot 3 \cdot 5 \cdot 7) (1.1)^4}{(4.92)^4 (2!)^2} + \right. \\ &\quad \left. \frac{(1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11) (1.1)^6}{(4.92)^6 (3!)^2} \right] \end{aligned} \quad (\text{L.19})$$

Table L.2 gives the potential in terms of a Legendre polynomial expansion and in terms of a toroidal expansion. The toroidal expansion sums to a final value using less terms than the Legendre polynomial expansion.

An interesting feature of Equation (L.14) is that imbedded within it is the monopole potential or the “far-field” potential. This becomes apparent after ex-

$2n$	<i>Legendre Polynomial Expansion</i> $\Phi_P(R, \theta)$	n	<i>Q-function Expansion</i> $\Phi_P(\rho, z)$
0	0.0658587841	0	0.0583452420
8	0.0667063578	8	0.0665527633
16	0.0670832050	16	0.0669681023
24	0.0670021858	24	0.0670099058
32	0.0670182826	32	0.0670149208
40	0.0670151956	40	0.0670155754
48	0.0670157666	48	0.0670156653
56	0.0670156656	56	0.0670156780
64	0.0670156824	64	0.0670156799
72	0.0670156799	72	0.0670156802
80	0.0670156802	80	0.0670156802
88	0.0670156802	81	0.0670156802
96	0.0670156802	82	0.0670156802
104	0.0670156802	83	0.0670156802

Table L.2: Numerical comparison for EXAMPLE 2

panding Equation (L.13) as follows

$$\begin{aligned}
\Phi_P(\rho, z) &= \frac{q}{4\pi^2\epsilon_0\sqrt{\rho a}} Q_{-\frac{1}{2}}(\xi) \\
&= \frac{q}{4\pi^2\epsilon_0\sqrt{\rho a}} \left(\frac{\pi}{(2\xi)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(4n-1)!!}{2^{2n} [n!]^2} \frac{1}{(2\xi)^{2n}} \right) \Big|_{\xi=\frac{\rho^2+a^2+z^2}{2\rho a}} \\
&= \frac{q}{4\pi^2\epsilon_0\sqrt{\rho a}} \left\{ \frac{\pi\sqrt{\rho a}}{\sqrt{\rho^2+a^2+z^2}} \left[1 + \frac{1 \cdot 3}{2^2} \frac{(\rho a)^2}{(\rho^2+a^2+z^2)^2} + \right. \right. \\
&\quad \left. \left. \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^4(2!)^2} \frac{(\rho a)^4}{(\rho^2+a^2+z^2)^4} + \dots \right] \right\} \\
&= \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{\rho^2+a^2+z^2}} \left[1 + \frac{1 \cdot 3}{2^2} \frac{(\rho a)^2}{(\rho^2+a^2+z^2)^2} + \right. \\
&\quad \left. \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^4(2!)^2} \frac{(\rho a)^4}{(\rho^2+a^2+z^2)^4} + \dots \right] \tag{L.20}
\end{aligned}$$

Taking the limit of Equation (L.20) as $a \rightarrow 0$, yields;

$$\lim_{a \rightarrow 0} [\Phi_P(\rho, z)] = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{\rho^2+z^2}} \Big|_{\substack{\rho=R\sin(\phi) \\ z=R\cos(\phi)}} \tag{L.21}$$

$$\Phi_P(\rho, z)^{monopole} = \frac{q}{4\pi\epsilon_0} \frac{1}{R} \tag{L.22}$$

Equation (L.22) implies that the only Q-function which could exhibit a monopole field is $Q_{-\frac{1}{2}}(\xi)$. If there exists no monopole field then no $Q_{-\frac{1}{2}}(\xi)$ term can exist. This can be, by deduction, extended to higher order multipoles. For example, if a given magnetic charge distribution yields a dipole field far from the source then a $Q_{\frac{1}{2}}(\xi)$ term must exist as the first term in the series expansion for the potential function because this term has imbedded within it the dipole effect. If, on the other hand, the far field solution looks like a quadrupole then the first term in the series expansion must be a $Q_{\frac{3}{2}}(\xi)$ because this term has imbedded within it the quadrupole effect. In other words, for the far field to look like a dipole, a $Q_{\frac{1}{2}}(\xi)$ term must exist, and for the far field to look like a quadrupole, a $Q_{\frac{3}{2}}(\xi)$ must exist, etc. This has been explained in depth in chapter 7.

APPENDIX M

ROTATED CYLINDRICAL COORDINATE SYSTEM

M.1 Introduction

Many advanced texts do not directly discuss the computation of potentials or fields from a source that is rotated and translated from the origin of chosen coordinate system [87]. It may be assumed by the author of that text that a particular coordinate system transformation [136] is all that is needed to solve the problem and in principle this is correct. However, a number of pit-falls can occur and one needs to be careful.

This appendix tries to give a more detailed explanation which may shed some light on electromagnetic sources which have been rotated and translated relative to a global coordinate system. The problem of a current loop which has been rotated and translated relative to some global coordinates system is used to illustrate some of the more important aspects of the problem. It is also useful to use a current loop because it is able to give the characteristic dipole field which is useful for simulating how an external field can affect measurements from a known source.

M.2 Rotation and translation of an electromagnetic source

The orientation of an object located anywhere in space can be represented by a translation and a rotation of the local coordinate system relative to some inertial or global coordinate system. Consider the following rotations shown in Figure M.1

.

Figure M.1: Coordinate system rotations

The equations which represent the above sequence of rotations are given by

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (\text{M.1})$$

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} \cos(\beta) & 0 & -\sin(\beta) \\ 0 & 1 & 0 \\ \sin(\beta) & 0 & \cos(\beta) \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \quad (\text{M.2})$$

$$\begin{bmatrix} x''' \\ y''' \\ z''' \end{bmatrix} = \begin{bmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} \quad (\text{M.3})$$

where

$$R_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \quad (\text{M.4})$$

$$R_\beta = \begin{bmatrix} \cos(\beta) & 0 & -\sin(\beta) \\ 0 & 1 & 0 \\ \sin(\beta) & 0 & \cos(\beta) \end{bmatrix} \quad (\text{M.5})$$

$$R_\gamma = \begin{bmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{M.6})$$

Equations (M.7) through (M.12) are called the *Cardanian* rotations and Equations (M.13) through (M.18) are called the *Eulerian* rotations.

$$\mathbf{R}_{\alpha\beta\gamma} = R_\alpha R_\beta R_\gamma \quad (\text{M.7})$$

$$\mathbf{R}_{\alpha\gamma\beta} = R_\alpha R_\gamma R_\beta \quad (\text{M.8})$$

$$\mathbf{R}_{\beta\alpha\gamma} = R_\beta R_\alpha R_\gamma \quad (\text{M.9})$$

$$\mathbf{R}_{\beta\gamma\alpha} = R_\beta R_\gamma R_\alpha \quad (\text{M.10})$$

$$\mathbf{R}_{\gamma\alpha\beta} = R_\gamma R_\alpha R_\beta \quad (\text{M.11})$$

$$\mathbf{R}_{\gamma\beta\alpha} = R_\gamma R_\beta R_\alpha \quad (\text{M.12})$$

$$\mathbf{R}_{\alpha\beta\alpha} = R_\alpha R_\beta R_\alpha \quad (\text{M.13})$$

$$\mathbf{R}_{\alpha\gamma\alpha} = R_\alpha R_\gamma R_\alpha \quad (\text{M.14})$$

$$\mathbf{R}_{\beta\alpha\beta} = R_\beta R_\alpha R_\beta \quad (\text{M.15})$$

$$\mathbf{R}_{\beta\gamma\beta} = R_\beta R_\gamma R_\beta \quad (\text{M.16})$$

$$\mathbf{R}_{\gamma\alpha\gamma} = R_\gamma R_\alpha R_\gamma \quad (\text{M.17})$$

$$\mathbf{R}_{\gamma\beta\gamma} = R_\gamma R_\beta R_\gamma \quad (\text{M.18})$$

A sequence of successive rotations can be chosen from among the twelve rotation

matrices shown above. The *Cardanian* rotations or the *Eulerian* rotations could be used to compute an arbitrary rotation of a physical system. In fact, the *Cardanian* rotations can be considered a subset of the *Eulerian* rotations. Only the *Cardanian* rotations will be employed in the analysis that follows.

Equations (M.4) through (M.6), when multiplied together, do not obey, in general, the commutative law and therefore care must be taken when considering a particular rotation. Of course, there are different sequences of rotations which may produce the same final rotation matrix, but in general, one must choose a particular sequence in order to arrive at a desired result.

One could have just as easily used the *Eulerian* system to produce the various angular rotations, and in fact, the analysis that follows can very easily be modified to handle this system. The *Euler* angles are used quite often in dynamic or aerodynamic problems and are related to the *pitch*, *roll*, and the *yaw*. In aerodynamics, for example, the *pitch* is a rotation around an axis parallel to the wings, thus the nose and tail both pitch up or down. The *roll* is a rotation around an axis drawn through the body of the vehicle from tail to nose. Finally, the *yaw* is a rotation around an axis orthogonal to the pitch and the roll axes. If an airplane model is placed on a flat surface and if it is spun or pivoted around the center of mass (coordinate origin), it would be described as yawing.

Consider the following current loop oriented arbitrarily in space as shown in Figure M.2 . The following equation describes the field point in terms of the inertial coordinates and the translation from the origin:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \mathbf{R} \begin{bmatrix} x - x_o \\ y - y_o \\ z - z_o \end{bmatrix} \quad (\text{M.19})$$

where

$$\mathbf{R} = \begin{bmatrix} \mathbf{r}_{11} & \mathbf{r}_{12} & \mathbf{r}_{13} \\ \mathbf{r}_{21} & \mathbf{r}_{22} & \mathbf{r}_{23} \\ \mathbf{r}_{31} & \mathbf{r}_{32} & \mathbf{r}_{33} \end{bmatrix} \quad (\text{M.20})$$

The components of the **R**-matrix are determined from one of the Equations (M.7)

Figure M.2: A rotated and translated current loop

through (M.12).

EXAMPLE 1:

Consider a current loop with a loop radius of a as shown in Figure M.3 . The general equation for the magnetic vector potential of the non-rotated current loop in cylindrical coordinates and in terms of Elliptic integrals is given by the well-known result:

$$\mathbf{A}(\rho, z) = \frac{\mu_0 I}{\pi} \frac{a}{k^2 \sqrt{\rho^2 + z^2 + a^2 + 2a\rho}} [(2 - k^2)K(k) - 2E(k)] \hat{\phi} \quad (\text{M.21})$$

where

$$k^2 = \frac{4a\rho}{\rho^2 + z^2 + a^2 + 2a\rho} \quad (\text{M.22})$$

Figure M.3: Non-rotated current loop

The components of the magnetic flux density in cylindrical coordinates are

$$B_\rho = \frac{\mu_0 I}{2\pi} \frac{z}{\rho(\rho^2 + z^2 + a^2 + 2a\rho)} \left\{ -K(k) + \left(\frac{\rho^2 + z^2 + a^2}{\rho^2 + z^2 + a^2 - 2a\rho} \right) E(k) \right\} \quad (\text{M.23})$$

$$B_\phi = 0 \quad (\text{M.24})$$

$$B_z = \frac{\mu_0 I}{2\pi} \frac{1}{(\rho^2 + z^2 + a^2 + 2a\rho)} \left\{ K(k) + \left(\frac{-(\rho^2 + z^2) + a^2}{\rho^2 + z^2 + a^2 - 2a\rho} \right) E(k) \right\} \quad (\text{M.25})$$

In terms of Cartesian coordinates, the magnetic vector potential is given by

$$\mathbf{A}(x, y, z) = \frac{\mu_0 I}{\pi} \frac{a}{k^2 \sqrt{x^2 + y^2 + z^2 + a^2 + 2a\sqrt{x^2 + y^2}}} \left[(2 - k^2)K(k) - 2E(k) \right] \hat{\phi} \quad (\text{M.26})$$

where

$$k^2 = \frac{4a\sqrt{x^2 + y^2}}{x^2 + y^2 + z^2 + a^2 + 2a\sqrt{x^2 + y^2}} \quad (\text{M.27})$$

$$\hat{\phi} = -\frac{y}{\sqrt{x^2 + y^2}}\hat{x} + \frac{x}{\sqrt{x^2 + y^2}}\hat{y} \quad (\text{M.28})$$

The components of the magnetic flux density in Cartesian coordinates are

$$B_x = B_\rho \left(\frac{x}{\sqrt{x^2 + y^2}} \right) \quad (\text{M.29})$$

$$B_y = B_\rho \left(\frac{y}{\sqrt{x^2 + y^2}} \right) \quad (\text{M.30})$$

$$B_z = B_z \quad (\text{M.31})$$

where B_ρ and B_z on the right hand sides of Equations (M.29) through (M.31) are the cylindrical components given in Equations (M.23) and (M.25) with ρ replaced by $\sqrt{x^2 + y^2}$. In terms of spherical coordinates, the magnetic vector potential is given by

$$\mathbf{A}(r, \theta) = \frac{\mu_0 I}{\pi} \frac{a}{k^2 \sqrt{r^2 + a^2 + 2ar \sin(\theta)}} \left[(2 - k^2)K(k) - 2E(k) \right] \hat{\phi} \quad (\text{M.32})$$

where

$$k^2 = \frac{4ar \sin(\theta)}{r^2 + a^2 + 2ar \sin(\theta)} \quad (\text{M.33})$$

The components of the magnetic flux density in spherical coordinates are

$$B_r = B_\rho \sin(\theta) + B_z \cos(\theta) \quad (\text{M.34})$$

$$B_\phi = 0 \quad (\text{M.35})$$

$$B_\theta = B_\rho \cos(\theta) - B_z \sin(\theta) \quad (\text{M.36})$$

where B_ρ and B_z are the cylindrical components given in Equations (M.23), and (M.25) with ρ replaced by $r \sin(\theta)$ and z replaced by $r \cos(\theta)$.

Figure M.4: Rotated current loop

EXAMPLE 2:

Consider the same current loop rotated about the x -axis through an angle α as shown in Figure M.4 . The general equation for the magnetic vector potential of the rotated current loop in terms of the local cylindrical coordinates is given by

$$\mathbf{A}(\rho, z) = \frac{\mu_0 I}{\pi} \frac{a}{k'^2 \sqrt{\rho'^2 + z'^2 + a^2 + 2a\rho'}} \left[(2 - k'^2) K(k') - 2E(k') \right] \hat{\phi}' \quad (\text{M.37})$$

where

$$k'^2 = \frac{4a\rho'}{\rho'^2 + z'^2 + a^2 + 2a\rho'} \quad (\text{M.38})$$

$$\rho'^2 = \rho^2 [\cos^2(\phi) + \sin^2(\phi) \cos^2(\alpha)] + z[z \sin^2(\alpha) + \rho \sin(\phi) \sin(2\alpha)] \quad (\text{M.39})$$

$$z' = -\rho \sin(\phi) \sin(\alpha) + z \cos(\alpha) \quad (\text{M.40})$$

$$\hat{\phi}' = c_{21} \hat{\rho} + c_{22} \hat{\phi} + c_{23} \hat{z} \quad (\text{M.41})$$

The components of the C -matrix are found from

$$C = R'_{cylinder} R_{\alpha} R_{cylinder}^{-1} \quad (\text{M.42})$$

where

$$R'_{cylinder} = \begin{bmatrix} \cos(\phi') & \sin(\phi') & 0 \\ -\sin(\phi') & \cos(\phi') & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (M.43)$$

$$R_{cylinder}^{-1} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (M.44)$$

Using Equation (M.4) and Equations (M.42) through (M.44) yield:

$$c_{21} = \frac{x \sin(\phi) \cos(\alpha) - \cos(\phi) [y \cos(\alpha) + z \sin(\alpha)]}{\sqrt{x^2 + [y \cos(\alpha) + z \sin(\alpha)]^2}} \quad (M.45)$$

$$c_{22} = \frac{x \cos(\phi) \cos(\alpha) + \sin(\phi) [y \cos(\alpha) + z \sin(\alpha)]}{\sqrt{x^2 + [y \cos(\alpha) + z \sin(\alpha)]^2}} \quad (M.46)$$

$$c_{23} = \frac{x \sin(\phi) \sin(\alpha)}{\sqrt{x^2 + [y \cos(\alpha) + z \sin(\alpha)]^2}} \quad (M.47)$$

The components of the magnetic flux density in terms of the global cylindrical coordinates are

$$B_\rho = (B_{\rho'} c_{11} + B_{z'} c_{31}) \quad (M.48)$$

$$B_\phi = (B_{\rho'} c_{12} + B_{z'} c_{32}) \quad (M.49)$$

$$B_z = (B_{\rho'} c_{13} + B_{z'} c_{33}) \quad (M.50)$$

where

$$B_{\rho'} = \frac{\mu_0 I}{2\pi} \frac{z'}{\rho' (\rho'^2 + z'^2 + a^2 + 2a\rho')} \left\{ -K(k') + \left(\frac{\rho'^2 + z'^2 + a^2}{\rho'^2 + z'^2 + a^2 - 2a\rho'} \right) E(k') \right\} \quad (\text{M.51})$$

$$B_{z'} = \frac{\mu_0 I}{2\pi} \frac{1}{(\rho'^2 + z'^2 + a^2 + 2a\rho')} \left\{ K(k') + \left(\frac{-(\rho'^2 + z'^2) + a^2}{\rho'^2 + z'^2 + a^2 - 2a\rho'} \right) E(k') \right\} \quad (\text{M.52})$$

The magnetic vector potential in terms of the Cartesian local coordinates is given by

$$\vec{A}(x, y, z) = \frac{\mu_0 I}{\pi} \frac{a}{k'^2 \sqrt{x'^2 + y'^2 + z'^2 + a^2 + 2a\sqrt{x'^2 + y'^2}}} \left[(2 - k'^2) K(k') - 2E(k') \right] \hat{\phi}' \quad (\text{M.53})$$

where

$$k'^2 = \frac{4a\sqrt{x'^2 + y'^2}}{x'^2 + y'^2 + z'^2 + a^2 + 2a\sqrt{x'^2 + y'^2}} \quad (\text{M.54})$$

$$x'^2 + y'^2 = x^2 + [y \cos(\alpha) + z \sin(\alpha)]^2 \quad (\text{M.55})$$

$$z' = -y \sin(\alpha) + z \cos(\alpha) \quad (\text{M.56})$$

$$\hat{\phi}' = t_{21}\hat{x} + t_{22}\hat{y} + t_{23}\hat{z} \quad (\text{M.57})$$

The components of the *T-matrix* are found from

$$T = R'_{cylinder} R_\alpha \quad (\text{M.58})$$

Using Equation (M.4), Equations (M.43), and (M.58) yield:

$$t_{21} = -\frac{[y \cos(\alpha) + z \sin(\alpha)]}{\sqrt{x^2 + [y \cos(\alpha) + z \sin(\alpha)]^2}} \quad (\text{M.59})$$

$$t_{22} = \frac{x \cos(\alpha)}{\sqrt{x^2 + [y \cos(\alpha) + z \sin(\alpha)]^2}} \quad (\text{M.60})$$

$$t_{23} = \frac{x \sin(\alpha)}{\sqrt{x^2 + [y \cos(\alpha) + z \sin(\alpha)]^2}} \quad (\text{M.61})$$

The components of the magnetic flux density in terms of the Cartesian global coordinates are

$$B_x = (B_{\rho'} t_{11} + B_{z'} t_{31}) \quad (\text{M.62})$$

$$B_y = (B_{\rho'} t_{12} + B_{z'} t_{32}) \quad (\text{M.63})$$

$$B_z = (B_{\rho'} t_{13} + B_{z'} t_{33}) \quad (\text{M.64})$$

The magnetic vector potential in terms of the local spherical coordinates is given by

$$\vec{A}(r, \theta) = \frac{\mu_0 I}{\pi} \frac{a}{k^2 \sqrt{r'^2 + a^2 + 2ar' \sin(\theta')}} \left[(2 - k'^2) K(k') - 2E(k') \right] \hat{\phi}' \quad (\text{M.65})$$

where

$$k'^2 = \frac{4ar \sin(\theta')}{r'^2 + a^2 + 2ar' \sin(\theta')} \quad (\text{M.66})$$

$$r' \sin(\theta') = \rho' \quad (\text{M.67})$$

$$\rho'^2 = r^2 \sin^2(\theta) [\cos^2(\phi) + \sin^2(\phi) \cos^2(\alpha)] + r \cos(\theta) [r \cos(\theta) \sin^2(\alpha) + r \sin(\theta) \sin(\phi) \sin(2\alpha)] \quad (\text{M.68})$$

$$\hat{\phi}' = s_{21} \hat{r} + s_{22} \hat{\phi} + s_{23} \hat{\theta} \quad (\text{M.69})$$

The components of the S -matrix are found from

$$S = R'_{sphere} R_{\alpha} R_{sphere}^{-1} \quad (\text{M.70})$$

where

$$R'_{sphere} = \begin{bmatrix} \cos(\phi') \sin(\theta') & \sin(\phi') \sin(\theta') & \cos(\theta') \\ -\sin(\phi') & \cos(\phi') & 0 \\ \cos(\phi') \cos(\theta') & \sin(\phi') \cos(\theta') & -\sin(\theta') \end{bmatrix} \quad (\text{M.71})$$

$$R_{sphere}^{-1} = \begin{bmatrix} \cos(\phi) \sin(\theta) & -\sin(\phi) & \cos(\phi) \cos(\theta) \\ \sin(\phi) \sin(\theta) & \cos(\phi) & \sin(\phi) \cos(\theta) \\ \cos(\theta) & 0 & -\sin(\theta) \end{bmatrix} \quad (\text{M.72})$$

Using (M.4), Equations (M.70) through (M.72) yield:

$$s_{21} = \frac{1}{\sqrt{x^2 + [y \cos(\alpha) + z \sin(\alpha)]^2}} \left\{ -[y \cos(\alpha) + x \sin(\alpha)] \cos(\phi) \sin(\theta) + x [\cos(\alpha) \sin(\phi) \sin(\theta) + \sin(\alpha) \cos(\theta)] \right\} \quad (\text{M.73})$$

$$s_{22} = \frac{[y \cos(\alpha) + x \sin(\alpha)] \sin(\phi) + x \cos(\alpha) \cos(\phi)}{\sqrt{x^2 + [y \cos(\alpha) + z \sin(\alpha)]^2}} \quad (\text{M.74})$$

$$s_{23} = \frac{1}{\sqrt{x^2 + [y \cos(\alpha) + z \sin(\alpha)]^2}} \left\{ -[y \cos(\alpha) + x \sin(\alpha)] \cos(\phi) \cos(\theta) + x [\cos(\alpha) \sin(\phi) \cos(\theta) - \sin(\alpha) \sin(\theta)] \right\} \quad (\text{M.75})$$

The components of the magnetic flux density in terms of the spherical global coordinates are

$$B_r = (B_{\rho'} v_{11} + B_{z'} v_{31}) \quad (\text{M.76})$$

$$B_{\phi} = (B_{\rho'} v_{12} + B_{z'} v_{32}) \quad (\text{M.77})$$

$$B_{\theta} = (B_{\rho'} v_{13} + B_{z'} v_{33}) \quad (\text{M.78})$$

where the components of the *V-Matrix* are found from

$$V = R'_{cylinder} R_{\alpha} R_{sphere}^{-1} \quad (M.79)$$

The matrices which are used in deriving the magnetic vector potential and the magnetic flux density of a rotated current loop about the *x*-axis are given below.

$$\begin{aligned} C &= R'_{cylinder} R_{\alpha} R_{cylinder}^{-1} \\ &= \begin{bmatrix} \cos(\phi') & \sin(\phi') & 0 \\ -\sin(\phi') & \cos(\phi') & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \cdot \\ &\quad \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \quad (M.80)$$

$$\begin{aligned} T &= R'_{cylinder} R_{\alpha} \\ &= \begin{bmatrix} \cos(\phi') & \sin(\phi') & 0 \\ -\sin(\phi') & \cos(\phi') & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \end{aligned} \quad (M.81)$$

$$\begin{aligned} S &= R'_{sphere} R_{\alpha} R_{sphere}^{-1} \\ &= \begin{bmatrix} \cos(\phi') \sin(\theta') & \sin(\phi') \sin(\theta') & \cos(\theta') \\ -\sin(\phi') & \cos(\phi') & 0 \\ \cos(\phi') \cos(\theta') & \sin(\phi') \cos(\theta') & -\sin(\theta') \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \cdot \\ &\quad \begin{bmatrix} \cos(\phi) \sin(\theta) & -\sin(\phi) & \cos(\phi) \cos(\theta) \\ \sin(\phi) \sin(\theta) & \cos(\phi) & \sin(\phi) \cos(\theta) \\ \cos(\theta) & 0 & -\sin(\theta) \end{bmatrix} \end{aligned} \quad (M.82)$$

Figure M.5: Non-rotated charged line segment

$$\begin{aligned}
 V &= R'_{cylinder} R_{\alpha} R_{sphere}^{-1} \\
 &= \begin{bmatrix} \cos(\phi') & \sin(\phi') & 0 \\ -\sin(\phi') & \cos(\phi') & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \cdot \\
 &\quad \begin{bmatrix} \cos(\phi) \sin(\theta) & -\sin(\phi) & \cos(\phi) \cos(\theta) \\ \sin(\phi) \sin(\theta) & \cos(\phi) & \sin(\phi) \cos(\theta) \\ \cos(\theta) & 0 & -\sin(\theta) \end{bmatrix} \quad (M.83)
 \end{aligned}$$

EXAMPLE 3:

Consider a charged line segment of length, L , located on the x -axis as shown in Figure M.5 . The charged line segment has a total charge, q , placed on it and the linear charge density is assumed constant. The differential scalar potential at point $P(x, y, z)$ can be written as

$$\begin{aligned}
 d\Phi_P(x, y, z) &= \frac{\lambda}{4\pi\epsilon_0} \frac{dx'}{|\mathbf{r}|} \\
 &= \frac{q}{4\pi\epsilon_0 L} \frac{dx'}{\sqrt{(x-x')^2 + y^2 + z^2}} \quad (M.84)
 \end{aligned}$$

Figure M.6: Rotated charged line segment

The scalar potential can be written as

$$\Phi_P(x, y, z) = \frac{q}{4\pi\epsilon_0 L} \int_0^L \frac{dx'}{\sqrt{(x-x')^2 + y^2 + z^2}} \quad (\text{M.85})$$

Integrating Equation (M.85) yields

$$\Phi_P(x, y, z) = \frac{q}{4\pi\epsilon_0 L} L n \left[\frac{\sqrt{(x-L)^2 + y^2 + z^2} - (x-L)}{\sqrt{x^2 + y^2 + z^2} - x} \right] \quad (\text{M.86})$$

Now consider the same charged line segment except that it is rotated about the z -axis through an angle ϕ , and it remains in the x - y plane as shown in Figure M.6 .

The equation of the line representing the charged line segment is given by

$$y' = \tan(\phi)x' \quad (\text{M.87})$$

and the differential length along the charged line segment is given by

$$dl' = \sqrt{(dy')^2 + (dx')^2} \quad (\text{M.88})$$

where

$$dl' = \sec(\phi)dx' \quad (\text{M.89})$$

or

$$dl' = \csc(\phi) dy' \quad (\text{M.90})$$

Using the same approach for the rotated system as was used for the non-rotated system, one can write the scalar potential at point $P(x, y, z)$ as

$$\Phi_P(x, y, z) = \frac{q}{4\pi\epsilon_0 L} \int_{l_1}^{l_2} \frac{dl'}{\sqrt{(x - x')^2 + (y - y')^2 + z^2}} \quad (\text{M.91})$$

where, in terms of the x' variable, one obtains

$$\begin{aligned} dl' &= \sqrt{[\tan(\phi)dx']^2 + (dx')^2} \\ &= \sec(\phi)dx' \end{aligned} \quad (\text{M.92})$$

and

$$l_1 = 0 \quad (\text{M.93})$$

$$l_2 = L \cos(\phi) \quad (\text{M.94})$$

$$y' = \tan(\phi)x' \quad (\text{M.95})$$

Substituting Equations (M.92) through (M.95) into Equation (M.91) yields:

$$\Phi_P(x, y, z) = \frac{q \sec(\phi)}{4\pi\epsilon_0 L} \int_0^{L \cos(\phi)} \frac{dx'}{\sqrt{(x - x')^2 + [y - \tan(\phi)x']^2 + z^2}} \quad (\text{M.96})$$

or, in terms of the y' variable, one obtains

$$\Phi_P(x, y, z) = \frac{q \csc(\phi)}{4\pi\epsilon_0 L} \int_0^{L \sin(\phi)} \frac{dy'}{\sqrt{[x - y' \cot(\phi)]^2 + (y - y')^2 + z^2}} \quad (\text{M.97})$$

Choose Equation (M.96) for the analysis. Integrating Equation (M.96) gives

$$\begin{aligned} \Phi_P(x, y, z) = & \frac{q}{4\pi\epsilon_0 L} \text{Ln} \left[\frac{1}{\sqrt{x^2 + y^2 + z^2} - [x \cos(\phi) + y \sin(\phi)]} \right. \\ & \left. \left\{ \sqrt{[x - L \cos(\phi)]^2 + [y - l \sin(\phi)]^2 + z^2} - \right. \right. \\ & \left. \left. [x \cos(\phi) + y \sin(\phi) - L] \right\} \right] \end{aligned} \quad (\text{M.98})$$

or

$$\Phi_P(u, v, z) = \frac{q}{4\pi\epsilon_0 L} \text{Ln} \left[\frac{\sqrt{(u - L)^2 + v^2 + z^2} - (u - L)}{\sqrt{u^2 + v^2 + z^2} - u} \right] \quad (\text{M.99})$$

in the rotated coordinate system. Check to see if Equation (M.98) includes, as a special case, the solution to the non-rotated system.

$$\begin{aligned} \lim_{\phi \rightarrow 0} \Phi_P(x, y, z) = & \lim_{\phi \rightarrow 0} \frac{q}{4\pi\epsilon_0 L} \text{Ln} \left[\frac{1}{\sqrt{x^2 + y^2 + z^2} - [x \cos(\phi) + y \sin(\phi)]} \right. \\ & \left\{ \sqrt{[x - L \cos(\phi)]^2 + [y - l \sin(\phi)]^2 + z^2} - \right. \\ & \left. [x \cos(\phi) + y \sin(\phi) - L] \right\} \Big] \\ = & \frac{q}{4\pi\epsilon_0 L} \text{Ln} \left[\frac{\sqrt{(x - L)^2 + y^2 + z^2} - (x - L)}{\sqrt{x^2 + y^2 + z^2} - x} \right] \end{aligned} \quad (\text{M.100})$$

Equation (M.100) is just Equation (M.86). This is, of course, what should be expected, if indeed, Equation (M.98) were the correct solution. Equation (M.98) yields the correct limiting case, and because of this, one should have some confidence that it is correct. Now, compare Equation (M.98) with Equation (M.99). Consider the coordinate transformation given by

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (\text{M.101})$$

Applying Equation (M.101) to Equation (M.99) will yield Equation (M.98). This says that one needs to only find the solution to the non-rotated source and then apply the proper transformation to get the solution to the rotated source. This is

Figure M.7: Comparison between a rotated charged line segment and a non-rotated charged line segment

usually much easier to do.

EXAMPLE 4:

The following example will illustrate, pictorially, the difference between Equation (M.86), and Equation (M.98). Let $\frac{q}{4\pi\epsilon_0} = 1$, $L = 1.0 \text{ m}$, $y = 1.0 \text{ m}$, $z = 0.0 \text{ m}$, and set $\phi = +30^\circ$ positive rotation which is a counterclockwise rotation. Equation (M.86) becomes

$$\Phi_P(x, 1, 0) = Ln \left[\frac{\sqrt{(x-1)^2 + 1} - (x-1)}{\sqrt{x^2 + 1} - x} \right] \quad (\text{M.102})$$

and Equation (M.98) becomes

$$\Phi_P(x, 1, 0) = Ln \left[\frac{\sqrt{\left[x - \frac{\sqrt{3}}{2}\right]^2 + \left[1 - \frac{1}{2}\right]^2} - \left[\frac{\sqrt{3}}{2}x + \frac{1}{2} - 1\right]}{\sqrt{x^2 + 1} - \left[\frac{\sqrt{3}}{2}x + \frac{1}{2}\right]} \right] \quad (\text{M.103})$$

Plotting Equations (M.102) and (M.103) yield Figure M.7 . The plots represent the potential at the field points $(x, 1, 0)$ as x varies while y and z are held fixed for the

Figure M.8: Comparison between a rotated charged line segment and a non-rotated charged line segment

rotated and non-rotated sources.

EXAMPLE 5:

Let $\frac{q}{4\pi\epsilon_0} = 1$, $L = 1.0$ m, $y = -1.0$ m, $z = 0.0$ m, and set $\phi = +30^\circ$. Equation (M.86) becomes

$$\Phi_P(x, -1, 0) = Ln \left[\frac{\sqrt{(x-1)^2 + 1} - (x-1)}{\sqrt{x^2 + 1} - x} \right] \quad (\text{M.104})$$

and (M.98) becomes

$$\Phi_P(x, -1, 0) = Ln \left[\frac{\sqrt{\left[x - \frac{\sqrt{3}}{2}\right]^2 + \left[-1 - \frac{1}{2}\right]^2} - \left[\frac{\sqrt{3}}{2}x - \frac{1}{2} - 1\right]}{\sqrt{x^2 + 1} - \left[\frac{\sqrt{3}}{2}x - \frac{1}{2}\right]} \right] \quad (\text{M.105})$$

Plotting Equations (M.104) and (M.86) yield Figure M.8 . The plots represent the potential at the field points $(x, -1, 0)$ as x varies while y and z are held fixed for the rotated and non-rotated sources.

APPENDIX N

BRONZAN'S METHOD

N.1 Introduction

In 1971, J.B. Bronzan published a paper [12] which showed how the magnetic scalar potential, under certain circumstances, could be used where one would normally have used a vector potential formulation. This appendix attempts to illustrate some of the salient features of his method and also attempts to make his derivation more accessible to the engineer. As a special case of Bronzan's method, one can derive a more simplified version of the magnetic scalar potential that is applicable to filamentary circuits.

N.2 General derivation of Bronzan's method

The derivation of Bronzan's magnetic scalar potential can be accomplished through a direct application of Maxwell's equations. One begins with

$$\nabla \times \mathbf{H} = \mathbf{J} \quad (\text{N.1})$$

and

$$\nabla \cdot \mathbf{H} = 0 \quad (\text{N.2})$$

Equation (N.2) is derived from

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\mu \mathbf{H}) = \mu (\nabla \cdot \mathbf{H}) + \mathbf{H} \cdot (\nabla \mu) \quad (\text{N.3})$$

and if $\mu = \text{constant}$ then $\nabla \cdot \mathbf{H} = 0$. Taking the *curl* of both sides of Equation (N.1) yields:

$$\nabla \times \nabla \times \mathbf{H} = \nabla \times \mathbf{J} \quad (\text{N.4})$$

where

$$\nabla \times \nabla \times \mathbf{H} = \nabla(\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H} = -\nabla^2 \mathbf{H} \quad (\text{N.5})$$

Figure N.1: A general magnetic source

Using Equations (N.4) and (N.5) yield:

$$\nabla^2 \mathbf{H} = -\nabla \times \mathbf{J} \quad (\text{N.6})$$

Employing an approach developed by Bouwkamp and Casimir [116], one obtains

$$\begin{aligned} \nabla^2(\mathbf{r} \cdot \mathbf{H}) &= 2(\nabla \cdot \mathbf{H}) + \mathbf{r} \cdot \nabla^2(\mathbf{H}) \\ &= \mathbf{r} \cdot \nabla^2(\mathbf{H}) \end{aligned} \quad (\text{N.7})$$

Employing Equations (N.6) and (N.7) yield:

$$\nabla^2(\mathbf{r} \cdot \mathbf{H}) = -\mathbf{r} \cdot (\nabla \times \mathbf{J}) \quad (\text{N.8})$$

Equation (N.8) is just Poisson's equation. The solution to Poisson's equation is given by

$$\mathbf{r} \cdot \mathbf{H} = \frac{1}{4\pi} \int \frac{\mathbf{r}' \cdot (\nabla' \times \mathbf{J})}{|\mathbf{r} - \mathbf{r}'|} d^3 r' \quad (\text{N.9})$$

where the integration is carried out over all space where $\mathbf{J} \neq 0$. If the source $\mathbf{J}$ can be bounded by some closed surface, $\partial\Omega$, as shown in Figure N.1, then $\mathbf{r} \cdot \mathbf{H}$ outside

Figure N.2: A magnetic source enclosed in a hypothetical sphere

of $\partial\Omega$ satisfies Laplace's equation where $\mathbf{r}'$ is the location vector of the source point. Consider the case when the bounding surface, $\partial\Omega$, is a sphere of radius R as shown in Figure N.2 .

We know that

$$\nabla^2(\mathbf{r} \cdot \mathbf{H}) = 0 \quad (\text{N.10})$$

for $r > R$ and in spherical coordinates one also knows that

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{4\pi}{2l+1} \frac{r'^l}{r^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \quad (\text{N.11})$$

Substituting Equation (N.11) into Equation (N.9) yields:

$$\begin{aligned} \mathbf{r} \cdot \mathbf{H} &= \frac{1}{4\pi} \int \mathbf{r}' \cdot (\nabla' \times \mathbf{J}) \left\{ \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{4\pi}{2l+1} \frac{r'^l}{r^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \right\} d^3 r' \\ &= \frac{1}{4\pi} \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \int \mathbf{r}' \cdot (\nabla' \times \mathbf{J}) r'^l Y_{lm}^*(\theta', \phi') d^3 r' \quad (\text{N.12}) \end{aligned}$$

where

$$\mathbf{r} \cdot \mathbf{H} = r H_r = -r \frac{\partial \Phi}{\partial r} \quad (\text{N.13})$$

Substituting Equation (N.13) into Equation (N.12), dividing both sides by r , and then integrating on r yields:

$$\begin{aligned} \Phi(\mathbf{r}) &= -\frac{1}{4\pi} \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{4\pi}{2l+1} Y_{lm}(\theta, \phi) \left(\int \frac{1}{r^{l+2}} dr \right) \int \mathbf{r}' \cdot (\nabla' \times \mathbf{J}) r'^l Y_{lm}^*(\theta', \phi') d^3 r' \\ &= \frac{1}{4\pi} \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta, \phi)}{(l+1)r^{l+1}} \int \mathbf{r}' \cdot (\nabla' \times \mathbf{J}) r'^l Y_{lm}^*(\theta', \phi') d^3 r' \end{aligned} \quad (\text{N.14})$$

In order to obtain Bronzan's result, one needs to use the vector identity

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}) \quad (\text{N.15})$$

with $\mathbf{F} = \mathbf{r}'$ and $\mathbf{G} = \mathbf{J}$ in Equation (N.15) to obtain

$$\Phi(\mathbf{r}) = -\frac{1}{4\pi} \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta, \phi)}{(l+1)r^{l+1}} \int \left(\nabla' \cdot (\mathbf{r}' \times \mathbf{J}) \right) r'^l Y_{lm}^*(\theta', \phi') d^3 r' \quad (\text{N.16})$$

This is exactly what Bronzan derives. As Bronzan writes

$$\Phi(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{4\pi}{2l+1} \frac{M_{lm} Y_{lm}(\theta, \phi)}{r^{l+1}} \quad (\text{N.17})$$

where

$$M_{lm} = -\frac{1}{4\pi} \int \frac{r'^l Y_{lm}^*(\theta', \phi')}{l+1} \left(\nabla' \cdot (\mathbf{r}' \times \mathbf{J}) \right) d^3 r' \quad (\text{N.18})$$

Also, using Equation (N.9) and Equation (N.13) one obtains the general magnetic scalar potential

$$\begin{aligned} -r \frac{\partial \Phi}{\partial r} &= \frac{1}{4\pi} \int \frac{\mathbf{r}' \cdot (\nabla' \times \mathbf{J})}{|\mathbf{r} - \mathbf{r}'|} d^3 r' \\ \Phi(\mathbf{r}) &= \Phi(\mathbf{a}) - \frac{1}{4\pi} \int \left(\mathbf{r}' \cdot (\nabla' \times \mathbf{J}) \right) d^3 r' \int \frac{1}{r |\mathbf{r} - \mathbf{r}'|} dr \end{aligned} \quad (\text{N.19})$$

As Bronzan states, one can only apply this only in certain regions of space.

Figure N.3: Prohibitive zones in Bronzan's method

Consider, for example, the case where the reference point is far from the origin as shown in Figure N.3 . Using Bronzan's notation, one writes $\nabla \cdot \mathbf{H} = 0$ where $\mathbf{J} = 0$, therefore

$$\Phi(\mathbf{x}) = \Phi(\mathbf{a}) - \int_{\mathbf{a}}^{\mathbf{x}} \mathbf{H}(\xi) \cdot d\xi \quad (\text{N.20})$$

The integral in Equation (N.20) must be chosen in such a way that it avoids all regions where $\mathbf{J} \neq 0$. One is at liberty to choose any $\mathbf{a}$; therefore $\mathbf{a}$ is an arbitrary reference point. $\Phi(\mathbf{a})$ has no effect on the computation of $\mathbf{H}$ as long as $\mathbf{J}(\mathbf{a}) = 0$. If $\mathbf{a}$ is chosen to be very far away from the origin, say ∞ , and allowing $\Phi(\mathbf{a}) = 0$ then $\Phi(\mathbf{x})$ is now a single-valued function. In other words, it is impossible for the integration path to enclose a current source without crossing the current source or a cross-hatched region as illustrated in Figure N.3 . This is the most important requirement for Bronzan's method to be applicable. One can now write

$$\Phi(\mathbf{x}) = - \int_{\mathbf{a}}^{\mathbf{x}} \mathbf{H}(\xi) \cdot d\xi \quad (\text{N.21})$$

where

$$\mathbf{H}(\xi) = \frac{1}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') \times (\xi \hat{x} - \mathbf{x}') d^3 x'}{|\xi \hat{x} - \mathbf{x}'|^3} \quad (\text{N.22})$$

Using Equation (N.22), one can rewrite Equation (N.21) as

$$\begin{aligned} \Phi(\mathbf{x}) &= -\frac{1}{4\pi} \int_{-\infty}^x d\xi \int \frac{\hat{x} \cdot (\mathbf{J}(\mathbf{x}') \times (\xi \hat{x} - \mathbf{x}')) d^3 x'}{|\xi \hat{x} - \mathbf{x}'|^3} \\ &= -\frac{1}{4\pi} \int_{-\infty}^x d\xi \int \frac{(\hat{x} \cdot (\mathbf{J}(\mathbf{x}') \times \xi \hat{x}) - \hat{x} \cdot (\mathbf{J}(\mathbf{x}') \times \mathbf{x}')) d^3 x'}{|\xi \hat{x} - \mathbf{x}'|^3} \\ &= \frac{1}{4\pi} \int_{-\infty}^x d\xi \int \frac{\hat{x} \cdot (\mathbf{J}(\mathbf{x}') \times \mathbf{x}') d^3 x'}{|\xi \hat{x} - \mathbf{x}'|^3} \\ &= \frac{1}{4\pi} \int \hat{x} \cdot (\mathbf{J}(\mathbf{x}') \times \mathbf{x}') d^3 x' \int_{-\infty}^x \frac{d\xi}{|\xi \hat{x} - \mathbf{x}'|^3} \end{aligned} \quad (\text{N.23})$$

but

$$\begin{aligned} (\xi \hat{x} - \mathbf{x}')^2 &= (\xi \hat{x} - \mathbf{x}') \cdot (\xi \hat{x} - \mathbf{x}') \\ &= \xi \hat{x} \cdot \xi \hat{x} - 2\xi \hat{x} \cdot \mathbf{x}' + \mathbf{x}' \cdot \mathbf{x}' \\ &= \xi^2 - 2\xi \hat{x} \cdot \mathbf{x}' + x'^2 \end{aligned} \quad (\text{N.24})$$

therefore

$$\Phi(\mathbf{x}) = \frac{1}{4\pi} \int \hat{x} \cdot (\mathbf{J}(\mathbf{x}') \times \mathbf{x}') d^3 x' \int_{-\infty}^x \frac{d\xi}{(\xi^2 - 2\xi \hat{x} \cdot \mathbf{x}' + x'^2)^{\frac{3}{2}}} \quad (\text{N.25})$$

Employ the identity

$$\mathbf{F} \cdot (\mathbf{G} \times \mathbf{V}) = -\mathbf{G} \cdot (\mathbf{F} \times \mathbf{V}) \quad (\text{N.26})$$

with $\mathbf{F} = \hat{x}$, $\mathbf{G} = \mathbf{J}(x')$, and $\mathbf{V} = \mathbf{x}'$ in Equation (N.26). Equation (N.25) is now written as

$$\begin{aligned} \Phi(\mathbf{x}) &= -\frac{1}{4\pi} \int \mathbf{J}(\mathbf{x}') \cdot (\hat{x} \times \mathbf{x}') d^3 x' \int_{-\infty}^x \frac{d\xi}{(\xi^2 - 2\xi \hat{x} \cdot \mathbf{x}' + x'^2)^{\frac{3}{2}}} \\ &= \frac{1}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') \cdot (\mathbf{x} \times \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'| [x |\mathbf{x} - \mathbf{x}'| + x^2 - \mathbf{x} \cdot \mathbf{x}']} d^3 x' \end{aligned} \quad (\text{N.27})$$

Equation (N.27) is Bronzan's result.

N.3 Bronzan's method in the presence of magnetized media

Can one derive an expression using Bronzan's method which includes the contribution from magnetized material? One begins by writing Maxwell's equations in the presence of magnetized material. The applicable equations are

$$\nabla \times \mathbf{H} = \mathbf{J} \quad (\text{N.28})$$

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \quad (\text{N.29})$$

$$\nabla \cdot \mathbf{B} = 0 \quad (\text{N.30})$$

Also, one can define the following

$$\mathbf{H} = \mathbf{H}_C + \mathbf{H}_M \quad (\text{N.31})$$

where $\mathbf{H}_C$ is the magnetic field intensity due to the current source and $\mathbf{H}_M$ is the magnetic field intensity due to the magnetized material. One also has that

$$\nabla \times \mathbf{H} = \nabla \times (\mathbf{H}_C + \mathbf{H}_M) = \mathbf{J} \quad (\text{N.32})$$

which implies that

$$\nabla \times \mathbf{H}_M = 0 \quad (\text{N.33})$$

Also, one knows that

$$\begin{aligned} \nabla \cdot \mathbf{B} &= \mu_0 \nabla \cdot (\mathbf{H}_C + \mathbf{H}_M + \mathbf{M}) \\ &= \mu_0 (\nabla \cdot \mathbf{H}_C + \nabla \cdot \mathbf{H}_M + \nabla \cdot \mathbf{M}) \end{aligned} \quad (\text{N.34})$$

Equation (N.34) must be identically zero, therefore

$$\nabla \cdot \mathbf{H}_M = -\nabla \cdot \mathbf{M} \quad (\text{N.35})$$

since $\nabla \cdot \mathbf{H}_C = 0$. One can now write the following:

$$\mathbf{H}_C = -\nabla \Phi_C \quad (\text{N.36})$$

and

$$\mathbf{H}_M = -\nabla\Phi_M \quad (\text{N.37})$$

From Equation (N.37), one can write

$$\begin{aligned} \nabla^2\Phi_M &= -\nabla \cdot \mathbf{H}_M \\ &= \nabla \cdot \mathbf{M} \end{aligned} \quad (\text{N.38})$$

Equation (N.38) is just Poisson's equation. The solution of Poisson's equation is given by

$$\begin{aligned} \Phi_M &= -\frac{1}{4\pi} \int \frac{\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r' \\ &= \frac{1}{4\pi} \int \mathbf{M}(\mathbf{r}') \cdot \nabla' \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) d^3r' \\ &= \frac{1}{4\pi} \int \frac{\mathbf{M}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3r' \end{aligned} \quad (\text{N.39})$$

We now have the total potential

$$\begin{aligned} \Phi_T &= \Phi_C + \Phi_M \\ &= \frac{1}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \cdot (\mathbf{r} \times \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'| [r |\mathbf{r} - \mathbf{r}'| + r^2 - \mathbf{r} \cdot \mathbf{r}']} d^3r' + \\ &\quad \frac{1}{4\pi} \int \frac{\mathbf{M}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3r' \end{aligned} \quad (\text{N.40})$$

where

$$\mathbf{H} = -\nabla\Phi_T \quad (\text{N.41})$$

Equation (N.40) is the complete scalar potential. The spherical harmonic expansion can be written as

$$\Phi(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{4\pi}{2l+1} \frac{M_{lm} Y_{lm}(\theta, \phi)}{r^{l+1}} \quad (\text{N.42})$$

where

$$M_{lm} = -\frac{1}{4\pi} \left(\int \frac{r'^l Y_{lm}^*(\theta', \phi')}{l+1} \left(\nabla' \cdot (\mathbf{r}' \times \mathbf{J}(\mathbf{r}')) \right) d^3r' + \int r'^l Y_{lm}^*(\theta', \phi') \left(\nabla' \cdot \mathbf{M}(\mathbf{r}') \right) d^3r' \right) \quad (\text{N.43})$$

N.4 Scalar potential of filamentary circuits

A method is developed which is a special case of Bronzan's method, and which uses the magnetic field intensity and the magnetic vector potential to derive from first principles the magnetic scalar potential due to a filamentary current-carrying conductor. The method is a general method which applies to filamentary conductors of arbitrary shape.

One knows that the *curl* of a gradient is always zero and that the *curl* of the magnetostatic field, $\mathbf{H}$, in a current-free region of space is always zero. This is given by

$$\nabla \times \mathbf{H} = 0 \quad (\text{N.44})$$

Because of Equation (N.44), it is valid to write

$$\mathbf{H} = -\nabla \Phi \quad (\text{N.45})$$

What about the magnetostatic field produced by a closed filamentary current at points outside the current carrying region? Apply the integral formulation for the magnetic vector potential given by

$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint \frac{d\mathbf{l}'}{r} \quad (\text{N.46})$$

Use the vector identity given by Equation (A.13).

$$\oint U d\mathbf{l} = \iint d\mathbf{S} \times \nabla U \quad (\text{N.47})$$

to reformulate Equation (N.46) as follows:

$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint \frac{d\mathbf{l}'}{r} = \frac{\mu_0 I}{4\pi} \iint d\mathbf{S}' \times \nabla' \left(\frac{1}{r} \right) \quad (\text{N.48})$$

where $U = \frac{1}{r}$. Taking the *curl* of both sides of Equation (N.48) yields the magnetic flux density.

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (\text{N.49})$$

Substituting Equation (N.49) into Equation (N.48) and dividing by μ_0 yields :

$$\mathbf{H} = \frac{1}{\mu_0} \nabla \times \mathbf{A} = \frac{I}{4\pi} \nabla \times \iint d\mathbf{S}' \times \nabla' \left(\frac{1}{r} \right) \quad (\text{N.50})$$

The ∇ operator outside the integral in Equation (N.50) can be brought under the integral operator because it does not operate on the primed coordinates. One can rewrite Equation (N.50) as follows:

$$\mathbf{H} = \frac{I}{4\pi} \iint \nabla \times \left[d\mathbf{S}' \times \nabla' \left(\frac{1}{r} \right) \right] \quad (\text{N.51})$$

but

$$\nabla' \left(\frac{1}{r} \right) = -\nabla \left(\frac{1}{r} \right) \quad (\text{N.52})$$

Employing Equation (N.52) enables one to rewrite Equation (N.51) as

$$\begin{aligned} \mathbf{H} &= -\frac{I}{4\pi} \iint \nabla \times \left[d\mathbf{S}' \times \nabla \left(\frac{1}{r} \right) \right] \\ &= \frac{I}{4\pi} \iint \nabla \times \left[\nabla \left(\frac{1}{r} \right) \times d\mathbf{S}' \right] \end{aligned} \quad (\text{N.53})$$

Consider the vector identity given by Equation (A.6).

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = (\mathbf{G} \cdot \nabla) \mathbf{F} + \mathbf{F} (\nabla \cdot \mathbf{G}) - (\mathbf{F} \cdot \nabla) \mathbf{G} - \mathbf{G} (\nabla \cdot \mathbf{F}) \quad (\text{N.54})$$

Using Equation (N.54), one can reformulate Equation (N.53) as

$$\begin{aligned}
\mathbf{H} &= \frac{I}{4\pi} \iint \nabla \times \left[\nabla \left(\frac{1}{r} \right) \times d\mathbf{S}' \right] \\
&= \frac{I}{4\pi} \iint \left[(d\mathbf{S}' \cdot \nabla) \nabla \left(\frac{1}{r} \right) + \nabla \left(\frac{1}{r} \right) (\nabla \cdot d\mathbf{S}') - \right. \\
&\quad \left. (\nabla \left(\frac{1}{r} \right) \cdot \nabla) d\mathbf{S}' - d\mathbf{S}' (\nabla \cdot \nabla \left(\frac{1}{r} \right)) \right] \\
&= \frac{I}{4\pi} \iint (d\mathbf{S}' \cdot \nabla) \nabla \left(\frac{1}{r} \right)
\end{aligned} \tag{N.55}$$

where $\mathbf{F} = \nabla \left(\frac{1}{r} \right)$ and $\mathbf{G} = d\mathbf{S}'$. Equation (N.55) is valid because $\nabla \cdot d\mathbf{S}'$ is always zero and $\nabla^2 \left(\frac{1}{r} \right) = 0$. One must recognize that $\nabla \cdot d\mathbf{S}'$ and $\nabla \times d\mathbf{S}'$ are always zero. Using the vector identity given by Equation (A.2), one can write

$$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{F} \times (\nabla \times \mathbf{G}) + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{G} \times (\nabla \times \mathbf{F}) \tag{N.56}$$

with $\mathbf{F} = d\mathbf{S}'$ and $\mathbf{G} = \nabla \left(\frac{1}{r} \right)$, one can reformulate Equation (N.55) as follows:

$$\begin{aligned}
\mathbf{H} &= \frac{I}{4\pi} \iint (d\mathbf{S}' \cdot \nabla) \nabla \left(\frac{1}{r} \right) \\
&= \frac{I}{4\pi} \iint \left[\nabla \left(d\mathbf{S}' \cdot \nabla \left(\frac{1}{r} \right) \right) - d\mathbf{S}' \times (\nabla \times \nabla \left(\frac{1}{r} \right)) - \right. \\
&\quad \left. (\nabla \left(\frac{1}{r} \right) \cdot \nabla) d\mathbf{S}' - \nabla \left(\frac{1}{r} \right) \times (\nabla \times d\mathbf{S}') \right] \\
&= \frac{I}{4\pi} \iint \nabla \left(d\mathbf{S}' \cdot \nabla \left(\frac{1}{r} \right) \right)
\end{aligned} \tag{N.57}$$

Equation (N.57) is valid because $\nabla \times \nabla \left(\frac{1}{r} \right) = 0$, $\nabla \times d\mathbf{S}' = 0$, and $(\nabla \left(\frac{1}{r} \right) \cdot \nabla) d\mathbf{S}' = 0$. Equation (N.57) can be rewritten as

$$\begin{aligned}
\mathbf{H} &= \frac{I}{4\pi} \iint \nabla \left(d\mathbf{S}' \cdot \nabla \left(\frac{1}{r} \right) \right) \\
&= \nabla \left[\frac{I}{4\pi} \iint \nabla \left(\frac{1}{r} \right) \cdot d\mathbf{S}' \right]
\end{aligned} \tag{N.58}$$

Comparing Equation (N.45) with Equation (N.58) shows that

$$\Phi = -\frac{I}{4\pi} \iint \nabla \left(\frac{1}{r} \right) \cdot d\mathbf{S}' \quad (\text{N.59})$$

Equation (N.59) is the equation for the magnetic scalar potential due to a filamentary current loop of arbitrary shape and it has a simple geometrical meaning. The quantity

$$-\iint \nabla \left(\frac{1}{r} \right) \cdot d\mathbf{S}' = \iint \frac{\hat{\mathbf{r}} \cdot d\mathbf{S}'}{r^2} \quad (\text{N.60})$$

represents the solid angle Ω subtended at the point of observation by the loop formed by the current filament. Therefore, the magnetostatic scalar potential can now be written as

$$\Phi = \frac{I}{4\pi} \Omega \quad (\text{N.61})$$

where the solid angle Ω is considered positive if the direction of the current in the filament, as seen from the point of observation, is counterclockwise. A few examples will clarify the use of Equations (N.60) and (N.61).

EXAMPLE 1:

Using Equation (N.61), compute the magnetostatic scalar potential and the magnetic field intensity on the axis of a current-carrying circular loop of radius a carrying a current I as shown in Figure N.4 . The magnetic scalar potential is written as

$$\begin{aligned} \Phi(z) &= \frac{I}{4\pi} \Omega \\ &= \frac{I}{2} [1 - \cos(\beta)] \\ &= \frac{I}{2} \left(1 - \frac{z}{\sqrt{a^2 + z^2}} \right) \end{aligned} \quad (\text{N.62})$$

Figure N.4: Magnetic scalar potential of a current loop valid at all points on the z axis

The magnetic field intensity is given by

$$\begin{aligned}
 \mathbf{H} &= -\nabla\Phi \\
 &= -\frac{I}{2}\nabla\left(1 - \frac{z}{\sqrt{a^2 + z^2}}\right) \\
 &= \frac{I}{2}\left[\frac{1}{\sqrt{a^2 + z^2}} - \frac{z^2}{(a^2 + z^2)^{\frac{3}{2}}}\right]\hat{z}, \\
 &= \frac{Ia^2}{2(a^2 + z^2)^{\frac{3}{2}}}\hat{z} \\
 &= \frac{I}{2a}\sin^3(\beta)\hat{z}
 \end{aligned} \tag{N.63}$$

Equation (N.63) is a well known result that can be obtained by the Biot-Savart law.

EXAMPLE 2:

Find the magnetic scalar potential due to current-carrying circular loop of radius a carrying a current I , as seen in Figure N.5, for all points in space not coincident with the conductor. Using the results of EXAMPLE 1, one knows that

on the axis of the loop the potential is given by

$$\Phi(z) = \frac{I}{2} \left(1 - \frac{z}{\sqrt{a^2 + z^2}} \right) \quad (\text{N.64})$$

where z is the distance from the center of the loop to the field point $(0, 0, z)$. Expand Equation (N.64) in powers of z for $z > a$ and then for $z < a$. For $z > a$, one obtains:

$$\Phi_{axis}(z) = \frac{I}{2} \left[\frac{1}{2} \left(\frac{a}{z} \right)^2 - \frac{1 \cdot 3}{2^2 \cdot 2!} \left(\frac{a}{z} \right)^4 + \frac{1 \cdot 3 \cdot 5}{2^3 3!} \left(\frac{a}{z} \right)^6 - \dots \right] \quad (\text{N.65})$$

and for $z < a$, one obtains:

$$\Phi_{axis}(z) = \frac{I}{2} \left[1 - \frac{z}{a} + \frac{1}{2} \left(\frac{z}{a} \right)^3 - \frac{1 \cdot 3}{2^2 2!} \left(\frac{z}{a} \right)^5 + \frac{1 \cdot 3 \cdot 5}{2^3 3!} \left(\frac{z}{a} \right)^7 - \dots \right] \quad (\text{N.66})$$

In order to obtain an expression for the magnetic scalar potential anywhere in space not coincident with the conductor, just replace $\frac{1}{z^n}$ in Equation (N.65) with $\frac{1}{r^n} P_{n-1}[\cos(\theta)]$ and replace z^n in Equation (N.66) with $r^n P_n[\cos(\theta)]$ (see Jackson 3rd ed. pages 102-103). Equation (N.65) becomes

$$\begin{aligned} \Phi(r, \theta) = & \frac{I a^2}{4 r^2} \left[P_1(\cos(\theta)) - \frac{1 \cdot 3}{2^1 \cdot 2!} \left(\frac{a^2}{r^2} \right) P_3(\cos(\theta)) + \right. \\ & \left. \frac{1 \cdot 3 \cdot 5}{2^2 \cdot 3!} \left(\frac{a^2}{r^2} \right)^2 P_5(\cos(\theta)) \dots \right] \end{aligned} \quad (\text{N.67})$$

and Equation (N.66) becomes

$$\begin{aligned} \Phi(r, \theta) = & \frac{I}{2} \left[1 - \left(\frac{r}{a} \right) P_1(\cos(\theta)) + \frac{1}{2} \left(\frac{r}{a} \right)^3 P_3(\cos(\theta)) - \right. \\ & \left. \frac{1 \cdot 3}{2^2 2!} \left(\frac{r}{a} \right)^5 P_5(\cos(\theta)) \dots \right] \end{aligned} \quad (\text{N.68})$$

Equations (N.67) and (N.68) can be rewritten in a more general form as

$$\Phi(r, \theta) = \frac{I}{2} \left(\frac{a}{r} \right)^2 \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n (n+1)!} \left(\frac{a}{r} \right)^{2n} P_{2n+1}(\cos(\theta)) \quad (\text{N.69})$$

Figure N.5: Magnetic scalar potential of a current loop valid at all points in space

and

$$\Phi(r, \theta) = \frac{I}{2} \left[1 + \frac{1}{2} \left(\frac{r}{a} \right) \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+2)(2n+1)!!}{2^n (n+1)! (2n+1)} \left(\frac{r}{a} \right)^{2n} P_{2n+1}(\cos(\theta)) \right] \quad (\text{N.70})$$

A discontinuity in the potential exists when applying Ampere's law to a contour that surrounds the current. Consider Figure N.5 . Applying Ampere's law to the blue contour yields:

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{enclosed}} = 0 \quad (\text{N.71})$$

where

$$\mathbf{H} = -\nabla\Phi = -\left(\frac{\partial\Phi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial\Phi}{\partial \theta} \hat{\theta} \right) \quad (\text{N.72})$$

and

$$d\mathbf{l} = dr\hat{r} + r d\theta\hat{\theta} \quad (\text{N.73})$$

One can now rewrite Equation (N.71) as

$$\oint \left[\left(\frac{\partial \Phi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \hat{\theta} \right) \cdot (dr \hat{r} + r d\theta \hat{\theta}) \right] = 0 \quad (\text{N.74})$$

Equation (N.74) can now be expressed as

$$\int_1^2 \left(\frac{\partial \Phi}{\partial \theta} \right) d\theta + \int_2^3 \left(\frac{\partial \Phi}{\partial r} \right) dr + \int_3^4 \left(\frac{\partial \Phi}{\partial \theta} \right) d\theta + \int_4^1 \left(\frac{\partial \Phi}{\partial r} \right) dr = 0 \quad (\text{N.75})$$

Each integral in Equation (N.75) can be evaluated as follows:

$$\int_1^2 \left(\frac{\partial \Phi}{\partial \theta} \right) d\theta = [\Phi_2(a^+, \theta_2) - \Phi_1(a^+, \theta_1)] \quad (\text{N.76})$$

$$\int_2^3 \left(\frac{\partial \Phi}{\partial r} \right) dr = [\Phi_3(a^+, \theta_2) - \Phi_2(a^-, \theta_2)] \quad (\text{N.77})$$

$$\int_3^4 \left(\frac{\partial \Phi}{\partial \theta} \right) d\theta = [\Phi_4(a^-, \theta_2) - \Phi_3(a^-, \theta_1)] \quad (\text{N.78})$$

and

$$\int_4^1 \left(\frac{\partial \Phi}{\partial r} \right) dr = [\Phi_1(a^+, \theta_1) - \Phi_4(a^-, \theta_1)] \quad (\text{N.79})$$

If $a^+ \rightarrow a^-$ then one is allowing the loop to become very narrow. Under this condition, one finds that

$$\Phi_3(a, \theta_2) - \Phi_2(a, \theta_2) = 0 \quad (\text{N.80})$$

and

$$\Phi_1(a, \theta_1) - \Phi_4(a, \theta_1) = 0 \quad (\text{N.81})$$

What remains from Equation (N.76) through Equation (N.79) is

$$\Phi_2(a, \theta_2) - \Phi_1(a, \theta_1) + \Phi_4(a, \theta_2) - \Phi_3(a, \theta_1) = 0 \quad (\text{N.82})$$

or

$$\Phi_2(a, \theta_2) - \Phi_3(a, \theta_1) = \Phi_1(a, \theta_1) - \Phi_4(a, \theta_2) \quad (\text{N.83})$$

Equation (N.83) implies that any discontinuity in the potential in the upper hemi-

sphere must be the same for all points on the upper hemisphere. Due to symmetry, one can make the same statement for the bottom hemisphere. Of course, this is only valid at $r = a$ and if $0 \leq \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \leq \pi$. In other words, the contour doesn't enclose the current source.

Performing the same analysis on the red contour yields:

$$\Phi_3(a, \theta_2) - \Phi_2(a, \theta_1) + \Phi_1(a, \theta_2) - \Phi_4(a, \theta_1) = I \quad (\text{N.84})$$

which is valid for any contour in which $\frac{\pi}{2}^+ < \theta < \frac{\pi}{2}^-$ and enclosing the current, I . The discontinuity in the potential given in Equation (N.84) can be easily checked by applying the appropriate boundary conditions to Equations (N.69) and (N.70).

The magnet field intensity at any point in space not coincident with the current loop is

$$\mathbf{H} = -\nabla\Phi = -\left[\left(\frac{\partial\Phi}{\partial r}\right)\hat{r} + \left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right)\hat{\theta}\right] \quad (\text{N.85})$$

Employing Equation (N.85) for $r > a$ yields:

$$\begin{aligned} H_r = & \frac{I a^2}{2 r^3} \left[P_1(\cos(\theta)) - \frac{1 \cdot 3}{2} \left(\frac{a^2}{r^2}\right) P_3(\cos(\theta)) + \right. \\ & \left. \frac{1 \cdot 3 \cdot 5}{2^2 \cdot 2!} \left(\frac{a^2}{r^2}\right)^2 P_5(\cos(\theta)) - \dots \right] \end{aligned} \quad (\text{N.86})$$

and

$$\begin{aligned} H_\theta = & -\frac{I a^2}{4 r^3} \left[P_1^1(\cos(\theta)) - \frac{1 \cdot 3}{2 \cdot 2!} \left(\frac{a^2}{r^2}\right) P_3^1(\cos(\theta)) + \right. \\ & \left. \frac{1 \cdot 3 \cdot 5}{2^2 \cdot 3!} \left(\frac{a^2}{r^2}\right)^2 P_5^1(\cos(\theta)) - \dots \right] \end{aligned} \quad (\text{N.87})$$

Employing Equation (N.85) for $r < a$ yields:

$$\begin{aligned} H_r = & \frac{I}{2a} \left[P_1(\cos(\theta)) - \frac{1 \cdot 3}{2} \left(\frac{r^2}{a^2}\right) P_3(\cos(\theta)) + \right. \\ & \left. \frac{1 \cdot 3 \cdot 5}{2^2 \cdot 2!} \left(\frac{r^2}{a^2}\right)^2 P_5(\cos(\theta)) - \dots \right] \end{aligned} \quad (\text{N.88})$$

and

$$H_\theta = \frac{I}{2a} \left[P_1^1(\cos(\theta)) - \frac{1}{2} \left(\frac{r^2}{a^2} \right) P_3^1(\cos(\theta)) + \frac{1 \cdot 3}{2^2} \left(\frac{r^2}{a^2} \right)^2 P_5^1(\cos(\theta)) - \dots \right] \quad (\text{N.89})$$

Equations (N.86) and (N.87) can be rewritten in a more general form (see Jackson 3rd edition pages 183-184) as

$$H_r = \frac{I a^2}{2 r^3} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n n!} \left(\frac{a}{r} \right)^{2n} P_{2n+1}(\cos(\theta)) \quad (\text{N.90})$$

and

$$H_\theta = -\frac{I a^2}{4 r^3} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n (n+1)!} \left(\frac{a}{r} \right)^{2n} P_{2n+1}^1(\cos(\theta)) \quad (\text{N.91})$$

Equations (N.88) and (N.89) can be rewritten, in a more general form, as

$$H_r = \frac{I}{2a} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n n!} \left(\frac{r}{a} \right)^{2n} P_{2n+1}(\cos(\theta)) \quad (\text{N.92})$$

and

$$H_\theta = -\frac{I}{4a} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+2)(2n+1)!!}{2^n (n+1)!(2n+1)} \left(\frac{r}{a} \right)^{2n} P_{2n+1}^1(\cos(\theta)) \quad (\text{N.93})$$

EXAMPLE 3:

Find the magnetic scalar potential due to a square filamentary loop of side $2a$ carrying a current I for all points on the z -axis not coincident with the conductor. In order to tackle this problem by the method developed in this paper, one can employ Equation (N.59) or Equation (N.61). If Equation (N.61) is employed, then

$$\begin{aligned} \Phi &= \frac{I}{4\pi} \Omega \\ &= \frac{I}{4\pi} \iint \frac{\hat{\mathbf{r}} \cdot d\mathbf{S}'}{r^2} \end{aligned} \quad (\text{N.94})$$

where the integral expression in Equation (N.94) represents the solid angle. Figure

Figure N.6: Magnetic scalar potential of a rectangular current loop valid at all points on the z axis

N.6 illustrates the basic configuration. One can write the unit vector $\hat{r}$ as follows:

$$\hat{r} = \frac{x' \hat{x} + y' \hat{y} + z \hat{z}}{\sqrt{x'^2 + y'^2 + z^2}} \quad (\text{N.95})$$

and the differential surface $d\mathbf{S}'$ is given by

$$d\mathbf{S} = \hat{z} dx' dy' \quad (\text{N.96})$$

Employing Equations (N.95) and (N.96), one can form the dot-product $\hat{r} \cdot d\mathbf{S}'$. It is given by

$$\hat{r} \cdot d\mathbf{S}' = \cos(\alpha) = \frac{z dx' dy'}{\sqrt{x'^2 + y'^2 + z^2}} \quad (\text{N.97})$$

Figure N.7: Magnetic flux density of a rectangular current loop valid at all points on the z axis by using the Biot-Savart law

One can now write the magnetic scalar potential as

$$\begin{aligned}
 \Phi &= \frac{I}{4\pi} \iint \frac{\hat{\mathbf{r}} \cdot d\mathbf{S}'}{r^2} \\
 &= \frac{Iz}{4\pi} \int_{-a}^a \int_{-a}^a \frac{dy' dx'}{(x'^2 + y'^2 + z^2)^{\frac{3}{2}}} \\
 &= \frac{Iz}{\pi} \int_0^a \int_0^a \frac{dy' dx'}{(x'^2 + y'^2 + z^2)^{\frac{3}{2}}} \\
 &= \frac{I}{\pi} \tan^{-1} \left(\frac{a^2}{z\sqrt{z^2 + 2a^2}} \right)
 \end{aligned} \tag{N.98}$$

The magnetic field intensity is given by

$$\begin{aligned}
 \mathbf{H} &= -\nabla\Phi, \\
 &= \frac{2Ia^2}{\pi} \left[\frac{1}{(z^2 + a^2)\sqrt{z^2 + 2a^2}} \right] \hat{z}
 \end{aligned} \tag{N.99}$$

Apply the Biot-Savart law in order to check Equation (N.99). Consider Figure N.7 . The differential magnetic flux density at point $(0, 0, z)$ due to element 1 is written

as

$$d\mathbf{B}_1 = \frac{\mu_0 I}{4\pi} \frac{d\mathbf{l}' \times \hat{r}}{r^2} \quad (\text{N.100})$$

where

$$d\mathbf{l}' = dx' \hat{x} \quad (\text{N.101})$$

$$\hat{r} = \frac{-x' \hat{x} + a \hat{y} + z \hat{z}}{\sqrt{x'^2 + y'^2 + z^2}} \quad (\text{N.102})$$

and

$$\frac{d\mathbf{l}' \times \hat{r}}{r^2} = \frac{-z dx' \hat{y} + a dx' \hat{z}}{(x'^2 + y'^2 + z^2)^{\frac{3}{2}}} \quad (\text{N.103})$$

Employing Equations (N.100), (N.101), (N.102), and (N.103) yields:

$$\mathbf{B}_1 = \frac{\mu_0 I}{2\pi} \left\{ - \left[z \int_0^a \frac{dx'}{(x'^2 + y'^2 + z^2)^{\frac{3}{2}}} \right] \hat{y} + \left[a \int_0^a \frac{dx'}{(x'^2 + y'^2 + z^2)^{\frac{3}{2}}} \right] \hat{z} \right\} \quad (\text{N.104})$$

Equation (N.104) evaluates to

$$\mathbf{B}_1 = \frac{\mu_0 I a}{2\pi} (-z \hat{y} + a \hat{z}) \frac{1}{(z^2 + a^2) \sqrt{z^2 + 2a^2}} \quad (\text{N.105})$$

Because of symmetry, one can immediately write down the contribution to the magnetic flux density from the remaining three sides.

$$\mathbf{B}_2 = \frac{\mu_0 I a}{2\pi} (z \hat{x} + a \hat{z}) \frac{1}{(z^2 + a^2) \sqrt{z^2 + 2a^2}} \quad (\text{N.106})$$

$$\mathbf{B}_3 = \frac{\mu_0 I a}{2\pi} (z \hat{y} + a \hat{z}) \frac{1}{(z^2 + a^2) \sqrt{z^2 + 2a^2}} \quad (\text{N.107})$$

$$\mathbf{B}_4 = \frac{\mu_0 I a}{2\pi} (-z \hat{x} + a \hat{z}) \frac{1}{(z^2 + a^2) \sqrt{z^2 + 2a^2}} \quad (\text{N.108})$$

Summing the contributions from each element yields:

$$\begin{aligned} \mathbf{B} &= \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3 + \mathbf{B}_4 \\ &= \frac{2Ia^2\mu_0}{\pi} \left[\frac{1}{(z^2 + a^2) \sqrt{z^2 + 2a^2}} \right] \hat{z} \end{aligned} \quad (\text{N.109})$$

Figure N.8: Magnetic scalar potential of a rectangular current loop valid at all points in space

and

$$\mathbf{H} = \frac{2Ia^2}{\pi} \left[\frac{1}{(z^2 + a^2)\sqrt{z^2 + 2a^2}} \right] \hat{z} \quad (\text{N.110})$$

Equation (N.110) is identical to Equation (N.99).

EXAMPLE 4:

Find the magnetic scalar potential at any point in space due to a rectangular filamentary loop lying in the x - y plane and centered at the origin. The dimensions of the loop are $-a < x' < a$ and $-b < y' < b$. The uniform current I circulates in the counterclockwise sense when viewed from some point on the positive z -axis as illustrated in Figure N.8. The cross-hatched region represents the area enclosed by the current loop. Why is this a concern? Instead of employing Equation (N.61), one can directly employ Equation (N.59). Equation (N.59) can be employed by recognizing that the differential surface area $d\mathbf{S}'$ is a vector which is directed normal to the area enclosed by the current loop and whose direction is specified by the right-hand-rule. The counter-clockwise direction of current, as viewed from some

point on the positive z -axis dictates that

$$d\mathbf{S}' = dy' dx' \hat{z} \quad (\text{N.111})$$

The vector $\mathbf{r}$ describes the distance vector from the differential “source area” to the field point. This is given by

$$\mathbf{r} = (x - x')\hat{x} + (y - y')\hat{y} + z\hat{z} \quad (\text{N.112})$$

and

$$\begin{aligned} \nabla \left(\frac{1}{r} \right) &= \nabla \left(\frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + z^2}} \right) \\ &= - \frac{(x - x')\hat{x} + (y - y')\hat{y} + z\hat{z}}{[(x - x')^2 + (y - y')^2 + z^2]^{\frac{3}{2}}} \end{aligned} \quad (\text{N.113})$$

Using Equation (N.111) and Equation (N.113), one can compute the dot-product as

$$\nabla \left(\frac{1}{r} \right) \cdot d\mathbf{S}' = - \frac{z dy' dx'}{[(x - x')^2 + (y - y')^2 + z^2]^{\frac{3}{2}}} \quad (\text{N.114})$$

From Equation (N.59), the magnetic scalar potential is written as

$$\begin{aligned} \Phi(x, y, z) &= - \frac{I}{4\pi} \iint \nabla \left(\frac{1}{r} \right) \cdot d\mathbf{S}' \\ &= \frac{Iz}{4\pi} \int_{-a}^a \int_{-b}^b \frac{dy' dx'}{[(x - x')^2 + (y - y')^2 + z^2]^{\frac{3}{2}}} \\ &= \frac{Iz}{4\pi} \int_{x-a}^{x+a} \int_{y-b}^{y+b} \frac{du dv}{[u^2 + v^2 + z^2]^{\frac{3}{2}}} \end{aligned} \quad (\text{N.115})$$

Evaluating Equation (N.115) is fairly straightforward. It evaluates to

$$\begin{aligned} \Phi(x, y, z) = & \frac{I}{4\pi} \left\{ \tan^{-1} \left[\frac{(x+a)(y+b)}{z\sqrt{(x+a)^2 + (y+b)^2 + z^2}} \right] - \right. \\ & \tan^{-1} \left[\frac{(x-a)(y+b)}{z\sqrt{(x-a)^2 + (y+b)^2 + z^2}} \right] - \\ & \tan^{-1} \left[\frac{(x+a)(y-b)}{z\sqrt{(x+a)^2 + (y-b)^2 + z^2}} \right] + \\ & \left. \tan^{-1} \left[\frac{(x-a)(y-b)}{z\sqrt{(x-a)^2 + (y-b)^2 + z^2}} \right] \right\} \quad (\text{N.116}) \end{aligned}$$

If one allows $a = b$ and $x = y = 0$, then one obtains the magnetic scalar potential on the positive z -axis for a square current loop. This is computed as follows:

$$\begin{aligned} \lim_{\substack{b \rightarrow a \\ y=x \rightarrow 0}} \Phi(x, y, z) = & \lim_{\substack{b \rightarrow a \\ y=x \rightarrow 0}} \frac{I}{4\pi} \left\{ \tan^{-1} \left[\frac{(x+a)(y+b)}{z\sqrt{(x+a)^2 + (y+b)^2 + z^2}} \right] - \right. \\ & \tan^{-1} \left[\frac{(x-a)(y+b)}{z\sqrt{(x-a)^2 + (y+b)^2 + z^2}} \right] - \\ & \tan^{-1} \left[\frac{(x+a)(y-b)}{z\sqrt{(x+a)^2 + (y-b)^2 + z^2}} \right] + \\ & \left. \tan^{-1} \left[\frac{(x-a)(y-b)}{z\sqrt{(x-a)^2 + (y-b)^2 + z^2}} \right] \right\} \\ = & \frac{I}{\pi} \tan^{-1} \left(\frac{a^2}{z\sqrt{z^2 + 2a^2}} \right) \quad (\text{N.117}) \end{aligned}$$

Of course, one can compute the magnetic field intensity anywhere in space not coincident with the rectangular loop by taking the negative gradient of Equation (N.116).