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**NUMERICAL SOLUTION OF
TRANSIENT AND STEADY-STATE NEUTRON
TRANSPORT PROBLEMS**

by

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ABSTRACT

This report describes briefly a general numerical procedure, called the discrete S_n method, for solving the neutron transport equation. The main topics relate to the derivation of suitable difference equations, and to the problem of solving these, while maintaining generality, accuracy, and reasonable computing speed. A few comparisons with other methods are made.

CONTENTS

	Page
Abstract	3
1. Introduction	7
2. The Transport Equation	9
3. The S_n Difference Equations	11
4. Accuracy Considerations	15
5. Generalizations	17
6. Numerical Procedures	18
7. Conclusions	22
References	30

TABLES

I. Transport Equations for Various Geometries	24
II. Transport Difference Equations	25
III. Table of Comparisons	27
IV. Table of μ_m , $\bar{\mu}_m$, and Related Quantities	28
V. Table of ϕ_{ml} , $\bar{\phi}_{ml}$, and Related Quantities	29

1. INTRODUCTION

During the past fifteen to twenty years a large number of methods have been developed pertaining to the solution of neutron diffusion and similar particle migration problems, mostly in connection with the design and operation of nuclear reactors. These methods, together with the assumptions and ideas on which they are based, form the bulk of what is called transport theory; the conservation law at the heart of this, an integro-differential equation, is known as the transport equation.

Most of the methods are approximate and require, except in the simplest cases, a large amount of numerical work. This is due to the nature of the transport equation and the variety and complexity of the applications, which also account for the large number of methods developed and the continuing work in this direction. The theory provides, however, many fundamental, purely analytical results, which are, for one thing, quite helpful in the development of approximations.

In the early investigations, emphasis was placed on expansions and other analytical techniques, to simplify and gain an understanding of the problems and to make the computational work as light as possible. Several of the methods developed then are still very important, such as ordinary diffusion theory, the spherical harmonics method, and the

variational techniques. These and other methods are described in considerable detail by Davison [1], and basic notions and results from transport theory are discussed by Case, de Hoffmann, and Placzek [2], as well as in [1].

During the last ten years most of the efforts have gone toward exploiting the existence of fast electronic computers. As a result several new methods have been developed, such as the S_n difference methods [3, 4] and the "Monte Carlo" statistical approach. In addition, several of the earlier methods, for example, those mentioned above, have been adapted to the new situation. Many of these more recent investigations are discussed by Richtmyer [5] and others were reported on at the Second International Conference on the Peaceful Uses of Atomic Energy (see for instance Bareiss [6] and Gelbard, et al [7]).

The methods which belong to transport theory include, besides the diffusion approximation and a variety of methods for particular problems, the transport theory methods proper, capable of yielding -- at least in theory -- solutions as precise as one may desire. Relatively few problems in practical reactor work require an accuracy beyond that attainable by ordinary diffusion calculations, and since these are rapid compared to other procedures, they are preferred. Outside this field and when particles other than neutrons are involved, the need for transport theory methods is much greater. Briefly stated, this need arises when the logarithmic difference ($\Delta N/N$) of the particle density is locally large over significant portions of the system under study, for it is in

such cases that diffusion theory results may be unreliable

There are several subheadings in transport theory depending mainly on how the time and energy variables are treated, and how the source term in the equation is defined. It is not possible, in this paper, to discuss in much detail more than one transport theory method, and this only under one of the many headings, the multigroup isotropic theory [1], pp. 255-261. This is, however, fairly typical since the basic considerations and numerical procedures do not vary a great deal from one method or situation to another. The selected method is a general one, recently completed and called the discrete S_n method, which represents a considerable simplification over the original S_n technique described in [3]. It also represents, as will become clear, an extension of the Wick-Chandrasekhar method of discrete ordinates [1], pp. 174-182, including the Yvon modification of that method (see [8]).

A typical S_n calculation involves many numerical processes. Some of these have a firm mathematical basis; the others rest - by design - on physical analogies, i.e., permit a more or less direct physical interpretation. Although this analogy approach has proved to be quite interesting and useful, a clear need exists here for further mathematical analysis.

2. THE TRANSPORT EQUATION

The transport equation is linear in the neutron flux N and of first order in the variables t , R , and Ω . More precisely, if N is the number

of neutrons flowing in the direction Ω at position R , per unit time and unit area, and if v is the neutron velocity (assumed constant), then the equation is given by:

$$(1) \quad (1/v)D_t N + \Omega \cdot \text{grad } N + \sigma N = S, \quad N = N(t, R, \Omega),$$

where S is the source term, and σ the collision probability per unit length. Equation (1) states that the total derivative of N in the direction Ω equals the rate (S) at which neutrons are introduced (in that direction), less the rate of removal (σN) by collision.

Here the quantities σ and S are not functions of Ω (the isotropic assumption); an equation like (1) is present for each neutron velocity group (the multigroup approach), so that v , σ , N , and S receive subscripts g , $g = 1, 2, \dots, G$. The equations are coupled through the source S , here defined as a linear combination of the average fluxes $\bar{N}_g$, plus an independent source Q_g :

$$(2) \quad S_g = \sum_{g'} \sigma_{g'g} \bar{N}_{g'} + Q_g \equiv \sigma_{gg} \bar{N}_g + \chi_g F + H_g + Q_g$$

where $\bar{N}_g$ is a simple average of N_g over Ω , and $\sigma_{g'g}$ the transfer probability ($g' \rightarrow g$) per unit length.

The first three terms on the right hand side of (2) represent a decomposition, convenient in the numerical work, of the sum on the left into: a) contributions ($\sigma_{gg} \bar{N}_g$) from "weak" scattering (the change in velocity is small, the neutrons are placed in the same group), b) contributions ($\chi_g F$) resulting from fission events, χ_g being the

spectrum of the fission neutrons, and c) contributions (H_g) from other groups, mainly by scattering.

If $D_t N \neq 0$ in (1) and the problem is well posed, we have a typical initial-value problem and may study transient or time-dependent phenomena. If $D_t N = 0$, we have a steady-state formulation, homogeneous or inhomogeneous depending on whether $Q_g = 0$ or not.

In the homogeneous case we divide N_g in (1) and the $\sigma_{gg} \bar{N}_g$ - term contained in S by a coefficient λ , and non-trivial solutions of (1), if any, exist only for special values of λ . One is normally concerned only with the largest positive λ (λ_0) and the corresponding positive N_g 's, though often not with λ_0 itself, but rather with some parameter p (if one exists), corresponding to $\lambda_0 = 1$, for instance the critical size of a reactor.

For further details regarding transport theory, the multigroup formulation, the physical assumptions involved here, and the problem of constructing appropriate cross sections ($\sigma_g, \sigma_{g'g}$), reference is made to [1] and to a recent paper by Wilkins [9].

3. THE S_n DIFFERENCE EQUATIONS

We assume, until Section 5, a steady state situation and a geometry with spherical symmetry. The transport equation becomes then:

$$(3) \quad \mu D_r N + (1/r)(1-\mu^2) D_\mu N + \sigma N = S, \quad N = N(r, \mu)$$

where r is the radial variable and μ the direction cosine with respect

to r .¹ We introduce a set of cells A_{im} formed by representing r and μ discretely, r by a set $\{r_i\}$, and μ by a set $\{\mu_m\}$, with $0 \leq r \leq a$, and $-1 \leq \mu \leq 1$. We choose for simplicity an even number of equal μ -intervals, $\mu_m = -1 + 2m/n$, $m = 0, 1, \dots, n$, and denote the interval midpoints by $\bar{\mu}_m$, $\bar{\mu}_m = -1 + (2m-1)/n$, $m = 1, 2, \dots, n$. These $\bar{\mu}_m$ are basically the discrete directions of the method and are given equal weights $(1/n)$, so that when $\bar{N}_{im}$, the average angular flux in A_{im} , has been determined, $\bar{N}_i$ will be given by $(1/n) \sum_m \bar{N}_{im}$.²

The intervals in r are also chosen in accordance with the resolution wanted, but otherwise quite arbitrarily, except that the actual boundaries specified in the problem are included in $\{r_i\}$.

Next, we let N_i denote the flux at r_i for a particular $\bar{\mu}$ and similarly N_m the flux at μ_m for a particular $\bar{r}$, where $\bar{r}_i = (r_i + r_{i-1})/2$. Further, we introduce the definitions: $\Delta_i = r_i - r_{i-1}$, $\Delta_m = \mu_m - \mu_{m-1}$, and $s_i = \Delta_i/r'_i$, where

$$(4) \quad r'_i = 2(r_i^3 - r_{i-1}^3)/3(r_i^2 - r_{i-1}^2)$$

1) To simplify the notation, a selection of the letters r, x, y, z , and θ (up to three) denotes the components of R , and i, j , and k the associated sub-scripts, as required. Similarly, we use ϕ and μ for the components of Ω , and l and m to indicate particular values. Subscripts not needed in the discussion are dropped freely.

2) The calculations are simpler and faster if intervals of equal weight are chosen. Other intervals may be used as long as the weights are given and the $\bar{\mu}$'s are constructed properly (see Section 4).

We denote the average of $1-\mu^2$ (to be defined in Section 4) by γ_m .

Finally, we assume the relations: $N_m = 2\bar{N}_{im} - N_{m-1}$ and

$$\bar{N}_{im} = (N_i + N_{i-1})/2.$$

We have thus postulated N to be linear over A_{im} in all the variables, a characteristic feature of the S_n approach. Equation (3) may now be averaged over A_{im} in a simple manner to form a difference equation, as follows (after multiplying through by Δ_i):

$$\begin{aligned} (5) \quad \bar{\mu}_m(N_i - N_{i-1}) + \gamma_m s_i (N_i + N_{i-1} - N_{m-1}) - \gamma_m s_i N_{m-1} + \sigma_i \Delta_i (N_i + N_{i-1})/2 = \\ = \Delta_i S_i \end{aligned}$$

where σ_i and S_i are midpoint values for σ and S (equal for all m) in A_{im} .

The difference equations for the discrete S_n method are now obtained from (5) by collecting terms:

$$\begin{aligned} (6) \quad (\bar{\mu}_m + \gamma_m s_i + \sigma_i \Delta_i/2)N_i + (-\bar{\mu}_m + \gamma_m s_i + \sigma_i \Delta_i/2)N_{i-1} = \\ = \Delta_i S_i + 2\gamma_m s_i N_{m-1} \end{aligned}$$

and to solve (6) means to evaluate N_i over i , given S_i and the boundary conditions, first doing this for $m = 0$ and then for other m ,

$m = 1, 2, \dots, n$. Note that we have $\bar{\mu} = -1$ and $\gamma = 0$ for $m = 0$, and can thus get started.

In order to evaluate N_i recursively over i , we solve (6) for N_{i-1} if $\bar{\mu}_m$ is negative and for N_i if $\bar{\mu}_m$ is positive. We note then that

the coefficient of N_i (of N_{i-1} in the second case) is less than or equal to unity in magnitude, since $\gamma_m s_i$ and $\sigma_i \Delta_i$ are inherently positive. With the additional comments that the right hand side of (6) is positive and that the proposed calculations inevitably involve round-off and other errors, it can be shown that it is necessary to solve (6) as described. The opposite procedure leads, in fact, to an exponential growth of the errors. Generally we have found, in all geometries examined, that the difference equations can be solved, usually with acceptable results, but the integration directions are prescribed for the reasons just discussed.

As was indicated above, equation (6) is not always satisfactory, this when $\sigma_i \Delta_i$ happens to be large, which often occurs in practice for some velocity groups, in some regions of the reactor. The calculated flux, as a function of r_i , may then oscillate between positive and negative values, which is physically unrealistic. To eliminate this difficulty, it will be necessary at times to use a more rudimentary difference equation, based on step functions rather than connected line segments.

If one now follows steps similar to those which led to equation (5) above, one obtains for the case of step functions and positive $\bar{\mu}_m$, the following alternates to (5) and (6):

$$(5') \quad \bar{\mu}_m(N_i - N_{i-1}) + \gamma_m s_i (N_i - N_{m-1}) + \sigma_i \Delta_i N_i = \Delta_i S_i$$

$$(6') \quad (\bar{\mu}_m + \gamma_m s_i + \sigma_i \Delta_i) N_i - \bar{\mu}_m N_{i-1} = \Delta_i S_i + \gamma_m s_i N_{m-1}$$

and for negative $\bar{\mu}_m$ the same equations, except that N_i and N_{i-1} are interchanged and the sign of $\bar{\mu}_m$ reversed.

4. ACCURACY CONSIDERATIONS

In the previous section we defined $\bar{\mu}_m$ as the midpoints of the chosen μ -intervals. If generality is maintained, we are clearly free to make modifications here, and finding that accuracy is thereby improved, we do this. In general, the modified $\bar{\mu}$'s are determined so that one or several moment conditions of the form $(1/n) \sum_m |\bar{\mu}_m|^s = 1/(s+1)$ are satisfied. Here we describe one such procedure, simple and fairly satisfactory, based on an argument stemming from diffusion theory (see [4]). In that procedure the $\bar{\mu}$'s are modified by a constant factor so that the second moment ($s = 2$) is correct. If the basic μ -intervals are of unequal weight, the moment conditions should of course reflect this. Values of modified $\bar{\mu}_m$ are given in Table IV for $n = 2, 4, 6$, and 8 .

To make further progress in the direction of higher accuracy we write (3), which is a conservation law, in the form of one:

$$(7) \quad \mu D_r N + (1/r) D_\mu (1-\mu^2) N + (2\mu/r) N + \sigma N = S$$

The important property here is that each term involving a derivative may be integrated (over the variable of differentiation) to yield a precise difference. Thus, for example, integration of the first term in (7) over (r_i, r_{i-1}) yields $\mu(N_i - N_{i-1})$.

It should be noted here that what we are after is an integral error

condition (the error in flow, integrated over the boundary of a cell A_{im} , to be zero). This type of condition, if one can be found, is in general more powerful than differential conditions which seldom establish more than that the difference equations are correct to some order in the differentials.

Equation (5), from which the S_n difference equation (6) was derived, clearly reflects the conservation properties of the transport equation (3), except that it remains to determine the coefficients γ_m . For this purpose we integrate (7) over μ from -1 to +1, denoting $(1/2) \cdot \int \mu N(r, \mu) d\mu$ by $J(r)$, obtaining:

$$(8) \quad D_r J + (2/r)J + \sigma \bar{N} = S, \quad \text{or in difference form}$$

$$(9) \quad J_i - J_{i-1} + s_i (J_i + J_{i-1}) + \sigma_i \Delta_i \bar{N}_i = \Delta_i S_i, \quad \text{where}$$

$$(10) \quad J_i + J_{i-1} = (1/n) \sum_m \bar{\mu}_m (N_i + N_{i-1}) = (1/n) \sum_m \bar{\mu}_m (N_m + N_{m-1})$$

Now for (6) to exhibit conservation over μ it is necessary that (6), when summed over m , $m = 1, 2, \dots, n$, and then divided by n , be equal to (9) above. This is readily shown for all terms except the ones involving s_i and γ_m . Here we examine two successive equations in (6) and the corresponding terms in (10), and collect all terms in N_{m-1} , and find as a result the following relation between the γ 's and $\bar{\mu}$'s:

$$(11) \quad \gamma_m - \gamma_{m-1} = -\bar{\mu}_m - \bar{\mu}_{m-1}, \quad \gamma_1 = -\bar{\mu}_1$$

Finally, it is desirable to make another correction in (6), which applies to spherical geometries only and relates to conservation. Clearly we can replace in (8) above, $D_r J + (2/r)J$ by $(1/r^2)D_r r^2 J$, which then changes $J_i - J_{i-1} + s_i(J_i + J_{i-1})$ in (9) to $(3\Delta_i / (r_i^3 - r_{i-1}^3)) \times (r_i^2 J_i - r_{i-1}^2 J_{i-1})$, using the averaging procedures now familiar. For accuracy near the origin these last two expressions should be identities, which can be achieved if $J_i - J_{i-1}$ in (9) is multiplied by $u_i = 3\Delta_i(r_i^2 + r_{i-1}^2) / 2(r_i^3 - r_{i-1}^3)$. This means then that the two $\bar{\mu}_m$'s in (6) should be multiplied by u_i .

If the alternate equation (6') is used in parts of a calculation, the neutron balance will not be complete, but the discrepancy may be used as a measure of whether a finer mesh spacing is called for or not.

5. GENERALIZATIONS

Time and space variables, no matter how many, are treated in very much the same way, and along the general lines described in the previous two sections, where the references to spherical symmetry were mainly for illustration. It is, therefore, quite simple to derive difference equations for other geometries, in transient as well as steady state situations, and to go beyond the framework of the multigroup isotropic theory if desired. It would be mainly repetition to give further details here except in tabular form. Transport equations for several geometries are listed in Table I and the corresponding difference equations are given in Table II.

It is, however, necessary to discuss briefly how the variable Ω is treated when two components, μ and ϕ , are present. Reference is here made to a unit sphere on which a (μ, ϕ) -point represents a direction. Here μ is the cosine of the angle measured from the pole and ϕ the longitude in radians. It is sufficient to consider a quadrant of the sphere, e.g., half of the upper hemisphere. To construct directions of equal weight, we divide the quadrant first into $n/2$ bands with areas proportional to $1, 2, \dots, n/2$, which define μ_m , $m = 0, 1, \dots, n/2$. Second, we section the bands along longitudes into $2, 4, \dots, n$ equal parts, respectively, and this defines $\phi_{m\ell}$, $\ell = 0, 1, \dots, 2m$. Each sector is then provided with a simple average $\bar{\phi}_m$ and each band with a $\bar{\mu}_m$, where $\bar{\mu}_m$ is adjusted (here with weights) as described in Section 4. The $\bar{\phi}_{m\ell}$ satisfy the relation $(1/2m) \sum_{\ell} \cos^2 \bar{\phi}_{m\ell} = 1/2$ for all m . Adjustment of $\bar{\phi}_{m\ell}$ has not been found useful nor necessary. Values of $\bar{\mu}_m$ and $\bar{\phi}_{m\ell}$ are given in Tables IV and V.

6. NUMERICAL PROCEDURES

The procedure for solving the S_n difference equations was discussed in Section 3. Here we outline the overall process, which is iterative. Each velocity group is treated separately, starting usually with the highest energy group, the slowing down of neutrons being the typical process. A pass through all the groups is called an outer iteration and repeated solutions for a particular group, inner iterations. The eigenvalue λ_0 is estimated after each outer iteration. The fission

density F , part of the source S defined by (2), is normally also evaluated at this time. Other parts of S are computed as needed from the latest figures available. After completing the calculations for a particular group one has, presumably, improved values for N_g (the angular flux), and can then calculate $\bar{N}_g$ (the average flux), divide by the latest value of λ_0 , and proceed to the next group.

The whole process is started from initial guesses for λ_0 and the flux distributions and terminated when some convergence criterion, usually on λ_0 , has been satisfied. The steps involving λ_0 are of course omitted for inhomogeneous and time-dependent problems. Other steps are added when a parameter p is to be determined. The details are too numerous to be given here. Reference is therefore made to the report on the DSN code, LAMS-2346*, which applies to geometries of one space dimension and to a variety of problem situations.

In steady-state problems, methods for accelerating the computational work are quite important, for the iterative process may not converge very rapidly. Some methods come to mind immediately such as starting the work with good guesses. These may come either from another problem, similar and previously solved, or from faster preliminary calculations, based on diffusion theory or a lower order S_n . Two other methods are now fairly well established in S_n calculations and others may be developed in the course of time.

The first method, which is due to G. Bell, of this Laboratory, saves repetitive work when an $\bar{N}_1$ -distribution has acquired about the

*In preparation.

correct shape, and may be off essentially by a factor C. The quantity C can be estimated from equation (9), if the terms in (9) are multiplied by C and the right hand side (divided by Δ_i) is written as follows:

$$(12) \quad \dots + C\sigma_i \bar{N}_i = S_i - \bar{\sigma}_i \bar{N}'_i + C\bar{\sigma}_i \bar{N}_i + (C-1)S_i + \bar{\sigma}_i \bar{N}_i - C\bar{\sigma}_i \bar{N}_i$$

where $\bar{\sigma}_i$ denotes σ_{ggi} and $\bar{N}'_i$ is the average flux from the previous cycle (outer iteration). If one examines (12), where the first three terms after the equality sign represent the desired form of S_i , one can at least argue that the last three terms should be zero in the mean, i.e., be zero if first weighted with volume elements V_i and then summed over i. These operations give the following formula for C:

$$(13) \quad C = (\sum S_i V_i - \sum \bar{\sigma}_i \bar{N}'_i V_i) / (\sum S_i V_i - \sum \bar{\sigma}_i \bar{N}_i V_i)$$

It should be noted, however, that the method should be used with care, for under certain conditions C may vanish.

The second method may be described this way: For a specified group, usually a high-velocity one, one solves the difference equations just once and computes then a distance between the previous flux $\bar{N}'_i$ (part of S_i) and the computed flux $\bar{N}_i$. For succeeding groups one or several inner iterations are then performed until the equivalent precision has been achieved in these groups. This saves repetitive work on neutron groups which converge more rapidly compared to others. Various definitions of distance and equivalent precision are possible

here.

Special methods have also been developed for smoothing out possible oscillations in the results, in particular in the direction of restraining the neutron flux to positive values. These are based on equation (6'), the so-called "exponential method," which is called on to repeat an integration step, should the regular equation give a negative flux. It should be pointed out that this is a very important technique in the discrete S_n method.

In steady state problems, the quantity λ_0 is corrected at the end of each cycle by the factor $\sum_{gi} V_i \bar{N}'_{gi} / \sum_{gi} V_i \bar{N}_{gi}$, where $\bar{N}'_{gi}$ and $\bar{N}_{gi}$ are the neutron distributions available at, respectively, the beginning and the end of the cycle.

A few results are given in Table III in the case of a uniform medium without reflector, representing one-group calculations to determine critical dimensions. These dimensions are measured in units of the mean free path between collisions, and depend on the parameter c only, c being the average number of neutrons emerging per collision. Results are given for three geometries and various S_n approximations (as detailed here) and comparisons are made with other specifications of the discrete method, the P_{n-1} (Wick-Chandrasekhar) method and the Double- $P_{\frac{n}{2}-1}$ (Yvon) method, n denoting the number of ordinates.

Generally speaking, diffusion theory is applicable when $c - 1$ is small. Here, however, it would be misleading to give results for the diffusion method, since this method is essentially constructed to give

good results in the cases represented by Table III [1] , pp. 94-108. More complete accuracy studies are planned for the future, in particular with regards to the diffusion approximation.

7. CONCLUSIONS

The discrete S_n method represents, from the point of view of the numerical analyst, a general solution of the transport equation. It includes as special cases the earlier methods which have been referred to, also based on discrete ordinates but of limited applicability. In practice the S_n method has proved both accurate and versatile, especially in the formulation given here, and has been tested extensively in geometries of one dimension and recently also in the case of finite cylinders (two space variables). With a better understanding of the processes involved, one can undoubtedly make further progress here and perhaps solve more complicated particle flow problems than those represented by the transport equation.

In developing numerical procedures we have not only kept accuracy and generality in mind but also various properties of modern electronic computers, e.g., the speed of their operations, and their memory and register capacities.

Of the many new features now part of the S_n method the following should be emphasized:

1. The representation of the angular variable is simple and general, and chosen to give good accuracy and to enhance the speed of calculation.

2. The "diamond" difference scheme is very effective here, since it involves few terms, is simple to generalize, and can readily be interpreted in terms of conservation.

3. With the aid of an alternate difference equation various difficulties are overcome, and physically meaningful results may be obtained even with very few intervals.

TABLE I. TRANSPORT EQUATIONS FOR VARIOUS GEOMETRIES

In time-dependent cases, add the term $(1/v)D_t N$ to the equations (on the left) and include the time variable.

1. Plane $(x;\mu)$ geometry:

$$\mu D_x N + \sigma N = S(x), \quad N = N(x, \mu), \quad -1 \leq \mu \leq 1$$

2. Spherical $(r;\mu)$ geometry:

$$\mu D_r N + (1/r)(1-\mu^2)D_\mu N + \sigma N = S(r), \quad N = N(r, \mu), \quad -1 \leq \mu \leq 1$$

3. Cylindrical $(r;\mu, \phi)$ geometry:

$$\sqrt{1-\mu^2} [\cos\phi \cdot D_r N - (1/r)\sin\phi \cdot D_\phi N] + \sigma N = S(r), \quad N = N(r, \mu, \phi)$$

$$0 \leq \mu \leq 1, \quad 0 \leq \phi \leq \pi$$

4. Finite cylindrical $(r, z; \mu, \phi)$ geometry:

$$\sqrt{1-\mu^2} [\cos\phi \cdot D_r N - (1/r)\sin\phi \cdot D_\phi N] + \mu D_z N + \sigma N = S(r, z), \quad N = N(r, z, \mu, \phi)$$

$$-1 \leq \mu \leq 1, \quad 0 \leq \phi \leq \pi$$

5. Cylindrical $(r, \theta; \mu, \phi)$ geometry:

$$\sqrt{1-\mu^2} [\cos\phi \cdot D_r N - (1/r)\sin\phi \cdot D_\phi N + (1/r)\sin\phi \cdot D_\theta N] + \sigma N = S(r, \theta)$$

$$N = N(r, \theta, \mu, \phi), \quad 0 \leq \mu \leq 1, \quad -\pi \leq \phi \leq \pi$$

6. Plane $(x, y; \mu, \phi)$ geometry:

$$\sqrt{1-\mu^2} (\cos\phi \cdot D_x N + \sin\phi \cdot D_y N) + \sigma N = S(x, y), \quad N = N(x, y, \mu, \phi)$$

$$0 \leq \mu \leq 1, \quad -\pi \leq \phi \leq \pi$$

7. Plane $(x, y, z; \mu, \phi)$ geometry:

$$\sqrt{1-\mu^2} (\cos\phi \cdot D_x N + \sin\phi \cdot D_y N) + \mu D_z N + \sigma N = S(x, y, z), \quad N = N(x, y, z, \mu, \phi)$$

$$-1 \leq \mu \leq 1, \quad -\pi \leq \phi \leq \pi$$

TABLE II. TRANSPORT DIFFERENCE EQUATIONS

Notation: σ_i denotes the average cross section and S_i the average source in the cell over which integration is performed. The differences Δ_i , Δ_j , and Δ_k refer to the space variables, in the order that these are given in Table I and below. The quantities $\bar{r}_i = (r_i + r_{i+1})/2$, $h_i = \sigma_i \Delta_i/2$, and $s_i = \Delta_i/\bar{r}_i$ are also referred to.

N_i denotes the angular flux at cell boundary i , at the midpoint of the cell surface defined by the remaining variables, with similar definitions for N_j , N_k and N_m . The average angular flux $\bar{N}_{im}$ in the cell is defined by $(N_i + N_{i-1})/2$, with these added assumptions:

$$N_j = 2\bar{N}_{im} - N_{j-1}, \quad N_k = 2\bar{N}_{im} - N_{k-1}, \quad \text{and} \quad N_m = 2\bar{N}_{im} - N_{m-1}$$

Further, the quantities γ_m are given by:

$$\gamma_m - \gamma_{m-1} = -\bar{\mu}_m - \bar{\mu}_{m-1}, \quad \gamma_1 = -\bar{\mu}_1 \quad (\text{spheres})$$

$$\gamma_{ml} - \gamma_{m, l-1} = -(\cos \bar{\phi}_{ml} + \cos \bar{\phi}_{m, l-1})/2, \quad \gamma_{m1} = -\cos \bar{\phi}_{m1}/2$$

(cylinders)

1. Plane ($x; \mu$) geometry:

$$(\bar{\mu}_m + h_i)N_i + (-\bar{\mu}_m + h_i)N_{i-1} = \Delta_i S_i, \quad m = 1, 2, \dots, n$$

2. Spherical ($r; \mu$) geometry:

$$(\bar{\mu}_m u_i + \gamma_m s_i + h_i)N_i + (-\bar{\mu}_m u_i + \gamma_m s_i + h_i)N_{i-1} - 2\gamma_m s_i N_{m-1} = \Delta_i S_i,$$

$$m = 0, 1, \dots, n$$

For this case, u_i and s_i are defined in the text.

3. Cylindrical $(r;\mu,\phi)$ geometry:

$$(\eta \cos \phi_{\ell} + \eta \gamma_{\ell} s_i + h_i) N_i + (-\eta \cos \phi_{\ell} + \eta \gamma_{\ell} s_i + h_i) N_{i-1} - 2 \eta \gamma_{\ell} s_i N_{\ell-1} = \Delta_i S_i,$$

$$\ell = 0, 1, \dots, 2m, \quad \eta_m = \sqrt{1 - \bar{\mu}_m^2}, \quad m = 1, 2, \dots, n/2.$$

For $\ell = 0$ $\bar{\phi} = -\pi$ and $\gamma = 0$.

4. Finite cylindrical $(r, z; \mu, \phi)$ geometry:

As for (3) above except subtract $2bN_{j-1}$ on the left hand side,

$b = \bar{\mu} \Delta_i / \Delta_j$, and add b also after h_i : Integrations as for (3)

but for the full range of m .

5. Cylindrical $(r, \theta; \mu, \phi)$ geometry:

As for (3) above except subtract $2bN_{j-1}$ on the left, $b = \eta s_i \sin \bar{\phi}_m / \Delta_j$,

and add b also after h_i . Integrations as for (3) but for both positive and negative ϕ -directions.

6. Plane $(x, y; \mu, \phi)$ geometry:

$$(\eta \cos \bar{\phi}_m + \eta \Delta_i \sin \bar{\phi}_{\ell} / \Delta_j + h_i) N_i + (-\eta \cos \bar{\phi}_m + \eta \Delta_i \sin \bar{\phi}_{\ell} / \Delta_j + h_i) N_{i-1} \\ - 2(\eta \Delta_i \sin \bar{\phi}_{\ell} / \Delta_j) N_{j-1} = \Delta_i S_i$$

Integrations as for (5) but with no $\ell = 0$

7. Plane $(x, y, z; \mu, \phi)$ geometry:

As for (6) above except subtract $2bN_{k-1}$ on the left, $b = \bar{\mu} \Delta_i / \Delta_k$,

and add b also after h_i . The integrations are performed as for (6)

but for the full range of m .

Basic number of integration directions: n for cases (1) and (2),

$n(n+2)/4$ for (3), $n(n+2)/2$ for (4), (5), and (6), and $n(n+2)$ for (7).

TABLE III. TABLE OF COMPARISONS

A. Critical half-thickness of uniform infinite slabs without reflectors, in units of the total mean free path, versus the number of neutrons (c) emerging per collision, comparing various S_n -approximations with exact values (see also [4]).

c	P_3	S_4	DP_1	P_7	S_8	DP_3	Exact
1.02	5.683	5.696	5.665	5.669	5.671	5.665	5.665
1.05	3.319	3.332	3.297	3.304	3.307	3.300	3.300
1.1	2.136	2.144	2.106	2.117	2.120	2.113	2.113
1.2	1.3191	1.3185	1.2770	1.2939	1.2957	1.2893	1.2893
1.4	.7773	.7655	.7229	.7427	.7422	.7367	.7366
1.6	.5590	.5426	.5027	.5202	.5173	.5119	.5120
1.8	.4387	.4210	.3848	.3992	.3943	.3884	.3887
2.0	.3619	.3443	.3116	.3234	.3170	.3102	.3108

B. Critical radii of uniform cylinders and spheres as above.

c	S_4	S_6	Exact	S_4	S_8	Exact
1.02	9.034	9.041	9.043	12.028	12.028	12.027
1.05	5.397	5.405	5.412	7.270	7.276	7.277
1.1	3.560	3.569	3.578	4.861	4.870	4.873
1.2	2.267	2.278	2.288	3.157	3.168	3.172
1.4	1.3829	1.3877	1.3973	1.9698	1.9817	1.9854
1.6	1.0103	1.0124	1.0209	1.4619	1.4727	1.4761
1.8	.7998	.8000	.8067	1.1704	1.1801	1.1833
2.0	.6633	.6622	.6673	.9790	.9877	.9906

Note the very good accuracy above of the Double-P method. More extensive tests do not, however, single out one method as more accurate overall than the others.

TABLE IV. TABLE OF μ_m , $\bar{\mu}_m$, AND RELATED QUANTITIES

n	m	Plane & Sphere		Other Geometries		
		μ_m	$\bar{\mu}_m$	μ_m	$\bar{\mu}_m$	$\sqrt{1-\bar{\mu}_m^2}$
2	0	-1.0000000	-	-1.0000000	-	-
	1	.0000000	-.5773503	.0000000	-.5773503	.8164966
4	0	-1.0000000	-	-1.0000000	-	-
	1	-.5000000	-.7745967	-.6666667	-.8703883	.4923659
	2	.0000000	-.2581989	.0000000	-.3481553	.9374369
6	0	-1.0000000	-	-1.0000000	-	-
	1	-.6666667	-.8451543	-.8333333	-.9363822	.3509820
	2	-.3333333	-.5070926	-.5000000	-.6810052	.7322786
	3	.0000000	-.1690309	.0000000	-.2553770	.9668416
8	0	-1.0000000	-	-1.0000000	-	-
	1	-.7500000	-.8819171	-.9000000	-.9621024	.2726884
	2	-.5000000	-.6299408	-.7000000	-.8101915	.5861653
	3	-.2500000	-.3778645	-.4000000	-.5570066	.8305081
	4	.0000000	-.1259882	.0000000	-.2025479	.9792724
16	0	-1.0000000	-	-1.0000000	-	-
	1	-.8750000	-.9393364	-.9722222	-.9895531	.1441689
	2	-.7500000	-.8140916	-.9166667	-.9477410	.3190407
	3	-.6250000	-.6888467	-.8333333	-.8780541	.4785614
	4	-.5000000	-.5636019	-.7222222	-.7804925	.6251651
	5	-.3750000	-.4383570	-.5833333	-.6550562	.7555802
	6	-.2500000	-.3131121	-.4166667	-.5017452	.8650155
	7	-.1250000	-.1878673	-.2222222	-.3205594	.9472284
	8	.0000000	-.0626224	.0000000	-.1114989	.9937646

Plane & Sphere: $\bar{\mu}_m = \frac{1}{2}(\mu_m + \mu_{m-1})n/\sqrt{n^2-1}$, $m = 1, 2, \dots, n$

Other Geometries: $\bar{\mu}_m = \frac{1}{2}(\mu_m + \mu_{m-1})\sqrt{n^2+2n}/\sqrt{n^2+2n-2}$, $m = 1, 2, \dots, n/2$

$\bar{\mu}_m = -\mu_{n-m+1}$, $m = (n/2) + 1, \dots, n$

TABLE V. TABLE OF ϕ_{ml} , $\bar{\phi}_{ml}$, AND RELATED QUANTITIES

$2m$	l	ϕ_{ml}	$\bar{\phi}_{ml}$	$\cos \bar{\phi}_{ml}$	$\sin \bar{\phi}_{ml}$
2	0	π	-	-	-
	1	$\pi/2$	$3\pi/4$	-.7071068	.7071068
4	0	π	-	-	-
	1	$3\pi/4$	$7\pi/8$	-.9238795	.3826834
	2	$2\pi/4$	$5\pi/8$	-.3826834	.9238795
6	0	π	-	-	-
	1	$5\pi/6$	$11\pi/12$	-.9659258	.2588190
	2	$4\pi/6$	$9\pi/12$	-.7071068	.7071068
	3	$\pi/2$	$7\pi/12$	-.2588190	.9659258
8	0	π	-	-	-
	1	$7\pi/8$	$15\pi/16$	-.9807853	.1950903
	2	$6\pi/8$	$13\pi/16$	-.8314696	.5555702
	3	$5\pi/8$	$11\pi/16$	-.5555702	.8314696
	4	$\pi/2$	$9\pi/16$	-.1950903	.9807853
16	0	π	-	-	-
	1	$15\pi/16$	$31\pi/32$	-.9951847	.0980171
	2	$14\pi/16$	$29\pi/32$	-.9569403	.2902847
	3	$13\pi/16$	$27\pi/32$	-.8819213	.4713967
	4	$12\pi/16$	$25\pi/32$	-.7730105	.6343933
	5	$11\pi/16$	$23\pi/32$	-.6343933	.7730105
	6	$10\pi/16$	$21\pi/32$	-.4713967	.8819213
	7	$9\pi/16$	$19\pi/32$	-.2902847	.9569403
	8	$\pi/2$	$17\pi/32$	-.0980171	.9951847

$$\phi_{ml} = \pi - l \frac{\pi}{2m}, \quad l = 0, 1, \dots, 2m$$

$$\bar{\phi}_{ml} = \pi - (2l - 1) \frac{\pi}{4m}, \quad l = 1, 2, \dots, 2m$$

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