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The Destruction of Magnetic Surfaces in Tokamaks
by Current Perturbations

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ABSTRACT

We report investigations of the feasibility of the hypothesis that the disruptive instability in tokamaks is caused by the breakup of magnetic surfaces. The envisioned breakup is caused by the interaction of $m = 2$, $n = 1$ and $m = 3$, $n = 1$ helical perturbations or by the interaction of the $m = 2$, $n = 1$ perturbation and the $m = 1$, $n = 0$ toroidal field (with $R/a = 5$). Both numerical and analytic work has been done for two MHD equilibrium current profiles: parabolic [$q \sim (1 - r^2/2a^2)^{-1}$] and the peaked model ($q \sim 1 + \lambda r^2/a^2$). For the parabolic profile, perturbation strengths of 1.5% are required to break up surfaces when $m = 2$ and $m = 3$ perturbations are present. When the $m = 2$ perturbation alone interacts with the toroidal field, perturbation strengths greater than 4% are required. For the peaked model with $q(0) = 1.1$, $q(a) = 3.7$, perturbation strengths of 0.5% were found to be sufficient to break the surfaces when both $m = 2$ and $m = 3$ perturbations are present, whereas strengths of 1.5% are required when only the $m = 2$ perturbation interacts with the toroidal field. These perturbation amplitudes are in reasonable agreement with experimental measurements. The peaked model permits breakup of surfaces for smaller perturbation strengths primarily because of its greater shear.

I. INTRODUCTION

The disruptive instability, characterized by a decrease in major radius, an increase in minor radius, and a negative voltage spike, is a limit to present-day tokamak operation. In this paper, we pursue the hypothesis that the disruptive instability is caused by the ergodic wandering of magnetic field lines. There seem to be two distinct types of disruptions: strong disruptions in which the plasma interacts strongly with the limiter and weaker disruptions which are independent of the limiter [1]. Because phenomena involving the limiter complicate the picture unnecessarily, we will be concerned with the latter type of disruption.

The wandering field lines caused by the so-called destruction of magnetic surfaces can lead to rapid change in the electron density and temperature profiles [2]. The most natural assumption is that this process would occur by means of a free expansion along magnetic field lines.

As a symmetry breaking perturbation such as a tearing mode grows, the field surfaces are destroyed quite suddenly and therefore it is possible that the transport we have described could be responsible for the disruptive instability. In fact, flattening of the T_e profile near the $q = 2$ surface has been observed before the onset of the disruption [3]. However, the presence of large magnetic islands would lead to the same flattening of the T_e profile, so that on the basis of electron temperature measurements, it does not seem to be possible to

distinguish between large, well-formed islands and islands which exhibit some wandering near their boundaries.

We treat the surface breakup problem by assuming a given MHD equilibrium and tearing mode instability and then by calculating the criterion for the magnetic surfaces to be destroyed by the breaking of symmetry. We do not treat the consequent modification of the equilibrium, i.e., our model is not self-consistent. The model is nevertheless expected to describe the early stages of the phenomenon. Indeed, the characteristics of the disruptive instability, notably the negative voltage spike, are in this framework considered to be secondary effects of surface breakup.

Magnetic surface breakup is in general due to the destruction of a symmetry in a system. For example, the discreteness of the toroidal field coils destroys axisymmetry in tokamaks, as does the presence of helical windings in a stellarator. These effects have been studied in detail [4,5]. In this paper, we concentrate on symmetry destroying magnetic perturbations which occur within the plasma, in particular the helical magnetic fields caused by one or more tearing modes [6].

A cylindrically symmetric system which has a helical perturbation $\sim t(r) \cos (m\theta + kz)$ superimposed upon it still possesses magnetic surfaces. This is due to the fact that the system still possesses helical symmetry, i.e., can be described in terms of the two variables r and $\tau_m \equiv m\theta + kz$. In fact, in the helically symmetric case the magnetic field can be described in terms of a "helical flux" function $\chi(r, \tau_m)$ by

$$rB_r = - \partial \chi / \partial \tau_m ,$$

$$mB_\theta + krB_z \equiv r\vec{k} \cdot \vec{B} = \partial \chi / \partial r . \quad (1)$$

From these equations, it is clear that $\vec{B} \cdot \nabla \chi = 0$, and therefore, the magnetic field lines lie on χ surfaces, i.e., are integrable. In fact, for a pure m, k perturbation, we have $\chi(r, \tau_m) = \chi^{(0)}(r) + \epsilon_m \chi_m(r) \cos \tau_m$. From (1), $\nabla \chi^{(0)}$ vanishes on the surface $q = m$ and therefore islands surrounding this surface are expected to appear.

A cylindrical system with two such helical perturbations with different helicities (i.e., $m/k \neq m'/k'$) no longer possesses such a symmetry. The same is true for a toroidal system with only one helical perturbation. Islands will still occur, since they are resonant phenomena and thus local to the surfaces $q = m$, $q = m'$. These are called primary resonances. In addition, smaller islands caused by the so-called "secondary resonances" will occur. These are essentially caused by mode-coupling of the helical perturbations present. If the perturbations are large enough that the individual island structures interact strongly, i.e., when we have an "overlapping of resonances," the field lines seem to lose much of their regularity and appear to wander "ergodically" or "stochastically" within some region. In this region, we say that the magnetic surfaces are destroyed. As the amplitude of the perturbation is increased, the secondary resonances begin to interact first, causing ergodic behavior in

a small region around the X , or hyperbolic, points at the intersection of the island separatrices. At a somewhat larger perturbation, the primary resonances begin to interact among themselves, leading to ergodic behavior in a much larger region. The latter, more drastic destruction of magnetic surfaces will be of more concern to us for the following reason: the volume of the ergodic region depends very strongly on the perturbation parameters ϵ_m and therefore the values of ϵ_m required for moderate and for extensive destruction of surfaces will not differ by more than about a factor of two [7]. In addition, as mentioned earlier, a small amount of local surface breakup near the X points is not expected to affect the temperature and density profiles much more than the islands themselves.

It should be mentioned that the interacting of neighboring island chains can be related to the occurrence of homoclinic points [8,9]. A homoclinic point is a point where the invariant curves passing through the hyperbolic points do not meet smoothly to form a separatrix, but intersect transversally. It is easy to imagine that the existence of a nearby chain of islands (with a different number of islands per unit length) would perturb a separatrix sufficiently that it would form two curves which intersect transversally. As a matter of fact, surface breakup can be studied by a thorough investigation of homoclinic points and their angles of intersection. However, treatment by overlap of resonances provides estimates of the criteria for ergodic behavior which are accurate enough for our purposes. (Nevertheless,

for more detailed information regarding the transition between integrable and ergodic behavior, we must resort to numerical integration.)

The degree of ergodic behavior depends upon the strength of the helical symmetry breaking perturbations and on the shear of the unperturbed field lines. For a given amount of shear, we can estimate the perturbation required as follows: by a familiar formula [10], island widths are given approximately by $\delta \sim (\epsilon/s)^{1/2}$, where ϵ is the perturbation amplitude and s is the shear. However, the distance between rational surfaces, d , is inversely proportional to the shear, and therefore the condition for overlapping of resonances is $\delta = d$ or $\epsilon \sim s^{-1}$. Thus, we expect shear to be a destabilizing factor, contrary to earlier expectations [5]. For this reason we use two equilibrium current profiles, one with low shear and the other with moderately high shear.

In this paper, we calculate the perturbation strengths needed to cause overlap of resonances in cylindrical geometry when two tearing modes are present ($m = 1 - m = 2$ and $m = 2 - m = 3$). We also calculate the $m = 2$ perturbation strength required when the effects of toroidal geometry are taken into account. In addition, we report results of numerical integrations of the magnetic field lines for various perturbation strengths--and for the models with differing shear. These computations serve as an independent check of the analytic calculations.

II. EQUILIBRIUM AND PERTURBATION MODELS

We consider two MHD equilibrium current profiles with cylindrical symmetry, the one-parameter parabolic model

$$J_z^{(0)} = \frac{2B_z^{(0)}}{Rq^{(0)}} \left(1 - r^2/a^2\right),$$

$$q = q(0) \left(1 - r^2/2a^2\right)^{-1}, \quad (2)$$

and the "peaked model" introduced by Furth et al. [11]

$$J_z^{(0)} = \frac{2B_z^{(0)}}{Rq^{(0)}} \left(1 + \lambda r^2/a^2\right)^{-2},$$

$$q = q(0) \left(1 + \lambda r^2/a^2\right). \quad (3)$$

In (2) and (3), $B_z^{(0)}$ is assumed to be constant (the $\beta_\theta = 1$ model). Hereafter, we deal in dimensionless variables, and therefore we can set $a = 1$, $B_z^{(0)} = 1$. The peaked model was chosen in order to accomodate higher shear; an attempt to introduce a parameter similar to λ to increase the shear of the parabolic profile leads to a model with reverse current near the plasma boundary $r = a$.

Toroidal corrections to these cylindrical equilibria are discussed in Section IV.

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The perturbations upon the equilibrium are assumed to arise from a tearing mode instability, with perturbed quantities varying as $\exp(\gamma t + im\theta - inz/R)$. However, since the e-folding time γ^{-1} is roughly 500 times the transit time for an electron $\tau_e = 2\pi R/v_{te}$, the tearing mode may be assumed to be static. (However, γ^{-1} is comparable to the ion transit time $2\pi R/v_{ti}$ and therefore this approximation is not as valid for ion motion. Also, since the ion gyroradius is larger, the ion drift surfaces are not approximated as well by the magnetic surfaces as the electron drift surfaces.)

For reasons of simplicity, the tearing modes are those associated with the MHD equilibrium with J_z constant within a channel $0 < r < r_0$ and zero outside [12], and therefore in terms of the helical flux function χ take the form $\chi_m^{(1)}(r, \tau_m) = \chi_m(r) \cos \tau_m$, with

$$\chi_m(r) = \epsilon_m (r/s_m)^m \quad r < s_m$$

$$= \epsilon_m [(s_m/r)^m - (s_m/r)^m] (1 - s_m^{2m})^{-1} \quad r > s_m. \quad (4)$$

Here, $\tau_m = m\theta - z/R$ ($n = 1$), s_m is the radius of the $q = m$ surface and ϵ_m is a small parameter. (Our accuracy is limited not only because χ_m is calculated from a simplified current profile; for wide islands we also need to consider the nonlinear contributions to the tearing mode. However, χ as given by (4) has two important properties: it is resonant at $q = m$ and has a sheet current component there. Also note that in (4) $\chi_m(1) = 0$, i.e. the conducting shell is located at $r=a=1$. We have calculated

the effect of moving the conducting shell to $r = 1.5a$ and have found that this changes the required perturbation strengths by less than 10%.)

In terms of the helical flux function χ , the magnetic field is given by (1). Therefore, if we assume $\vec{B}^{(0)} \cdot \vec{B}^{(1)} = 0$ at the rational surface $q = m$ (since islands are produced by perturbed fields perpendicular to the equilibrium field), the perturbed field for a pure m mode is given by

$$\begin{aligned} B_r^{(1)} &= \frac{\epsilon_m \chi_m(r) \sin \tau_m}{r} \\ B_\theta^{(1)} &= \frac{\epsilon_m \chi_m'(r) \cos \tau_m}{2(1 + \alpha_m^2)} \\ B_z^{(1)} &= \frac{\epsilon_m \alpha_m \chi_m'(r) \cos \tau_m}{2(1 + \alpha_m^2)}, \end{aligned} \quad (5)$$

where $\alpha_m = -r/mR$. Since $B_r^{(0)} = 0$, the equilibrium field can also be expressed in terms of a zeroth order helical flux:

$$\begin{aligned} \chi^{(0)} &= \int dr (m B_\theta^{(0)}(r) - r B_z^{(0)}/R) \\ &= \frac{1}{R} \int r dr \left(\frac{m}{q(r)} - 1 \right). \end{aligned} \quad (6)$$

The perturbed current density associated with the field (5) is calculated by $\vec{J}^{(1)} = \nabla \times \vec{B}^{(1)}$ to be

$$J_r^{(1)} = 0$$

$$J_\theta^{(1)} = -\epsilon_m \cos \tau_m \left[\frac{\chi_m'(r)}{rR} + \frac{\chi_m'(r)}{m} \frac{d}{dr} \left(\frac{\alpha_m}{1 + \alpha_m^2} \right) + \frac{\alpha_m \chi_m''(r)}{m(1 + \alpha_m^2)} \right]$$

$$J_z^{(1)} = \epsilon_m \cos \tau_m \left[\frac{\chi_m'(r)}{mr} \frac{d}{dr} \left(\frac{r}{1 + \alpha_m^2} \right) - \frac{m\chi_m(r)}{r^2} + \frac{\chi_m''(r)}{m(1 + \alpha_m^2)} \right] \quad (7)$$

Notice that with χ_m given by (4), χ_m'' behaves like $\delta(r - s_m)$ and therefore the current (7) has a sheet current component

$$\vec{J}_s^{(1)} = \frac{\epsilon_m \cos \tau_m \chi_m''(r)}{m(1 + \alpha_m^2)^{1/2}} \left(\cos \beta \hat{e}_z + \sin \beta \hat{e}_\theta \right), \quad (8)$$

where $\tan \beta = -\alpha_m = r/Rq(s_m) = B_\theta^{(0)}(s_m)/B_z^{(0)}$, the pitch of the unperturbed magnetic field lines on the rational surface $q = m$.

When the islands around the surface $q = m$ are thin, most of the perturbed current flowing through them is due to the sheet current. We therefore take as our definition of perturbation strength (or perturbation amplitude) the amount of sheet current passing through one of the islands of the chain in the direction of $\vec{B}^{(0)}$, divided by the total unperturbed current. That is, perturbation strength is I_m/I_0 , where

$$\begin{aligned} I_m &= \int_{s_m - \delta}^{s_m + \delta} r dr \int_0^{\pi/m} d\theta \vec{J}_s^{(1)} \cdot \vec{B}^{(0)} / |\vec{B}^{(0)}| \\ &= -\frac{2\epsilon_m}{m^2} \int_{s_m - \delta}^{s_m + \delta} \frac{\chi_m''(r) r}{(1 + \alpha_m^2)^{1/2}} dr = \frac{4\epsilon_m}{m[1 + \alpha_m^2(s_m)]^{1/2} (1 - s_m^2)} \end{aligned}$$

and

$$I_0 = 2\pi \int_0^a J_z^{(0)}(r) r dr .$$

III. SURFACE BREAKUP IN THE CYLINDRICAL TOKAMAK

The differential equations for the magnetic field lines are

$$\frac{dr}{dz} = \frac{B_r}{B_z} ,$$

$$\frac{d\theta}{dz} = \frac{B_\theta}{rB_z} . \quad (9)$$

If the magnetic fields have a helical symmetry, i.e., are derivable from a helical flux function $\chi(r, \tau_m)$, then $\vec{B} \cdot \nabla \chi = 0$. Thus, χ is a first integral for (9), since $d\chi/dz = 0$. Therefore, for a perturbation by a pure m , n mode, the islands are merely level lines of χ .

In order to calculate the perturbation strengths necessary to produce overlap of the $m = 2$ and $m = 3$ primary resonances, we calculate the width of each set of islands separately, i.e., as if the other perturbation were not present. The rationale for this approach is that, as mentioned in the previous section, the near-resonant invariant curves are shifted by an amount $\sim \epsilon^{1/2}$, whereas off-resonance curves are shifted by a much smaller amount, $\sim \epsilon$. The equations for the separatrices are

$$\chi^{(0)}(r) + \chi_m(r) \cos \tau_m = \chi^{(0)}(s_m) - \chi_m(s_m) . \quad (10)$$

We now use (4) and (6) to see that for the parabolic model, in the region $s_2 < r < s_3$, the $m = 2$ islands extend to a maximum radius r given by

$$(s_2^2 - r^2)^2 = 4q(0)R\epsilon_2 \left(1 + \frac{s_2^2/r^2 - s_2^2 r^2}{1 - s_2^4} \right) , \quad (11)$$

and the $m = 3$ islands extend to a minimum radius given by

$$(s_3^2 - r^2)^2 = \frac{8q(0)R\epsilon_3}{3} \left(1 + \frac{r^3}{s_3^3} \right) . \quad (12)$$

These formulas reduce to the familiar $(\epsilon/s)^{1/2}$ island half-width [9]

$$w_m = \frac{2s_m}{m^{1/2}} \left(\frac{B_{r,\max}^{(1)}}{B_{\theta}^{(0)}(s_m)} \frac{q(s_m)}{s_m q'(s_m)} \right)^{1/2} \quad (13)$$

for $|r - s_m| \ll 1$, but take into account the radial dependence of the perturbation and the change in the shear over the island width. For $m = 2$ and $m = 3$ islands wide enough to intersect each other, these formulas differ from (13) by as much as a factor of three in the value of ϵ_m .

For the peaked model, these equations become

$$\begin{aligned}
 & - \frac{1}{\lambda R q(0)} \ln(1 + \lambda r^2) + \frac{r^2}{2R} + \frac{1}{2\lambda R} \left(\frac{2}{q(0)} \ln \frac{2}{q(0)} + 1 - \frac{2}{q(0)} \right) \\
 & = \epsilon_2 \left(1 + \frac{s_2^2/r^2 - s_2^2 r^2}{1 - s_2^4} \right) \quad (14)
 \end{aligned}$$

for the $m = 2$ islands and

$$\begin{aligned}
 & - \frac{3}{2\lambda R q(0)} \ln(1 + \lambda r^2) + \frac{r^2}{2R} + \frac{1}{2\lambda R} \left(\frac{3}{q(0)} \ln \frac{3}{q(0)} + 1 - \frac{3}{q(0)} \right) \\
 & = \epsilon_3 \left(1 + \frac{r^3}{s_3^3} \right) \quad (15)
 \end{aligned}$$

for the $m = 3$ islands.

For the parabolic model we used $q(0) = 1.75$ in order that the $q = 2$ and $q = 3$ surfaces be at reasonable radii ($s_2 = 0.5$ and $s_3 = 0.91$). Also, it should be noted that $q(a) = 2q(0) = 3.5$. For the peaked model, we were able to specify $q(0)$ and $q(a)$ independently. (We used 1.1 and 3.7 respectively, as suggested by von Goeler [3] as values typical just before the onset of the disruptive instability. These give $s_2 = 0.59$ and $s_3 = 0.85$.) The results are that if $I_2 = I_3$, the parabolic profile requires $I_2/I_0 = 1.5\%$ for overlap of these $m = 2$ and $m = 3$ islands, whereas the peaked model requires $I_2/I_0 = 0.5\%$. From the electron temperature profile in ST, it is estimated that there are 4 cm. $m = 2$ islands present just before the

disruption [3]. The parabolic profile requires a perturbation strength $I_2/I_0 = 0.3\%$, whereas the peaked model requires $I_2/I_0 = 0.6\%$. It is clear, therefore, that magnetic surface breakup due to simultaneous $m = 2$ and $m = 3$ helical perturbations can occur for realistic perturbation strengths if a current profile similar to the peaked model is assumed. As has been noted in previous work [10], surface breakup in the parabolic model requires perturbation amplitudes which are much too large.

Similar calculations were performed with $m = 1$ and $m = 2$ helical perturbations. Of course, for these calculations we must have $q(0) < 1$. Because the shear near $r = 0$ is very small, one expects that the criterion for breakup of surfaces will depend strongly upon $q(0)$. Therefore, these results are plotted against $q(0)$ for various values of $q(a)$ in Fig. 1, for the particular case when $I_1/I_0 = I_2/I_0$. We see that, roughly speaking, a perturbation strength of about a percent or more is required.

In addition to the above calculations based upon the overlap of primary resonances, numerical work has been performed. The differential equations (9) have been integrated numerically for various perturbation strengths for both the parabolic and peaked profiles. Figures 2 - 5 show the intersection of the magnetic field trajectories with the plane $z = 2n\pi R$ (i.e., $\phi = 0$) for the parabolic model and for four different perturbation strengths ($I/I_0 \equiv I_2/I_0 = I_3/I_0$). (The points were plotted by a line printer and therefore are quantized to a 100×60 grid. However, the essential nature of the graphs is still clear.) The situation is rather simple in Fig. 2, for $I/I_0 = 3.75 \times 10^{-3}$. There

are $m = 2$ and $m = 3$ islands near the $q = 2$ and $q = 3$ surfaces, respectively. However, the other surfaces ($q = 7/3$ and $5/2$ are shown) are rather undistorted by the perturbation. In Fig. 3, with $\tilde{I}/I_0 = 7.5 \times 10^{-3}$, the $m = 2$ and $m = 3$ islands are somewhat larger and we can observe a small amount of stochastic behavior near the X-points of the $m = 2$ islands. At $\tilde{I}/I_0 = 1.5 \times 10^{-2}$ (Fig. 4), where the analytic calculations show island overlap, the points seem to be placed at random, except that the $m = 2$ and $m = 3$ islands persist. For the larger value of $\tilde{I}/I_0$ shown in Fig. 5, namely, 2.25×10^{-2} , the randomness is somewhat more pronounced and the $m = 2$ and $m = 3$ islands have begun to shrink. The results shown in Figs. 6 - 9, for the peaked model with $q(0) = 1.1$, $q(a) = 3.7$, are qualitatively similar. However, the magnetic surfaces begin to break while the islands are still thin (compared to those of the parabolic model), because the rational surfaces are closer to each other.

Since our findings agree with the idea that a model with lower shear requires larger perturbation amplitudes, we will restrict our attention in the rest of this paper to the peaked model.

IV. TOROIDAL EFFECTS

In this section we consider the following toroidal effects:

(i) when $B_z^{(0)} = B_\phi^{(0)} \sim (R + r \cos \theta)^{-1}$, magnetic surface break-up can occur with only one, e.g., the $m = 2$ helical perturbation, and (ii) in toroidal geometry, the flux surfaces are displaced outward by an amount $\sim a/R$. This effect causes the $m = 2$ and $m = 3$ island chains to overlap--for points on the outside of the magnetic axis--for smaller perturbation strengths. The latter effect will only be estimated. For all these calculations we use an aspect ratio $R/a = 5$.

When a single mode with helical $m = 2$, $n = 1$ symmetry is superimposed upon an equilibrium with toroidal symmetry, both symmetries are broken, and therefore magnetic surfaces should be expected to disappear, at least for a large enough perturbation amplitude. In terms of the differential Eqs. (9), this occurs by mode coupling of the $m = 2$, $n = 1$ terms with the $m = 1$, $n = 0$ terms of $B_z^{(0)} \sim (R + r \cos \theta)^{-1}$. If $1 < q(0) < q(a) < 4$, the major effect of this coupling is to form $m = 3$, $n = 1$ islands about the $q = 3$ surface. However, we will also be interested in cases where $q(0) < 1$, so we consider the form that Eqs. (9) take near $q = m$ for $m = 1$ or $m = 3$. By (5),

$$\begin{aligned} r \frac{dr}{dz} &= \frac{\epsilon_2 \chi_2(r) \sin \tau_2}{(1 + r \cos \theta/R)^{-1} + \epsilon_2 \alpha_2 \chi_2(r) \cos \tau_2/2(1 + \alpha_2^2)}, \\ r \frac{d\theta}{dz} &= \frac{B_\theta^{(0)}(r) + \epsilon_2 \chi_2(r) \cos \tau_2/2(1 + \alpha_2^2)}{(1 + r \cos \theta/R)^{-1} + \epsilon_2 \alpha_2 \chi_2(r) \cos \tau_2/2(1 + \alpha_2^2)}. \end{aligned} \quad (16)$$

We expand these expressions, dropping terms of order ϵ_2^2 or higher. Also, since we are concerned with behavior near the $q = 1$ or the $q = 3$ surface, we neglect terms of order ϵ_2 which are not resonant at the appropriate surface [i.e., the only terms of order ϵ_2 which we keep are those proportional to $\sin(m\theta - z/R)$ or $\cos(m\theta - z/R)$]. We obtain

$$\frac{dr}{dz} \approx \frac{\epsilon_2 \chi_2'(r) \sin \tau_m}{2R}$$

$$\frac{d\theta}{dz} \approx \frac{1}{r} \left(1 + \frac{r}{R} \cos \theta \right) B_\theta^{(0)}(r) + \frac{\epsilon_2 \chi_2'(r) \cos \tau_m}{4R(1 + \alpha_2^2)}$$

$$- \frac{1}{R} \frac{B_\theta^{(0)}(r) \epsilon_2 \alpha_2 \chi_2'(r) \cos \tau_m}{2(1 + \alpha_2^2)} \quad (17)$$

Finally, since $B_\theta^{(0)}$ and α_2 are both of order a/R , we can make further approximations to obtain

$$\frac{dr}{dz} \approx \frac{\epsilon_2 \chi_2'(r) \sin \tau_m}{2R}$$

$$\frac{d\theta}{dz} \approx \frac{B_\theta^{(0)}(r)}{r} + \frac{\epsilon_2 \chi_2'(r) \cos \tau_m}{4R} \quad (18)$$

The second of these can also be written in the form

$$\frac{d\tau_m}{dz} = m \left(\frac{B_\theta^{(0)}(r)}{r} + \frac{\epsilon_2 \chi_2'(r) \cos \tau_m}{4R} \right) - \frac{1}{R} \quad (18-c)$$

As can be verified directly, the function

$$H = \int \chi_2(r)^{\frac{m}{2}-1} \left(\frac{1}{R} - \frac{m}{r} B_\theta^{(0)}(r) \right) dr - \frac{\epsilon_2 \chi_2(r)^{\frac{m}{2}} \cos \tau_m}{2R} \quad (19)$$

is an integral of motion for the system (18), i.e., satisfies $dH/dz = 0$. [We have calculated by first order perturbation theory the correction to H obtained by including the nonresonant but unperturbed term $\cos \theta B_\theta^{(0)}(r)/R$ of (17). The effect is a negligible 5% change in perturbation strength required for surface breakup.] Therefore, H can be used in the same manner as the helical flux χ is used in (10) to calculate the width of the $m = 1$ or $m = 3$ islands. We obtain

$$\int_{s_m}^r \chi_2(r')^{\frac{m}{2}-1} \left(\frac{1}{R} - \frac{m}{r'} B_\theta^{(0)}(r') \right) dr' = \frac{\epsilon_2}{2R} \left(\chi_2(s_m)^{\frac{m}{2}} + \chi_2(r)^{\frac{m}{2}} \right) \quad (20)$$

For $m = 3$ and for the peaked model with $q(0) = 1.1$, $q(a) = 3.7$, we find that the $m = 2$ islands intersect the $m = 3$ islands given by (20) for $\epsilon_2 = 4 \times 10^{-2}$, or $I_2/I_0 = 1.5\%$. (For the parabolic model the value is 4%.) Thus, the interaction of an $m = 2$ mode and the toroidal field can produce breakup for reasonably small perturbation strengths.

The results of (20) for the $m = 1$ islands and (14) for the $m = 2$ islands are plotted as a function of $q(0)$ for various values of $q(a)$ in Fig. 10. We see that very large perturbation strengths are required to cause destruction of surfaces by the creation of $m = 1$ islands. This result should not be surprising, since toroidal effects, which are of order r/R , are small at the $q = 1$ surface. Also, the shear goes to zero as $r \rightarrow 0$, again requiring larger perturbation strengths.

As in the cylindrical case, numerical integrations were performed in toroidal geometry. The results agree quite well with the analytic calculations based upon (20).

If we take into account an $m = 3$ helical perturbation as well as the toroidal effect already considered, Eqs. (18) become

$$\frac{dr}{dz} \approx \left(\frac{\epsilon_2 \chi_2(r)}{2R} + \frac{\epsilon_3 \chi_3(r)}{r} \right) \sin \tau_3 ,$$

$$\frac{d\tau_3}{dz} = \frac{3B_\theta^{(0)}(r)}{r} + \left(\frac{3\epsilon_2 \chi_2'(r)}{4R} + \frac{\epsilon_3 \chi_3'(r)}{r} \right) \cos \tau_3 - \frac{1}{R} . \quad (21)$$

These equations have an integral of motion $H = H_0(r) + H_1(r) \cos \tau_3$, where

$$H_1(r) = \exp \int \left(\frac{3\epsilon_2 \chi_2'(r)}{4R} + \frac{\epsilon_3 \chi_3'(r)}{r} \right) \left(\frac{\epsilon_2 \chi_2(r)}{2R} + \frac{\epsilon_3 \chi_3(r)}{r} \right)^{-1} dr \quad (22)$$

and

$$H_0(r) = \int H_1(r) \left(\frac{3B_{\theta}^{(0)}(r)}{r} - \frac{1}{R} \right) \left(\frac{\epsilon_2 \chi_2(r)}{2R} + \frac{\epsilon_3 \chi_3(r)}{r} \right)^{-1} dr \quad (23)$$

In the limits $\epsilon_2 \rightarrow 0$, $\epsilon_3 \rightarrow 0$, we recover the integrals of motion $\chi^{(0)}(r) + \epsilon_3 \chi_3(r) \cos \tau_3$ and (19), respectively. In the region $s_2 < r < s_3$, we can use the expressions (4) for $\chi_2(r)$ and $\chi_3(r)$ to find, after some algebra

$$H_1(r) = \left(\delta_2 (1/r^2 - r^2) + \delta_3 r^2 \right)^{3/2}$$

$$H_0(r) = \int H_1(r)^{1/2} \left(\frac{1}{R} - \frac{3B_{\theta}^{(0)}(r)}{r} \right) dr, \quad (24)$$

where $\delta_2 = \epsilon_2 s_2^2 / 2R(1 - s_2^4)$ and $\delta_3 = \epsilon_3 / s_3^3$. Therefore, the equation giving the minimum radius of the $m = 3$ island chain is

$$\int_{s_3}^r H_1(r')^{1/2} \left(\frac{1}{R} - \frac{3B_{\theta}^{(0)}(r')}{r'} \right) dr' = H_1(s_3) + H_1(r). \quad (25)$$

The result of this calculation for the peaked model is that the value for ϵ_3 (or I_3/I_0) required for these $m = 3$ islands to overlap the $m = 2$ islands is reduced about 10% from the value reported in Section 3 (which was $I_2/I_0 = I_3/I_0 = 0.5\%$). Thus, the effect of mode coupling with the toroidal field is noticeable but not large when an $m = 3$ mode is also present.

The final toroidal effect that we consider is the modification of the equilibrium flux surfaces by the toroidal geometry. We assume that the flux function $\psi = (R + r \cos \theta) A_\phi$ can be expanded in powers of the inverse aspect ratio $\epsilon_T \equiv a/R$, i.e., $\psi = \psi_0(r) + \epsilon_T \psi_1(r, \theta) + \dots$. Next, we use a method of obtaining ψ_1 , given a cylindrically symmetric equilibrium $\psi_0(r)$ and a pressure distribution $p = p(\psi)$. The method, developed by Yoshikawa [13], gives $\epsilon_T \psi_1(r, \theta) = \psi_0'(r) [f(r) - f(1)] \cos \theta$, assuming that ψ is regular at the origin, where

$$f(r) = \int_0^r dr' \frac{1}{r' (\psi_0')^2} \int_0^{r'} dr'' \left(\frac{r''}{R} (\psi_0'')^2 - 2r''^2 R \frac{dp}{dr''} \right). \quad (26)$$

(The equilibrium poloidal field, as calculated from this expression, couples in (16) in the same manner as the toroidal field. However, the poloidal field terms are smaller than the toroidal field terms by a factor of a/R , and therefore have been neglected.) We have used the boundary condition corresponding to a conducting wall at the plasma boundary, i.e., $\psi(a) = \text{constant}$. For the peaked model, we have

$$\psi_0(r) = - \frac{\ln(1 + \lambda r^2)}{2\lambda q(0)}.$$

We also assume the linear pressure distribution $p(\psi) = p_0[1 - \psi/\psi(a)]$ [14]. After some algebra, we find

$$f(r) = -\frac{1}{4\lambda R} \left(\frac{1}{\lambda r^2} (1 - \lambda^2 r^4) \ln(1 + \lambda r^2) + 2\lambda r^2 + 2d\ell(1 + \lambda r^2) \right) \\ + \frac{\beta_\theta}{2R(1 + \lambda)^2 \ln(1 + \lambda)} \left(3\lambda r^2 + \lambda^2 r^4 + \frac{1}{\lambda r^2} (1 - \lambda^2 r^4) \ln(1 + \lambda r^2) \right. \\ \left. + 2d\ell(1 + \lambda r^2) \right), \quad (27)$$

where $d\ell(x) = -\int_1^x dt(t-1)^{-1} \ln t$ is the dilogarithm function [15].

As is well known, the level curves of ψ are shifted to the outside of the torus by an amount $\sim a/R$. In fact, the axis (i.e., the point where $\nabla\psi = 0$) is shifted by an amount

$$\left(\frac{r}{a}\right)_{\text{axis}} \approx \frac{a}{R} \left[\frac{\lambda^2 - 1}{4\lambda} \log(1 + \lambda) - \frac{1}{2} + \left(\frac{\beta_\theta}{(1 + \lambda)^2 \log(1 + \lambda)} - \frac{1}{2\lambda} \right) d\ell(1 + \lambda) \right. \\ \left. + (\lambda^2 + 6\lambda - 2) \frac{\beta_\theta}{4(1 + \lambda)^2 \log(1 + \lambda)} - \frac{\beta_\theta(\lambda^2 - 1)}{2\lambda(1 + \lambda)^2} + \frac{1}{4\lambda} \right], \quad (28)$$

or for $q(0) = 1.1$, $q(a) = 3.7$

$$\left(\frac{r}{a}\right)_{\text{axis}} \approx \frac{a}{R} (0.1995 + 0.1192 \beta_\theta).$$

Because of the conducting shell the ψ -surfaces near the plasma boundary are shifted only slightly, and therefore the toroidal

effect calculated here is merely one of bunching up the ψ -surfaces on the outside of the torus (and spreading them out on the inside). Since the rotational transform is a function of ψ , the shear $\partial q / \partial x$ will be increased on the outer side (and decreased on the inner side) approximately by the factor $[1 - (r/a)_{\text{axis}}]^{-1}$; for $\beta_\theta = 1$ this ratio is 1.07. (It should be recognized that we are predicting thinner islands on the outside of the axis--where the toroidal field is weaker. This does not contradict flux conservation because the shape of the islands is also changed. In fact, this effect has been observed numerically.) We therefore conclude that the toroidal character of the equilibrium flux surfaces has a negligible effect on the destruction of magnetic surfaces.

A more accurate treatment of toroidal effects on the breakup of surfaces is possible. However, this would require some fundamental changes; for example, the toroidal effects on MHD modes (4) as well as on the MHD equilibrium would be needed.

IV. CONCLUSIONS

We have seen evidence that higher shear reduces the strength of the perturbation required to break magnetic surfaces. In fact, for the peaked model with moderate shear [$q(0) = 1.1$, $q(a) = 3.7$] this threshold perturbation (0.5%) is the same required to produce the observed 4 cm. $m = 2$ islands in ST. With the same model, toroidal effects do not seem to be quite large enough to destroy a pure $m = 2$, $n = 1$ symmetry, but with more shear they should have a larger effect.

This behavior can be related to the disruptive instability in the following manner: as ohmic heating causes the current profile to become more peaked, stable MHD modes become unstable. (For a quantitative discussion, see Ref. [14].) If the current is sufficiently peaked at this time, magnetic surfaces can begin to break up, with subsequent loss of particles and energy. Otherwise, the modes may saturate below the threshold for destruction of magnetic surfaces. However, the current profile will continue contracting, and at a certain point, conditions for destruction of magnetic surfaces will occur. Which of these phenomena will occur depends upon the stability criteria for the various modes and the level at which they saturate. The former occurs on a somewhat faster time scale and would therefore seem to be more consistent with observations.

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Fig. 1. Helical perturbation strength required for overlap of $m = 1$ and $m = 2$ resonances ($\tilde{I} \equiv I_2 = I_3$) vs. $q(0)$. Straight tokamak, peaked model.

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Fig. 2. Intersections of magnetic field lines with the $\phi = 0$ surface. Parabolic model, $\tilde{I}/I_0 = 3.75 \times 10^{-3}$.

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Fig. 3. Parabolic model, $\tilde{I}/I_0 = 7.5 \times 10^{-3}$.

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Fig. 4. Parabolic model, $\tilde{I}/I_0 = 1.50 \times 10^{-2}$.

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Fig. 5. Parabolic model, $\tilde{I}/I_0 = 2.25 \times 10^{-2}$.

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Fig. 6. Peaked model, $q(0) = 1.1$, $q(a) = 3.7$,
 $\tilde{I}/I_0 = 1.25 \times 10^{-3}$.

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Fig. 7. Peaked model, $\tilde{I}/I_0 = 2.5 \times 10^{-3}$.

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Fig. 8. Peaked model, $\tilde{I}/I_0 = 5.0 \times 10^{-3}$.

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Fig. 9. Peaked model, $\tilde{I}/I_0 = 7.5 \times 10^{-3}$.

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Fig. 10. Helical perturbation strength required for overlap of $m = 1$ and $m = 2$ resonances vs. $q(0)$. $m = 1$ islands formed by mode coupling with the toroidal field. $R/a = 5$. peaked model.

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