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Cavity Stability: Finite Element Analysis for Steady and Transient Creep

by

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CAVITY STABILITY: FINITE ELEMENT ANALYSIS FOR STEADY AND TRANSIENT CREEP

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ABSTRACT

Long-term steady-state creep and creep rupture of underground structures is a problem of increasing importance by virtue of the implications to storage of radioactive waste materials and to energy extraction methods which employ underground structures or cavities. This paper describes the axisymmetric finite element code CSAAS which has been developed at the Los Alamos Scientific Laboratory in order to look at these problems. Special attention is directed to the "nearly incompressible" element formulation in CSAAS and to the stable method of time stepping which is used to obtain an estimate of the incompressible steady creep state from an initial elastic solution. Applications are made to thermal creep problems involving pure halite which obeys a well documented creep law of Weertman's form; the flow rule for axisymmetric stress states is based on the Prandtl-Reuss law of metal plasticity. The resulting discretized equations are highly nonlinear.

Comparison of the numerical results from CSAAS with the few known analytical solutions to one-dimensional steady creep problems indicates a high degree of accuracy for this finite element method. When the code is applied to the examination of the long-term stability of a spherical cavity in halite, various steady motions of the cavity (such as buoyancy or slumping) can be observed, depending on the depth, the temperature field, and the internal pressurization of the cavity.

I. INTRODUCTION

At sufficiently high temperatures metals and certain other crystalline materials show progressive plastic deformation, or creep, when loaded under a constant, nonhydrostatic state of stress. The situation becomes exaggerated when either the temperature or the stress level is raised. Three stages of creep are often distinguishable. These are primary, secondary, and tertiary creep, during which the creep strain-rate is decreasing,

constant, and increasing with time, respectively.¹

It is the secondary creep stage with which we are mainly concerned here.

A nonmetallic substance whose steady-state creep behavior has been thoroughly characterized is halite (NaCl). The steady-state creep behavior of isotropic polycrystalline aggregates of halite has been determined in the temperature range from 20°C to over 500°C and at confining (hydrostatic) pressures of zero² and 2 kbars.³ It is well

described by Weertman's formula which is based on dislocation climb mechanisms,

$$\dot{\epsilon} = A\sigma^N \exp(-E/RT) \quad (1)$$

where σ , $\dot{\epsilon}$, and T are the stress, creep strain-rate, and absolute temperature, respectively, and where A , E , and N are material constants, and R is the universal gas constant. Considering that N usually ranges from 3 to 7 for most crystalline materials ($N \approx 5.5$ for halite), it is clear from Eq. (1) that the steady creep problem is highly nonlinear, and numerical techniques must be undertaken to solve real problems.

Extension of the one-dimensional observations for metallic creep is carried out to multiaxial stress states by use of the Prandtl-Reuss flow rule of plasticity; thus, the creep strain-rates are proportional to the stress deviators. This constitutive relation, together with the equilibrium equations and strain-rate-velocity equations, constitutes a problem of steady-state creep. It is clear from these equations that the steady creep problem is not unlike the nonlinear elastic problem for incompressible materials; thus, numerical methods capable of solving the nonlinear elastic problem, such as the method of successive elastic solutions,⁴ may be applicable to the steady creep problem.

The main conclusion of this study and recommendations for further work are contained in Section II.

The numerical solution of basic creep equations starting from an initial elastic solution is discussed in Section III. A numerical algorithm is developed which allows the calculation of incremental creep strains, stresses, and displacements as in traditional initial strain methods.^{5,6} However, in the method of this report we use the midinterval value of the stress which appears to lead to a higher order of accuracy and, more importantly, to numerical stability, thus permitting the use of very large time steps as steady-state creep is approached. On the

other hand, this method requires the formation of the elastic-creep stiffness matrix at each time step. In addition, as steady creep is approached, the deformation becomes nearly incompressible. To describe this feature of creeping flow accurately, we have used the finite element formulation of Ref. 7 for cylindrically orthotropic and nearly incompressible materials. We believe this feature to be especially important to the numerical solution of long-term steady creep--a physical phenomenon that maintains both the features of fluid flow and the deformation of solids possessing finite shear strength.

Section IV contains a brief description of the axisymmetric creep code CSAAS which incorporates the numerical methods of Section III into a working code. Results from use of this numerical method are compared to those from one-dimensional steady creep test problems for which analytical creep rates can be obtained. There is good agreement between exact and calculated creep rates.

The stability of cavities in massive salt bodies is discussed in Section V. Recent proposals to use cavities for storage of radioactive wastes and as a means of energy extraction from thermonuclear explosives⁷ have made necessary an accurate prediction of the creep behavior of *in situ* salt at geothermal and elevated temperatures. When the numerical method was applied to the examination of the long-term stability of a spherical cavity in halite, various steady motions of the cavity could be observed, depending on the depth, the temperature field, and the cavity pressurization. Thus, for a 200-m-radius cavity at a depth of 2 km in a salt dome with geothermal temperature field of 27.5°C/km of depth, the steady creep rate ranged from about 0.1 mm/yr for full cavity pressurization (which balanced the overburden pressure) to about 30 m/yr for no cavity pressurization. In the former case the deformation mode was that of a buoyant (rising) cavity; with no cavity pressure, the deformation mode was that of a collapsing

cavity with a maximum and minimum of radial inward velocity at the base and the top of the cavity, respectively. For a localized radially symmetric "hot spot" temperature field of 500°C on the wall of the same cavity, with full pressurization and with the interior temperature field that of the 20-yr transient field, the steady creep rate was calculated to be 100 mm/yr; the deformation mode was slumping of the cavity wall with salt flow down the vertical face of the cavity.

A geomechanical model for the formation of halite domes is also discussed in Section V, and steady creep rates are calculated.

II. CONCLUSIONS AND RECOMMENDATIONS

This report is concerned primarily with the description of a finite element code capable of treating long-term steady-state creep problems. Application of the code to the stability of Pacer cavities⁹ in salt domes has led us to a number of conclusions concerning these cavities as well as to certain recommendations for further numerical work on the behavior of salt-dome cavities in particular and creep flow problems in general. In addition, certain observations are made about the numerical method of this report, and modifications are recommended.

The following are our main conclusions:

- (1) The time stepping method described here, together with a stiffness formulation for an incompressible finite element, provides a stable and accurate method for predicting long-term steady creep.
- (2) Although we use the recommended method of successive approximation (see Refs. 4 or 9) to improve on our long-term creep solution, we have seldom observed convergence of this method when used with finite element discretization for bona fide two-dimensional problems. The method of successive approximation appears to break down when rigid or very slowly

creeping regions occur along with regions of high creep rate. Other iterative methods, such as the Newton-Raphson method, may converge; however, the increased amount of programming, together with a nonsymmetric matrix in the solution routine, will be costly.

- (3) Deep (greater than several kilometers) unpressurized cavities in domes of pure halite will creep at such high rates as to preclude their use for Pacer or for long-term storage of radioactive materials.
- (4) The internal pressurization (and the modest swings in pressure) proposed for Pacer cavities will stabilize these cavities so that long-term creep--characterized by squashing of the cavity--will not be a problem.
- (5) Vertical slumping of the cavity wall will occur at a rate of about 100 mm/yr at the 20-yr temperature profile (500°C on the wall of a halite dome that was initially 100°C) and consequently should not be a problem.
- (6) Buoyant cavities rise at trivial rates, and this deformation mode occurs when the overburden pressure at the equator of the cavity is balanced by the internal pressure.
- (7) Defects in the cavity wall should not present a problem unless there is also a pressure-free surface associated with the defect. If such a differential surface loading exists then steady creep will occur. An example of a steady creeping flow from such a loading state is that of the mechanism advanced for formation of the great salt domes.

The following are four recommendations for analytical work to provide better understanding of the creep behavior of cavities in salt domes:

- (1) The present version of the creep code CSAAS should be modified to take into

account the geometry changes that occur during transient and steady creep.

- (2) Nonspherical cavities, often the result of solution mining in salt domes, should be looked at by this finite element method, with particular attention directed to the effect of the asymmetry on creep rate.
- (3) Although collapse (such as a falling roof) of a cavity in a salt dome for Pacer seems to be a remote possibility because of the internal pressurization, such a phenomenon can be investigated as a brittle creep rupture problem using our numerical technique.
- (4) Anhydrite (CaSO_4) seams occur in the salt domes that have been proposed for Pacer. The presence of the nearly vertical seams can enhance or degrade cavity stability by stiffening or weakening the cavity walls against long-term and transient creep. The cavity stability problem should be further explored using a finite element model with embedded anhydrite seams.

III. FINITE ELEMENT SOLUTION OF CREEP PROBLEMS

A. Basic Equations

Several general accounts of the finite element method are available, and we will employ the notation used in Ref. 5. Thus, we designate vectors by $\{v\}$, matrices by $[M]$, etc. In brief, we assume that the region of interest Ω is divided into a finite number of subregions--called finite elements--as shown by the triangular elements of Fig. 1. The subdivided region usually will not correspond to the region Ω because of disparities along the boundary, although the use of isoparametric elements allows an accurate description of curvilinear boundaries. At the interconnections, called nodes, of the assemblage of elements, values of the dependent variables of the problem are defined; for the axisymmetric displacement formulation that we will

Fig. 1. The finite element idealization.

employ, the nodal point variables are the radial and axial displacements (u, w) . Once nodal point values of u and w are given, their variation is completely defined within any element by the shape functions $N_i(r, z)$, $i = 1, 2, \dots, L$,

$$\begin{aligned} u(r, z) &= \sum_{i=1}^L N_i(r, z) u_i, \\ w(r, z) &= \sum_{i=1}^L N_i(r, z) w_i \end{aligned} \quad (2)$$

where L is the number of element nodal points and $\{u_i, w_i\}$ are the nodal values of u and w . For the three-node triangular element, the shape functions $N_i(r, z)$, $i = 1, 2, 3$, are linear functions of r and z . Once elemental shape functions are taken, there remains the determination of the nodal values of u and w that satisfy the field equations and boundary conditions of the problem. For the creep problem, these field equations are the strain-displacement relations, the equilibrium equations, and the constitutive relationship. We represent the components of the stress tensor $(\sigma_r, \sigma_\theta, \sigma_z, \tau_{rz})$ and strain tensor $(\epsilon_r, \epsilon_\theta, \epsilon_z, \gamma_{rz})$ by the vectors $\{\sigma\}$ and $\{\epsilon\}$, respectively. The strains within any

element are then given, in terms of the nodal displacements, by the relation

$$\{\epsilon\} = \left\{ \begin{array}{c} \frac{\lambda u}{\partial r} \\ \frac{u}{r} \\ \frac{\lambda w}{\partial z} \\ \frac{\lambda u}{\partial z} + \frac{\lambda w}{\partial r} \end{array} \right\} = [B] \{\delta\} \quad (3)$$

where $[B]$ is the strain-displacement matrix and $\{\delta\}$ is the vector of nodal displacements

$$\{\delta\}^T = (u_i, w_i, u_j, w_j, u_k, w_k) \quad (4)$$

The interested reader should consult Chapter 5 of Ref. 5 for details. The equilibrium of any element Δ^n then requires that

$$\int_{\Delta^n} [B]^T \{\sigma\} dV - \{F\}^n = 0 \quad (5)$$

where $\{F\}^n$ is the six-component vector of nodal point forces acting on Δ^n . Equation (5) is valid regardless of the constitutive relation, and Eq. (3) applies for small strains only.

For the constitutive equation, we assume, as in plastic flow of metals, that the elastic and creep strain increments are additive

$$\{d\epsilon\} = \{d\epsilon\}_e + \{d\epsilon\}_c \quad (6)$$

where the elastic strain increments are given by

$$\{d\epsilon\}_e = [D]^{-1} \{d\sigma\} \quad (7)$$

In Eq. (7), $[D]$ is the elasticity matrix which for axisymmetric elasticity takes the form

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \quad (8)$$

E being Young's modulus and ν Poisson's ratio.

The creep constitutive equation, Eq. (1), is generalized for multiaxial stress states to the form

$$\{d\epsilon\}_c = \frac{3}{2} \frac{d\epsilon_{ef}}{\sigma_{ef}} \{s\} \quad (9)$$

where $\{s\}$ is the deviator tensor which for axisymmetric stress states takes the form

$$\{s\} = \{\sigma\} - p \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix}$$

and where p is the average pressure, $p = 1/3$

$(\sigma_r + \sigma_\theta + \sigma_z)$. In Eq. (9), σ_{ef} is the square root of the second invariant of the deviator tensor which for axisymmetric stress states is given by (see Ref 4, for example)

$$\sigma_{ef} = \sqrt{\frac{1}{2}[(\sigma_r - \sigma_\theta)^2 + (\sigma_r - \sigma_z)^2 + (\sigma_\theta - \sigma_z)^2 + 6\tau_{rz}^2]} \quad (10)$$

and $d\epsilon_{ef}$ is directly given from Eq. (1)

$$d\epsilon_{ef} = A \sigma_{ef}^N \exp(-E/RT) dt \quad (11)$$

Alternatively, Eq. (9) can be written in the matrix form

$$\{d\epsilon\}_c = \frac{d\epsilon_{ef}}{\sigma_{ef}} [\mu] \{\sigma\} \quad (12)$$

where $[\mu]$ is a matrix with components

$$[\mu] = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad (13)$$

Equations (10)-(12) reduce to Eq. (1) for the case of uniaxial stress, they constrain the creep flow to be incompressible (since $s_1 + s_2 + s_3 = 0$), and they are based on the Prandtl-Reuss law of metal plasticity.

In the usual initial strain method for solving transient creep problems, one proceeds by calculating an initial elastic solution for displacements and stresses for the given applied loads at time equal to zero. Using the computed stress field, one can then determine incremental creep strains from Eqs. (10)-(12). Substituting Eq. (7) into Eq. (6) and solving for $\{d\sigma\}$ then gives

$$\{d\sigma\} = [D] \left(\{d\epsilon\} - \frac{d\epsilon_{ef}}{\sigma_{ef}} [\mu] \{\sigma\} \right). \quad (14)$$

Using Eq. (3) for $\{d\epsilon\}$ in terms of incremental displacements $\{d\delta\}$, and then substituting into Eq. (5) for element equilibrium gives

$$[k]_n \{d\delta\} = \{F\}^n + \int_{\Delta^n} [B]^T [D] \{d\epsilon\}_c dV \quad (15)$$

where $[k]_n$ is the element stiffness matrix,

$$[k]_n = \int_{\Delta^n} [B]^T [D] [B] dV. \quad (16)$$

Assembling the equations for element equilibrium and annihilating the internal nodal forces then gives (for constant external loads)

$$[K] \{d\delta\} = \{dF\}_c \quad (17)$$

where $\{dF\}_c$ are the nodal forces caused by the incremental creep strains and $[K]$ is the master stiffness matrix. The system of equations (17) is then resolved for the incremental displacements, and the procedure can then be repeated for the next time increment using the new stress state. Note

that the master stiffness matrix, once formed, is used in all succeeding re-solutions.

When initial strain methods are used, numerical instability occurs as long-term steady creep is approached. For the sake of economy, these problems need to be solved using relatively long time intervals, but such large intervals dt yield solutions where creep increments calculated from Eqs. (11) and (12) are unstable--in this context instability refers to an oscillating divergence of stresses as creep continues (see Refs. 6 and 10). To overcome this difficulty, we propose to use the average stress values during the time increment dt for computation of the incremental creep strains. Thus, from Eqs. (11) and (12) we write

$$\{d\epsilon\}_c = A \bar{\sigma}_{ef}^{(N-1)} \exp(-E/RT) [\mu] \{\bar{\sigma}\} dt \quad (18)$$

where a bar indicates the midinterval value,

$$\bar{\sigma} = \{\sigma\} + \frac{1}{2} \{d\sigma\}, \quad \bar{\sigma}_{ef} = \sigma_{ef} + \frac{1}{2} d\sigma_{ef}. \quad (19)$$

The increment in σ_{ef} can be computed from the expression

$$d\sigma_{ef} = \{\nabla \sigma_{ef}\}^T \{d\sigma\} \quad (20)$$

where

$$\{\nabla \sigma_{ef}\}^T = \left[\frac{\partial \sigma_{ef}}{\partial \sigma_r}, \frac{\partial \sigma_{ef}}{\partial \sigma_\theta}, \frac{\partial \sigma_{ef}}{\partial \sigma_z}, \frac{\partial \sigma_{ef}}{\partial \tau_{rz}} \right].$$

Using the definition of σ_{ef} from Eq. (10),

$$\{\nabla \sigma_{ef}\} = \frac{1}{\sigma_{ef}} [\mu] \{\sigma\}. \quad (21)$$

Substituting Eq. (21) into Eq. (20) and the result into the second equation in (19) and the results into (18) then yields, after use of the binomial expansion and neglect of term of order $\{d\sigma\}^T \{d\sigma\}$, the following expression for the incremental creep strains:

$$\{d\epsilon\}_c = \frac{d\epsilon_{ef}}{\sigma_{ef}} ([\mu] \{\sigma\} + \frac{1}{2} [C] \{d\sigma\}) \quad (22)$$

where $[C]$ is a matrix of the form

$$[C] = [\mu] + \frac{(N-1)}{\sigma_{ef}} \{v\} \{v\}^T \quad (23)$$

and where the vector $\{v\}$ is given by

$$\{v\} = [\mu] \{\sigma\} . \quad (24)$$

If now the expression in Eq. (22) for incremental creep strains and that in Eq. (7) for incremental elastic strains are substituted into Eq. (6) and the resulting equation is solved for incremental stresses, one obtains an equation analogous to Eq. (14)

$$\{d\sigma\} = [D]_{ec} \left(\{d\epsilon\} - \frac{d\epsilon_{ef}}{\sigma_{ef}} [\mu] \{\sigma\} \right) . \quad (25)$$

Again $d\epsilon_{ef}$ is given by Eq. (11) and the symmetric matrix $[D]_{ec}$, henceforth called the elastic-creep matrix, is given by

$$[D]_{ec} = \left([D]^{-1} + \frac{d\epsilon_{ef}}{2\sigma_{ef}} [C] \right)^{-1} . \quad (26)$$

Formation of the element stiffness matrix follows as before [Eqs. (15) and (16)], and assembling gives a system of linear equations to be solved for incremental displacements over any time increment

$$[K]_c \{d\delta\} = \{dF\}_c . \quad (27)$$

The master stiffness matrix $[K]_c$ now depends on the state of stress in each element and must be reformulated for each time increment.

B. Incompressibility Considerations

We rewrite the elastic-creep matrix in the form

$$[D]_{ec} = [D^*]^{-1} = ([D]^{-1} + \lambda_1 [\mu] + \lambda_2 \{v\} \{v\}^T)^{-1} \quad (28)$$

where λ_1 and λ_2 are now proportional to dt . The elastic constitutive matrix $[D]$ is directly invertible for all materials where Poisson's ratio is not equal to 0.5 and the resulting matrix is also non-singular. On the other hand, both $[\mu]$ and $\{v\} \{v\}^T$ are singular matrices; in fact, it can be quickly shown that the vector $\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix}$ is a null space solution for both $[\mu]$ and $\{v\} \{v\}^T$ and that this solution represents an incompressible flow. Thus, as steady-state creep is approached through the use of larger and larger dt , the matrix to be inverted in Eq. (28) becomes more nearly singular (as it will be dominated by $[\mu]$ and $\{v\} \{v\}^T$) and the solution of the assembled equations (27) will become more and more numerically sensitive.

Various attempts have been made^{7,11} to circumvent this problem of numerical sensitivity in nearly incompressible elastic problems. In general, the methods introduce a new stresslike variable (e.g., pressure) and employ a variational principle (such as the Hellinger-Reissner principle) which incorporates the new variable and whose Euler equations are the field equations of the problem.

We will describe briefly the alterations to the stiffness matrix that we have incorporated for creep analysis. SAAS^{1,2} is the computer code on which our method is based. It uses a quadrilateral composed of the four triangular elements (in which the shape functions are linear in r and z) shown in Fig. 2. A common pressure variable is taken for all four triangles. The matrix $[D^*]$ is formed and then reduced according to the analysis of Ref. 7 for cylindrically orthotropic materials (no shear corrections are made). In this manner, a new 5-by-5 constitutive matrix is generated. Formation of the subelement stiffness matrix is carried out, followed by elimination of the central node and the common pressure term for the four subtriangles

Fig. 2. Quadrilateral finite element.

to give the resultant quadrilateral stiffness matrix. Assembly of the element stiffness matrices is then carried out as usual; the resultant set of equations numbers only $2M$, where M is the number of nodes. However, this system of equations was not nearly so sensitive to incompressibility effects as the usual displacement formulation.

C. Steady-State Creep

If elastic effects are neglected, we can write a finite element description of steady creep flow incorporating the dependent field variables, stress, strain-rate, and velocity. For Eq. (3) we have an analogous (linear) expression for strain rates in terms of nodal velocities

$$\{\dot{\epsilon}\} = [B] \{\dot{\delta}\} \quad (29)$$

Equilibrium, Eq. (5), applies as before, while the constitutive relationship becomes

$$\{\dot{\epsilon}\} = \frac{\dot{\epsilon}_{ef}}{\sigma_{ef}} [\mu] \{\sigma\} \quad (30)$$

where σ_{ef} and $\dot{\epsilon}_{ef}$ are given by Eqs. (10) and (11), respectively. If the stress state at steady-state

creep is known, then the system of equations (5), (29), and (30) is not unlike a linear elastic problem although the constitutive matrix $[\mu]$ is now singular. This suggests a further numerical calculation for predicting steady creep. At the terminus of the transient creep calculation, the creep stresses are used to predict element constitutive matrices from Eqs. (10) and (11). Using the nearly incompressible matrix $[\mu]^*$

$$[\mu]^* = \begin{bmatrix} 1 & -\frac{1}{2} + \gamma & -\frac{1}{2} + \gamma & 0 \\ -\frac{1}{2} + \gamma & 1 & -\frac{1}{2} + \gamma & 0 \\ -\frac{1}{2} + \gamma & -\frac{1}{2} + \gamma & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

where γ is small ($\approx 10^{-6}$), elastic solution is then carried out using the nearly incompressible formulation described previously for a new set of velocities, strain-rates, and creep stresses. Iteration can then be performed with the new creep stresses until convergence is obtained.

IV. FINITE ELEMENT PROGRAM CSAAS

A. Description of the Computer Program CSAAS

The processes of numerical analysis for transient and steady creep described in Section III have been incorporated into the two-dimensional axisymmetric stress and thermal analysis program TSAAS. The TSAAS finite element code is an extensively modified version of the widely known SAAS code¹² and uses the quadrilateral element of Fig. 2, together with the linear triangle. The Los Alamos code is compiled for CDC-7600s and has the capability to handle 1600 nodes and elements; in addition, a mesh generator and plotting and editing package have been written. A complete description of the TSAAS code is given in Ref 13.

Because of the existence of the TSAAS code with its elaborate pre- and post-processors, it was decided to modify this code for transient and steady creep rather than to develop an entirely new creep code which possibly could have higher order

isoparametric elements.⁵ The flow chart, Fig. 3, illustrates the workings of the stress analysis portion of this modified code, called CSAAS (for Creep Stress Analysis of Axisymmetric Solids). The main features that differ from a conventional transient, finite element, initial strain creep code are:

- (1) Formation of an elastic-creep stiffness matrix for nearly incompressible elements.
- (2) An automatic time step adjustment feature based on the maximum and minimum changes in element effective stresses.
- (3) A steady-state creep solution, by iteration, at the terminus of the transient creep calculations using the values of the element effective stresses as the variables to be iterated on.

Although the finite element analysis in Section III employed a creep law of the Weertman form (see Eq. (1)) and assumed incompressible creep flow, in the CSAAS code we have, in addition, allowed for a general time-hardening power law of the form

$$\dot{\epsilon}_{ef} = \sum_{i=1}^M A_i T^{p_i} \sigma_{ef}^{q_i} t^{r_i} \quad (31)$$

where A_i , p_i , q_i , and r_i , ($i = 1, 2, \dots, M$) are a set of material constants. Compressible creeping flow can be calculated with CSAAS by inputting an arbitrary value of Poisson's ratio for creep.

B. Application of CSAAS Code to One-Dimensional Test Problems

As mentioned previously, the equations for steady creep flow (Eqs. (5), (29), and (30)) are highly nonlinear because of the presence of σ_{ef}^N ($N > 1$) in the constitutive equation; consequently analytical solutions to anything but one-dimensional problems should not be expected. Analytical solutions for the two problems of radially symmetric deformation of long cylindrical tubes and hollow spheres can be compared with the results of a finite element analysis.

Fig. 3. Flow chart of CSAAS.

Figure 4 shows the finite element mesh for a typical section of a thick-walled tube under the condition of plane strain and loaded by a uniform internal pressure of 365 Pa. The material of the tube is assumed to have an elastic modulus of 20 MPa with a Poisson's ratio of 0.3. The creep behavior was assumed to be described by

$$\dot{\epsilon}_{ef} = A \sigma_{ef}^N$$

where for this calculation $A = 6.4 \times 10^{-19} \text{ s}^{-1}$ and $N = 4.4$. In this case, the effective stress is simply the absolute value of the difference of σ_r and σ_θ . The starting value of dt was 0.01 s.

Figure 4 also illustrates the variation of the effective stress at the inner and outer elements as a function of time, starting with the values

computed from the initial elastic solution. Steady-state creep is attained, approximately, for $t > 3$. The numerical method used in CSAAS is stable over very long times, in contrast to that used in the usual initial strain procedure where instability occurs as steady state is approached. Table I illustrates this numerical stability feature in more detail. The analytical value of the steady-state radial velocity at the inner radius of the tube was

computed from the example given in Chapter 7 of Ref. 1 as 0.4777×10^{-5} m/s. Considering the high degree of nonlinearity of this problem, we regard this to be excellent agreement between our computed value and the analytical value.

The mesh for the second test problem, a thick-walled halite sphere, is shown in Fig. 5. The sphere is assumed to be under an internal pressure of 4 MPa. Material properties used in all

Fig. 4. Tube mesh and variation of effective stress with time for a thick-walled tube.

TABLE I
LONG TIME VARIATION OF SELECTED STRESS
AND VELOCITY VALUES FOR TUBE TEST PROBLEM

Time (s)	Value of dt Used (s)	σ_{ef} at Inner- most Element (Pa)	σ_{ef} at Outer- most Element (Pa)	Radial Velocity at $r = 0.16$ (m/s)
3.03	1.28	778.7	553.8	0.4819×10^{-5}
82.4	40.9	773.6	642.6	0.4753×10^{-5}
2620	1310	773.1	642.1	0.4739×10^{-5}
21000	10500	772.7	642.9	0.4730×10^{-5}
Equilibrium		773	642	0.4741×10^{-5}

Fig. 5. Sphere mesh and variation of effective stress with time for a thick-walled halite sphere.

calculations to follow are given in Table II. The creep behavior was taken to be of the form of Eq. (1) in terms of effective stress and strain-rate; the creep constants are those of Heard.³ We emphasize again that the elastic properties are used primarily to generate the initial elastic stress state. We treated this problem as one of steady-state thermal creep, with the temperature field being that of the steady state with 500°C on the inner surface and 100°C on the outer surface of the sphere. The variation of effective stress at the inner and outer elements with time, starting with the elastic values at $t = 0$, is shown in Fig. 5. Note the complete turnabout of the inner and outer

TABLE II

MATERIAL PROPERTIES OF HALITE USED IN FINITE ELEMENT CREEP CALCULATIONS

Density	2243 kg/m ³
Modulus	10 ⁶ MPa
Poisson's ratio	0.45-0.495
A, Eq. (1)	3 x 10 ⁻⁶ s ⁻¹
E/R, Eq. (1)	12 500 K
N, Eq. (1)	5.5

effective stresses from the initial elastic solution to the steady-state creep solution.

The steady-state creep of the inflated sphere can also be calculated analytically. The dependent variables of the problem are the radial and hoop stresses σ_r and σ_θ and the radial velocity u . In spherical polar coordinates (r, θ, ϕ), we have the equilibrium and strain-rate velocity relations of the form

$$\frac{\partial \sigma_r}{\partial r} + \frac{2}{r}(\sigma_r - \sigma_\theta) = 0, \quad \dot{\epsilon}_r = \frac{\partial u}{\partial r}, \quad \dot{\epsilon}_\theta = \dot{\epsilon}_\phi = \frac{u}{r}.$$

Incompressibility ($\dot{\epsilon}_r + \dot{\epsilon}_\theta + \dot{\epsilon}_\phi = 0$) then determines the velocity field to be of the form $u(r) = C/r^2$, where C is a constant. From $u(r)$ the strain-rates can be determined.

The constitutive relations for the problem of spherical symmetry from Eq. (9) are

$$\dot{\epsilon}_r = \frac{3}{2} \frac{\dot{\epsilon}_{ef}}{\sigma_{ef}} s_r, \quad \dot{\epsilon}_\theta = \frac{3}{2} \frac{\dot{\epsilon}_{ef}}{\sigma_{ef}} s_\theta.$$

It is easily shown from Eqs. (9) through (11) and $\sigma_\theta = \sigma_\phi$ that $s_r = \frac{2}{3}(\sigma_r - \sigma_\theta) = -2s_\theta$ and that $\sigma_{ef} = \sigma_\theta - \sigma_r$ if we anticipate that $\sigma_\theta > \sigma_r$ throughout the sphere. With an effective strain-rate of the form of Eq. (1), we then obtain

$$\dot{\epsilon}_r = -A(\sigma_\theta - \sigma_r)^N \exp[-E/RT(r)].$$

Substituting $\dot{\epsilon}_r = -2C/r^3$ into the above equation one can solve for the effective stress $\sigma_\theta - \sigma_r$. This can be put into the equilibrium equation which, followed by integration and use of the boundary conditions $\sigma_r(a) = -p$ and $\sigma_r(b) = 0$, gives an expression from which C can be determined by a single quadrature

$$p = 2 \frac{(2C)^{\frac{1}{N}}}{A} \int_a^b \frac{\exp[E/NRT(r)]}{r^{(1+3/N)}} dr.$$

Knowing C , the velocity of the inner radius of the sphere can be determined.

Using the halite creep properties of Table II and $a = 0.2$ and $b = 0.5$, the above integral was evaluated for the steady temperature field

$$T(r) = 500 - \frac{400b}{(b-a)}(1 - a/r) .$$

For an internal pressure of 4 MPa, the interior velocity was computed from the above to be 0.593×10^{-7} m/s; the value of the steady-state interior velocity computed by means of the finite element method was 0.583×10^{-7} m/s after five steady-state iterations. Figure 6 illustrates the analytical and finite element velocity and the effective stress distribution through the sphere for the steady-state creep situation. Agreement between calculated and exact values is very close, considering the use of a coarse mesh of only seven elements through the thickness and the wide variation between the inner and outer surface temperatures.

Fig. 6. Comparison of analytical and CSAAS effective stress and velocity for the sphere problem.

V. HALITE FLOW PROBLEMS AND CAVITY STABILITY

In this section, we present the results of the application of the previously described numerical method to the calculation of creep behavior of halite in massive salt bodies such as those described in Ref. 14. Sections A and B deal with the stability of spherical cavities under geothermal and "hot spot" temperature conditions. The latter thermal situation was calculated after 20-yr operation of a Pacer cavity.⁹ The final problem, discussed in Section C, is a model that has been put forth to explain the formation of the great rock-salt domes.¹⁵

A. Underground Cavities with Geothermal Temperature Fields

The great salt domes of the Gulf Coastal Plains are massive, vertical, nearly circular cylinders of salt with diameters as large as 4 or 5 km and even greater heights. The formation of these outstanding geological structures is attributed to the creep behavior of salt--particularly at elevated effective stress or temperature--as discussed in Section C. Certain of these salt domes have cavities now being used for storage of natural gas, and one has been used to contain the explosion of a nuclear device.¹⁶ The usage of cavities in these geological formations requires the prediction of their long-term stability as far as creep flow is concerned.

Figure 7 illustrates one of the finite element meshes used to investigate long-term cavity stability in a salt dome. The salt dome itself was modeled as an equicylinder of 2-km radius; the central spherical cavity had a 200-m radius. A geothermal gradient of 27.5°C/km was used to calculate the temperature boundary condition on the cylindrical surface of the salt dome; the upper surface $z = 0$ (2 km) was assumed to have a temperature of 20°C and that of the cavity surface 75°C. Figure 8 illustrates the contours of constant temperature throughout the salt dome for the steady-state temperature field calculated with these boundary conditions.

The overburden loading at the equator of this cavity is equivalent to a pressure of 440 bars. On the premise that the creep stability of such a cavity is dependent on the presence of internal pressurization of the cavity, we carried out creep calculations with values of internal pressure of 0, 110, 220, 330, and 440 bars. The halite properties used in the calculations were those given in Table II; the mechanical boundary conditions on the salt dome are indicated in Fig. 7.

The results of these calculations are summarized in Table III where values of the steady-state velocity at selected points of the salt dome are listed. V_z and V_r are the axial and radial velocities, respectively; positive V_z is upward and positive V_r is radially outward with respect to Fig. 7. Figures 9 and 10 are vector plots for the velocity field at the steady state for the two cases of zero

and 440-bar internal pressurization, and Figs. 11 and 12 illustrate contours of the effective stress distributions.

The influence of the cavity depth on the steady-state creep rate was investigated using the finite element meshes shown in Figs. 13, 14, and 15 together with Fig. 7. Again the salt dome was taken to be a circular cylinder of 2-km radius with a spherical cavity of 200-m radius centered on the axis of the cylinder. The previously described geothermal gradient was used to set the temperature field in the salt dome, and the cavity was taken to be unpressurized. The mechanical boundary conditions were again taken as shown in Fig. 7.

Fig. 7. Salt-dome mesh for centered cavity.

Fig. 8. Geothermal temperature field.

TABLE III
TYPICAL STEADY-STATE VELOCITIES AT SELECTED
LOCATIONS IN THE GEOTHERMAL SALT-DOME MODEL

Cavity Pressure (bar)	V_z at Top of Cavity	V_r at Equator of Cavity	V_z at Base of Cavity	Surface Subsidence Velocity
0	-6.6 m/yr	-14 m/yr	30 m/yr	-0.6 m/yr
110	-1.2 m/yr	-3 m/yr	7 m/yr	-0.13 m/yr
220	-0.2 m/yr	-0.6 m/yr	1.5 m/yr	-0.024 m/yr
330	-2.4 mm/yr	-11 mm/yr	42 mm/yr	-0.6 mm/yr
440	0.05 mm/yr	-0.002 mm/yr	0.05 mm/yr	-0.00007 mm/yr

Fig. 9. Velocity plot at steady state (0 internal pressure).

Fig. 10. Velocity plot at steady state (440-bar cavity pressure).

Table IV summarizes the results of these calculations. For the creep calculations relating to the deep (2-and 4-km) cavities summarized in Table IV, the subsidence velocity was constant across the top surface of the salt dome. The material flow rate across the top surface of the salt dome agreed with the material flow rate across the cavity surface (based on the radial velocity at the cavity equator) to within one or two percent. This provided a

check on the incompressibility of the steady creep flow.

We have concluded from this study that the steady creep rate of cavities in salt domes at geothermal temperatures depends strongly on the difference between the overburden pressure and the internal pressure in the cavity. For instance, for unpressurized cavities the creep rate will depend strongly on the distance from the surface to the

Fig. 11. Effective stress contour plot (0 internal pressure).

Fig. 12. Effective stress contour plot (440-bar cavity pressure).

TABLE IV
STEADY-STATE VELOCITIES AT SELECTED LOCATIONS
IN UNPRESSURIZED GEOTHERMAL SALT CAVITIES

Cavity Depth (km)	Cavity Temperature (°C)	V_z at Top of Cavity	V_r at Equator of Cavity	V_z at Base of Cavity	Surface Subsidence Velocity
0.5	33.7	-0.09 mm/yr	-0.12 mm/yr	0.28 mm/yr	-0.06 mm/yr
1.0	47.5	-7 mm/yr	-20 mm/yr	58 mm/yr	-1.3 mm/yr
2.0	75	-6.6 m/yr	-14 m/yr	30 m/yr	-0.6 m/yr
4.0	130	-45 km/yr	-68 km/yr	120 km/yr	-2.8 km/yr

center of the cavity. Furthermore, for deep cavities in which the difference between the overburden and the internal pressure is small, the dominant deformation mode will be buoyancy of the cavity, and the buoyancy rate will be very small. Thus, an increase of internal pressure approaching the overburden pressure will tend to stabilize deep cavities in salt domes. When the overburden pressure exceeds substantially the internal cavity pressure, the cavity undergoes a squashing deformation similar to that depicted by Fig. 9. Such deformations have been observed for deep ellipsoidal cavities in salt domes.¹⁷

B. Underground Cavity with "Hot-Spot" Temperature Field

The proposal⁸ for power generation by thermonuclear explosives calls for a salt-dome cavity in which a fluid would be used to store the energy released from the detonation of a thermonuclear device; the energy would be continuously extracted from the cavity by using the hot fluid to drive a power generating station. The presence of the hot fluid in the cavity, together with the creep tendency of halite even at ambient temperatures, has raised questions about the creep stability of such cavities.

The radially symmetric temperature field around a spherical cavity of radius a in an infinite medium of ambient temperatures T_a is given by¹⁸

$$T(r) = T_a + \frac{a}{r}(T_0 - T_a) \operatorname{erfc} \left(2\sqrt{\frac{r-a}{Dt}} \right) \quad (32)$$

where T_0 is the constant temperature of the wall, D is the thermal diffusivity (assumed constant) of the medium, and t is time. erfc is the complementary error function. Equation (32) was evaluated for a cavity temperature of 500°C, an ambient salt temperature of 100°C, and an average diffusivity of $1.5 \times 10^{-7} \text{ m}^2/\text{s}$; the resulting temperature field at the end of 20 yr is shown as one of the curves in Fig. 16. Again the cavity has a 200-m radius.

In Fig. 17 the local polar coordinate finite element mesh of the region of the salt dome near

the cavity was used to calculate the creep rate of the heated halite under gravitational loading alone. The nodes were initialized to the radially symmetric 20-yr temperature field described previously. The outer nodes were constrained, simulating rigid salt behavior beyond the outer radius of the hollow sphere. The large number of nodes in this mesh were felt to be necessary because of the highly localized temperature field.

The results of this calculation are given in Fig. 16 which shows the variation with distance

Fig. 13. Salt-dome mesh for deep cavity.

into the halite of the vertical velocity and effective stress on the equatorial plane of the cavity. As can be seen in Fig. 16, the first 15 m of halite slump with an almost constant velocity of about 100 mm/yr; from 15 to about 35 m of depth, the vertical velocity gradient is constant, while beyond 35 m the halite is effectively rigid.

C. Salt-Dome Formation

Halite's great plasticity and tendency toward creep may well be responsible for the formation of the great salt domes. Rock salt, lying deposited in vast beds between brittle sedimentary layers of rock, possesses an unstable equilibrium with respect to defects in the overlying rock strata.¹⁶ Flow of the less dense salt into the defective rock would enhance the difference between the overburden pressure in the regions of defective and more competent rock, thereby tending to accentuate the

degree of instability. Of course, the effect would be greatly enhanced by the higher temperatures prevailing at the depths of the rock-salt layers.

Figure 18 illustrates a finite element mesh of a halite disc $0 \leq r \leq a$, $0 \leq z \leq h$ which was employed to study this phenomenon. We assumed that the disc was at a uniform temperature of 100°C. For mechanical boundary conditions we used:

$$\text{on } r = 0 : u = \tau_{rz} = 0$$

$$\text{on } z = 0 : w = \tau_{rz} = 0$$

$$\text{on } r = a : u = \tau_{rz} = 0$$

$$\text{on } z = h : \tau_{rz} = 0, \sigma_z = \begin{cases} 0, & 0 \leq r \leq c \\ -p, & c \leq r \leq a \end{cases}$$

Fig. 14. Salt-dome mesh for high cavity.

Fig. 15. Salt-dome mesh for surface cavity.

The presence of the free boundary in our model roughly simulates the differential overburden loading of a defective overlaying rock stratum. Halite mechanical properties (Table II), a pressure p equal to 10 bars, a disc radius of 5 km, and c equal to 1 km were used in the calculations; a number of runs were made with varying values of h .

A vector plot of the nodal velocities for a thick disc (1000 m) is shown in Fig. 19. The flow is primarily radial under the far field pressure loading, and at any given radius, the radial velocity is nearly constant and the vertical velocity is linear with z . Near $r = c$ the flow turns and becomes a nearly uniform vertical upward velocity on the disc ($0 \leq r \leq c, z = h$). This instantaneous velocity field, if applied over a sufficiently long time, would produce a vertical cylinder of deformed halite as conjectured in Ref. 15. Of course, a detailed calculation of the evolution of the salt dome would require consideration of the changing geometry of

Fig. 16. Twenty-year temperature profile and effective stress and vertical velocity at the equator of the "hot spot" cavity.

Fig. 17. The local cavity mesh.

Fig. 18. Geomechanical model for the origin of salt domes.

Fig. 19. Velocity field for the geomechanical model.

our model problem as time passes. The rise rates calculated for the three disc thicknesses are summarized in Table V.

TABLE V
RISE RATES CALCULATED FROM MODEL
PROBLEM FOR SALT-DOME FORMATION

Disc Thickness (m)	Rise Rate (mm/yr)
1000	2.7×10^{-3}
250	0.8×10^{-3}
100	0.3×10^{-3}

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