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THE RELATION BETWEEN REGRESSION COEFFICIENTS
IN A MULTIPLE-REGRESSION ANALYSIS

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The Relation between Regression Coefficients
in a Multiple-Regression Analysis*

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ABSTRACT

In a multiple-regression analysis it is often necessary to assign a value, different from the least-squares value, to one or more of the regression coefficients. A simple method is described which eliminates the matrix algebra which, in such cases, is usually necessary in order to re-evaluate the remaining coefficients, together with the associated variances and covariances. The method is generalized to give an equally simple procedure for testing possible linear relations among the regression coefficients.

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I. INTRODUCTION

Consider the regression equation

$$F(x) = \sum_{\mu=1}^N \phi_{\mu}(x) a_{\mu}, \quad (1)$$

where $F(x)$ is the dependent variate, the a_{μ} are the regression coefficients, and the $\phi_{\mu}(x)$ constitute a set of N independent variates. The only condition imposed upon these independent variates is that with each distinct value of x is associated a unique single value of $\phi_{\mu}(x)$ for each value of μ .

In the fitting of observed values $F(x_j) \equiv F_j$ to the curvilinear regression equation, Eq. (1), the $\phi_{\mu}(x_j) \equiv \phi_{\mu j}$ are usually chosen as a finite subset of a complete set of base functions for the expansion of $F(x)$. However, the order N of the subset which provides a sufficiently accurate representation for the regression curve is usually not known. The usual procedure is to fit curves of successively higher order until further terms do not improve the "goodness of fit." This means either that one must invert matrices of successively higher order for each new value N or that "orthogonal" functions $\phi_{\mu}(x)$ must be used in the original regression equation. Either alternative presupposes a great deal of tedious arithmetic. Even with modern high-speed computers the repeated inversion of high-order matrices is time consuming, and the use of "orthogonal" base functions is not usually feasible in practice since in this context "orthogonal" depends not only upon the points x_j at which the observations F_j are made but also upon the relative accuracy of these observations.

W. G. Cochran¹ has developed a very convenient method for the elimination of regression coefficients, i.e., for the evaluation of the regression coefficients in Eq. (1) for decreasing values of N , which avoids these computational difficulties. In this report Cochran's method is generalized to the case in which a regression coefficient is assigned an arbitrary value, not necessarily zero. This extends the applicability of this technique to the many cases of physical interest where there

¹ W. G. Cochran, Supp. J. R. S. S. 5, 171 (1938).

exists tentative information (in the form of linear relations) about one or more of these coefficients in addition to that derived from the experimental data which is being analyzed. For example, in the analysis of neutron angular-distribution data in terms of a series of Legendre polynomials, the regression equation takes the form

$$\sigma(\mu) = \sum_{\ell=0}^N P_{\ell}(\mu) a_{\ell},$$

where $\sigma(\mu)$ is the differential cross section evaluated at the scattering angle $\theta = \cos^{-1} \mu$. Suppose we desire to test a relation among the coefficients a_{ℓ} of the form

$$\sum_{\ell=0}^N \lambda_{\ell} a_{\ell} = c,$$

where the λ_{ℓ} and c are a set of preassigned constants. Such a relation may be suggested, for example, by general symmetry assumptions. By a relatively simple extension of Cochran's method, it is possible to incorporate such "tests" into the regression analysis with very little additional computation.

While the emphasis in this report will be on the utilitarian aspects of the method, the relations which will be derived also have a pedagogic value. The equations which are essential to this method provide an insight into the significance of the elements of the error matrix which is not possible in the conventional analysis.

II. THE METHOD

We assume at the outset that the value of $F(x)$ has been measured at s distinct points x_j and that $s > N$. Further we assume that the data F_j are observationally independent and that the variance associated with a single observation is Λ_j^{-2} , i. e.,

$$\langle (F_j - \langle F_j \rangle) (F_k - \langle F_k \rangle) \rangle = \delta_{jk} \Lambda_j^{-2}, \quad j, k=1, \dots, s, \quad (2)$$

where $\langle F_j \rangle$ denotes the value of F_j which would be expected if an infinite number of determinations of F_j were properly averaged. It will prove convenient to introduce

the following matrices:

$$\Phi = (\phi_{j\mu}) \quad \text{where} \quad \phi_{j\mu} \equiv \phi_{\mu}(x_j) \quad , \quad j=1, \dots, s; \mu=1, \dots, N.$$

$$F = (F_j) \quad \text{where} \quad F_j \equiv F(x_j) \quad , \quad j=1, \dots, s.$$

$$a = (a_{\mu}) \quad , \quad \mu=1, \dots, N.$$

$$\Lambda^2 = (\Lambda_j^2 \delta_{jk}) \quad \text{where} \quad \Lambda_j^2 \text{ is defined by Eq. (2), } j, k=1, \dots, s.$$

The regression equation, Eq. (1), now can be written in the form

$$F_j = \sum_{\mu=1}^N \phi_{j\mu} a_{\mu} \quad , \quad j = 1, \dots, s. \quad (3)$$

Since $s > N$, the relations constitute an overdetermined set for the evaluation of the regression coefficient vector a . The normal equations for the determination of these coefficients are

$$\sum_{j=1}^s \phi_{j\mu} \Lambda_j^2 F_j = \sum_{j=1}^s \sum_{\nu=1}^N \phi_{j\mu} \Lambda_j^2 \phi_{j\nu} a_{\nu} \quad , \quad \mu = 1, \dots, N. \quad (4)$$

The solution a of Eq. (4) is the column vector whose components minimize the quadratic Q_n , where

$$Q_n = \sum_{j=1}^s \Lambda_j^2 \left[F_j - \sum_{\mu=1}^N \phi_{j\mu} a_{\mu} \right]^2. \quad (5)$$

Suppose now that to a given component, say a_N , of a we assign a value ω which is not equal to the value a_N obtained for the solution of Eq. (4). The N th component of the vector a is singled out only for convenience of notation; the results are valid for any component of this vector. In place of Eq. (3) we now have the set of regression equations

$$F_j - \phi_{jN} \omega = \sum_{\mu=1}^{N-1} \phi_{j\mu} \bar{a}_{\mu} \quad , \quad j=1, \dots, s. \quad (6)$$

The regression coefficients $\bar{a}_\mu$, $\mu=1, \dots, N-1$, are not equal in general to the corresponding a_μ . To obtain the $\bar{a}_\mu$ the usual procedure would be to solve the normal equations obtained from Eq. (6), namely,

$$\sum_{j=1}^s \phi_{j\mu} \Lambda_j^2 (F_j - \phi_{jN} \omega) = \sum_{j=1}^s \sum_{\nu=1}^{N-1} \phi_{j\nu} \Lambda_j^2 \phi_{j\nu} \bar{a}_\nu, \quad \mu=1, \dots, N-1, \quad (7)$$

by inverting the $(N-1) \times (N-1)$ matrix $\widetilde{(\phi \Lambda^2 \phi)}$ which is obtained from the original matrix $\widetilde{(\phi \Lambda^2 \phi)}$ in Eq. (4) by deleting the Nth row and Nth column. The notation $\widetilde{\phi}$ is used to denote the transpose of the matrix ϕ . The solutions $\bar{a}_\nu$ of Eq. (7) minimize the quadratic Q_{N-1} , where

$$Q_{N-1} = \sum_{j=1}^s \Lambda_j^2 \left[(F_j - \phi_{jN} \omega) - \sum_{\mu=1}^{N-1} \phi_{j\mu} \bar{a}_\mu \right]^2. \quad (8)$$

It is possible to avoid this repeated matrix inversion by the following procedure. From the first $N-1$ equations of the set (4) we subtract the corresponding equations of the set (7). This gives

$$\omega \sum_{j=1}^s \phi_{j\mu} \Lambda_j^2 \phi_{jN} = \sum_{j=1}^s \left\{ \sum_{\nu=1}^{N-1} \phi_{j\mu} \Lambda_j^2 \phi_{j\nu} (a_\nu - \bar{a}_\nu) + \phi_{j\mu} \Lambda_j^2 \phi_{jN} a_N \right\}, \quad \mu=1, \dots, N-1. \quad (9)$$

Equations (9) can be written in the somewhat more convenient form

$$(\omega - a_N) (\widetilde{\phi \Lambda^2 \phi})_{\mu N} = \sum_{\nu=1}^{N-1} (\widetilde{\phi \Lambda^2 \phi})_{\mu \nu} (a_\nu - \bar{a}_\nu), \quad \mu=1, \dots, N-1. \quad (9')$$

Denoting by $(\widetilde{\phi \Lambda^2 \phi})^{-1}$ the inverse of the $N \times N$ matrix $\widetilde{\phi \Lambda^2 \phi}$ we have, by definition of an inverse matrix,

$$\sum_{\nu=1}^N (\widetilde{\phi \Lambda^2 \phi})_{\mu \nu} [(\widetilde{\phi \Lambda^2 \phi})^{-1}]_{\nu \rho} = \delta_{\mu \rho}, \quad \mu, \rho=1, \dots, N. \quad (10)$$

Multiplying Eq. (9') from the right by $[(\widetilde{\phi \Lambda^2 \phi})^{-1}]_{\mu \rho}$, $\rho=1, \dots, N$, and summing

the result for each value of ρ over μ from $\mu=1$ to $\mu=N-1$, we obtain

$$\begin{aligned} & [\omega - a_N] \sum_{\mu=1}^{N-1} (\tilde{\phi} \Lambda^2 \phi)_{N\mu} [(\tilde{\phi} \Lambda^2 \phi)^{-1}]_{\mu\rho} \\ &= \sum_{\nu=1}^{N-1} [a_\nu - \bar{a}_\nu] \sum_{\mu=1}^{N-1} (\tilde{\phi} \Lambda^2 \phi)_{\nu\mu} [(\tilde{\phi} \Lambda^2 \phi)^{-1}]_{\mu\rho}, \quad \rho=1, \dots, N. \end{aligned} \quad (11)$$

With the help of Eq. (10), the set of equations (11) can be written in the form

$$\begin{aligned} & [\omega - a_N] \{ \delta_{N\rho} - (\tilde{\phi} \Lambda^2 \phi)_{NN} [(\tilde{\phi} \Lambda^2 \phi)^{-1}]_{N\rho} \} \\ &= \sum_{\nu=1}^{N-1} [a_\nu - \bar{a}_\nu] \{ \delta_{\nu\rho} - (\tilde{\phi} \Lambda^2 \phi)_{\nu N} [(\tilde{\phi} \Lambda^2 \phi)^{-1}]_{N\rho} \}, \quad \rho=1, \dots, N. \end{aligned} \quad (12)$$

For $\rho=N$, Eq. (12) becomes

$$\begin{aligned} & [\omega - a_N] \{ 1 - (\tilde{\phi} \Lambda^2 \phi)_{NN} [(\tilde{\phi} \Lambda^2 \phi)^{-1}]_{NN} \} \\ &= - [(\tilde{\phi} \Lambda^2 \phi)^{-1}]_{NN} \sum_{\nu=1}^{N-1} [a_\nu - \bar{a}_\nu] (\tilde{\phi} \Lambda^2 \phi)_{\nu N}, \end{aligned} \quad (12a)$$

while for $\rho \neq N$, Eq. (12) gives

$$\begin{aligned} & [\omega - a_N] \{ -(\tilde{\phi} \Lambda^2 \phi)_{NN} [(\tilde{\phi} \Lambda^2 \phi)^{-1}]_{N\rho} \} \\ &= [a_\rho - \bar{a}_\rho] - [(\tilde{\phi} \Lambda^2 \phi)^{-1}]_{N\rho} \sum_{\nu=1}^{N-1} [a_\nu - \bar{a}_\nu] (\tilde{\phi} \Lambda^2 \phi)_{\nu N}, \quad \rho=1, \dots, N-1. \end{aligned} \quad (12b)$$

The use of Eq. (12a) to eliminate the summation over ν from the set of equations (12b) leads to

$$\bar{a}_\rho = a_\rho - (a_N - \omega) \frac{[(\tilde{\phi} \Lambda^2 \phi)^{-1}]_{N\rho}}{[(\tilde{\phi} \Lambda^2 \phi)^{-1}]_{NN}}, \quad \rho=1, \dots, N-1. \quad (13)$$

The evaluation of the regression coefficients $\bar{a}_\rho$ by use of Eqs. (13) is considerably simpler than the usual procedure of solving the set (7) of normal equations. We shall also see that these relations, Eqs. (13), make possible a

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Equating the right-hand sides of corresponding equations in the families (16) and (17) and regrouping terms, we obtain

$$\sum_{\nu=1}^{N-1} \left\{ \left[\overline{(\tilde{\phi} \Lambda^2 \phi)^{-1}} \right]_{\mu\nu} - \left[(\tilde{\phi} \Lambda^2 \phi)^{-1} \right]_{\mu\nu} + \frac{\left[(\tilde{\phi} \Lambda^2 \phi)^{-1} \right]_{N\mu} \left[(\tilde{\phi} \Lambda^2 \phi)^{-1} \right]_{N\nu}}{\left[(\tilde{\phi} \Lambda^2 \phi)^{-1} \right]_{NN}} \right\} \sum_{j=1}^s \phi_{j\nu} \Lambda_j^2 F_j$$

$$= \omega \left\{ \sum_{\nu=1}^{N-1} \left[\overline{(\tilde{\phi} \Lambda^2 \phi)^{-1}} \right]_{\mu\nu} \left[(\tilde{\phi} \Lambda^2 \phi) \right]_{\nu N} + \frac{\left[(\tilde{\phi} \Lambda^2 \phi)^{-1} \right]_{N\mu}}{\left[(\tilde{\phi} \Lambda^2 \phi)^{-1} \right]_{NN}} \right\}, \quad \mu=1, \dots, N-1. \quad (18)$$

The set of equations (18) must be independent of the value of ω ; however, the necessary condition for this is that the coefficients of ω in Eq. (18) vanish. Thus we obtain the relations

$$\sum_{\nu=1}^{N-1} \left[\overline{(\tilde{\phi} \Lambda^2 \phi)^{-1}} \right]_{\mu\nu} (\tilde{\phi} \Lambda^2 \phi)_{\nu N} + \frac{\left[(\tilde{\phi} \Lambda^2 \phi)^{-1} \right]_{N\mu}}{\left[(\tilde{\phi} \Lambda^2 \phi)^{-1} \right]_{NN}} = 0, \quad \mu=1, \dots, N-1. \quad (19)$$

Replacing the summation over the s values of j by use of equality (4) and making use of Eqs. (19) reduces the set of equations (18) to

$$\sum_{\rho=1}^N \alpha_{\rho} \sum_{\nu=1}^{N-1} \left\{ \left[\overline{(\tilde{\phi} \Lambda^2 \phi)^{-1}} \right]_{\mu\nu} \left[(\tilde{\phi} \Lambda^2 \phi)^{-1} \right]_{\mu\nu} + \frac{\left[(\tilde{\phi} \Lambda^2 \phi)^{-1} \right]_{N\nu} \left[(\tilde{\phi} \Lambda^2 \phi)^{-1} \right]_{N\mu}}{\left[(\tilde{\phi} \Lambda^2 \phi)^{-1} \right]_{NN}} \right\} (\tilde{\phi} \Lambda^2 \phi)_{\nu\rho} = 0,$$

$$\mu=1, \dots, N-1. \quad (20)$$

It might be mentioned that the two sets of equations (19) and (20) are not independent since Eqs. (19) can be obtained from Eqs. (20) by use of Eqs. (10). Since the coefficients of the α_{ρ} in Eqs. (20) do not depend on the data F_j , these equations are independent of the set of values α_{ρ} . Thus the coefficients of α_{ρ} in Eqs. (20) must

vanish separately and we have the relations

$$\begin{aligned} \sum_{\nu=1}^{N-1} [(\tilde{\phi}\Lambda^2\phi)^{-1}]_{\mu\nu} (\tilde{\phi}\Lambda^2\phi)_{\nu\rho} \\ = \sum_{\nu=1}^{N-1} [(\tilde{\phi}\Lambda^2\phi)^{-1}]_{\mu\nu} (\tilde{\phi}\Lambda^2\phi)_{\nu\rho} - \frac{[(\tilde{\phi}\Lambda^2\phi)^{-1}]_{N\mu}}{[(\tilde{\phi}\Lambda^2\phi)^{-1}]_{NN}} \sum_{\nu=1}^{N-1} [(\tilde{\phi}\Lambda^2\phi)^{-1}]_{N\nu} (\tilde{\phi}\Lambda^2\phi)_{\nu\rho}, \\ \rho=1, \dots, N; \mu=1, \dots, N-1. \end{aligned}$$

Using Eqs. (10) to evaluate the sums on the right-hand side of these last relations, we obtain

$$\sum_{\nu=1}^{N-1} [(\tilde{\phi}\Lambda^2\phi)^{-1}]_{\mu\nu} (\tilde{\phi}\Lambda^2\phi)_{\nu\rho} = \delta_{\mu\rho} - \frac{[(\tilde{\phi}\Lambda^2\phi)^{-1}]_{N\mu}}{[(\tilde{\phi}\Lambda^2\phi)^{-1}]_{NN}} \delta_{N\rho}, \rho=1, \dots, N; \mu=1, \dots, N-1. \quad (21)$$

Multiplying Eqs. (21) from the right by $[(\tilde{\phi}\Lambda^2\phi)^{-1}]_{\rho\sigma}$, with $\sigma=1, \dots, N-1$ in turn, and summing over all values of ρ , we obtain for each value of σ an expression of the form

$$[(\tilde{\phi}\Lambda^2\phi)^{-1}]_{\mu\sigma} = [(\tilde{\phi}\Lambda^2\phi)^{-1}]_{\mu\sigma} - \frac{[(\tilde{\phi}\Lambda^2\phi)^{-1}]_{N\sigma} [(\tilde{\phi}\Lambda^2\phi)^{-1}]_{N\mu}}{[(\tilde{\phi}\Lambda^2\phi)^{-1}]_{NN}}, \\ \mu, \sigma = 1, \dots, N-1. \quad (22)$$

Equations (22) make possible the evaluation of the components of the $(N-1) \times (N-1)$ error matrix $(\tilde{\phi}\Lambda^2\phi)^{-1}$ in terms of the components of the original $N \times N$ error matrix $(\tilde{\phi}\Lambda^2\phi)^{-1}$. Again, the evaluation of Eqs. (22) is considerably less arduous than the inversion of the matrix $(\tilde{\phi}\Lambda^2\phi)$, particularly for large values of N . It also should be noted that both matrices, $(\tilde{\phi}\Lambda^2\phi)^{-1}$ and $(\tilde{\phi}\Lambda^2\phi)$, are symmetric so that only half of the nondiagonal elements of Eqs. (22) need be evaluated.

Finally, in order to evaluate the relative "goodness of fit" obtained from the regression equations (3) and (6), we obtain an expression for the quadratic Q_{N-1} in terms of Q_N . By use of Eqs. (4), the definition (5) of Q_N may be written as

$$Q_N = \sum_{j=1}^s \Lambda_j^2 F_j^2 - \sum_{\mu=1}^N a_{\mu} \sum_{\nu=1}^N (\tilde{\phi} \Lambda^2 \phi)_{\mu\nu} a_{\nu}. \quad (23)$$

Similarly expanding Eq. (8), the definition of Q_{N-1} , and using Eq. (7), we obtain

$$Q_{N-1} = \sum_{j=1}^s \Lambda_j^2 F_j^2 - 2\omega (\tilde{\phi} \Lambda^2 F)_N - \sum_{\mu=1}^{N-1} \bar{a}_{\mu} \left\{ \sum_{\nu=1}^N (\tilde{\phi} \Lambda^2 \phi)_{\mu\nu} a_{\nu} - (\phi \Lambda^2 \phi)_{\mu N} \omega \right\} + \omega^2 (\tilde{\phi} \Lambda^2 \phi)_{NN}. \quad (24)$$

Eliminating the coefficients $\bar{a}_{\mu}$ with the help of Eqs. (13) and comparing the resulting equation with Eq. (23), we obtain after a little algebraic manipulation

$$Q_{N-1} = Q_N + \frac{(\omega - a_N)^2}{[(\tilde{\phi} \Lambda^2 \phi)^{-1}]_{NN}} \quad (25)$$

It is obvious that Q_N and Q_{N-1} are measures of the "goodness of fit" of the regression equations (3) and (6), respectively. If the fit is too good relative to the expected uncertainties Λ_j^{-1} , one might suspect that terms which are statistically not significant are included in the regression equations. In this case Q_N and/or Q_{N-1} would be "very small." On the other hand, the value of these quadratic expressions would be "very large" if significantly non-zero terms were excluded from the regression equations. Thus for the "correct" regression expansion, the value of Q should be neither "too large" nor "too small."

In order for these qualitative notions to be of any practical value, it is necessary to obtain some criteria on which to base our judgment of the "correct" value for Q . If it is assumed that the errors Λ_j^{-1} are normally distributed with zero mean, it can be shown that Q_N has a chi-squared distribution with $(s-N-1)$

degrees of freedom. Thus the expectation value and variance of Q_N are given by the expressions

$$\langle Q_N \rangle = s - N - 1, \quad (26)$$

and

$$\langle (Q_N - \langle Q_N \rangle)^2 \rangle = 2(s - N - 1). \quad (27)$$

The statistical significance of the fit to the regression equation (3) is obtained from the probability $P(\chi^2 > Q_N)$ for obtaining a value of χ^2 greater than the value calculated for Q_N .

From Eqs. (26) and (27) it is possible to obtain a crude criterion for the relative merits of the regression equations (3) and (6). Let ΔQ_N denote the positive difference $Q_N - Q_{N-1}$. We then have

$$\langle \Delta Q \rangle = 1, \quad \langle (\Delta Q - \langle \Delta Q \rangle)^2 \rangle = 4(s - N - 1) - 2.$$

Thus, by preassigning a value to a single regression coefficient, we might expect, on the average, to increase the value of the relevant Q by unity. If Q is increased by a much larger amount, this is an indication that the value ω , assigned to the coefficient a_N , say, is not consistent with the data F_j .

III. A SUMMARY OF RESULTS

The simplicity of the results of the preceding section is somewhat obscured by the involved algebraic steps necessary to the derivation. Actually, the only relations which are required for the use of this technique are the following: If the k th regression coefficient a_k is "arbitrarily" set equal to the value ω_1 , the new regression coefficients $\bar{a}_\rho$ are given by the equations

$$\bar{a}_\rho = a_\rho - (a_k - \omega_1) \frac{[(\tilde{\phi} \Lambda^2 \phi)^{-1}]_{\rho k}}{[(\tilde{\phi} \Lambda^2 \phi)^{-1}]_{kk}}, \quad \rho = 1, \dots, N. \quad (13')$$

The associated quadratic Q_{N-1} becomes

$$Q_{N-1} = Q_N + \frac{(\omega - a_k)^2}{[\tilde{(\phi \Lambda \phi)^{-1}}]_{kk}} \quad (25')$$

The elements $[\tilde{(\phi \Lambda \phi)^{-1}}]_{\mu\sigma}$ of the new error matrix are independent of the value ω_1 assigned to the coefficient a_k and are given by the relations

$$[\tilde{(\phi \Lambda \phi)^{-1}}]_{\mu\sigma} = [\tilde{(\phi \Lambda \phi)^{-1}}]_{\mu\sigma} - \frac{[\tilde{(\phi \Lambda \phi)^{-1}}]_{k\sigma} [\tilde{(\phi \Lambda \phi)^{-1}}]_{k\mu}}{[\tilde{(\phi \Lambda \phi)^{-1}}]_{kk}},$$

$$\mu, \sigma = 1, 2, \dots, k-1, k+1, \dots, N.$$

(22')

Obviously this process can be continued, i.e., it is now possible to preassign the value ω_2 to a coefficient $\bar{a}_l$, $l \neq k$, and, from the above relations, to evaluate the new regression coefficients $\bar{a}_p$ (say), the associated quadratic Q_{N-2} , and the new error matrix $\tilde{(\phi \Lambda \phi)^{-1}}$.

IV. SUBSIDIARY LINEAR RELATIONS AMONG REGRESSION COEFFICIENTS

Often, particularly in a physical problem, various assumptions about the nature of the phenomenon being measured lead to different linear relations among the regression coefficients. By use of the techniques developed in Sec. II the amount of calculation required for the statistical evaluation of such relations is reduced to a minimum.

Suppose, for example, that on the basis of a given set of data we wish to investigate the validity of the relation

$$\sum_{\mu=1}^N \lambda_{\mu} \bar{a}_{\mu} = c, \quad (28)$$

where the λ_μ and c are known constants. When $\bar{a}_\mu$ is replaced by use of equality (13'), Eq. (28) becomes

$$\sum_{\mu=1}^N \lambda_\mu \left\{ a_\mu - (a_k - \omega_1) \frac{[(\tilde{\phi} \Lambda^2 \phi)^{-1}]_{\mu k}}{[(\tilde{\phi} \Lambda^2 \phi)^{-1}]_{kk}} \right\} = c.$$

Solving for ω_1 , we obtain

$$\bar{a}_k = \omega_1 = a_k + \frac{c - \sum_{\mu} \lambda_\mu a_\mu}{\sum_{\mu} \lambda_\mu [(\tilde{\phi} \Lambda^2 \phi)^{-1}]_{\mu k}} [(\tilde{\phi} \Lambda^2 \phi)^{-1}]_{kk}. \quad (29)$$

Once ω_1 is obtained, the remaining regression coefficients, subject to Eq. (28), are evaluated by use of Eqs. (13').

It is perhaps unnecessary to point out that a system of subsidiary conditions of the form

$$\sum_{\mu} \lambda_{i\mu} \bar{a}_\mu = c_i, \quad i=1, \dots, t < N, \quad (30)$$

can also be handled within the framework of this method by continuing the process of assigning values ω_i to different regression coefficients, as indicated in Sec. III. A set of equations of the form of Eq. (29) is then used to evaluate the ω_i which satisfy Eqs. (30).

V. A NUMERICAL EXAMPLE

It is customary to analyze neutron angular-distribution data in terms of a series of Legendre polynomials, i.e., the regression equation takes the form

$$\sigma(x_j) = \sum_{\mu=1}^N \phi_{j\mu} a_\mu, \quad (31)$$

where $\sigma(x_j)$ is the differential scattering cross section measured at the scattering angle $\theta_j = \cos^{-1} x_j$ and $\phi_{j\mu}$ is the Legendre polynomial of order μ evaluated at x_j . For the scattering of 2-Mev neutrons by carbon, relative (not normalized) values of the cross section $\sigma(x_j)$ were measured² at eight distinct values of x_j . The regression curve (31) with $N=6$ was used in fitting these data and the following result was obtained:

$$a = \begin{Bmatrix} 0.1741 \\ 0.0141 \\ 0.2723 \\ 0.0205 \\ 0.0985 \\ 0.0309 \end{Bmatrix}, \quad (32)$$

$$(\tilde{\phi} \Lambda^2 \phi)^{-1} = \begin{Bmatrix} 0.0293 & -0.0336 & 0.0651 & -0.0631 & 0.0587 & -0.0481 \\ -0.0336 & 0.1864 & -0.1463 & 0.2497 & -0.1500 & 0.1931 \\ 0.0651 & -0.1463 & 0.3311 & -0.2647 & 0.2750 & -0.2208 \\ -0.0631 & 0.2497 & -0.2647 & 0.5680 & -0.2759 & 0.4137 \\ 0.0587 & -0.1500 & 0.2750 & -0.2759 & 0.4480 & -0.2132 \\ -0.0481 & 0.1931 & -0.2208 & 0.4137 & -0.2132 & 0.5288 \end{Bmatrix}, \quad (33)$$

and

$$Q_6 = 6.408. \quad (34)$$

This relatively large value of Q_6 is not unexpected since the data $\sigma(x_j)$ used for this analysis were those values which were measured in the laboratory frame of reference whereas, strictly speaking, Eqs. (31) can be justified as a representation only of center-of-mass values $\sigma(x_j)$. In this experiment, this regression analysis is preliminary to a much more refined analysis which takes account of corrections for resolution and multiple scattering. For our present purpose, however, these considerations need not concern us.

Suppose that we wish to test various relations, of the form of Eq. (28), among the Legendre coefficients in Eqs. (31).

² Lane, Langsdorf, and Monahan (to be published).

Elimination of a regression coefficient.— First let us eliminate the coefficient a_6 , i. e., we set $\bar{a}_6 = \omega = 0$. Substituting this value of ω into Eqs. (13), we obtain for the new regression coefficients the values

$$\bar{a} = \begin{Bmatrix} 0.1769 \\ 0.0028 \\ 0.2852 \\ - 0.0037 \\ 0.1110 \end{Bmatrix}. \quad (35)$$

Using Eq. (25) to evaluate the associated quadratic Q_5 , we obtain

$$Q_5 = 8.19. \quad (36)$$

Comparing this last result with the value of Q_6 given by Eq. (34), we see that for the data at hand the value of a_6 given in Eq. (32) is statistically significant.

Preassigning a value to a regression coefficient.— Next let us assign to the coefficient $\bar{a}_6$ the arbitrary value ω . In this case we obtain from Eqs. (13) the new regression coefficients

$$\bar{a} = \begin{Bmatrix} 0.1741 - (0.0309 - \omega) & (0.0481 / 0.5288) \\ 0.0141 - (0.0309 - \omega) & (0.1931 / 0.5288) \\ 0.2723 - (0.0309 - \omega) & (0.2208 / 0.5288) \\ 0.0205 - (0.0309 - \omega) & (0.4137 / 0.5288) \\ 0.0985 - (0.0309 - \omega) & (0.2132 / 0.5288) \end{Bmatrix}. \quad (37)$$

Equation (37) shows quite clearly the ease with which this type of substitution can be made. Furthermore, this relation gives the explicit dependence of each of the coefficients $\bar{a}_p$ on the "eliminated" variable $\bar{a}_6$. For any value of ω , Q_5 is evaluated by use of Eq. (25).

Linear relations among regression coefficients.— Suppose now that a relation of the form of Eq. (28) existed among the regression coefficients. Again let us assign to $\bar{a}_6$ an arbitrary value ω and determine ω as that value which satisfies the condition (28). This value is obtained from Eq. (29) which, in the

present case, becomes

$$\bar{a}_6 = \omega = 0.0309 + 0.5288 \frac{c - \sum_{\mu} \lambda_{\mu} a_{\mu}}{\sum_{\mu} \lambda_{\mu} [(\tilde{\phi} \Lambda^2 \phi)^{-1}]_{\mu 6}}, \quad (38)$$

where the a_{μ} and the $[(\tilde{\phi} \Lambda^2 \phi)^{-1}]_{\mu 6}$ are given in Eqs. (32) and (33), respectively.

As an example, let us investigate the specific case in which the values of λ_{μ} and c in Eq. (28) are

$$\lambda_2 = \lambda_4 = -\lambda_5 = 1; \lambda_6 = 2; \lambda_1 = \lambda_3 = 0; c = 0.$$

Thus we are considering the probability of the existence of a relation of the form

$$\bar{a}_2 + \bar{a}_4 - \bar{a}_5 + 2\bar{a}_6 = 0 \quad (39)$$

among the regression coefficients. Evaluating $\bar{a}_6 = \omega$ with the help of Eq. (38), we obtain

$$\bar{a}_6 = \omega = 0.0368.$$

From Eq. (25) we see that the new value of Q is essentially the same as that given in Eq. (34). Thus on the basis of these data alone we cannot reject the hypothesis stated by Eq. (39).

For each of the above cases, the "reduced" error matrix $(\tilde{\phi} \Lambda^2 \phi)^{-1}$ is obtained from the set of equations (22). These give

$$(\tilde{\phi} \Lambda^2 \phi)^{-1} = \begin{Bmatrix} 0.0249 & -0.0160 & 0.0450 & -0.0255 & 0.0393 \\ -0.0160 & 0.1159 & -0.0657 & 0.0987 & -0.0721 \\ 0.0450 & -0.0657 & 0.2388 & -0.0919 & 0.1860 \\ -0.0255 & 0.0987 & -0.0918 & 0.2442 & -0.1092 \\ 0.0393 & -0.0721 & 0.1860 & -0.1092 & 0.3620 \end{Bmatrix}. \quad (40)$$