MASTER

DEPARTMENT OF COMPUTER SCIENCES
THE TECHNOLOGICAL INSTITUTE
NORTHWESTERN UNIVERSITY
EVANSTON, ILLINOIS 60201

DISTRIBUTION OF THIS DOCUMENT UNLIMITED

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

Progress Report C00-2280-17 November 1974

THE INTERIOR CORNER PROBLEM
FOR THE NEUTRON DIFFUSION EQUATION

by

Erwin H. Bareiss and Steven R. Vickery

CS-75-3

ES-75-3

NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Energy Research and Development Administration, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

MASTER

Prepared For The U. S. Atomic Energy Commission
Under Contract No. AT(11-1)2280

DISTRIBUTION OF THIS DOCUMENT UNLIMITED

Abstract

The qualitative and quantitative behavior of the solution of the neutron diffusion equation at interior corners of two different materials is investigated. This is done by assuming regionwise constant coefficients and sources, reducing the problem to dimensionless form and solving the Neumann problem over a circular disk. The general analytic solution is given, as well as special solutions. A series of parameter studies have been made which include the extreme cases that can occur in neutron diffusion. The results including eigenvalues, eigenfunctions, Fourier coefficients and flux profiles are tabulated and presented in graphical form. It is shown how to select appropriate function spaces to obtain good numerical methods for coarse network calculations.

TABLE OF CONTENTS

	Page
Abstract	
1. Introduction	6
2. Formulation of Problem and Dimensional Analysis	10
3. An Eigenvalue Problem and Special Cases	15
4. The Analytic Solution of the Inhomogeneous Problem	22
Appendix A	39
5. Discussion of Figures and Tables	42
6. Conclusions	128
Appendix B	133
7. Operating Instructions for Computer Programs	151
Appendix C: Code Listings and Flowchart for BLKIT	158
8. References	182

List of Tables

	Page
I. Eigenvalue Table	60
II. Computed Source Table	65
III. Eigenvalue Table	70
IV. Eigenvalue Table	71
V. Eigenvalue Table	72
VI. Eigenvalue Table	73
VII. Eigenvalue Table	74
VIII. Eigenvalue Table	75
IX. Eigenvalue Table	76
X. Eigenvalue Table	77
XI. Parameter Study	80
XII. Parameter Study	84
XIII. Parameter Study	88
XIV. Parameter Study	92
XV. Parameter Study	95
XVI. Parameter Study	99
XVII. Parameter Study	103
XVIII. Parameter Study	107
XIX. Parameter Study	111
XX. Parameter Study	115
XXI. Parameter Study	119
XXII. Parameter Study	122
XXIII. Parameter Study	125
XXIV. Zeros of Dimensionless Error functions	149
XXV. Extreme Values of Dimensionless Error functions.	150

List of Figures

	Page
Figure 1 - The Model Problem	11
2 - Characteristic Equation for $\pi/4$	46
3 - Characteristic Equation for $3\pi/4$	47
4 - Characteristic Equation for $\pi/3$	48
5 - Characteristic Equation for $2\pi/3$	49
6 - Characteristic Equation for $\pi/6$	50
7 - Characteristic Equation for $5\pi/6$	51
8 - Characteristic Equation for $\pi/8$	52
9 - Characteristic Equation for $3\pi/8$	53
10 - Characteristic Equation for $5\pi/8$	54
11 - Eigenfunctions	56
12 - Eigenfunctions	58
13 - Profile of Neutron Flux	68
14 - Three Dimensional Parameter Cube	78
15 to 22 - Profiles of Neutron Flux	83
23 to 33 - Dimensionless Error functions	138
34 to 36 - Flow Charts for BLKIT	167

1. Introduction

This report is part of a larger project on the computational complexity in multidimensional neutron transport calculations. In recent years, a number of attempts have been made to implement higher order polynomial approximation methods for the numerical solution of both the neutron diffusion and the neutron transport equations. The general approach has been to find solutions contained in higher order Sobolev spaces and to justify the approach mathematically by demonstrating that in the limit, as the diameters of the finite elements or meshes converge to zero, some specific Sobolev norm of the error in the approximate solution converges to zero. The major benefit of such approximation methods was intended to reduce the overall size of the global problem by a significant factor by using large finite elements or coarse networks. The inconsistency in this reasoning has been widely recognized, since in nuclear reactor calculations we deal mostly with highly inhomogeneous domains. It is well known, that the solution of the diffusion and transport equations exhibit a "singular" behavior at corners and interfaces where different materials meet, as demonstrated by Abu-Shumays and Bareiss (1973), Babuska and Kellog (1973), Birkhoff (1972), Boffi and Molinari (1968), Strang and Fix (1973) and many more researchers quoted by these authors. Concerning the neutron diffusion equation, the behavior of the solution is quite similar to that of stress concentrations in structural mechanics. Strang and Fix (1973) remark that the neglect

of dealing properly with these singularities can lead numerically to catastrophic results. On the other hand, we know of no production code for neutron diffusion or transport calculations that takes these effects into account except perhaps by reducing the size of finite elements or meshes near corners and interfaces. For transport theory calculations, which mostly deal with detail calculations, the reason for this omission is obvious: The behavior at corners is so complicated that at this time no complete and rigorous mathematical formulation exists. For the diffusion equation, the usual argument is that the diffusion theory approximation is not valid at corners, so why bother.

In our opinion, this kind of reasoning is not correct. We distinguish between the approximation of the mathematical model for the description of the physical process (in this case the neutron diffusion equation) and the numerical approximation method to solve the equations for the mathematical model. Obviously, if the mathematical model permits only a prediction within 10 per cent, say, of the actual physical situation, there is no reason to seek a numerical solution that solves the model equation to, say, 10^{-6} per cent accuracy. If the numerical solution to the model is only of 10 per cent accuracy, then the cumulative error is of the order of 20 per cent which may or may not be sufficient for engineering applications. As a general rule however, we prefer to obtain numerical answers to the model equation that yields one extra decimal digit, i.e., in this example, the cumulative error would be about 11 per cent. This is usually quite acceptable.

Classical error analysis is "qualitative". It provides the important information for consistency and convergence, usually by stating that the error behaves like $O(h^\alpha)$ for $h \rightarrow 0$. This is not sufficient information for practical applications where the parameter h is usually large. We are interested in "quantitative" error analysis. This means we aim to provide formulas or numerical tables that permit reasonably accurate predictions (in numerical form) of the error in the computed solution for coarse networks of finite elements. We contend that this is possible by proper systematic analysis and systematic use of the computer at very low cost. The basic idea is as follows. First, we locate possible trouble spots. These are analyzed within a "computational cell". The best numerical methods are selected. Second, the cumulative errors of the global problem are determined. Third, the total efficiency is investigated, and the method with the best overall performance is selected. The success of the method depends clearly on the fact that many small scale calculations are performed, and that the number of cases to be studied can be drastically reduced by dimensional analysis. In addition, the entire process can be automated on a computer, Bareiss (1973), Derstine (1973).

In this report we study the significance of the possible singularities of the solution of the neutron diffusion equation at corners of the embedded wedge problem. We obtain the behavior of the solution using the traditional approach and show how "realistic" benchmark problems can be created. As expected, the solution

in the neighborhood of the vertex behaves like a fractional power of the distance from the vertex of the wedges. But as summarized in the section "Conclusions", the analytic behavior is significantly different from that of a single wedge imbedded in a vacuum, i.e., the corner problem.

2. Formulation of Problem and Dimensional Analysis

For fast and slow nuclear reactor core calculations, the greatest amount of computer time is spent to solve the two dimensional diffusion equation

$$-\nabla D \nabla \phi + \sigma_a \phi = S \quad (1)$$

where ϕ is the scalar flux, σ_a the absorption coefficient and S the external source within a given energy group. Typical core calculations are performed over a domain which contains some 500 to 1000 subregions of rectangular, hexagonal, circular, and odd sized shapes, and from one to 100 energy groups. Mostly each of these subregions are assigned regionwise constant material coefficients D and σ_a . Thus, there may be thousands of corner effects to be studied. A case, where the corner effect has its greatest effect occurs when a control rod is removed, and the void is filled with water. In order to simplify our problem, we draw a circle with center at the vertex of a cone which divides different regions, and consider only the regions of the disk bounded by the circle (Figure 1). Typical material constants for this problem are

$$\begin{array}{llll} D_1 = 1/6 & \sigma_{a,1} = .02 & \alpha = \pi/4 & \sigma_t = 2 \\ D_2 = 1/3 & \sigma_{a,2} = .10 & \pi - \alpha = 3\pi/4 & \sigma_t = 1 \end{array}$$

Figure 1. The model problem.

For comparison with transport theory calculations, note that

$D \doteq \lambda_t/3 = 1/3 \sigma_t$, where λ_t is the transport mean free path and σ_t the total cross section.

The diffusion length $L = \sqrt{D/\sigma_a}$ is in the respective regions D_1 and D_2

$$L_1 = (.12)^{-\frac{1}{2}} = 2.887 \quad L_2 = (.30)^{-\frac{1}{2}} = 1.826$$

In engineering practice, it is assumed that a point disturbance has no noticeable effect at a distance greater than $3L$. For later use we introduce the usual notation

$$\kappa^2 = \sigma_a/D = 1/L^2 \quad (2)$$

The mathematical problem is defined by Eq. (1), the radius R of the circle, the vertex angles 2α , and $2\alpha'$, where $\alpha' = \pi - \alpha$, the regionwise constants D_i , $\sigma_{ai} = \sigma_i$, S_i ($i = 1, 2$), the interface conditions

$$\phi^- = \phi^+ \quad (3)$$

$$D^- \frac{\partial \phi^-}{\partial n} = -D^+ \frac{\partial \phi^+}{\partial n} \quad (4)$$

along the interfaces in the normal direction n and the boundary condition

$$\frac{\partial \phi}{\partial n} = 0 \quad (5)$$

on the circle, where n points in the radial direction. Condition (5) has been chosen to simulate the connection with the remainder of the core. It does not introduce additional corner effects, and poses no essential restriction on the flux shape of ϕ . We consider this mathematical formulation a reasonably realistic definition of a benchmark problem. Another reasonable choice instead of (5) would be $\phi = 0$ on the circle of radius $2R$. We would expect quite similar results, but the numerical solution would require about $4^2 = 16$ times as much computer time.

To solve this problem we use polar coordinates (ρ, θ) . Obviously, D , σ_a , S are functions of θ only. The leakage operator in (1) is now

$$-\nabla D(\theta) \nabla = -\frac{1}{\rho} \frac{\partial}{\partial \rho} D(\theta) \rho \frac{\partial}{\partial \rho} - \frac{1}{\rho^2} \frac{\partial}{\partial \theta} D(\theta) \frac{\partial}{\partial \theta} \quad (6)$$

If we let $\rho = Rr$, $\phi(\rho, \theta) \equiv \bar{\phi}(Rr, \theta) \equiv \bar{\phi}(r, \theta)$ then

$$R^2(\nabla D \nabla)_{\rho} \phi = (\nabla D \nabla)_{r} \bar{\phi} \quad (7)$$

Since $D(\theta) \neq 0$ in our applications, we can write (1) and (5) as

$$-L_{\rho} \phi + \kappa^2(\theta) \phi = S(\theta)/D(\theta)$$

$$\frac{\partial \phi}{\partial \rho} = 0 \text{ for } \rho = R$$

where $-L_{\rho}$ is (6) divided by $D(\theta)$. Multiplying equation (1) by R^2 yields from (7)

$$-L_r \bar{\phi} + R^2 \kappa^2(\theta) \bar{\phi} = R^2 S(\theta)/D(\theta)$$

$$\frac{\partial \bar{\phi}}{\partial r} = 0 \text{ for } r = 1$$

Let further $\bar{\phi}(r, \theta) = u(r, \theta) + \frac{S_2}{D_2 \kappa_2^2}$, then these equations yield

$$-L_r u + R^2 \kappa^2(\theta) u = \begin{cases} R^2 \kappa_1^2 \left(\frac{S_1}{\sigma_1} - \frac{S_2}{\sigma_2} \right) & \text{for } |\theta| < \alpha \\ 0 & \text{for } \alpha < |\theta| < \pi \end{cases}$$

$$\frac{\partial u}{\partial r} = 0 \quad \text{for } r = 1$$

It is now obvious, that we have to consider only the following problem for $r < 1$

$$-\nabla^2 u_1 + c_1 u_1 = c_1 \quad (|\theta| < \alpha) \quad (8)$$

$$-\nabla^2 u_2 + c_2 u_2 = 0 \quad (\alpha < |\theta| < \pi) \quad (9)$$

$$u_1 = u_2 \quad (\theta = \alpha) \quad (10)$$

$$d \frac{\partial u_1}{\partial \theta} = \frac{\partial u_2}{\partial \theta} \quad (\theta = \alpha) \quad (11)$$

$$\frac{\partial u_1}{\partial r} = \frac{\partial u_2}{\partial r} = 0 \quad (r=1) \quad (12)$$

$$\frac{\partial u_1}{\partial \theta} = 0 \quad (\theta=0), \quad \frac{\partial u_2}{\partial \theta} = 0 \quad (\theta=\pi) \quad (13)$$

where

$$c_i = R^2 \kappa_i^2 = R^2 \sigma_i / D_i \geq 0 \quad (i=1,2) \quad (14)$$

$$d = D_1 / D_2 \geq 0 \quad (15)$$

The solution u depends only on the three parameters $c_1 \geq 0$, $c_2 \geq 0$, $d \geq 0$ and is bounded by unity and zero ($u(1,0) = 1 - \epsilon$, $u(1,\pi) = 0 + \epsilon$ for $\min(c_1, c_2) > 0$).

To solve the original problem, we determine c_1, c_2 and d by (14) and (15), respectively, and solve (8) to (13) for u . Then

$$\phi(\rho, \theta) = u\left(\frac{\rho}{R}, \theta\right) \left(\frac{S_1}{\sigma_{a1}} - \frac{S_2}{\sigma_{a2}} \right) + \frac{S_2}{\sigma_{a2}}$$

We note also that interchanging the regional indices $i = 1, 2$, has the same effect as replacing α ($\pi/2 > \alpha > 0$) by $\pi - \alpha$ and d by $1/d$ in (8) to (13). This property is useful in checking the correctness of the computer code.

3. An Eigenvalue Problem and Special Cases

We consider briefly the eigenvalue problem for (8) to (15) when $c_1 = c_2 = 0$. Then (8) and (9) are replaced by

$$-\nabla^2 u_i = -\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}\right) u_i = \mu^2 u_i \quad (i=1,2) \quad (16)$$

Obviously, (16) leads to a Sturm-Liouville problem. We try

$$\begin{aligned} u_1 &= v_1(r) \cos v\theta \cdot \cos v(\pi - \alpha) \\ u_2 &= v_2(r) \cos v(\pi - \theta) \cdot \cos v\alpha \end{aligned} \quad (17)$$

since (16) can be solved by separation of variables. Thus, (16) becomes after simplification,

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \mu^2 - \frac{v^2}{r^2}\right) v_i(r) = 0 \quad (i=1,2) \quad (18)$$

The general solution of (18) which remains finite at the origin is the Bessel function $J_v(\mu r)$. We note that if $v_1 = v_2$, (17) satisfies the continuity condition (10) for every μ and v at $\theta = \alpha$.

A restriction on v is placed by (11) which yields, after substitution of (17) into (11) and simplifying

$$d \sin(v\alpha) \cdot \cos v(\pi - \alpha) = -\sin v(\pi - \alpha) \cos(v\alpha) \quad (19)$$

$$-\frac{\tan v(\pi - \alpha)}{\tan v\alpha} = d \quad (\alpha \neq 0) \quad (20)$$

which is the characteristic equation for v .

We see from (19) that $v = 0$ is always a solution without regard to the value of d . Furthermore, (20) reveals, that v must always be real if $d > 0$. We can also consider d as a function of v and α . Figures 2 to 5 represent $d(v; \alpha)$ for $\alpha = \pi/4, 3\pi/4, \pi/3$ and $2\pi/3$. Obviously, $d(v; 3\pi/4) = 1/d(v; \pi/4)$, and $d(v; 2\pi/3) = 1/d(v; \pi/3)$. Figures 6 and 10 represent additional illuminating examples. From (19) and (20) we see that a solution v_0 of this equation induces other solutions v_k given by

$$v_k = v_0 + k \left[\text{l.c.m.} \left(\frac{\pi}{\alpha}, \frac{\pi}{\pi - \alpha} \right) \right]$$

(where l.c.m. stands for least common multiple), if α is a rational multiple of π , such as $\alpha = \frac{2\pi}{3}$. In this case, the calculations of the larger roots v_k in (20) are greatly simplified. On the other hand, since $v = 0$ is always a root of (19), extraneous partial eigenvalues v_k are introduced at the locations where $v = n \left[\text{l.c.m.} \left(\pi/\alpha, \pi/(\pi - \alpha) \right) \right]$. Another set of extraneous values results from the cosine terms in (19) which vanish when $v = (2\ell + 1)\pi/2\alpha = (2\ell + 1)\pi/2(\pi - \alpha)$. Having established that $v \geq 0$ always, the eigenvalues μ^2 are now determined by (12), i.e.,

$$d J_v(\mu r)/dr = \mu J'_v(x) = 0 \quad (r=1, x=\mu) \quad (21)$$

Hence, the eigenvalues μ^2 are the squares of the zeros of the derivatives of $J_v(x)$ and are positive. Hence the set of eigen functions of (16) is given by

$$u_{i,j} = J_{\nu_i}(\mu_{ij}r) \begin{cases} \cos \nu_i(\pi - \alpha) \cos \nu_i \theta & (0 \leq \theta \leq \alpha) \\ \cos \nu_i \alpha \cos \nu_i(\pi - \theta) & (\alpha \leq \theta \leq \pi) \end{cases} \quad (22)$$

where the ν_i 's are the solutions of (20) and the μ_{ij} the solutions of (21) when $\nu = \nu_i$.

Figures 11 and 12 illustrate the functions

$$\mathcal{U}_i(\theta) = \begin{cases} \cos \nu_i(\pi - \alpha) \cos \nu_i \theta & 0 \leq \theta \leq \alpha \\ \cos \nu_i \alpha \cos \nu_i(\pi - \theta) & \alpha \leq \theta \leq \pi \end{cases} \quad (23)$$

and the corresponding Bessel functions $J_{\nu_i}(x)$ for $\alpha = \pi/4$ and $\alpha = \pi/3$ for $d = \frac{1}{2}$. Note, that the μ_{ij} can be estimated from the relative maxima of the functions $J_{\nu_i}(x)$. We obtain the eigen function by a simple coordinate transformation such that $x = r\mu_{ij}$.

The actual calculation of the "partial" eigenvalues ν_i was done by solving the transcendental equation (19) by the secant method to 10 significant figures and checking the results by substituting ν_i into (20). The calculation of the μ_{ij} was done by use of McMahon's expansion for large zeros (Abramowitz and Stegun, (1964), p. 371, and Bickley et al. (1952), p. xxxvii).

$$\mu_{ij} = B - \sum_{r=1}^{\infty} \frac{A_{2r-1}}{(2r-1)! 2^3 r^{2r-1}} \quad (24)$$

where

$$B = \frac{1}{2}(2\nu_i + 4j - 3)\pi, \quad A_1 = u + 3$$

$$A_3 = 7u^2 + 82u - 9, \quad A_5 = 4(83u^3 + 2075u^2 - 3039u + 3537),$$

$$A_7 = 6(6949u^4 + 296492u^3 - 1248002u^2 + 7414380u - 5853627) \dots$$

$$\text{and } u = 4\nu_i^2$$

and subsequent iterative refinement by the secant method to 10 significant figures. The results were checked by calculating $J_{\nu_i}(\mu_{ij} + \epsilon)$ for $\epsilon = -2^{-20}$, $\epsilon = 0$, and $\epsilon = 2^{-20}$.

It is easily seen or verified that the functions $\mathcal{Y}_i(\theta)$ of (23) satisfy the orthogonality relations

$$\begin{aligned} \frac{1}{D_2} \int_0^\pi D(\theta) \mathcal{Y}_i \mathcal{Y}_j d\theta &= \\ &= d \cos \nu_i(\pi - \alpha) \cos \nu_j(\pi - \alpha) \int_0^\alpha \cos \nu_i \theta \cos \nu_j \theta d\theta \\ &\quad + \cos \nu_i \alpha \cos \nu_j \alpha \int_\alpha^\pi \cos \nu_i(\pi - \theta) \cos \nu_j(\pi - \theta) d\theta \\ &= \delta_{ij} N_i \end{aligned} \quad (25)$$

where for $\nu_i \neq 0$

$$N_i = d \frac{\cos^2 \nu_i(\pi - \alpha)}{2} \left(\alpha + \frac{\sin 2\nu_i \alpha}{2\nu_i} \right) + \frac{\cos^2 \nu_i \alpha}{2} \left(\pi - \alpha + \frac{\sin 2\nu_i(\pi - \alpha)}{2\nu_i} \right) \quad (26)$$

and for $\nu_i = 0$

$$N_0 = d\alpha + \pi - \alpha \quad (27)$$

Similarly, it is known that (18) yields the orthogonality relations (Luke 1962, p.255)

$$\int_0^1 r J_{\nu_i}(\mu_{ik} r) J_{\nu_i}(\mu_{il} r) = \delta_{k,l} M_{i,k} \quad (28)$$

where

$$M_{ik} = \frac{1}{2} \left[1 - (\nu_i / \mu_{ik})^2 \right] J_{\nu_i}^2(\mu_{ik}) \quad (29)$$

Unfortunately if $v_i \neq v_j$, there is no orthogonality relation of use to us. In searching the literature (Luke 1962, 1969; Watson 1966, p. 148; Bateman 1953, p. 10; Abramowitz and Stegun 1969, Jahnke-Emde 1960, McLachlan 1955) we found many interesting expressions but they were not too well suited for computation. We found that the following modification of the published ones served us best, also far from being satisfactory (Bateman 1953, p. 10)

$$\int_0^z r J_p(\alpha r) J_q(\beta r) dr = \quad (30)$$

$$\sum_{m=0}^{\infty} (-1)^m \frac{(\alpha/2)^{p+2m} (\beta/2)^{q+p+1+2m}}{p+q+1+2m} \sum_{n=0}^m \frac{(\beta/\alpha)^{2n}}{n! \Gamma(q+n+1) (m-n)! \Gamma(p+m-n+1)}$$

We note that in the eigenvalue problem given by (16) and (10) to (15), the eigenvalues $\mu_i^2 \geq 0$. Otherwise, the Bessel functions $J_\nu(\mu r)$ would have to be replaced by the modified Bessel functions $I_\nu(\mu r)$. These, however, are not admissible functions since (12) cannot be satisfied. Therefore, the special problem (8) to (14) where $c_1 = c_2 = c$ and (8) and (9) are replaced by

$$(-\nabla^2 + c) u_i = 0 \quad (i=1,2) \quad (31)$$

has necessarily only the unique trivial solution.

The corresponding inhomogeneous problem

$$\begin{aligned} (-\nabla^2 + c) u_1 &= c \\ (-\nabla^2 + c) u_2 &= 0 \end{aligned} \quad (32)$$

subject to conditions (10) to (14) has also a unique solution. It is given by

$$u = \sum_{i,j=0}^{\infty} a_{ij} J_{\nu_i}^{(2)}(\mu_{ij}r) \quad (33)$$

The Fourier coefficients a_{ij} are obtained by substituting (33) into (32). This yields

$$\sum_{i,j} a_{ij} (\mu_{ij}^2 + c) J_{\nu_i}^{(2)}(\mu_{ij}r) = \begin{cases} c & 0 \leq \theta < \alpha \\ 0 & \alpha < \theta \leq \pi \end{cases} \quad (34)$$

Using the orthogonality relations (25) to (29) we obtain a_{ij} ,

$$a_{ij} (\mu_{ij}^2 + c) N_{i,M_{ij}} = cd \begin{cases} \cos \nu_i \alpha \sin \nu_i \alpha & (\nu_i = 0) \\ \frac{\sin \nu_i \alpha}{\nu_i} & (\nu_i \neq 0) \end{cases} I_{ij} \quad (35)$$

where (Luke 1969, Vol. II, p. 54)

$$I_{ij} = \int_0^1 r J_{\nu_i}(\mu_{ij}r) dr = \frac{1}{\nu_i + 2} \sum_{n=0}^{\infty} \frac{J_{\nu_i+n}(\mu_{ij})}{2^n [(\nu_i + 4) / 2]_n} \quad (36)$$

$$[z]_n = z(z+1)\dots(z+n-1)$$

The problem indicated by (32) to (36) has been computed for $c=.24$ and $d=.50$. Table I lists the necessary parameters ν_i with the corresponding eigenvalues μ_{ij} and the coefficients a_{ij} . The function u of (33) has been computed at the points indicated in Figure 10, and the profiles have been plotted along $(\theta=0, \theta=\pi)$, $(\theta=\alpha$,

$\theta = \alpha + \pi$), and $r=1$. In Table II, the corresponding values are tabulated, and also the right side quantities obtained when the truncated calculated solution u of (33) is inserted in (32). The derivation from c and o is a measure for the local accuracy of the numerical solution.

It remains to consider the limit case when $c \rightarrow 0$. From (35) we conclude that $a_{ij} \rightarrow 0$ if $\mu_{ij} \neq 0$. However, the eigenvalue problem (16) leads to (18) which has as a solution $v = J_\nu(o \cdot r) = 0$ if $\nu \neq 0$. For $\nu = 0$, $J_0(o \cdot r) = \text{constant}$. Indeed $u = \text{constant}$ satisfies (16) and all interface and boundary conditions. Thus for $c=0$, u is not uniquely determined. It follows from (35) that for $\mu_{ij} = 0$ we obtain

$$\lim_{c \rightarrow 0} a_{00} = \alpha d / N_0 M_{00} = 2 / \left[1 + (\pi - \alpha) / \alpha d \right] \quad (37)$$

The other coefficients $\lim_{c \rightarrow 0} \alpha_{ij}$ are bounded, but irrelevant since $J_\nu(o \cdot r) = 0$ for $\nu_i > 0$. Thus

$$\lim_{c \rightarrow 0} \left[u(r, \theta) - a_{00} \right] = 0 \quad (38)$$

4. The Analytic Solution of the Inhomogeneous Problem

In this section we shall solve the problem posed by (8) to (15). However, we replace temporarily (8) and (9) by

$$-\nabla^2 u + c_i u = Q_i \quad (39)$$

where Q_i ($i=1,2$) is constant. The solution for the special case $c_1=c_2=c$ has already been solved and is for $Q_1 = \text{constant}$, $Q_2=0$ given by (23) and (33) to (36) after adjusting (32) to (39), i.e.,

$$u = \sum_{i,j=0}^{\infty} a_{ij} \mathcal{Y}_i(\theta) J_{\nu_i}(\mu_{ij} r) \quad (40)$$

$$a_{ij} = \frac{d_1 Q_1}{(\mu_{ij}^2 + c) N_{iM_{ij}}} \begin{cases} \alpha & (\nu_i=0) \\ \sin 2\nu_i \alpha I_{ij}/2\nu_i & (\nu_i \neq 0) \end{cases} \quad (41)$$

By considering (36), we see that for large $\nu_i \mu_{ij}$

$$|a_{ij}| \leq \frac{1}{\nu_i(\nu_i+2)(\mu_{ij}^2+c)} \quad (42)$$

Since $|\mathcal{Y}_i| \leq 1$, $|J_{\nu_i}(x)| \leq 1$, this estimate gives a good indication of the rate of convergence for the series expansion (40) of u .

To solve the problem (8) to (15), we can find the eigen functions of the operator. Since the operator is of the Sturm-Liouville type, these eigen functions form a complete orthonormal system. Hence the usual approach is as follows. Let $\mathcal{L}u = Q$ define the operator. Then the eigen functions satisfy $\mathcal{L}\omega_k = \lambda_k \omega_k$. Let $Q = \sum b_k \omega_k$, $u = \sum a_k \omega_k$ and obtain $a_k = b_k / \lambda_k$. Unfortunately, the eigen-

value problem for $\mathcal{L}$ cannot be solved by the method of separation of variables. This is shown as follows. Let

$$\begin{aligned} A_1(r) &= \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} & B_1(\theta) &= 1 \\ A_2(r) &= 1/r^2 & B_2(\theta) &= \frac{\partial}{\partial \theta^2} \quad (\theta \neq \alpha) \\ A_3(r) &= -1 & B_3(\theta) &= c_i \quad (i=1,2) \end{aligned}$$

Now, the left side of (39) becomes

$$-(-\nabla^2 + c_i) u = \sum_{m=1}^3 A_m(r) B_m(\theta) u \quad (43)$$

If either A_m or B_m have a set of common eigen vectors, separation is possible. This can occur only for $B_3(\theta) = c_i = c$, a constant, i.e., for the case we have already solved. (Note that the case $A_3(r) = c(r)$, $B_3(\theta) = 1$ is also separable).

Since we know that (40) represents the general solution to our problem if in (39) $c_i = 0$ and Q_i is replaced by $Q(r, \theta) \in \mathcal{L}_2$ (i.e., a square integrable function) or by $Q(r, \theta) \in \mathcal{L}_1$, we could seek the solution to our problem by solving

$$-\nabla^2 u = Q_i - c_i u \quad (i=1,2) \quad (44)$$

where necessarily $u \in \mathcal{L}_2$ (or $\mathcal{L}_1$). Thus, substituting (40) into (39) we have

$$\sum_{k,j=0}^{\infty} a_{kj} (\mu_{kj}^2 + c_i) J_k(\mu_{kj} r) J_k(\mu_{kj} r) = Q_i \quad (45)$$

We prove that (40) is not the general solution of (44), but that (40) and (45) lead to the appropriate behavior near the point $r=0$. Multiplying (44) by r^2 , we obtain from (43)

$$\left(r \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2}\right) u = r^2(Q_i - c_i u) = O(r^2) \quad (46)$$

Obviously, all eigen functions $u_{ij} = Y_i J_{\nu_i}(\mu_{ij} r)$ satisfy

$$\left(r \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2}\right) u_{ij} = r^2 \mu_{ij}^2 u_{ij} = O(r^2)$$

Thus, the u_{ij} are expected to yield at least an approximation of order one, and therefore the linear combination (40) should yield an approximation of order two. What is the actual analytic behavior?

Let

$$u = \sum_{l=0}^{\infty} Y_l(\theta) v_l(r) \quad (47)$$

Then, from (44),

$$r^2 \nabla^2 u = \sum_{l=0}^{\infty} \left(r \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \nu_l^2\right) v_l Y_l = r^2 \left(c_i \sum_{l=0}^{\infty} v_l Y_l - Q_i\right)$$

Using the orthogonality relation (25) yields an infinite system of ordinary differential equations, which is after dividing by N_k ,

$$\left(r \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \nu_k^2\right) v_k = r^2 \left(\sum_{l=0}^{\infty} p_{kl} v_l - q_k\right) \quad (k=0,1,2,\dots) \quad (48)$$

where $p_{k\ell}$ and q_k are defined by

$$N_{k\ell} p_{k\ell} = c_1 \int_0^\alpha \mathcal{U}_k \mathcal{U}_\ell d\theta + c_2 \int_\alpha^\pi \mathcal{U}_k \mathcal{U}_\ell d\theta \quad (49)$$

$$N_{k\ell} q_k = Q_1 \int_0^\alpha \mathcal{U}_k d\theta + Q_2 \int_\alpha^\pi \mathcal{U}_k d\theta \quad (50)$$

The general solution of (48) is known if we give a particular solution $w(r) = (w_0, w_1, w_2, \dots)^T$ for the inhomogeneous equation, and the homogeneous solution $y(r) = (y_0, y_1, y_2, \dots)^T$ which satisfies (48) for $q = (q_0, q_1, q_2, \dots)^T = 0$. For the particular solution w we try

$$w_\ell = \sum_{j=0}^{\infty} b_{\ell j} r^{2j} \quad (\ell=0,1,2,\dots) \quad (51)$$

and substitute (51) for v_ℓ in (48). To simplify our notation, we introduce the following vectors and matrices.

$$b_j^T = (b_{0j}, b_{1j}, b_{2j}, \dots) \quad (52)$$

$$w = \sum_{j=0}^{\infty} b_j r^{2j}$$

The diagonal matrix D_j has the elements ($k, \ell = 0, 1, 2, \dots$)

$$(D_j)_{k\ell} = [(2j)^2 - v_k^2] \delta_{k\ell} \quad (53)$$

while P is defined by the elements of (49)

$$P_{k\ell} = p_{k\ell} \quad (54)$$

Note that by (49) the matrix $NP = (n_k p_{kl})$ is symmetric, which property is useful for estimating bounds for b_j and for computational purposes.

Now, the substitution (51) transforms (48) into

$$\sum_{j=0}^{\infty} D_j b_j r^{2j} = \sum_{j=0}^{\infty} P b_j r^{2j+2} - q r^2 \quad (55)$$

Comparing equal powers in r determines the vectors b_j . We note that $v_0 = 0$, and hence D_0 is singular. Thus, it follows from (52) to (55) for the coefficients of $r^0 = 1$ in (55) that $D_0 b_0 = 0$. Hence, the components of b_0 are $b_{00} \neq 0$, but are arbitrary, while $b_{10} = b_{20} = \dots = 0$, i.e.,

$$b_0^T = b_{00} [1, 0, 0, 0, \dots] / \quad (56)$$

The vector b_1 is defined by comparing the coefficients of r^2 in (55), $D_1 b_1 = P b_0 - q$. In general,

$$b_1 = D_1^{-1} (P b_0 - q) \quad (57)$$

$$b_j = D_j^{-1} P b_{j-1} \quad (j > 1) \quad (58)$$

In particular, we have for given b_{00} , and $k = 0, 1, 2, \dots$

$$b_{k1} = (4 \cdot 1^2 - v_k^2)^{-1} (p_{k0} b_{00} - q_k)$$

$$b_{kj} = (4 \cdot j^2 - v_k^2)^{-1} \sum_{\ell=0}^{\infty} p_{k\ell} b_{\ell j-1} \quad (j > 1) \quad (59)$$

Hence the particular solutions (51) are $w^T = (w_0, w_1, \dots)$ where

$$w_o = \sum_{j=0}^{\infty} b_{oj} r^{2j}, w_{\ell} = \sum_{j=1}^{\infty} b_{\ell j} r^{2j} \quad (\ell > 0) \quad *) \quad (60)$$

The magnitude of w can be estimated from (52) and

$$b_j = D_j^{-1} P D_{j-1}^{-1} \dots P D_1^{-1} (P b_o - q) \quad (61)$$

by taking the appropriate norms.

For the homogeneous solutions we try

$$y_{\ell} = \sum_{j=0}^{\infty} a_{\ell j} r^{\alpha + 2j} \quad (62)$$

We let $q_k = 0$ in (48), and define the diagonal matrix D_j by its elements

$$(D_j)_{k\ell} = [(\alpha + 2j)^2 - v_k^2] \delta_{k\ell} \quad (63)$$

The matrix P is as before defined by $P_{k\ell} = p_{k\ell}$, while the vector a_j is defined by

$$a_j^T = [a_{oj}, a_{1j}, \dots, a_{\ell j}, \dots] \quad (64)$$

Now, instead of (55) we have after cancellation of the common factor r^{α} ,

$$\sum_{j=0}^{\infty} D_j a_j r^{2j} = \sum_{j=0}^{\infty} P a_j r^{2(j+1)} \quad (65)$$

*) The case when $2j = v_k$ is discussed in the appendix of this section.

This is an eigenvalue problem for α .

By comparing equal powers of r we obtain the relations

$$D_0 a_0 = 0$$

$$D_j a_j = P a_{j-1} \quad (j=1,2,\dots) \quad (66)$$

or

$$(\alpha^2 - v_k^2) a_{k0} = 0$$

$$\left[(\alpha + 2j)^2 - v_k^2 \right] a_{kj} = \sum_{\ell=0}^{\infty} p_{k\ell} a_{\ell, j-1} \quad (67)$$

$$(k = 0, 1, 2, \dots)$$

We see that we have a countable infinite set of homogeneous solutions given by $\alpha = \pm v_k$. For our problem we select $\alpha \geq 0$. Let $\alpha = v_k \geq 0$. Then by (67) a normalized solution is determined by the vector $a_o^{(k)}$ whose components are all zero, except the component $a_{k,0}^{(k)} \neq 0$; i.e., for $j = 0$, (64) has the form

$$a_o^{(k)} = [0, \dots, 0, 1, 0, \dots]^T \quad (68)$$

if $\alpha = v_k$. The other vectors are determined recursively by (66), and (63). For $\alpha = v_k$,

$$a_j^{(k)} = D_j^{-1} P a_{j-1}^{(k)} \quad (j > 0) \quad (69)$$

$$(D_i)_{kk} = 2j(2j - v_k) \quad (70)$$

$$(D_i)_{\ell\ell} = (v_k - 2j)^2 - v_\ell^2 \quad (\ell \neq k)$$

Convergence properties are obtained by taking norms of

$$a_j^{(k)} = D_j^{-1} P D_{j-1}^{-1} \dots P D_1^{-1} a_0^{(k)} \quad (71)$$

Thus, replacing y_ℓ of (62) by $y_\ell^{(k)}$, the homogeneous solutions are given by $y^{(k)} = (y_0^{(k)}, y_1^{(k)}, \dots)^T$ where

$$y_\ell^{(k)} = \sum_{j=1}^{\infty} a_{\ell j}^{(k)} r^{v_k + 2j} + \delta_{\ell k} r^{v_k} \quad (72)$$

Equation (68) has been used to reformulate (62). In (72) we have by (68) to (70),

$$a_{\ell j}^{(k)} = \left[(v_k - 2j)^2 - v_\ell^2 \right]^{-1} \sum_{\ell=0}^{\infty} p_{\ell m} a_{m, j-1}^{(k)} \quad (73)$$

$$\text{with} \quad a_{m,0} = \delta_{m,k} \quad (74)$$

The analytic solution to the Neumann problem is therefore solved after taking the following steps.

- (1) Use algorithm (61) or (56) to (59) to calculate the matrix

$$B = [b_0, b_1, \dots, b_j, \dots]$$

The particular solution is given by

$$w = B[1, r^2, \dots, r^{2j}, \dots]^T$$

Calculate the normal derivative at $r = 1$,

$$\partial w / \partial r = 2B[1, 2, \dots, j, \dots] \quad (r=1) \quad (75)$$

- (2) Use algorithm (71) or (68) to (70) to calculate for $k = 0, 1, 2, \dots$ the matrices

$$A^{(k)} = [a_0^{(k)}, a_1^{(k)}, \dots, a_j^{(k)}, \dots]^T$$

The homogeneous solutions are given by

$$y^{(k)} = r^{v_k} A^{(k)} [1, r^2, \dots, r^{2j}, \dots]^T \quad (76)$$

Calculate the normal derivatives at $r=1$

$$\partial y^{(k)} / \partial r = A^{(k)} [v_k, v_k + 2, \dots, v_k + 2j, \dots]^T \quad (r=1) \quad (77)$$

- (3) Solve for the γ_k in

$$\sum_{k=0}^{\infty} \frac{\partial y^{(k)}}{\partial r} \gamma_k + \frac{\partial w}{\partial r} = 0 \quad (r=1) \quad (78)$$

- (4) The solution to the problem is given by (47) if

$$v = \sum_{k=0}^{\infty} \gamma_k y^{(k)} + w \quad (79)$$

$$u = v^T \mathcal{U} \quad \mathcal{U} = (\mathcal{U}_0, \mathcal{U}_1, \dots)^T \quad (80)$$

Although the analytic solution of our problem given in the last paragraph gives considerable insight into the behavior of the solution to the embedded wedge problem, and although it is not hard to prove absolute convergence of the series involved, we had considerable doubts in our mind whether the actual convergence was fast enough for numerical calculations to obtain accurate solutions. We therefore decided to base our calculations on Bessel function expansions as indicated earlier in (45). In hindsight, we are not so sure that this is the best general approach since we encountered severe difficulties in convergence of the series involved.

As we noted above all series expansions for the homogeneous and the particular solutions have the general form $u = v^T \mathcal{U}$, where $\mathcal{U} = (\mathcal{U}_0, \mathcal{U}_1, \dots)^T$ is the vector of the partial eigen functions (23) and $v = (v_0, v_1, \dots)^T$ has components of the form

$$v_k = \sum_{j=0}^{\infty} r^{\nu_k} \sum_{j=0}^{\infty} a_{kj} r^{2j} \quad (81)$$

with ν_k^2 the eigen value parameter belonging to $\mathcal{U}_k$. Since the Bessel functions of the eigen value problem (18) have the expansion

$$J_{\nu}(\mu r) = (\mu r)^{\nu} \sum_{j=0}^{\infty} a_{\nu j} (\mu r)^{2j} \quad (82)$$

$$a_{\nu j} = (-1)^j \left[2^{\nu+2j} j! \Gamma(\nu+j) \right]^{-1}$$

it is obvious that v_k can be represented by a linear combination of Bessel functions. Because of (28),

$$r^{v_k+2j} = \sum_{i=0}^{\infty} b_{ijk} J_{v_k}(\mu_{ki}r) \quad (83)$$

$$b_{ijk} = M_{ki}^{-1} \int_0^1 r^{v_k+2j+1} J_{v_k}(\mu_{ki}r) dr$$

We can substitute (83) into (81), collect terms and obtain

$$v_k = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} c_{lki} J_{v_k}(\mu_{ki}r) \quad (84)$$

For $r \ll 1$, we expect the truncated series (84) to approximate (81) quite well.

Instead of deriving the coefficients c_{lki} in (84) from (81), we repeat the arguments of this section for the ansatz

$$u = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} c_{lki} J_{v_k}(\mu_{ki}r) \quad (85)$$

Here we have the advantage that library routines are available to calculate $J_v(x)$, and that it is relatively easy to determine the zeros μ_{ki} of $J'_{v_k}(x)$ by McMahon's expansion (24).

To solve (8) to (15) we note that

$$-\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} J_{\nu_k}(\mu_{ki} r) = (\mu_{ki}^2 - \frac{\nu_k^2}{r^2}) J_{\nu_k}(\mu_{ki} r) \quad (86)$$

$$-\frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \mathcal{U}_m(\theta) = \frac{\nu_m^2}{r^2} \mathcal{U}_m(\theta) \quad (87)$$

and that (10) to (13) are satisfied by (85). Using (25) to (27) and (85) to (87) transforms (8) and (9) into a system of linear equations,

$$\sum_{k=0}^{\infty} \sum_{i=0}^{\infty} c_{mki} (\mu_{ki}^2 - \frac{\nu_k^2 - \nu_m^2}{r^2}) J_{\nu_k}(\mu_{ki} r) \quad (88)$$

$$= q_m - \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} c_{ski} p_{ms} J_{\nu_k}(\mu_{ki} r)$$

$$(m = 0, 1, 2, \dots)$$

where p_{ms} is given by (49) and q_m by (50) when $Q_1 = c_1$, $Q_2 = 0$, i.e.,

$$q_m = c_1 d/N_m \int_0^\alpha \mathcal{U}_m d\theta \quad (89)$$

We note that $\mu_{k0} = 0$ implies $J_{\nu_k}(0) = 0$ except for $\nu_0 = 0$. Hence we

set $c_{lko} = 0$ for $k \neq 0$. Furthermore, we derive from (86) the relation

$$\int_0^1 [r(\mu_{ki}^2 - \mu_{pl}^2) - (\nu_k^2 - \nu_p^2)/r] J_{\nu_k}(\mu_{ki} r) J_{\nu_p}(\mu_{pl} r) dr = 0 \quad (90)$$

Multiplying (88) by $r J_{\nu_m}(\mu_{m\ell} r)$, integrating, denoting (see (30)) for $z = 1$)

$$I_{ki;m\ell} = \int_0^1 r J_{\nu_k}(\mu_{ki} r) J_{\nu_m}(\mu_{m\ell} r) dr \quad (91)$$

and using (90) yields

$$\begin{aligned} & \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} c_{mki} \mu_{m\ell}^2 I_{ki;m\ell} \\ &= q_m I_{00;m\ell} + \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} c_{ski} p_{ms} I_{ki;m\ell} \end{aligned} \quad (92)$$

$$(c_{sko} = 0 \text{ if } k \neq 0; \quad m, \ell = 0, 1, 2, \dots)$$

If we truncate (92) at $(i, k, \ell, m, s) = (I, K, L, M, M)$ we have a linear system of $(M+1)L+1$ equations for $(M+1)(K+1)L+1$ unknowns. If we multiply (88) by $r J_{\nu_p}(\mu_{p\ell} r)$ we obtain an additional set of equations, but we must now also evaluate integrals of the form $\int_0^1 r^{-1} J_p(\alpha r) J_q(\beta r) dr$. The solution of (92) in its augmented form represents the solution (85).

Thus we have proven our assertion stated following (45). The solution can be approximated by (40), but its accuracy is limited. As in (45) we obtain

$$\sum_{k,j=1}^{\infty} a_{kj} (\mu_{kj}^2 + c(\theta)) J_{\nu_k}(\mu_{kj} r) = Q(\theta) \quad (93)$$

Applying the orthogonality conditions (25) to (29) yields

$$(\mu_{m\ell}^2 + c_{m\ell}^{m\ell}) a_{m\ell} + \sum_{\substack{k,j=0 \\ k \neq m \\ j \neq \ell}}^{\infty} c_{kj}^{m\ell} a_{kj} = q_{m\ell} \quad (94)$$

where (see(49) and (50))

$$c_{kj}^{m\ell} = p_{km} \cdot B_{kj}^{m\ell}$$

$$B_{kj}^{m\ell} = M_{m\ell}^{-1} \int_0^1 r J_{\nu_k}(\mu_{kj} r) J_{\nu_m}(\mu_{m\ell} r) dr \quad (95)$$

$$q_{m\ell} = N_m^{-1} M_{m\ell}^{-1} q_m$$

$$m, \ell = 0, 1, 2, \dots$$

$$a_{ko} = a_{oo} \delta_{ko}$$

Except for $(k, j) = (m, \ell) = (0, 0)$, the coefficients of a_{ko} and a_{mo} are omitted since they correspond to $\mu_{k,0} = \mu_{m,0} = 0$.

An iterative approach to the solution of our problem (39) is as follows. Let us rewrite (39) as

$$Lu + c(\theta)u = Q(\theta) \quad (96)$$

and let $K_i (i = 0, 1, 2, \dots)$ be positive numbers. Then we solve the special problems (e.g. as in (40) to (41))

$$L u_0 + K_0 u_0 = Q$$

$$L u_1 + K_1 u_1 = (K_0 - c(\theta)) u_0$$

(97)

$$L u_k + K_k u_k = (K_{k-1} - c(\theta)) u_{k-1}$$

The solution of (96) is given by

$$u = \sum_{k=0}^{\infty} u_k \quad (98)$$

The differential operator belonging to L is $-\nabla^2$ in polar coordinates. Thus, substituting (40) in $(L + K_0)u_0$ yields

$$\sum_{k,j} a_{kj}^{(0)} (\mu_{kj}^2 + K_0) J_{\nu_k}(\mu_{kj} r) = Q(\theta) \quad (100)$$

Applying the moment method yields

$$a_{m\ell}^{(0)} = \frac{q_{m\ell}}{\mu_{m\ell}^2 + K_0} \quad (101)$$

For $(L + K_i)u_i = (K_{i-1} - c(\theta)) u_{i-1}$ ($i=1,2,\dots$) we obtain similarly

$$a_{m,\ell}^{(i)} = \left\{ K_{i-1} - \sum_{k,\ell=0}^{\infty} c_{kj}^{m\ell} a_{kj}^{(i-1)} \right\} (\mu_{m\ell}^2 + K_i)^{-1} \quad (102)$$

If we let $K_0 = K_1 = \dots = K$, we have asymptotically the best convergence if the spectral norm of the iteration operator is minimized with respect to K ,

$$\min_{(k)} \mathcal{P}[(L + K)^{-1}(K - c(\theta))] = \mathcal{P}_{\min}.$$

Because of the expense of evaluating (30) to obtain $c_{kj}^{m\ell}$ and the solution of (93) or (100), most of our calculations were done as follows. We took the moments of $\mathcal{U}_m(\theta)$ only. Then (93) becomes

$$\begin{aligned} & \sum_{j=0}^J a_{mj} (\mu_{mj}^2 + p_{mm}) J_{\nu_m}(\mu_{mj}r) \\ &= q_m - \sum_{\substack{k=0 \\ k \neq m}}^K \sum_{j=0}^J a_{kj} p_{mk} J_{\nu_k}(\mu_{kj}r) \\ & \quad (m = 0, 1, 2, \dots, M) \end{aligned} \tag{103}$$

where $J, K, M \rightarrow \infty$. For the numerical solution this equation is truncated, and discrete least square fits are made using the Householder method as explained below. To avoid input round-off, the coordinates are at

$$r_i = 2^{-P} i \quad (i = 1, 2, \dots, 2^P) \tag{104}$$

The solution is achieved by block iteration of the Gauss-Seidel type. In (103), the left side represents the m^{th} rectangular diagonal block. This block has $J(m)$ columns, except for $m = 0$, when $J = J(0) + 1$, since $a_{00} \neq 0$, ($a_{m0} = 0$ for $m \neq 0$). It has 2^p rows corresponding to (104). The iteration starts by letting $a_{kj}^{(0)} = 0$ on the right side of (103). Now $a_{mj}^{(1)}$ ($m = 0, 1, 2, \dots$) is obtained by applying the Householder method to the $(2^p \times J)$ blocks with immediate substitution on the right side. (No acceleration techniques are used, because experiments showed no increase in overall efficiency). The iteration is terminated when the change in the most significant coefficients, usually $a_{00}^{(i)}$ and $a_{10}^{(i)}$ become smaller than the requested tolerance.

We note that the system (103) is inconsistent for $r = 0$. We have

$$\sum_{j=0}^J a_{0j} p_{00} = q_0$$

$$\sum_{j=1}^J a_{0j} p_{m0} = q_m \quad (m = 1, 2, \dots)$$

which is a consequence of using (40) instead of (85) for our approximation space. They serve however their purpose, namely to generate reasonably realistic benchmark problems.

Appendix A

If $2j = v_k$ in (59), the algorithm (56) to (58) breaks down. Since for our applications, v_k is a transcendental number, it does not occur in our theoretical investigations. On the other hand we can specify a value d by (20) that yields a v_k arbitrarily close to $2j$. If $d = 1$, (not a realistic case) we have $v_k = 2j$ for certain α 's.

To complete our investigations we show how to modify algorithm (56) to (58) for $2j = v_k$. If this is the only occurrence of a singular matrix D_j , then (58) and (59) yield the condition

$$\sum_{\ell=0}^{\infty} p_{k\ell} b_{\ell j-1} = 0 \quad (A1)$$

Since b_{00} is an arbitrary parameter, (61) shows immediately how b_{00} must be modified to satisfy $(Pb_{j-1})_k = 0$.

A more general modification replaces (52) by

$$w = \sum_{j=0}^{\infty} b_j r^{2j} + \beta_j r^{2j} \ell_{nr} \quad (A2)$$

$\beta_j = [\beta_{0j}, \beta_{1j}, \dots]^T$ is a zero vector up to the index j in which D_j becomes singular. We note that

$$\left(r \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - v_k^2 \right) r^{2j} = \left[(2j)^2 - v_k^2 \right] r^{2j} \quad (A3)$$

$$\left(r \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - v_k^2 \right) r^{2j} \ell_{nr} = \left[(2j)^2 - v_k^2 \right] r^{2j} \ell_{nr} + 2(2j) r^{2j}$$

Now, let $\beta_{v_k/2} = [0, \dots, 0, \beta_{k, v_k/2}, 0, \dots]^T$. Then (56) to (58) are unchanged for $j < v_k/2$. For $j = v_k/2$,

$$(D_j)_{\ell\ell} = \begin{cases} (2j)^2 - v_\ell^2 & (\ell \neq k) \\ 0 & (\ell = k) \end{cases} \quad (A4)$$

because $2j = v_k$. From (48), (A2), (A3) follows

$$(D_j b_j + 4j \beta_j) r^j = P b_{j-1} r^{2j} \quad (A5)$$

Hence

$$\beta_{kj} = \sum_{s=0}^{\infty} p_{ks} b_{s, j-1} (2v_k)^{-1} \quad (j = v_k/2) \quad (A6)$$

$$b_{\ell j} = \sum_{s=0}^{\infty} p_{\ell s} b_{s, j+1} (v_k^2 - v_\ell^2)^{-1} \quad (\ell \neq k)$$

b_{kj} is a free parameter, say zero or one.

For $j = 1 + v_k/2$, we have

$$\begin{aligned} D_j b_j r^{2j} + \text{diag}(4j) \beta_j r^{2j} + D_j \beta_j r^{2j} \ell n r \\ = P(b_{j-1} r^{2j} + \beta_{j-1} r^{2j} \ell n r) \end{aligned} \quad (A7)$$

Thus

$$\beta_j = D_j^{-1} P \beta_{j-1} \quad (A8)$$

$$b_j = D_j^{-1} [P b_{j-1} - \text{diag}(4j) \beta_j]$$

This formula is valid for $j > v_k/2$, until the next occurrence where D_j becomes singular. We then either use the free parameter $b_{k, v_k/2}$ to satisfy the singularity condition, or introduce as in (A2) to (A6) a new vector $\beta_j^{(2)} r^{2j} \ell_n^2 r$ and note that

$$\begin{aligned} (r \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - v_k^2) r^{2j} \ell_n^2 r &= \\ &= (4j^2 - v_k^2) r^{2j} \ell_n^2 r + 8j r^{2j} \ell_{nr} + 2r^{2j} \end{aligned}$$

The general process is now clearly recognized. The implication is that for the cases treated here, logarithmic terms can appear, and the Bessel function approximation is not complete in all cases. We have

$$(r \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - v_k^2) r^{2j} (\ell_{nr})^m = [(4j^2 - v_k^2) \ell_n^m r + 4j m \ell_n^{m-1} r + m \ell_n^{m-2} r] r^{2j} \quad (A9)$$

from which follows by the arguments given above that the particular solution has, instead of (52) or (A2), the general vector form

$$w = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \beta_j^{(m)} r^{2j} (\ell_{nr})^m \quad (\beta_j^{(0)} = b_j) \quad (A10)$$

and the homogeneous solutions (62), (72) or (76) are in general the vectors

$$y^{(k)} = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \alpha_j^{(k,m)} r^{v_k + 2j} (\ell_{nr})^m \quad (A11)$$

5. Discussion of Figures and Tables

In observing the characteristic equation, we can easily notice the periodic property of the curves. The steepness of these curves imply that a small change in D causes only a slight change in the eigenvalue. We point out that the eigenvalues are bounded as D approaches either 0 or infinity. By observing the curves at these extremes, we can find the relative bounds of the eigenvalues. We note that at D equal 1, an eigenvalue is equal to 1 for all angles. The physical interpretation of this is that if there is not any difference in the diffusion lengths of the regions, the derivative of the angular eigenfunction is not discontinuous. The peaks in these plots and the relative maxima represent the extraneous eigenvalues that are deleted from computation. As mentioned in the text, the characteristic equation of α is the inverse of $\pi - \alpha$, which is easily seen in these examples.

In Figures 11 and 12, we notice that the first angular eigenfunction is a constant 1. In the following angular eigenfunctions, we can see the discontinuity at the interface. The radial eigenfunctions in the following figures illustrate the dampening effect of the Bessel functions. We can estimate the eigenvalues from the relative maxima of the Bessel function. As Nu increases in value, we can see that the Bessel function tends toward zero.

In Table I, we can see relative periodic behavior of both the radial and angular eigenfunction. It is interesting to note that for larger Nu 's, McMahon's series gives a smaller error while for smaller Nu 's, iterative improvement had to be used.

The error of the truncated expansion becomes apparent as we notice the difference of the expansion coefficients calculated by two different methods. The computed source (Tables II) illustrates this error at several points. We note that the error becomes much larger as we approach the origin and the interface. Since the origin and the interface are the only points common to both regions, we expect an average value rather than a certain source value. The Householder method tends to balance the error on both sides of the input source as the output source illustrates. From the profile plot, Figure 13, we can see the effect of the interface by noting the curves at points 17 and 13.

In Tables III to X, we can see a large number of eigenvalues at once. It is interesting to note that as D increases from .5 to 2.0, $Nu(2)$ changes from a value greater than 1 to a value less than 1. Since the Bessel function is defined as X^{Nu} times a series, this implies that a change in the slope of our expansion occurs near the origin. This is interesting because it implies that the flux behaves like x^c , $c > 0$.

In the parameter study we have several cases to study. In some of these cases, we notice negative flux values. Since this is physically impossible, we realize that it is caused by our expansion. In some other extreme cases, we notice little change in the flux values and a large error in our computed source. This is again a failure of our expansion. If more expansion coefficients were used in our calculations, the error would be somewhat reduced. But as explained in the text, the functions used are not

the best low order approximation and therefore we have a large relative error in some cases.

We note again that using the Householder method distributes the error throughout the domain rather than concentrating it around the nearly singular points. By using this expansion, we can study the behavior of the neutron flux, but we cannot make extremely accurate calculations.

Figures 2 to 10

Characteristic Equation/(20)

$$d(\nu, \alpha) = - \tan \nu(\pi - \alpha) / \tan \nu \alpha$$

for given vortex angle 2α . In the Figures, substitute D for d, Nu for ν , $\text{PI} \times \text{Fraction}$ for α ($\alpha = \pi/4, 3\pi/4, \pi/3, 2\pi/3$, etc.)

D CHARACTERISTIC EQUATION FOR NU
AT $PI * 1 / 4$

Figure 2

Figure 3

Figure 4

Figure 5

Figure 6

D CHARACTERISTIC EQUATION FOR NU
AT $\text{PI} \approx 5/6$

Figure 7

CHARACTERISTIC EQUATION FOR NU
AT $PI * 1/8$

Figure 8

Figure 9

D CHARACTERISTIC EQUATION FOR NU
AT $PI \approx 5/8$

Figure 10

Figures 11 to 12

The Angular Eigenfunctions $\mathcal{Y}_i(\theta)$ of Eq. (23) for $\nu_i = Nu$ and the corresponding Radial Eigenfunctions (Bessel functions) $J(X) = J_{\nu_i}(X)$ for $\nu_i = Nu$. The zeros of $dJ(X)/dX$ are indicated by $X = \mu_{ij} = Mu$. The corresponding coefficients a_{ij} of the approximate solution

$$u = \sum_{i=0}^4 \sum_{j=1}^4 a_{ij} \mathcal{Y}_{i\nu_i}(\mu_{ij}r)$$

are indicated by $a_{ij} = A_j$ opposit $Mu = \mu_{ij}$. The coefficients which determine the problem are given by

c_i : $R^2 \text{SIGMA}/D$

Q_i : SOURCE

d : D_1/D_2

α : INTERFACE AT $\pi/$.

DOMAIN 1 $R^2 \text{SIGMA}/D_1$ SOURCE UNIT CIR
 DOMAIN 2 .020 1.000 $D1/D2 = .500$
 .050 0.0 INTERFACE AT $P1/4.00$

Figure 11a

Figure 11 b

	R^2 SIGMA/D	SOURCE	UNIT CIR
DOMAIN 1	.020	1.000	D1/D2= .500
DOMAIN 2	.050	0.0	INTERFACE AT $\pi/3.00$

Figure 12a

Figure 12b

Table I

This table gives the angular eigen values $\nu_i = Nu$ and the corresponding eigenvalues $\mu_{ij} = Mu$, as well as the expansion coefficients $a_{ij} = A_j$ for the special case of equal diffusion length L in both regions. The coefficient $c = 0.24$ is connected to L by $c = (R/L)^2$. $d = D_1/D_2 = \frac{1}{2}$. The third column gives the residual $F(Nu)$ for equation (20), and the computed values $F(Mu) = J_{\nu_i}(\mu_{ij})$.

Table I a

EIGENVALUE TABLE

INTERFACE ANGLE AT $\pi/3.00$
 THE DIFFUSION COEFFICIENT IS .500

NU(1) =	0.00000000000	F(NU) =	0.
MU(1, 1) =	0.00000000000	F(MU) =	0.
MU(1, 2) =	3.83170597023	F(MU) =	9.17051E-12
MU(1, 3) =	7.01558666996	F(MU) =	-4.25337E-11
MU(1, 4) =	10.17346813126	F(MU) =	1.23257E-12
MU(1, 5) =	13.32369193640	F(MU) =	-6.63738E-14
MU(1, 6) =	16.47063005113	F(MU) =	-7.55817E-12
MU(1, 7) =	19.61585851052	F(MU) =	4.55309E-11
MU(1, 8) =	22.76008438061	F(MU) =	-4.91129E-11
MU(1, 9) =	25.90367208762	F(MU) =	4.76032E-11
NU(2) =	1.09841929080	F(NU) =	-1.42109E-13
MU(2, 1) =	1.96602350501	F(MU) =	5.32907E-15
MU(2, 2) =	5.47079673870	F(MU) =	1.86517E-13
MU(2, 3) =	8.68056853087	F(MU) =	-3.25784E-12
MU(2, 4) =	11.85275230770	F(MU) =	1.13798E-13
MU(2, 5) =	15.01185635368	F(MU) =	-1.43774E-14
MU(2, 6) =	18.16482078292	F(MU) =	8.09219E-12
MU(2, 7) =	21.31440142154	F(MU) =	-1.68199E-12
MU(2, 8) =	24.46191612939	F(MU) =	4.27686E-13
MU(2, 9) =	27.60807593926	F(MU) =	-1.55098E-13
NU(3) =	1.90158070920	F(NU) =	-1.04023E-11
MU(3, 1) =	2.93878825076	F(MU) =	-6.35936E-13
MU(3, 2) =	6.57429494088	F(MU) =	2.19025E-12
MU(3, 3) =	9.83128006831	F(MU) =	-2.77556E-14
MU(3, 4) =	13.02865531125	F(MU) =	8.32001E-12
MU(3, 5) =	16.20354168338	F(MU) =	-8.56204E-13
MU(3, 6) =	19.36734980424	F(MU) =	1.28231E-13
MU(3, 7) =	22.52484929339	F(MU) =	-2.79221E-14
MU(3, 8) =	25.67840442064	F(MU) =	-7.32747E-15
MU(3, 9) =	28.82932537881	F(MU) =	-2.62194E-11
NU(4) =	4.09841929080	F(NU) =	-3.55271E-13
MU(4, 1) =	5.42631016130	F(MU) =	-1.59872E-14
MU(4, 2) =	9.40535835317	F(MU) =	2.86926E-12
MU(4, 3) =	12.81161147811	F(MU) =	-7.30971E-13
MU(4, 4) =	16.09805675368	F(MU) =	7.19425E-14
MU(4, 5) =	19.33294631527	F(MU) =	-4.21885E-15
MU(4, 6) =	22.54015441047	F(MU) =	2.15371E-11
MU(4, 7) =	25.73058922540	F(MU) =	-5.57665E-12
MU(4, 8) =	28.91002935071	F(MU) =	1.67821E-12
MU(4, 9) =	32.08184770208	F(MU) =	-5.73985E-13

Table I a (cont'd)

NU(5) =	4.90158070920	F(NU) =	-1.06439E-11
MU(5, 1) =	6.30816605042	F(MU) =	-1.82965E-13
MU(5, 2) =	10.39915801767	F(MU) =	5.99520E-14
MU(5, 3) =	13.85988755845	F(MU) =	-1.09024E-12
MU(5, 4) =	17.18124603372	F(MU) =	1.59206E-13
MU(5, 5) =	20.44084310719	F(MU) =	-3.50830E-14
MU(5, 6) =	23.66658311349	F(MU) =	3.77476E-15
MU(5, 7) =	26.87148063533	F(MU) =	-1.75815E-11
MU(5, 8) =	30.06254352393	F(MU) =	5.58809E-12
MU(5, 9) =	33.24391715012	F(MU) =	-1.96554E-12
NU(6) =	7.09841929080	F(NU) =	-2.55795E-13
MU(6, 1) =	8.68338717083	F(MU) =	-8.85514E-12
MU(6, 2) =	13.04941978587	F(MU) =	2.70015E-11
MU(6, 3) =	16.65255150033	F(MU) =	-7.33191E-13
MU(6, 4) =	20.06926055451	F(MU) =	4.76064E-13
MU(6, 5) =	23.39860679978	F(MU) =	-1.32339E-13
MU(6, 6) =	26.67804507903	F(MU) =	3.01981E-14
MU(6, 7) =	29.92574601005	F(MU) =	-1.24345E-14
MU(6, 8) =	33.15181574022	F(MU) =	-2.77556E-15
MU(6, 9) =	36.36239767477	F(MU) =	-2.29049E-11
NU(7) =	7.90158070920	F(NU) =	-1.06013E-11
MU(7, 1) =	9.54242024581	F(MU) =	-2.29132E-11
MU(7, 2) =	13.99963811478	F(MU) =	4.02833E-11
MU(7, 3) =	17.65217445995	F(MU) =	-1.70974E-13
MU(7, 4) =	21.10306994071	F(MU) =	5.45786E-13
MU(7, 5) =	24.45806770337	F(MU) =	-1.75193E-13
MU(7, 6) =	27.75766192469	F(MU) =	7.17204E-14
MU(7, 7) =	31.02170185835	F(MU) =	7.10543E-15
MU(7, 8) =	34.26132447214	F(MU) =	-3.13083E-14
MU(7, 9) =	37.48335128225	F(MU) =	-4.47258E-11
NU(8) =	10.09841929080	F(NU) =	-7.67386E-13
MU(8, 1) =	11.87492901751	F(MU) =	0.
MU(8, 2) =	16.56163853715	F(MU) =	-4.98206E-11
MU(8, 3) =	20.34235334801	F(MU) =	-3.63709E-12
MU(8, 4) =	23.88401261583	F(MU) =	2.46025E-13
MU(8, 5) =	27.30839096698	F(MU) =	-2.90212E-13
MU(8, 6) =	30.66334922428	F(MU) =	1.23679E-13
MU(8, 7) =	33.97285926895	F(MU) =	-2.79776E-14
MU(8, 8) =	37.25062464350	F(MU) =	1.93179E-14
MU(8, 9) =	40.50517793033	F(MU) =	-9.54792E-15
NU(9) =	10.90158070920	F(NU) =	-1.15392E-11
MU(9, 1) =	12.72275124632	F(MU) =	0.
MU(9, 2) =	17.48719973418	F(MU) =	-1.24345E-14
MU(9, 3) =	21.31241541978	F(MU) =	-1.02980E-11
MU(9, 4) =	24.88618505696	F(MU) =	5.72875E-14
MU(9, 5) =	28.33548838695	F(MU) =	-2.73115E-13
MU(9, 6) =	31.71059388352	F(MU) =	1.46994E-13
MU(9, 7) =	35.03683378581	F(MU) =	-6.35048E-14
MU(9, 8) =	38.32876749134	F(MU) =	3.26406E-14
MU(9, 9) =	41.59550423261	F(MU) =	-8.88178E-16

Table I b

EXPANSION COEFFICIENTS FOR THE DIFFUSION EQUATION

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/3.00$
 DIFFUSION COEFFICIENT = .50

DOMAIN 1	DOMAIN 2
SIGMA/D = .24	SIGMA/D = .24
SOURCE = 1.00	SOURCE = 0.00

THIS APPROXIMATION USES 5 ANGULAR EIGENVALUES
 EACH HAVING 5 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
 USING THE SPECIAL CASE

A(1, 1) = .8333333333

A(2, 1) = -.47362414367

A(2, 2) = -.02107012530

A(2, 3) = -.00503764153

A(2, 4) = -.00199111444

A(2, 5) = -.00097273511

A(3, 1) = -.15279459254

A(3, 2) = -.01249255112

A(3, 3) = -.00349672660

A(3, 4) = -.00151100060

A(3, 5) = -.00077607405

A(4, 1) = .02799504066

A(4, 2) = .00456240040

A(4, 3) = .00163104095

A(4, 4) = .00082112348

A(4, 5) = .00046363898

A(5, 1) = .01853086990

A(5, 2) = .00344751499

A(5, 3) = .00130414610

A(5, 4) = .00068176012

A(5, 5) = .00039474440

Table I c

EXPANSION COEFFICIENTS FOR THE DIFFUSION EQUATION

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/3.00$
 DIFFUSION COEFFICIENT = .50

DOMAIN 1		DOMAIN 2	
SIGMA/D =	.24000	SIGMA/D =	.24000
SOURCE =	1.00000	SOURCE =	0.00000

THIS APPROXIMATION USES 5 ANGULAR EIGENVALUES
 EACH HAVING 5 RADIAL EIGENVALUES
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

A(1, 1) =	.79577471546
A(2, 1) =	-.47296961159
A(2, 2) =	-.02089867351
A(2, 3) =	-.00474712586
A(2, 4) =	-.00171997367
A(2, 5) =	-.00057928546
A(3, 1) =	-.15250954117
A(3, 2) =	-.01245337241
A(3, 3) =	-.00334142749
A(3, 4) =	-.00134863145
A(3, 5) =	-.00048546054
A(4, 1) =	.02789122952
A(4, 2) =	.00458016321
A(4, 3) =	.00158854056
A(4, 4) =	.00077837634
A(4, 5) =	.00032250424
A(5, 1) =	.01844881584
A(5, 2) =	.00346549888
A(5, 3) =	.00127363220
A(5, 4) =	.00065442856
A(5, 5) =	.00028078098

Table II

Function values $u(r,\theta)$ for the case of equal diffusion lengths in both regions for the parameters given in Table I and the points indicated in Figure 13, and computed source.

Table II a

RADIUS	ANGLE	INPUT SOURCE	OUTPUT SOURCE
0.000	0.00X PI	1.00	1.90986
.125	0.00X PI	1.00	.821309
.250	0.00X PI	1.00	1.062724
.500	0.00X PI	1.00	.858751
.750	0.00X PI	1.00	.869804
1.000	0.00X PI	1.00	.881087
0.000	.13X PI	1.00	1.90986
.125	.13X PI	1.00	.750650
.250	.13X PI	1.00	1.105921
.500	.13X PI	1.00	1.071855
.750	.13X PI	1.00	1.094932
1.000	.13X PI	1.00	1.098022
0.000	.25X PI	1.00	1.90986
.125	.25X PI	1.00	.528628
.250	.25X PI	1.00	.817698
.500	.25X PI	1.00	.896731
.750	.25X PI	1.00	.918686
1.000	.25X PI	1.00	.915873
0.000	.33X PI	1.00	1.90986
.125	.33X PI	1.00	.317737
.250	.33X PI	1.00	.328168
.500	.33X PI	1.00	.310324
.750	.33X PI	1.00	.310752
1.000	.33X PI	1.00	.311957
0.000	.50X PI	0.00	1.90986
.125	.50X PI	0.00	.124658
.250	.50X PI	0.00	-.033413
.500	.50X PI	0.00	-.050462
.750	.50X PI	0.00	-.063733
1.000	.50X PI	0.00	-.063096
0.000	.67X PI	0.00	1.90986
.125	.67X PI	0.00	.033405
.250	.67X PI	0.00	-.026949
.500	.67X PI	0.00	.024045
.750	.67X PI	0.00	.021281
1.000	.67X PI	0.00	.018461
0.000	.75X PI	0.00	1.90986
.125	.75X PI	0.00	.015390
.250	.75X PI	0.00	-.023832
.500	.75X PI	0.00	-.005924
.750	.75X PI	0.00	-.006093
1.000	.75X PI	0.00	-.007402
0.000	1.00X PI	0.00	1.90986
.125	1.00X PI	0.00	.000860
.250	1.00X PI	0.00	-.014788
.500	1.00X PI	0.00	.011979
.750	1.00X PI	0.00	.011337
1.000	1.00X PI	0.00	.009529

VALUES OF SELECTED POINTS

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/3.00$
 DIFFUSION COEFFICIENT = .50

DOMAIN 1	DOMAIN 2
SIGMA/D = .24	SIGMA/D = .24
SOURCE = 1.00	SOURCE = 0.00

THIS APPROXIMATION USES 5 ANGULAR EIGENVALUES
 EACH HAVING 5 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

ANGLE	RADIUS	POINT VALUE
0.000 X π	0.000	.79577472
0.000 X π	.250	.87865151
0.000 X π	.500	.94898284
0.000 X π	1.000	1.00805973
.250 X π	0.000	.79577472
.250 X π	.250	.84435754
.250 X π	.500	.88582132
.250 X π	1.000	.92329161
.333 X π	0.000	.79577472
.333 X π	.250	.82041772
.333 X π	.500	.83680786
.333 X π	1.000	.84877107
.500 X π	0.000	.79577472
.500 X π	.250	.79497681
.500 X π	.500	.78857512
.500 X π	1.000	.77960343
.667 X π	0.000	.79577472
.667 X π	.250	.77505616
.667 X π	.500	.75747364
.667 X π	1.000	.74270456
1.000 X π	0.000	.79577472
1.000 X π	.250	.75881023
1.000 X π	.500	.73422503
1.000 X π	1.000	.71628023

SERIES EXPANSION METHOD

$S(2) = 0.$
 $S(1) = 1.0000$
 $C(2) = .2400$
 $C(1) = .2400$
 $D = .5000$
 $ALPH = 60$

Figure 13

Tables III to X

These tables list the angular eigenvalues ν_i and the corresponding radial eigenvalues μ_{ij} for different vortex angles 2α and diffusion ratios $d = D_1/D_2$. To read the tables substitute

α : = INTERFACE ANGLE AT π /"INTEGER"

d : = THE DIFFUSION COEFFICIENT IS "NUMBER"

ν_i : = $Nu(I)$

μ_{ik} : = $Mu(I,K)$

$F(Nu)$ is the residue obtained from (20) by calculating $F = d + \tan \nu(\pi - \alpha)/\tan \nu \alpha$.

$F(Mu)$ is the Function value of $(dJ_{\nu_i}/dx)(\mu_{ij})$.

They confirm the accuracy of the computed eigenvalues.

EIGENVALUE TABLE

INTERFACE ANGLE AT $\theta = 0.0$
THE DIFFUSION COEFFICIENT IS .250

NU (1)	=	0.0000000000	F (NU)	=	0.0
NU (1, 1)	=	0.0000000000	F (MU)	=	0.0
NU (1, 2)	=	3.83176597623	F (MU)	=	9.17051E-12
NU (1, 3)	=	7.01558666996	F (MU)	=	-4.25337E-11
NU (1, 4)	=	10.17346813126	F (MU)	=	1.23257E-12
NU (1, 5)	=	13.32369193640	F (MU)	=	-6.63738E-14
NU (1, 6)	=	16.47063005113	F (MU)	=	-7.55817E-12
NU (1, 7)	=	19.61585851052	F (MU)	=	4.55309E-11
NU (1, 8)	=	22.76008438661	F (MU)	=	-4.91129E-11
NU (2)	=	1.19275085295	F (NU)	=	-2.70006E-13
NU (2, 1)	=	2.08412552230	F (MU)	=	1.77636E-15
NU (2, 2)	=	5.60338425180	F (MU)	=	3.45498E-13
NU (2, 3)	=	8.81809390709	F (MU)	=	-8.60823E-12
NU (2, 4)	=	11.99281701796	F (MU)	=	3.46723E-13
NU (2, 5)	=	15.15347699199	F (MU)	=	-2.55351E-14
NU (2, 6)	=	18.30749448024	F (MU)	=	3.06710E-11
NU (2, 7)	=	21.45783583529	F (MU)	=	-6.73633E-12
NU (2, 8)	=	24.6059292605780	F (MU)	=	1.81871E-12
NU (3)	=	1.80724914705	F (NU)	=	-7.84439E-12
NU (3, 1)	=	2.82754875214	F (MU)	=	-2.70006E-13
NU (3, 2)	=	6.44732767882	F (MU)	=	1.88249E-12
NU (3, 3)	=	9.69830733296	F (MU)	=	-2.81997E-14
NU (3, 4)	=	12.89237372224	F (MU)	=	6.14619E-12
NU (3, 5)	=	16.06514668625	F (MU)	=	-6.13620E-13
NU (3, 6)	=	19.22748375343	F (MU)	=	-1.13243E-14
NU (3, 7)	=	22.38389887363	F (MU)	=	-1.13243E-14
NU (3, 8)	=	25.53662098311	F (MU)	=	1.19349E-14
NU (4)	=	4.19275085295	F (NU)	=	-9.82645E-13
NU (4, 1)	=	5.53039427560	F (MU)	=	-2.48690E-14
NU (4, 2)	=	9.52295446546	F (MU)	=	2.43583E-12
NU (4, 3)	=	12.93565727875	F (MU)	=	-7.81597E-13
NU (4, 4)	=	16.22618171706	F (MU)	=	6.66134E-14
NU (4, 5)	=	19.463933010540	F (MU)	=	-2.22045E-15
NU (4, 6)	=	22.67326645622	F (MU)	=	2.47696E-11
NU (4, 7)	=	25.86535235000	F (MU)	=	-6.46561E-12
NU (4, 8)	=	29.04611289312	F (MU)	=	1.94333E-12
NU (5)	=	4.80724914705	F (NU)	=	-8.22808E-12
NU (5, 1)	=	6.20506367328	F (MU)	=	-1.45661E-13
NU (5, 2)	=	10.28326155342	F (MU)	=	2.03837E-13
NU (5, 3)	=	13.73764969914	F (MU)	=	-1.07914E-12
NU (5, 4)	=	17.054894287292	F (MU)	=	1.49436E-13
NU (5, 5)	=	20.31155076013	F (MU)	=	-1.2445E-14
NU (5, 6)	=	23.53507059762	F (MU)	=	-8.88178E-16
NU (5, 7)	=	26.73822568794	F (MU)	=	-1.55352E-11
NU (5, 8)	=	29.92788186269	F (MU)	=	4.88931E-12
NU (6)	=	7.19275085295	F (NU)	=	-4.26326E-13
NU (6, 1)	=	8.78449235316	F (MU)	=	-9.99734E-12
NU (6, 2)	=	13.16146537220	F (MU)	=	2.89155E-11
NU (6, 3)	=	16.77047341463	F (MU)	=	-6.76792E-13
NU (6, 4)	=	20.19122241103	F (MU)	=	4.96492E-13
NU (6, 5)	=	23.5235441984	F (MU)	=	-1.33449E-13
NU (6, 6)	=	26.80535237302	F (MU)	=	3.01981E-14
NU (6, 7)	=	30.05498936251	F (MU)	=	1.99840E-15
NU (6, 8)	=	33.28263536848	F (MU)	=	-6.66134E-16
NU (7)	=	7.80724914705	F (NU)	=	-8.79652E-12
NU (7, 1)	=	9.44172853899	F (MU)	=	-2.06786E-11
NU (7, 2)	=	13.88846058108	F (MU)	=	3.92379E-11
NU (7, 3)	=	17.53526755951	F (MU)	=	-2.41585E-13
NU (7, 4)	=	20.98217351700	F (MU)	=	5.24913E-13
NU (7, 5)	=	24.33416281123	F (MU)	=	-1.73861E-13
NU (7, 6)	=	27.65138393500	F (MU)	=	4.28546E-14
NU (7, 7)	=	30.849349700073	F (MU)	=	-1.48770E-14
NU (7, 8)	=	34.13150411420	F (MU)	=	-3.41949E-14
NU (8)	=	10.19275085295	F (NU)	=	-5.84645E-13
NU (8, 1)	=	11.7740255075	F (MU)	=	0.0
NU (8, 2)	=	15.67067059331	F (MU)	=	-9.44089E-15
NU (8, 3)	=	20.000708508740	F (MU)	=	2.11831E-13
NU (8, 4)	=	24.9299000000000	F (MU)	=	-3.02869E-13
NU (8, 5)	=	30.0980000000000	F (MU)	=	-2.73115E-14
NU (8, 6)	=	37.3716360770	F (MU)	=	5.95520E-15

Table III

INTERFACE ANGLE AT $\pi/3.00$
THE DIFFUSION COEFFICIENT IS .500

Table IV

NU(1) =	0.00000000000	F (NU) =	0.
MU(1, 1) =	0.00000000000	F (MU) =	0.
MU(1, 2) =	3.83170597023	F (MU) =	9.17051E-12
MU(1, 3) =	7.01558666996	F (MU) =	-4.25337E-11
MU(1, 4) =	10.17346813126	F (MU) =	1.23257E-12
MU(1, 5) =	13.32369193640	F (MU) =	-6.63738E-14
MU(1, 6) =	16.47063005113	F (MU) =	-7.55817E-12
MU(1, 7) =	19.61585851052	F (MU) =	4.55309E-11
MU(1, 8) =	22.76008438061	F (MU) =	-4.91129E-11
MU(1, 9) =	25.90367208762	F (MU) =	4.76032E-11
NU(2) =	1.09841929080	F (NU) =	-1.42109E-13
MU(2, 1) =	1.96602350501	F (MU) =	5.32907E-15
MU(2, 2) =	5.47079673870	F (MU) =	1.86517E-13
MU(2, 3) =	8.68056853087	F (MU) =	-3.25784E-12
MU(2, 4) =	11.85275230770	F (MU) =	1.13798E-13
MU(2, 5) =	15.01185635368	F (MU) =	-1.43774E-14
MU(2, 6) =	18.16482078292	F (MU) =	8.09219E-12
MU(2, 7) =	21.31440142154	F (MU) =	-1.68199E-12
MU(2, 8) =	24.46191612939	F (MU) =	4.27686E-13
MU(2, 9) =	27.60807593926	F (MU) =	-1.55098E-13
NU(3) =	1.90158070920	F (NU) =	-1.04023E-11
MU(3, 1) =	2.93878825076	F (MU) =	-6.35936E-13
MU(3, 2) =	6.57429494088	F (MU) =	2.19025E-12
MU(3, 3) =	9.83128006831	F (MU) =	-2.77556E-14
MU(3, 4) =	13.02865531125	F (MU) =	8.32001E-12
MU(3, 5) =	16.20354168338	F (MU) =	-8.56204E-13
MU(3, 6) =	19.36734980424	F (MU) =	1.28231E-13
MU(3, 7) =	22.52484929339	F (MU) =	-2.79221E-14
MU(3, 8) =	25.67840442064	F (MU) =	-7.32747E-15
MU(3, 9) =	28.82932537881	F (MU) =	-2.62194E-11
NU(4) =	4.09841929080	F (NU) =	-3.55271E-13
MU(4, 1) =	5.42631016130	F (MU) =	-1.59872E-14
MU(4, 2) =	9.40535835317	F (MU) =	2.86926E-12
MU(4, 3) =	12.81161147811	F (MU) =	-7.30971E-13
MU(4, 4) =	16.09805675368	F (MU) =	7.19425E-14
MU(4, 5) =	19.33294631527	F (MU) =	-4.21885E-15
MU(4, 6) =	22.54015441047	F (MU) =	2.15371E-11
MU(4, 7) =	25.73058922540	F (MU) =	-5.57665E-12
MU(4, 8) =	28.91002935071	F (MU) =	1.67821E-12
MU(4, 9) =	32.08184770208	F (MU) =	-5.73985E-13
NU(5) =	4.90158070920	F (NU) =	-1.06439E-11
MU(5, 1) =	6.30816605042	F (MU) =	-1.82965E-13
MU(5, 2) =	10.39915801767	F (MU) =	5.99520E-14
MU(5, 3) =	13.85988755845	F (MU) =	-1.09024E-12
MU(5, 4) =	17.18124603372	F (MU) =	1.59206E-13
MU(5, 5) =	20.44084310719	F (MU) =	-3.50830E-14
MU(5, 6) =	23.66658311349	F (MU) =	3.77476E-15
MU(5, 7) =	26.87148063533	F (MU) =	-1.75815E-11
MU(5, 8) =	30.06254352393	F (MU) =	5.58809E-12
MU(5, 9) =	33.24391715012	F (MU) =	-1.96554E-12
NU(6) =	7.09841929080	F (NU) =	-2.55795E-13
MU(6, 1) =	8.68338717083	F (MU) =	-8.85514E-12
MU(6, 2) =	13.04941978587	F (MU) =	2.70015E-11
MU(6, 3) =	16.65255150033	F (MU) =	-7.33191E-13
MU(6, 4) =	20.06926055451	F (MU) =	4.76064E-13
MU(6, 5) =	23.39860679978	F (MU) =	-1.32339E-13
MU(6, 6) =	26.67804507903	F (MU) =	3.01981E-14
MU(6, 7) =	29.92574601005	F (MU) =	-1.24345E-14
MU(6, 8) =	33.15181574022	F (MU) =	-2.77556E-15
MU(6, 9) =	36.36239767477	F (MU) =	-2.29049E-11
NU(7) =	7.90158070920	F (NU) =	-1.06013E-11
MU(7, 1) =	9.54242024581	F (MU) =	-2.29132E-11
MU(7, 2) =	13.99963811478	F (MU) =	4.02833E-11
MU(7, 3) =	17.65217445995	F (MU) =	-1.70974E-13
MU(7, 4) =	21.10306994071	F (MU) =	5.45786E-13
MU(7, 5) =	24.45806770337	F (MU) =	-1.75193E-13
MU(7, 6) =	27.75766192469	F (MU) =	7.17204E-14
MU(7, 7) =	31.02170185835	F (MU) =	7.10543E-15
MU(7, 8) =	34.26132447214	F (MU) =	-3.13083E-14
MU(7, 9) =	37.48335128225	F (MU) =	-4.47258E-11
NU(8) =	10.09841929080	F (NU) =	-7.67386E-13
MU(8, 1) =	11.87492901751	F (MU) =	0.
MU(8, 2) =	16.56163853715	F (MU) =	-4.98206E-11
MU(8, 3) =	20.34235334801	F (MU) =	-3.63709E-12
MU(8, 4) =	23.88401761583	F (MU) =	2.46025E-13
MU(8, 5) =	27.30839096698	F (MU) =	-2.90212E-13
MU(8, 6) =	30.66339922428	F (MU) =	1.23679E-13
MU(8, 7) =	33.97285976095	F (MU) =	-2.79776E-14
MU(8, 8) =	37.25062466350	F (MU) =	1.93179E-14
MU(8, 9) =	40.50517793033	F (MU) =	-9.54728E-15

INTERFACIAL ANGLE AT P_1 3.00
THE DIFFUSION COEFFICIENT IS 2.000

[illegible]

Table V

INTERFACE ANGLE AT P1/ 3.00
 THE DIFFUSION COEFFICIENT IS 4.000

Table VI

NU (1)	=	0.0000000000	F (NU)	=	0.	F (NU)	=	0.
MU (1, 1)	=	0.0000000000	F (MU)	=	0.	F (MU)	=	0.
MU (1, 2)	=	3.83170597023	F (MU)	=	9.17051E-12	F (MU)	=	9.17051E-12
MU (1, 3)	=	7.01558466960	F (MU)	=	-4.25337E-11	F (MU)	=	-4.25337E-11
MU (1, 4)	=	10.1736813126	F (MU)	=	1.23257E-12	F (MU)	=	1.23257E-12
MU (1, 5)	=	13.32369193640	F (MU)	=	-6.63738E-14	F (MU)	=	-6.63738E-14
MU (1, 6)	=	16.4706300513	F (MU)	=	-7.5517E-12	F (MU)	=	-7.5517E-12
MU (1, 7)	=	19.61585851052	F (MU)	=	4.55309E-11	F (MU)	=	4.55309E-11
MU (1, 8)	=	22.76008438061	F (MU)	=	-4.91129E-11	F (MU)	=	-4.91129E-11
NU (2)	=	8.6614132527	F (NU)	=	-8.52651E-14	F (NU)	=	-8.52651E-14
MU (2, 1)	=	1.64196711835	F (MU)	=	-2.92744E-11	F (MU)	=	-2.92744E-11
MU (2, 2)	=	5.111311969051	F (MU)	=	-1.81188E-13	F (MU)	=	-1.81188E-13
MU (2, 3)	=	8.30914780211	F (MU)	=	4.50306E-12	F (MU)	=	4.50306E-12
MU (2, 4)	=	11.47528710176	F (MU)	=	-1.27087E-12	F (MU)	=	-1.27087E-12
MU (2, 5)	=	14.63072266540	F (MU)	=	1.73583E-13	F (MU)	=	1.73583E-13
MU (2, 6)	=	17.78121973611	F (MU)	=	-4.80171E-14	F (MU)	=	-4.80171E-14
MU (2, 7)	=	20.92962597480	F (MU)	=	2.04003E-14	F (MU)	=	2.04003E-14
MU (2, 8)	=	24.07520279192	F (MU)	=	-3.12528E-14	F (MU)	=	-3.12528E-14
NU (3)	=	2.15385867473	F (NU)	=	-2.41585E-13	F (NU)	=	-2.41585E-13
MU (3, 1)	=	3.23358885502	F (MU)	=	-3.93285E-12	F (MU)	=	-3.93285E-12
MU (3, 2)	=	6.91101738335	F (MU)	=	3.06599E-12	F (MU)	=	3.06599E-12
MU (3, 3)	=	10.18442381439	F (MU)	=	-6.08402E-14	F (MU)	=	-6.08402E-14
MU (3, 4)	=	13.39098184413	F (MU)	=	1.75062E-11	F (MU)	=	1.75062E-11
MU (3, 5)	=	16.57178854950	F (MU)	=	-1.95810E-12	F (MU)	=	-1.95810E-12
MU (3, 6)	=	19.73974430172	F (MU)	=	3.26739E-13	F (MU)	=	3.26739E-13
MU (3, 7)	=	22.90031575712	F (MU)	=	-5.98410E-14	F (MU)	=	-5.98410E-14
MU (3, 8)	=	26.05623942248	F (MU)	=	3.13083E-14	F (MU)	=	3.13083E-14
NU (4)	=	3.84614132527	F (NU)	=	-2.20268E-13	F (NU)	=	-2.20268E-13
MU (4, 1)	=	5.14718415583	F (MU)	=	-7.10543E-15	F (MU)	=	-7.10543E-15
MU (4, 2)	=	9.08959761816	F (MU)	=	3.52797E-12	F (MU)	=	3.52797E-12
MU (4, 3)	=	12.47855035459	F (MU)	=	-6.03961E-13	F (MU)	=	-6.03961E-13
MU (4, 4)	=	15.75413201341	F (MU)	=	5.90639E-14	F (MU)	=	5.90639E-14
MU (4, 5)	=	18.98144716498	F (MU)	=	1.22125E-14	F (MU)	=	1.22125E-14
MU (4, 6)	=	22.18303851548	F (MU)	=	1.44994E-11	F (MU)	=	1.44994E-11
MU (4, 7)	=	25.36912938240	F (MU)	=	-3.65008E-12	F (MU)	=	-3.65008E-12
MU (4, 8)	=	28.54510415093	F (MU)	=	1.06382E-12	F (MU)	=	1.06382E-12
NU (5)	=	5.15385867473	F (NU)	=	-3.55271E-14	F (NU)	=	-3.55271E-14
MU (5, 1)	=	6.58335747811	F (MU)	=	-3.28626E-13	F (MU)	=	-3.28626E-13
MU (5, 2)	=	10.76812670932	F (MU)	=	1.08388E-13	F (MU)	=	1.08388E-13
MU (5, 3)	=	14.18573159296	F (MU)	=	-1.17373E-12	F (MU)	=	-1.17373E-12
MU (5, 4)	=	17.51810239203	F (MU)	=	1.84297E-13	F (MU)	=	1.84297E-13
MU (5, 5)	=	20.78559722568	F (MU)	=	-3.97460E-14	F (MU)	=	-3.97460E-14
MU (5, 6)	=	24.01732227422	F (MU)	=	-6.43929E-15	F (MU)	=	-6.43929E-15
MU (5, 7)	=	27.22696920633	F (MU)	=	-2.40266E-11	F (MU)	=	-2.40266E-11
MU (5, 8)	=	30.421802228034	F (MU)	=	7.75247E-12	F (MU)	=	7.75247E-12
NU (6)	=	6.84614132527	F (NU)	=	-6.18172E-13	F (NU)	=	-6.18172E-13
MU (6, 1)	=	8.41269419570	F (MU)	=	-6.31317E-12	F (MU)	=	-6.31317E-12
MU (6, 2)	=	12.74915046004	F (MU)	=	2.17497E-11	F (MU)	=	2.17497E-11
MU (6, 3)	=	16.33646933532	F (MU)	=	-9.14380E-13	F (MU)	=	-9.14380E-13
MU (6, 4)	=	19.74234423533	F (MU)	=	4.55636E-13	F (MU)	=	4.55636E-13
MU (6, 5)	=	23.0636264257	F (MU)	=	-9.17044E-14	F (MU)	=	-9.17044E-14
MU (6, 6)	=	26.33676563190	F (MU)	=	9.76996E-15	F (MU)	=	9.76996E-15
MU (6, 7)	=	29.57938910371	F (MU)	=	-7.10543E-15	F (MU)	=	-7.10543E-15
MU (6, 8)	=	32.80126634438	F (MU)	=	4.67112E-11	F (MU)	=	4.67112E-11
NU (7)	=	8.15385867473	F (NU)	=	-4.54747E-13	F (NU)	=	-4.54747E-13
MU (7, 1)	=	9.81146302152	F (MU)	=	-2.98321E-11	F (MU)	=	-2.98321E-11
MU (7, 2)	=	14.29644545490	F (MU)	=	4.19815E-11	F (MU)	=	4.19815E-11
MU (7, 3)	=	17.96420996988	F (MU)	=	-4.84057E-14	F (MU)	=	-4.84057E-14
MU (7, 4)	=	21.42573954798	F (MU)	=	5.35127E-13	F (MU)	=	5.35127E-13
MU (7, 5)	=	24.78877474629	F (MU)	=	-2.07390E-13	F (MU)	=	-2.07390E-13
MU (7, 6)	=	28.09472230250	F (MU)	=	3.08642E-14	F (MU)	=	3.08642E-14
MU (7, 7)	=	31.36393832077	F (MU)	=	-1.53211E-14	F (MU)	=	-1.53211E-14
MU (7, 8)	=	34.60787278895	F (MU)	=	-1.55431E-14	F (MU)	=	-1.55431E-14
NU (8)	=	9.84614132527	F (NU)	=	-6.44594E-13	F (NU)	=	-6.44594E-13
MU (8, 1)	=	11.60813741129	F (MU)	=	1.77636E-15	F (MU)	=	1.77636E-15
MU (8, 2)	=	14.24980274473	F (MU)	=	-2.05524E-11	F (MU)	=	-2.05524E-11
MU (8, 3)	=	20.03620042062	F (MU)	=	-2.46143E-12	F (MU)	=	-2.46143E-12
MU (8, 4)	=	23.56714274014	F (MU)	=	3.13971E-13	F (MU)	=	3.13971E-13
MU (8, 5)	=	26.98423722893	F (MU)	=	-2.48496E-13	F (MU)	=	-2.48496E-13
MU (8, 6)	=	30.33320403386	F (MU)	=	1.20747E-13	F (MU)	=	1.20747E-13
MU (8, 7)	=	33.45371074440	F (MU)	=	-2.50878E-14	F (MU)	=	-2.50878E-14
MU (8, 8)	=	36.45371074440	F (MU)	=	-1.11022E-15	F (MU)	=	-1.11022E-15

LITERATURE TABLE

INTERFACIAL ANGLE AT $Pi = 4.00$
THE DIFFUSION COEFFICIENT IS $\cdot 250$

NU (1) =	0.0000000000	F (NU) =	0.0
NU (1.1) =	0.0000000000	F (MU) =	0.0
MU (1.1) =	3.43170597023	F (MU) =	9.17051E-12
MU (1.2) =	7.01550666996	F (MU) =	-4.25337E-11
MU (1.3) =	10.17346813126	F (MU) =	1.23257E-12
MU (1.4) =	13.32369193640	F (MU) =	-6.63738E-14
MU (1.5) =	16.47063005113	F (MU) =	-7.55817E-12
MU (1.6) =	19.61585851052	F (MU) =	4.55309E-11
MU (1.7) =	22.76005438061	F (MU) =	-4.91129E-11
MU (1.8) =	25.90425000000	F (MU) =	-3.69482E-13
NU (2) =	1.19397336804	F (NU) =	0.0
MU (2.1) =	2.08564710569	F (MU) =	3.57492E-13
MU (2.2) =	5.6050654520	F (MU) =	-8.68772E-12
MU (2.3) =	8.81987161964	F (MU) =	-8.68772E-12
MU (2.4) =	11.99462859472	F (MU) =	3.41061E-13
MU (2.5) =	15.15530927856	F (MU) =	-4.20775E-14
MU (2.6) =	18.30934085059	F (MU) =	3.10199E-11
MU (2.7) =	21.45969423931	F (MU) =	-6.45008E-12
MU (2.8) =	24.60779033053	F (MU) =	1.85721E-12
NU (3) =	2.80602663196	F (NU) =	-5.55644E-12
MU (3.1) =	3.98168215023	F (MU) =	5.32907E-15
MU (3.2) =	7.76509578315	F (MU) =	5.02931E-12
MU (3.3) =	11.08247808749	F (MU) =	-2.05613E-13
MU (3.4) =	14.31448580021	F (MU) =	-5.10703E-15
MU (3.5) =	17.51206597461	F (MU) =	-1.05072E-11
MU (3.6) =	20.69196611531	F (MU) =	1.86984E-12
MU (3.7) =	23.86148567807	F (MU) =	-4.23106E-13
MU (3.8) =	27.02437155834	F (MU) =	1.04583E-13
NU (4) =	5.19397336804	F (NU) =	-7.24754E-13
MU (4.1) =	6.62704591073	F (MU) =	-3.60600E-13
MU (4.2) =	10.75712834211	F (MU) =	1.98064E-13
MU (4.3) =	14.23740497267	F (MU) =	-1.19194E-12
MU (4.4) =	17.57152718365	F (MU) =	1.80966E-13
MU (4.5) =	20.84028235154	F (MU) =	-9.99201E-15
MU (4.6) =	24.07296503012	F (MU) =	1.39888E-14
MU (4.7) =	27.28332710819	F (MU) =	-2.52055E-11
MU (4.8) =	30.47881221039	F (MU) =	8.14382E-12
NU (5) =	6.80602663196	F (NU) =	-5.55197E-12
MU (5.1) =	8.36960985300	F (MU) =	-5.97566E-12
MU (5.2) =	12.70131975209	F (MU) =	2.09184E-11
MU (5.3) =	16.28611114570	F (MU) =	-9.19709E-13
MU (5.4) =	19.69025945387	F (MU) =	4.39204E-13
MU (5.5) =	23.01025742504	F (MU) =	-1.05915E-13
MU (5.6) =	26.28233950884	F (MU) =	1.82077E-14
MU (5.7) =	29.52421802186	F (MU) =	-1.33227E-15
MU (5.8) =	32.74543185677	F (MU) =	4.50830E-11
NU (6) =	9.19397336804	F (NU) =	-5.61329E-13
MU (6.1) =	10.91726877379	F (MU) =	0.0
MU (6.2) =	15.51270442374	F (MU) =	2.44142E-11
MU (6.3) =	19.24183500410	F (MU) =	-9.44897E-13
MU (6.4) =	22.74663934696	F (MU) =	4.37872E-13
MU (6.5) =	26.14264162344	F (MU) =	-2.60680E-13
MU (6.6) =	29.47482022833	F (MU) =	8.01581E-14
MU (6.7) =	32.76552604880	F (MU) =	-2.15824E-14
MU (6.8) =	36.022744817946	F (MU) =	1.86517E-14
NU (7) =	10.80602663196	F (NU) =	-5.50671E-12
MU (7.1) =	12.42200077993	F (MU) =	3.55271E-15
MU (7.2) =	17.3735347062	F (MU) =	0.0
MU (7.3) =	21.19733680022	F (MU) =	-9.24327E-12
MU (7.4) =	24.76731674154	F (MU) =	8.26006E-14
MU (7.5) =	28.21368911875	F (MU) =	-2.80220E-13
MU (7.6) =	31.548634297054	F (MU) =	1.68976E-13
MU (7.7) =	34.76045233342	F (MU) =	-5.77316E-14
MU (7.8) =	38.2008769207	F (MU) =	1.59872E-14
NU (8) =	13.19397336804	F (NU) =	-7.24754E-13
MU (8.1) =	15.13166018704	F (MU) =	1.77636E-15
MU (8.2) =	20.10328193114	F (MU) =	-1.77636E-15
MU (8.3) =	24.09435130962	F (MU) =	7.54952E-15
MU (8.4) =	27.71100461075	F (MU) =	1.27278E-12
MU (8.5) =	31.23043660090	F (MU) =	-9.45410E-14
MU (8.6) =	34.6623566653	F (MU) =	1.76525E-13
MU (8.7) =	38.06014700017	F (MU) =	-1.04029E-13
MU (8.8) =	41.30877137951	F (MU) =	5.92859E-14

Table VII

Table VIII

EIGENVALUE TABLE		INTERFAC ANGLE AT $\theta = 4.00$		THE DIFFUSION COEFFICIENT IS		.500	
NU (1)	=	0.0000000000		NU (1)	=	0.0000000000	
NU (1, 1)	=	0.0000000000		NU (1, 1)	=	0.0000000000	
NU (1, 2)	=	0.0000000000		NU (1, 2)	=	0.0000000000	
NU (1, 3)	=	0.0000000000		NU (1, 3)	=	0.0000000000	
NU (1, 4)	=	0.0000000000		NU (1, 4)	=	0.0000000000	
NU (1, 5)	=	0.0000000000		NU (1, 5)	=	0.0000000000	
NU (1, 6)	=	0.0000000000		NU (1, 6)	=	0.0000000000	
NU (1, 7)	=	0.0000000000		NU (1, 7)	=	0.0000000000	
NU (1, 8)	=	0.0000000000		NU (1, 8)	=	0.0000000000	
NU (2)	=	1.0660075808		NU (2)	=	1.0660075808	
NU (2, 1)	=	1.97632334872		NU (2, 1)	=	1.97632334872	
NU (2, 2)	=	5.48233317309		NU (2, 2)	=	5.48233317309	
NU (2, 3)	=	8.69252427410		NU (2, 3)	=	8.69252427410	
NU (2, 4)	=	11.86492275346		NU (2, 4)	=	11.86492275346	
NU (2, 5)	=	15.02415800985		NU (2, 5)	=	15.02415800985	
NU (2, 6)	=	18.17721109572		NU (2, 6)	=	18.17721109572	
NU (2, 7)	=	21.32685570121		NU (2, 7)	=	21.32685570121	
NU (2, 8)	=	24.47441875874		NU (2, 8)	=	24.47441875874	
NU (3)	=	2.8933924493		NU (3)	=	2.8933924493	
NU (3, 1)	=	4.08070022543		NU (3, 1)	=	4.08070022543	
NU (3, 2)	=	7.877967233970		NU (3, 2)	=	7.877967233970	
NU (3, 3)	=	11.20133567418		NU (3, 3)	=	11.20133567418	
NU (3, 4)	=	14.4368934328		NU (3, 4)	=	14.4368934328	
NU (3, 5)	=	17.63685619552		NU (3, 5)	=	17.63685619552	
NU (3, 6)	=	20.81846476172		NU (3, 6)	=	20.81846476172	
NU (3, 7)	=	23.98927879417		NU (3, 7)	=	23.98927879417	
NU (3, 8)	=	27.15317869567		NU (3, 8)	=	27.15317869567	
NU (4)	=	5.10660075808		NU (4)	=	5.10660075808	
NU (4, 1)	=	6.53186545665		NU (4, 1)	=	6.53186545665	
NU (4, 2)	=	10.65035526192		NU (4, 2)	=	10.65035526192	
NU (4, 3)	=	14.12480849877		NU (4, 3)	=	14.12480849877	
NU (4, 4)	=	17.45511603983		NU (4, 4)	=	17.45511603983	
NU (4, 5)	=	20.72112761241		NU (4, 5)	=	20.72112761241	
NU (4, 6)	=	23.95172652636		NU (4, 6)	=	23.95172652636	
NU (4, 7)	=	27.16044599951		NU (4, 7)	=	27.16044599951	
NU (4, 8)	=	30.35460018263		NU (4, 8)	=	30.35460018263	
NU (5)	=	6.8933924493		NU (5)	=	6.8933924493	
NU (5, 1)	=	8.46343572181		NU (5, 1)	=	8.46343572181	
NU (5, 2)	=	12.80546808979		NU (5, 2)	=	12.80546808979	
NU (5, 3)	=	16.39575990541		NU (5, 3)	=	16.39575990541	
NU (5, 4)	=	19.80366752566		NU (5, 4)	=	19.80366752566	
NU (5, 5)	=	23.12645808584		NU (5, 5)	=	23.12645808584	
NU (5, 6)	=	26.46077783675		NU (5, 6)	=	26.46077783675	
NU (5, 7)	=	29.644335011417		NU (5, 7)	=	29.644335011417	
NU (5, 8)	=	32.86701006268		NU (5, 8)	=	32.86701006268	
NU (6)	=	9.10660075808		NU (6)	=	9.10660075808	
NU (6, 1)	=	10.82457413934		NU (6, 1)	=	10.82457413934	
NU (6, 2)	=	15.41096701193		NU (6, 2)	=	15.41096701193	
NU (6, 3)	=	19.13502850109		NU (6, 3)	=	19.13502850109	
NU (6, 4)	=	22.63623384808		NU (6, 4)	=	22.63623384808	
NU (6, 5)	=	26.02947931280		NU (6, 5)	=	26.02947931280	
NU (6, 6)	=	29.3595436071		NU (6, 6)	=	29.3595436071	
NU (6, 7)	=	32.64834790722		NU (6, 7)	=	32.64834790722	
NU (6, 8)	=	35.90874803418		NU (6, 8)	=	35.90874803418	
NU (7)	=	10.8933924493		NU (7)	=	10.8933924493	
NU (7, 1)	=	12.71412604370		NU (7, 1)	=	12.71412604370	
NU (7, 2)	=	17.4777730496		NU (7, 2)	=	17.4777730496	
NU (7, 3)	=	21.3025564071		NU (7, 3)	=	21.3025564071	
NU (7, 4)	=	24.87601111179		NU (7, 4)	=	24.87601111179	
NU (7, 5)	=	28.382506193581		NU (7, 5)	=	28.382506193581	
NU (7, 6)	=	31.69996271863		NU (7, 6)	=	31.69996271863	
NU (7, 7)	=	35.0276032719377		NU (7, 7)	=	35.0276032719377	
NU (7, 8)	=	38.317827274262		NU (7, 8)	=	38.317827274262	
NU (8)	=	13.10660075808		NU (8)	=	13.10660075808	
NU (8, 1)	=	15.040104229313		NU (8, 1)	=	15.040104229313	
NU (8, 2)	=	20.004418747103		NU (8, 2)	=	20.004418747103	
NU (8, 3)	=	23.94574704083		NU (8, 3)	=	23.94574704083	
NU (8, 4)	=	27.60476949683		NU (8, 4)	=	27.60476949683	
NU (8, 5)	=	31.12100889248		NU (8, 5)	=	31.12100889248	
NU (8, 6)	=	34.5507745076		NU (8, 6)	=	34.5507745076	
NU (8, 7)	=	37.92222157181		NU (8, 7)	=	37.92222157181	
NU (8, 8)	=	41.25388271592		NU (8, 8)	=	41.25388271592	
NU (9)	=	13.10660075808		NU (9)	=	13.10660075808	
NU (9, 1)	=	15.040104229313		NU (9, 1)	=	15.040104229313	
NU (9, 2)	=	20.004418747103		NU (9, 2)	=	20.004418747103	
NU (9, 3)	=	23.94574704083		NU (9, 3)	=	23.94574704083	
NU (9, 4)	=	27.60476949683		NU (9, 4)	=	27.60476949683	
NU (9, 5)	=	31.12100889248		NU (9, 5)	=	31.12100889248	
NU (9, 6)	=	34.5507745076		NU (9, 6)	=	34.5507745076	
NU (9, 7)	=	37.92222157181		NU (9, 7)	=	37.92222157181	
NU (9, 8)	=	41.25388271592		NU (9, 8)	=	41.25388271592	
NU (10)	=	13.10660075808		NU (10)	=	13.10660075808	
NU (10, 1)	=	15.040104229313		NU (10, 1)	=	15.040104229313	
NU (10, 2)	=	20.004418747103		NU (10, 2)	=	20.004418747103	
NU (10, 3)	=	23.94574704083		NU (10, 3)	=	23.94574704083	
NU (10, 4)	=	27.60476949683		NU (10, 4)	=	27.60476949683	
NU (10, 5)	=	31.12100889248		NU (10, 5)	=	31.12100889248	
NU (10, 6)	=	34.5507745076		NU (10, 6)	=	34.5507745076	
NU (10, 7)	=	37.92222157181		NU (10, 7)	=	37.92222157181	
NU (10, 8)	=	41.25388271592		NU (10, 8)	=	41.25388271592	
NU (11)	=	13.10660075808		NU (11)	=	13.10660075808	
NU (11, 1)	=	15.040104229313		NU (11, 1)	=	15.040104229313	
NU (11, 2)	=	20.004418747103		NU (11, 2)	=	20.004418747103	
NU (11, 3)	=	23.94574704083		NU (11, 3)	=	23.94574704083	
NU (11, 4)	=	27.60476949683		NU (11, 4)	=	27.60476949683	
NU (11, 5)	=	31.12100889248		NU (11, 5)	=	31.12100889248	
NU (11, 6)	=	34.5507745076		NU (11, 6)	=	34.5507745076	
NU (11, 7)	=	37.92222157181		NU (11, 7)	=	37.92222157181	
NU (11, 8)	=	41.25388271592		NU (11, 8)	=	41.25388271592	
NU (12)	=	13.10660075808		NU (12)	=	13.10660075808	
NU (12, 1)	=	15.040104229313		NU (12, 1)	=	15.040104229313	
NU (12, 2)	=	20.004418747103		NU (12, 2)	=	20.004418747103	
NU (12, 3)	=	23.94574704083		NU (12, 3)	=	23.94574704083	
NU (12, 4)	=	27.60476949683		NU (12, 4)	=	27.60476949683	
NU (12, 5)	=	31.12100889248		NU (12, 5)	=	31.12100889248	
NU (12, 6)	=	34.5507745076		NU (12, 6)	=	34.5507745076	
NU (12, 7)	=	37.92222157181		NU (12, 7)	=	37.92222157181	
NU (12, 8)	=	41.25388271592		NU (12, 8)	=	41.25388271592	
NU (13)	=	13.10660075808		NU (13)	=	13.10660075808	
NU (13, 1)	=	15.040104229313		NU (13, 1)	=	15.040104229313	
NU (13, 2)	=	20.004418747103		NU (13, 2)	=	20.004418747103	
NU (13, 3)	=	23.94574704083		NU (13, 3)	=	23.94574704083	
NU (13, 4)	=	27.60476949683		NU (13, 4)	=	27.60476949683	
NU (13, 5)	=	31.12100889248		NU (13, 5)	=	31.12100889248	
NU (13, 6)	=	34.5507745076		NU (13, 6)	=	34.5507745076	
NU (13, 7)	=	37.92222157181		NU (13, 7)	=	37.92222157181	
NU (13, 8)	=	41.25388271592		NU (13, 8)	=	41.25388271592	
NU (14)	=	13.10660075808		NU (14)	=	13.10660075808	
NU (14, 1)	=	15.040104229313		NU (14, 1)	=	15.040104229313	
NU (14, 2)	=	20.004418747103		NU (14, 2)	=	20.004418747103	
NU (14, 3)	=	23.94574704083		NU (14, 3)	=	23.94574704083	
NU (14, 4)	=	27.60476949683		NU (14, 4)	=	27.60476949683	
NU (14, 5)	=	31.12100889248		NU (14, 5)	=	31.12100889248	
NU (14, 6)	=	34.5507745076		NU (14, 6)	=	34.5507745076	
NU (14, 7)	=	37.92222157181		NU (14, 7)	=	37.92222157181	
NU (14, 8)	=	41.25388271592		NU (14, 8)	=	41.25388271592	
NU (15)	=	13.10660075808		NU (15)	=	13.10660075808	
NU (15, 1)	=	15.040104229313		NU (15, 1)	=	15.040104229313	
NU (15, 2)	=	20.004418747103		NU (15, 2)	=	20.004418747103	
NU (15, 3)	=	23.94574704083		NU (15, 3)	=	23.94574704083	
NU (15, 4)	=	27.60476949683		NU (15, 4)	=	27.60476949683	
NU (15, 5)	=	31.12100889248		NU (15, 5)	=	31.12100889248	
NU (15, 6)	=	34.5507745076		NU (15, 6)	=	34.5507745076	
NU (15, 7)	=	37.92222157181		NU (15, 7)	=	37.92222157181	
NU (15, 8)	=	41.25388271592		NU (15, 8)	=	41.25388271592	
NU (16)	=	13.10660075808		NU (16)	=	13.10660075808	
NU (16, 1)	=	15.040104229313		NU (16, 1)	=	15.040104229313	
NU (16, 2)	=	20.004418747103		NU (16, 2)	=	20.004418747103	
NU (16, 3)	=	23.94574704083		NU (16, 3)	=	23.94574704083	
NU (16, 4)	=	27.60476949683		NU (16, 4)	=	27.60476949683	
NU (16, 5)	=	31.12100889248		NU (16, 5)	=	31.12100889248	
NU (16, 6)	=	34.5507745076		NU (16, 6)	=	34.5507745076	
NU (16, 7)	=	37.92222157181		NU (16, 7)	=	37.92222157181	
NU (16, 8)	=	41.25388271592		NU (16, 8)	=	41.25388271592	
NU (17)	=	13.10660075808		NU (17)	=	13.10660075808	</

INCREASE ANGLE AT 21/4.00
THE DIFFUSION COEFFICIENT IS 2.000

[illegible]

Table IX

Table X

FIGURE TABLE
INTERFACE ANGLE AT $\theta = 4.00$
THE DIFFUSION COEFFICIENT IS 4.000

NU(1)	=	0.0000000000	F (NU)	=	0.
NU(1, 1)	=	0.0000000000	F (MU)	=	0.
NU(1, 2)	=	3.83170597023	F (MU)	=	9.17051E-12
NU(1, 3)	=	7.0155866696	F (MU)	=	-4.25337E-11
NU(1, 4)	=	10.17346813126	F (MU)	=	-1.23257E-12
NU(1, 5)	=	13.32369193640	F (MU)	=	-6.63738E-14
NU(1, 6)	=	16.4706380513	F (MU)	=	-7.55817E-12
NU(1, 7)	=	19.6158581052	F (MU)	=	-4.55309E-11
NU(1, 8)	=	22.76008438061	F (MU)	=	-4.91129E-11
NU(2)	=	4.80602663196	F (NU)	=	-3.05533E-13
NU(2, 1)	=	1.58904912217	F (MU)	=	-4.52900E-11
NU(2, 2)	=	5.05343825257	F (MU)	=	-2.29372E-13
NU(2, 3)	=	8.24957042861	F (MU)	=	-4.93894E-12
NU(2, 4)	=	11.41486159525	F (MU)	=	-1.13687E-12
NU(2, 5)	=	14.56976617747	F (MU)	=	1.52822E-13
NU(2, 6)	=	17.71994126216	F (MU)	=	-6.12843E-14
NU(2, 7)	=	20.86750243556	F (MU)	=	1.65423E-14
NU(2, 8)	=	24.01349497057	F (MU)	=	-7.35523E-15
NU(3)	=	3.19397336804	F (NU)	=	-7.60281E-13
NU(3, 1)	=	4.41959485407	F (MU)	=	3.55271E-15
NU(3, 2)	=	8.26382914509	F (MU)	=	5.37570E-12
NU(3, 3)	=	11.60786563846	F (MU)	=	-3.38396E-13
NU(3, 4)	=	14.85582172368	F (MU)	=	3.84137E-14
NU(3, 5)	=	18.06415613790	F (MU)	=	-2.33231E-11
NU(3, 6)	=	21.25182632071	F (MU)	=	4.41891E-12
NU(3, 7)	=	24.42724244627	F (MU)	=	-1.01852E-12
NU(3, 8)	=	27.59475971928	F (MU)	=	3.01870E-13
NU(4)	=	4.80602663196	F (NU)	=	-5.3640E-13
NU(4, 1)	=	6.20372673574	F (MU)	=	-1.42109E-13
NU(4, 2)	=	10.2815821069	F (MU)	=	2.05613E-13
NU(4, 3)	=	13.7366406471	F (MU)	=	-1.06448E-12
NU(4, 4)	=	17.05325391876	F (MU)	=	1.46105E-13
NU(4, 5)	=	20.30987376919	F (MU)	=	-1.79856E-14
NU(4, 6)	=	23.53336490111	F (MU)	=	-1.77636E-14
NU(4, 7)	=	26.73649747925	F (MU)	=	-1.55059E-11
NU(4, 8)	=	29.92613549016	F (MU)	=	4.89842E-12
NU(5)	=	7.19397336804	F (NU)	=	-8.81073E-13
NU(5, 1)	=	8.78580226318	F (MU)	=	-1.00151E-11
NU(5, 2)	=	13.16291665030	F (MU)	=	2.89447E-11
NU(5, 3)	=	16.77200071884	F (MU)	=	-6.61249E-13
NU(5, 4)	=	20.19280203085	F (MU)	=	4.89830E-13
NU(5, 5)	=	23.52520312692	F (MU)	=	-1.25455E-13
NU(5, 6)	=	26.80703167119	F (MU)	=	1.42109E-14
NU(5, 7)	=	30.0566338410	F (MU)	=	-6.21725E-15
NU(5, 8)	=	33.28432985176	F (MU)	=	-2.14273E-14
NU(6)	=	8.80602663196	F (NU)	=	-7.60281E-13
NU(6, 1)	=	10.50542800004	F (MU)	=	3.55271E-15
NU(6, 2)	=	15.06039195861	F (MU)	=	3.67351E-11
NU(6, 3)	=	18.76689634021	F (MU)	=	-1.14575E-13
NU(6, 4)	=	22.25566929885	F (MU)	=	5.03153E-13
NU(6, 5)	=	25.63941201255	F (MU)	=	-2.43139E-13
NU(6, 6)	=	28.96180513995	F (MU)	=	1.04139E-13
NU(6, 7)	=	32.24447257231	F (MU)	=	-4.44089E-15
NU(6, 8)	=	35.49965073864	F (MU)	=	2.44249E-15
NU(7)	=	11.19397336804	F (NU)	=	-1.10845E-12
NU(7, 1)	=	13.03085927896	F (MU)	=	-1.77636E-15
NU(7, 2)	=	17.84284472042	F (MU)	=	-2.66454E-15
NU(7, 3)	=	21.66401944138	F (MU)	=	-1.40723E-11
NU(7, 4)	=	25.29493371874	F (MU)	=	1.73195E-14
NU(7, 5)	=	28.70764283337	F (MU)	=	-2.68230E-13
NU(7, 6)	=	32.09005953248	F (MU)	=	1.71196E-13
NU(7, 7)	=	35.42233627031	F (MU)	=	-4.90719E-14
NU(7, 8)	=	38.719495319629	F (MU)	=	2.15383E-14
NU(8)	=	12.80407063196	F (NU)	=	-8.08466E-13
NU(8, 1)	=	14.72499615866	F (MU)	=	6.88178E-16
NU(8, 2)	=	19.60294431294	F (MU)	=	-1.96581E-14
NU(8, 3)	=	23.9994749503	F (MU)	=	1.50990E-14
NU(8, 4)	=	27.23847274767	F (MU)	=	6.67024E-13
NU(8, 5)	=	30.74901150783	F (MU)	=	-1.11024E-13
NU(8, 6)	=	34.1663766881	F (MU)	=	1.79190E-13
NU(8, 7)	=	37.53227333036	F (MU)	=	-7.59394E-14
NU(8, 8)	=	40.85790621008	F (MU)	=	3.88578E-14

Figure 14

The Three-Dimensional Parameter Space $\{c_1, c_2, d\}$. Calculations in the following Tables and Figures were done at points indicated by a black dot. They are helpful for interpolation. Note that $(c_1, c_2, d)_{\alpha} \cong (c_2, c_1, 1/d)_{\pi-\alpha}$ (equivalence)

Tables XI to XXIII and Figures 15 - 22

These tables give the expansion coefficients (a), computed flux values (b), input source and computed source (c) values for the problem specified. The figure following the table gives the location of the tabulated values and displays the flux shapes along selected lines.

Table XI a

EXPANSION COEFFICIENTS FOR THE DIFFUSION EQUATION

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/3.00$
 DIFFUSION COEFFICIENT = .25

DOMAIN 1	DOMAIN 2
SIGMA/D = 16.00	SIGMA/D = 16.00
SOURCE = 16.00	SOURCE = 0.00

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
 EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
 USING THE SPECIAL CASE

A(1, 1)=	.1111111111
A(2, 1)=	-1.39271724095
A(2, 2)=	-.20222115279
A(2, 3)=	-.06159764635
A(3, 1)=	-.89931104692
A(3, 2)=	-.14761886124
A(3, 3)=	-.04799461849
A(4, 1)=	.27071152716
A(4, 2)=	.05782392409
A(4, 3)=	.02233100933

Table XI b

VALUES OF SELECTED POINTS

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/3.00$
 DIFFUSION COEFFICIENT = .25

DOMAIN 1	DOMAIN 2
SIGMA/D = 16.00	SIGMA/D = 16.00
SOURCE = 16.00	SOURCE = 0.00

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
 EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

ANGLE	RADIUS	POINT VALUE
0.000 X π	0.000	.10610330
0.000 X π	.250	.50159301
0.000 X π	.500	.78865268
0.000 X π	1.000	.93331686
.250 X π	0.000	.10610330
.250 X π	.250	.29979440
.250 X π	.500	.46333141
.250 X π	1.000	.56998037
.333 X π	0.000	.10610330
.333 X π	.250	.16410591
.333 X π	.500	.19196859
.333 X π	1.000	.20768060
.500 X π	0.000	.10610330
.500 X π	.250	.09608369
.500 X π	.500	.05588100
.500 X π	1.000	.02165079
.667 X π	0.000	.10610330
.667 X π	.250	.05666803
.667 X π	.500	.02078437
.667 X π	1.000	.00269985
1.000 X π	0.000	.10610330
1.000 X π	.250	.03360010
1.000 X π	.500	-.00122818
1.000 X π	1.000	-.02086814

INPUT SOURCE COMPARED WITH OUTPUT SOURCE AT SELECTED POINTS

INPUT DATA
UNIT CIRCLE
DIFFUSION COEFFICIENT = .25

INTERFACE AT PI/ 3.00

Table XI c

DOMAIN 1
SIGMA/D = 16.00000
SOURCE = 16.00000

DOMAIN 2
SIGMA/D = 16.00000
SOURCE = 0.00000

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
USING THE SPECIAL CASE

RADIUS	ANGLE	INPUT SOURCE	OUTPUT SOURCE
0.000	0.00X PI	16.00	1.777778
.125	0.00X PI	16.00	9.272224
.250	0.00X PI	16.00	16.891876
.500	0.00X PI	16.00	17.417823
.750	0.00X PI	16.00	15.941592
1.000	0.00X PI	16.00	17.225062
0.000	.13X PI	16.00	1.777778
.125	.13X PI	16.00	8.243649
.250	.13X PI	16.00	15.235074
.500	.13X PI	16.00	18.167081
.750	.13X PI	16.00	16.164441
1.000	.13X PI	16.00	17.824442
0.000	.25X PI	16.00	1.777778
.125	.25X PI	16.00	5.395136
.250	.25X PI	16.00	9.421459
.500	.25X PI	16.00	13.264624
.750	.25X PI	16.00	11.547704
1.000	.25X PI	16.00	12.939328
0.000	.33X PI	16.00	1.777778
.125	.33X PI	16.00	2.877862
.250	.33X PI	16.00	3.454600
.500	.33X PI	16.00	4.031014
.750	.33X PI	16.00	3.793196
1.000	.33X PI	16.00	4.023708
0.000	.50X PI	0.00	1.777778
.125	.50X PI	0.00	1.624223
.250	.50X PI	0.00	.562638
.500	.50X PI	0.00	-1.203659
.750	.50X PI	0.00	-.554515
1.000	.50X PI	0.00	-1.019747
0.000	.67X PI	0.00	1.777778
.125	.67X PI	0.00	.840972
.250	.67X PI	0.00	-.111484
.500	.67X PI	0.00	-.177228
.750	.67X PI	0.00	.007301
1.000	.67X PI	0.00	-.153133
0.000	.75X PI	0.00	1.777778
.125	.75X PI	0.00	.638181
.250	.75X PI	0.00	.018450
.500	.75X PI	0.00	.987618
.750	.75X PI	0.00	.830718
1.000	.75X PI	0.00	.909579
0.000	1.00X PI	0.00	1.777778
.125	1.00X PI	0.00	.402672
.250	1.00X PI	0.00	-.318250
.500	1.00X PI	0.00	-1.038767
.750	1.00X PI	0.00	-.741495
1.000	1.00X PI	0.00	-1.029635

$S(2) = 0.$
 $S(1) = 16.0000$
 $C(2) = 16.0000$
 $C(1) = 16.0000$
 $D = .2500$
 $ALPH = 60$

Figure 15

Table XIII a

EXPANSION COEFFICIENTS FOR THE DIFFUSION EQUATION

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/3.00$
 DIFFUSION COEFFICIENT = 4.00

DOMAIN 1

SIGMA/D = 16.00000

SOURCE = 16.00000

DOMAIN 2

SIGMA/D = 16.00000

SOURCE = 0.00000

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
 EACH HAVING 3 RADIAL EIGENVALUES
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

A(1, 1) = .63661977237

A(2, 1) = -2.33015334098

A(2, 2) = -.28734739060

A(2, 3) = -.06078393699

A(3, 1) = -.88608249872

A(3, 2) = -.15051614300

A(3, 3) = -.03823118445

A(4, 1) = .38008214446

A(4, 2) = .07917848859

A(4, 3) = .02346473005

Table XII b

VALUES OF SELECTED POINTS

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/3.00$
 DIFFUSION COEFFICIENT = 4.00

DOMAIN 1	DOMAIN 2
SIGMA/D = 16.00	SIGMA/D = 16.00
SOURCE = 16.00	SOURCE = 0.00

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
 EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

ANGLE	RADIUS	POINT VALUE
0.000 X π	0.000	.63661977
0.000 X π	.250	.82131115
0.000 X π	.500	.91030668
0.000 X π	1.000	.95154202
.250 X π	0.000	.63661977
.250 X π	.250	.76744132
.250 X π	.500	.82929924
.250 X π	1.000	.86337329
.333 X π	0.000	.63661977
.333 X π	.250	.73008692
.333 X π	.500	.76013218
.333 X π	1.000	.77297995
.500 X π	0.000	.63661977
.500 X π	.250	.43875875
.500 X π	.500	.24854759
.500 X π	1.000	.11491773
.667 X π	0.000	.63661977
.667 X π	.250	.26724142
.667 X π	.500	.08924729
.667 X π	1.000	.00677285
1.000 X π	0.000	.63661977
1.000 X π	.250	.16928410
1.000 X π	.500	.01905788
1.000 X π	1.000	-.04518094

INPUT SOURCE COMPARED WITH OUTPUT SOURCE
AT SELECTED POINTS

INPUT DATA

UNIT CIRCLE

DIFFUSION COEFFICIENT = 4.00

INTERFACE AT $PI/3.00$

Table XII c

DOMAIN 1

SIGMA/D = 16.00000

SOURCE = 16.00000

DOMAIN 2

SIGMA/D = 16.00000

SOURCE = 0.00000

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
USING 8 EQUALLY SPACED COLLOCATION POINTS.

RADIUS	ANGLE	INPUT SOURCE	OUTPUT SOURCE
0.000	0.00X PI	16.00	10.185916
.125	0.00X PI	16.00	13.284266
.250	0.00X PI	16.00	15.386743
.500	0.00X PI	16.00	15.875882
.750	0.00X PI	16.00	15.532056
1.000	0.00X PI	16.00	15.705493
0.000	.13X PI	16.00	10.185916
.125	.13X PI	16.00	13.045859
.250	.13X PI	16.00	15.020612
.500	.13X PI	16.00	16.002373
.750	.13X PI	16.00	15.597909
1.000	.13X PI	16.00	15.813413
0.000	.25X PI	16.00	10.185916
.125	.25X PI	16.00	12.376384
.250	.25X PI	16.00	13.731189
.500	.25X PI	16.00	14.861340
.750	.25X PI	16.00	14.522431
1.000	.25X PI	16.00	14.718454
0.000	.33X PI	16.00	10.185916
.125	.33X PI	16.00	11.768667
.250	.33X PI	16.00	12.384564
.500	.33X PI	16.00	12.677623
.750	.33X PI	16.00	12.576463
1.000	.33X PI	16.00	12.649899
0.000	.50X PI	0.00	10.185916
.125	.50X PI	0.00	6.997596
.250	.50X PI	0.00	2.607968
.500	.50X PI	0.00	-3.120574
.750	.50X PI	0.00	-1.565719
1.000	.50X PI	0.00	-2.379144
0.000	.67X PI	0.00	10.185916
.125	.67X PI	0.00	3.989217
.250	.67X PI	0.00	-.215736
.500	.67X PI	0.00	-1.194014
.750	.67X PI	0.00	-.506364
1.000	.67X PI	0.00	-.853236
0.000	.75X PI	0.00	10.185916
.125	.75X PI	0.00	3.170401
.250	.75X PI	0.00	-.203188
.500	.75X PI	0.00	.999579
.750	.75X PI	0.00	1.174029
1.000	.75X PI	0.00	1.081004
0.000	1.00X PI	0.00	10.185916
.125	1.00X PI	0.00	2.272166
.250	1.00X PI	0.00	-.807324
.500	1.00X PI	0.00	-2.272616
.750	1.00X PI	0.00	-1.776818
1.000	1.00X PI	0.00	-2.133999

SERIES EXPANSION METHOD

$S(2) = 0.$
 $S(1) = 16.0000$
 $C(2) = 16.0000$
 $C(1) = 16.0000$
 $D = 4.0000$
 $ALPH = 60$

Figure 16

Table XIII a

EXPANSION COEFFICIENTS FOR THE DIFFUSION EQUATION

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/3.00$
 DIFFUSION COEFFICIENT = 4.00

DOMAIN 1	DOMAIN 2
SIGMA/D = 16.00000	SIGMA/D = .06250
SOURCE = 16.00000	SOURCE = 0.00000

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
 EACH HAVING 3 RADIAL EIGENVALUES
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

A(1, 1) =	.94432865616
A(2, 1) =	-.08625471603
A(2, 2) =	-.00456478731
A(2, 3) =	-.01046182281
A(3, 1) =	-.07677115651
A(3, 2) =	-.01150042837
A(3, 3) =	-.00315909553
A(4, 1) =	.02956618132
A(4, 2) =	.00553026017
A(4, 3) =	.00180205133

Table XIII b

VALUES OF SELECTED POINTS

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/3.00$
 DIFFUSION COEFFICIENT = 4.00

DOMAIN 1	DOMAIN 2
SIGMA/D = 16.00	SIGMA/D = .05
SOURCE = 16.00	SOURCE = 0.00

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
 EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

ANGLE	RADIUS	POINT VALUE
0.000 X π	0.000	.94432866
0.000 X π	.250	.95225197
0.000 X π	.500	.95558044
0.000 X π	1.000	.95962989
.250 X π	0.000	.94432866
.250 X π	.250	.94932905
.250 X π	.500	.95102952
.250 X π	1.000	.95408940
.333 X π	0.000	.94432866
.333 X π	.250	.94739046
.333 X π	.500	.94710949
.333 X π	1.000	.94818834
.500 X π	0.000	.94432866
.500 X π	.250	.93362497
.500 X π	.500	.92116069
.500 X π	1.000	.90879539
.667 X π	0.000	.94432866
.667 X π	.250	.92848208
.667 X π	.500	.92182472
.667 X π	1.000	.91372547
1.000 X π	0.000	.94432866
1.000 X π	.250	.92901962
1.000 X π	.500	.93042451
1.000 X π	1.000	.92503026

INPUT SOURCE COMPARED WITH OUTPUT SOURCE
AT SELECTED POINTS

INPUT DATA

UNIT CIRCLE

INTERFACE AT $P1/ 3.00$

DIFFUSION COEFFICIENT = 4.00

Table XIII c

DOMAIN 1

SIGMA/D = 16.00000

SOURCE = 16.00000

DOMAIN 2

SIGMA/D = .06250

SOURCE = 0.00000

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
USING 8 EQUALLY SPACED COLLOCATION POINTS.

RADIUS	ANGLE	INPUT SOURCE	OUTPUT SOURCE
0.000	0.00X PI	16.00	15.109258
.125	0.00X PI	16.00	15.290808
.250	0.00X PI	16.00	15.390938
.500	0.00X PI	16.00	15.318776
.750	0.00X PI	16.00	15.336739
1.000	0.00X PI	16.00	15.421847
0.000	.13X PI	16.00	15.109258
.125	.13X PI	16.00	15.275153
.250	.13X PI	16.00	15.367321
.500	.13X PI	16.00	15.334713
.750	.13X PI	16.00	15.346273
1.000	.13X PI	16.00	15.433373
0.000	.25X PI	16.00	15.109258
.125	.25X PI	16.00	15.231763
.250	.25X PI	16.00	15.283338
.500	.25X PI	16.00	15.273267
.750	.25X PI	16.00	15.281224
1.000	.25X PI	16.00	15.355977
0.000	.33X PI	16.00	15.109258
.125	.33X PI	16.00	15.193369
.250	.33X PI	16.00	15.196795
.500	.33X PI	16.00	15.139025
.750	.33X PI	16.00	15.154590
1.000	.33X PI	16.00	15.203703
0.000	.50X PI	0.00	.059021
.125	.50X PI	0.00	-.073745
.250	.50X PI	0.00	-.271619
.500	.50X PI	0.00	-.430878
.750	.50X PI	0.00	-.244055
1.000	.50X PI	0.00	-.326405
0.000	.67X PI	0.00	.059021
.125	.67X PI	0.00	-.159827
.250	.67X PI	0.00	-.251782
.500	.67X PI	0.00	-.001365
.750	.67X PI	0.00	.039387
1.000	.67X PI	0.00	-.078429
0.000	.75X PI	0.00	.059021
.125	.75X PI	0.00	-.172418
.250	.75X PI	0.00	-.181821
.500	.75X PI	0.00	.222067
.750	.75X PI	0.00	.177391
1.000	.75X PI	0.00	.046593
0.000	1.00X PI	0.00	.059021
.125	1.00X PI	0.00	-.177453
.250	1.00X PI	0.00	-.134674
.500	1.00X PI	0.00	.131787
.750	1.00X PI	0.00	.087270
1.000	1.00X PI	0.00	-.105632

SERIES EXPANSION METHOD

$S(2) = 0.$
 $S(1) = 16.0000$
 $C(2) = .0625$
 $C(1) = 16.0000$
 $D = 4.0000$
 $ALPH = 60$

Figure 17

Table XIV a

EXPANSION COEFFICIENTS FOR THE DIFFUSION EQUATION

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/3.00$
 DIFFUSION COEFFICIENT = .25

DOMAIN 1		DOMAIN 2	
SIGMA/D =	.06250	SIGMA/D =	16.00000
SOURCE =	.06250	SOURCE =	0.00000

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
 EACH HAVING 3 RADIAL EIGENVALUES
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

A(1, 1) =	.00109714490
A(2, 1) =	-.01274886220
A(2, 2) =	-.00121090007
A(2, 3) =	-.00047743988
A(3, 1) =	-.00940642555
A(3, 2) =	-.00110366676
A(3, 3) =	-.00038172878
A(4, 1) =	.00131751167
A(4, 2) =	.00030266500
A(4, 3) =	.00009947227

INPUT DATA

Table XIV b

UNIT CIRCLE INTERFACE AT $\pi/3.00$
 DIFFUSION COEFFICIENT = .25

DOMAIN 1	DOMAIN 2
SIGMA/D = .06	SIGMA/D = 16.00
SOURCE = .06	SOURCE = 0.00

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
 EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

ANGLE	RADIUS	POINT VALUE
0.000 X π	0.000	.00109714
0.000 X π	.250	.00462424
0.000 X π	.500	.00748006
0.000 X π	1.000	.00974614
.250 X π	0.000	.00109714
.250 X π	.250	.00275743
.250 X π	.500	.00412501
.250 X π	1.000	.00523011
.333 X π	0.000	.00109714
.333 X π	.250	.00155045
.333 X π	.500	.00171517
.333 X π	1.000	.00181325
.500 X π	0.000	.00109714
.500 X π	.250	.00097255
.500 X π	.500	.00060431
.500 X π	1.000	.00025829
.667 X π	0.000	.00109714
.667 X π	.250	.00065627
.667 X π	.500	.00029928
.667 X π	1.000	.00001601
1.000 X π	0.000	.00109714
1.000 X π	.250	.00053052
1.000 X π	.500	.00032462
1.000 X π	1.000	.00020202

INPUT SOURCE COMPARED WITH OUTPUT SOURCE
AT SELECTED POINTS

Table XIV c

INPUT DATA

UNIT CIRCLE

INTERFACE AT $PI/3.00$

DIFFUSION COEFFICIENT = .25

DOMAIN 1

SIGMA/D = .06250

SOURCE = .06250

DOMAIN 2

SIGMA/D = 16.00000

SOURCE = 0.00000

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
USING 8 EQUALLY SPACED COLLOCATION POINTS.

RADIUS	ANGLE	INPUT SOURCE	OUTPUT SOURCE
0.000	0.00X PI	.06	.000069
.125	0.00X PI	.06	.034030
.250	0.00X PI	.06	.066395
.500	0.00X PI	.06	.046663
.750	0.00X PI	.06	.027070
1.000	0.00X PI	.06	.039001
0.000	.13X PI	.06	.000069
.125	.13X PI	.06	.029051
.250	.13X PI	.06	.058162
.500	.13X PI	.06	.051225
.750	.13X PI	.06	.030194
1.000	.13X PI	.06	.040812
0.000	.25X PI	.06	.000069
.125	.25X PI	.06	.015449
.250	.25X PI	.06	.030758
.500	.25X PI	.06	.035746
.750	.25X PI	.06	.021379
1.000	.25X PI	.06	.027135
0.000	.33X PI	.06	.000069
.125	.33X PI	.06	.003642
.250	.33X PI	.06	.003668
.500	.33X PI	.06	.002565
.750	.33X PI	.06	.001467
1.000	.33X PI	.06	.002125
0.000	.50X PI	0.00	.017554
.125	.50X PI	0.00	.015892
.250	.50X PI	0.00	.007445
.500	.50X PI	0.00	-.005397
.750	.50X PI	0.00	-.003216
1.000	.50X PI	0.00	-.006649
0.000	.67X PI	0.00	.017554
.125	.67X PI	0.00	.010225
.250	.67X PI	0.00	.002237
.500	.67X PI	0.00	-.000986
.750	.67X PI	0.00	-.001644
1.000	.67X PI	0.00	-.004543
0.000	.75X PI	0.00	.017554
.125	.75X PI	0.00	.008912
.250	.75X PI	0.00	.003436
.500	.75X PI	0.00	.006859
.750	.75X PI	0.00	.004690
1.000	.75X PI	0.00	.002923
0.000	1.00X PI	0.00	.017554
.125	1.00X PI	0.00	.007745
.250	1.00X PI	0.00	.004024
.500	1.00X PI	0.00	.002121
.750	1.00X PI	0.00	.002132
1.000	1.00X PI	0.00	.000718

Table XV a

EXPANSION COEFFICIENTS FOR THE DIFFUSION EQUATION

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/3.00$
 DIFFUSION COEFFICIENT = 4.00

DOMAIN 1		DOMAIN 2	
SIGMA/D =	.06250	SIGMA/D =	16.00000
SOURCE =	.06250	SOURCE =	0.00000

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
 EACH HAVING 3 RADIAL EIGENVALUES
 USING 2 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

A(1, 1) =	.01180161767
A(2, 1) =	-.02648505627
A(2, 2) =	-.00289869651
A(2, 3) =	-.00200261059
A(3, 1) =	-.01227392337
A(3, 2) =	-.00218435200
A(3, 3) =	-.00094912975
A(4, 1) =	.00528972609
A(4, 2) =	.00119253948
A(4, 3) =	.00043699329

Table XV b

VALUES OF SELECTED POINTS

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/3.00$
 DIFFUSION COEFFICIENT = 4.00

DOMAIN 1	DOMAIN 2
SIGMA/D = .06	SIGMA/D = 16.00
SOURCE = .06	SOURCE = 0.00

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
 EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

ANGLE	RADIUS	POINT VALUE
0.000 X π	0.000	.01180162
0.000 X π	.250	.01408754
0.000 X π	.500	.01492778
0.000 X π	1.000	.01561618
.250 X π	0.000	.01180162
.250 X π	.250	.01337208
.250 X π	.500	.01395794
.250 X π	1.000	.01450116
.333 X π	0.000	.01180162
.333 X π	.250	.01287986
.333 X π	.500	.01308965
.333 X π	1.000	.01332745
.500 X π	0.000	.01180162
.500 X π	.250	.00915476
.500 X π	.500	.00683758
.500 X π	1.000	.00494762
.667 X π	0.000	.01180162
.667 X π	.250	.00722981
.667 X π	.500	.00554929
.667 X π	1.000	.00417244
1.000 X π	0.000	.01180162
1.000 X π	.250	.00641042
1.000 X π	.500	.00536147
1.000 X π	1.000	.00417245

2

INPUT SOURCE COMPARED WITH OUTPUT SOURCE
AT SELECTED POINTS

Table XV c

INPUT DATA
UNIT CIRCLE INTERFACE AT $PI/3.00$
DIFFUSION COEFFICIENT = 4.00

DOMAIN 1		DOMAIN 2
SIGMA/D = .06250		SIGMA/D = 16.00000
SOURCE = .06250		SOURCE = 0.00000

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
USING 8 EQUALLY SPACED COLLOCATION POINTS.

RADIUS	ANGLE	INPUT SOURCE	OUTPUT SOURCE
0.000	0.00X PI	.06	.000738
.125	0.00X PI	.06	.027453
.250	0.00X PI	.06	.040119
.500	0.00X PI	.06	.011552
.750	0.00X PI	.06	.002067
1.000	0.00X PI	.06	.013655
0.000	.13X PI	.06	.000738
.125	.13X PI	.06	.025030
.250	.13X PI	.06	.037176
.500	.13X PI	.06	.018632
.750	.13X PI	.06	.006829
1.000	.13X PI	.06	.017904
0.000	.25X PI	.06	.000738
.125	.25X PI	.06	.018280
.250	.25X PI	.06	.025078
.500	.25X PI	.06	.018862
.750	.25X PI	.06	.009269
1.000	.25X PI	.06	.017196
0.000	.33X PI	.06	.000738
.125	.33X PI	.06	.012324
.250	.33X PI	.06	.011668
.500	.33X PI	.06	.003806
.750	.33X PI	.06	.002820
1.000	.33X PI	.06	.006675
0.000	.50X PI	0.00	.188826
.125	.50X PI	0.00	.137064
.250	.50X PI	0.00	.067548
.500	.50X PI	0.00	.007901
.750	.50X PI	0.00	.046578
1.000	.50X PI	0.00	.010168
0.000	.67X PI	0.00	.188826
.125	.67X PI	0.00	.093025
.250	.67X PI	0.00	.037199
.500	.67X PI	0.00	.067551
.750	.67X PI	0.00	.073826
1.000	.67X PI	0.00	.041403
0.000	.75X PI	0.00	.188826
.125	.75X PI	0.00	.082509
.250	.75X PI	0.00	.045564
.500	.75X PI	0.00	.110997
.750	.75X PI	0.00	.099808
1.000	.75X PI	0.00	.074349
0.000	1.00X PI	0.00	.188826
.125	1.00X PI	0.00	.072588
.250	1.00X PI	0.00	.048253
.500	1.00X PI	0.00	.070843
.750	1.00X PI	0.00	.064827
1.000	1.00X PI	0.00	.037547

$S(2) = 0.$
 $S(1) = .0625$
 $C(2) = 10.0000$
 $C(1) = .0625$
 $D = 4.0000$
 $ALPH = 60$

Figure 18

Table XVI a

EXPANSION COEFFICIENTS FOR THE DIFFUSION EQUATION

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/4.00$
 DIFFUSION COEFFICIENT = 4.00

DOMAIN 1		DOMAIN 2	
SIGMA/D =	1.00000	SIGMA/D =	1.00000
SOURCE =	1.00000	SOURCE =	0.00000

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
 EACH HAVING 3 RADIAL EIGENVALUES
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

A(1, 1) =	.72756545413
A(2, 1) =	-.53219354661
A(2, 2) =	-.01916293784
A(2, 3) =	-.00312956681
A(3, 1) =	.03690248532
A(3, 2) =	.00503952427
A(3, 3) =	.00129440997
A(4, 1) =	-.01475928066
A(4, 2) =	-.00278001223
A(4, 3) =	-.00083106088

Table XVI b

EXPANSION COEFFICIENTS FOR THE DIFFUSION EQUATION

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/4$ 4.00
 DIFFUSION COEFFICIENT = 4.00

DOMAIN 1	DOMAIN 2
SIGMA/D = 1.00	SIGMA/D = 1.00
SOURCE = 1.00	SOURCE = 0.00

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
 EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
 USING THE SPECIAL CASE

A(1, 1)=	.57142857143
A(2, 1)=	-.53459979294
A(2, 2)=	-.02016926314
A(2, 3)=	-.00454931241
A(3, 1)=	.03722852469
A(3, 2)=	.00509037891
A(3, 3)=	.00168888392
A(4, 1)=	-.01493711108
A(4, 2)=	-.00277402853
A(4, 3)=	-.00104620485

INPUT SOURCE COMPARED WITH OUTPUT SOURCE
AT SELECTED POINTS

Table XVI c

INPUT DATA
UNIT CIRCLE INTERFACE AT $\pi/4.00$
DIFFUSION COEFFICIENT = 4.00

DOMAIN 1	DOMAIN 2
SIGMA/D = 1.00000	SIGMA/D = 1.00000
SOURCE = 1.00000	SOURCE = 0.00000

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
USING 12 EQUALLY SPACED COLLOCATION POINTS.

RADIUS	ANGLE	INPUT SOURCE	OUTPUT SOURCE
0.000	0.00X π	1.00	.727565
.125	0.00X π	1.00	.935226
.250	0.00X π	1.00	1.058837
.500	0.00X π	1.00	1.100436
.750	0.00X π	1.00	1.075663
1.000	0.00X π	1.00	1.087408
0.000	.13X π	1.00	.727565
.125	.13X π	1.00	.922529
.250	.13X π	1.00	1.032543
.500	.13X π	1.00	1.096503
.750	.13X π	1.00	1.073989
1.000	.13X π	1.00	1.087163
0.000	.25X π	.50	.727565
.125	.25X π	.50	.888361
.250	.25X π	.50	.957265
.500	.25X π	.50	.973081
.750	.25X π	.50	.966233
1.000	.25X π	.50	.972278
0.000	.33X π	0.00	.727565
.125	.33X π	0.00	.776013
.250	.33X π	0.00	.730118
.500	.33X π	0.00	.496578
.750	.33X π	0.00	.539228
1.000	.33X π	0.00	.513890
0.000	.50X π	0.00	.727565
.125	.50X π	0.00	.571820
.250	.50X π	0.00	.479112
.500	.50X π	0.00	.447912
.750	.50X π	0.00	.466492
1.000	.50X π	0.00	.457684
0.000	.67X π	0.00	.727565
.125	.67X π	0.00	.400787
.250	.67X π	0.00	.298445
.500	.67X π	0.00	.438007
.750	.67X π	0.00	.424013
1.000	.67X π	0.00	.430749
0.000	.75X π	0.00	.727565
.125	.75X π	0.00	.325576
.250	.75X π	0.00	.153317
.500	.75X π	0.00	.113776
.750	.75X π	0.00	.130896
1.000	.75X π	0.00	.115785
0.000	1.00X π	0.00	.727565
.125	1.00X π	0.00	.208415
.250	1.00X π	0.00	-.100614
.500	1.00X π	0.00	-.204612
.750	1.00X π	0.00	-.142680
1.000	1.00X π	0.00	-.172040

INPUT SOURCE COMPARED WITH OUTPUT SOURCE
AT SELECTED POINTS

INPUT DATA
UNIT CIRCLE INTERFACE AT PI/ 4.00
DIFFUSION COEFFICIENT = 4.00

Table XVI d

DOMAIN 1	DOMAIN 2
SIGMA/D = 1.00000	SIGMA/D = 1.00000
SOURCE = 1.00000	SOURCE = 0.00000

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
USING THE SPECIAL CASE

RADIUS	ANGLE	INPUT SOURCE	OUTPUT SOURCE
0.000	0.00X PI	1.00	.571429
.125	0.00X PI	1.00	.800735
.250	0.00X PI	1.00	.929660
.500	0.00X PI	1.00	.942972
.750	0.00X PI	1.00	.914291
1.000	0.00X PI	1.00	.939550
0.000	.13X PI	1.00	.571429
.125	.13X PI	1.00	.786577
.250	.13X PI	1.00	.900669
.500	.13X PI	1.00	.941847
.750	.13X PI	1.00	.911529
1.000	.13X PI	1.00	.940720
0.000	.25X PI	.50	.571429
.125	.25X PI	.50	.748602
.250	.25X PI	.50	.817544
.500	.25X PI	.50	.815572
.750	.25X PI	.50	.807116
1.000	.25X PI	.50	.821401
0.000	.33X PI	0.00	.571429
.125	.33X PI	0.00	.624210
.250	.33X PI	0.00	.568142
.500	.33X PI	0.00	.319790
.750	.33X PI	0.00	.396264
1.000	.33X PI	0.00	.341419
0.000	.50X PI	0.00	.571429
.125	.50X PI	0.00	.399449
.250	.50X PI	0.00	.302755
.500	.50X PI	0.00	.292771
.750	.50X PI	0.00	.314282
1.000	.50X PI	0.00	.295338
0.000	.67X PI	0.00	.571429
.125	.67X PI	0.00	.211640
.250	.67X PI	0.00	.114988
.500	.67X PI	0.00	.299728
.750	.67X PI	0.00	.263844
1.000	.67X PI	0.00	.276871
0.000	.75X PI	0.00	.571429
.125	.75X PI	0.00	.128494
.250	.75X PI	0.00	-.043860
.500	.75X PI	0.00	-.038930
.750	.75X PI	0.00	-.017791
1.000	.75X PI	0.00	-.053503
0.000	1.00X PI	0.00	.571429
.125	1.00X PI	0.00	-.001838
.250	1.00X PI	0.00	-.324149
.500	1.00X PI	0.00	-.357429
.750	1.00X PI	0.00	-.285728
1.000	1.00X PI	0.00	-.348875

Table XVII a

EXPANSION COEFFICIENTS FOR THE DIFFUSION EQUATION

INPUT DATA

UNIT CIRCLE

INTERFACE AT $\pi/4.00$

DIFFUSION COEFFICIENT = .25

DOMAIN 1

SIGMA/D = 1.00000

SOURCE = 1.00000

DOMAIN 2

SIGMA/D = 1.00000

SOURCE = 0.00000

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
 EACH HAVING 3 RADIAL EIGENVALUES
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

A(1, 1) =	.09794150344
A(2, 1) =	-.12465455984
A(2, 2) =	-.00672053025
A(2, 3) =	-.00122523878
A(3, 1) =	.02268222941
A(3, 2) =	.00277169405
A(3, 3) =	.00067830238
A(4, 1) =	-.00572995167
A(4, 2) =	-.00114127721
A(4, 3) =	-.00035091643

VALUES OF SELECTED POINTS

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/4.00$
 DIFFUSION COEFFICIENT = .25

DOMAIN 1	DOMAIN 2
SIGMA/D = 1.00	SIGMA/D = 1.00
SOURCE = 1.00	SOURCE = 0.00

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
 EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

ANGLE	RADIUS	POINT VALUE
0.000 X π	0.000	.09794150
0.000 X π	.250	.12364552
0.000 X π	.500	.14827579
0.000 X π	1.000	.16890729
.250 X π	0.000	.09794150
.250 X π	.250	.11180133
.250 X π	.500	.12298159
.250 X π	1.000	.13153761
.333 X π	0.000	.09794150
.333 X π	.250	.10955785
.333 X π	.500	.11839738
.333 X π	1.000	.12470656
.500 X π	0.000	.09794150
.500 X π	.250	.10276101
.500 X π	.500	.10737918
.500 X π	1.000	.11124759
.667 X π	0.000	.09794150
.667 X π	.250	.09386008
.667 X π	.500	.09199327
.667 X π	1.000	.09155432
1.000 X π	0.000	.09794150
1.000 X π	.250	.08187649
1.000 X π	.500	.06648257
1.000 X π	1.000	.05358789

INPUT SOURCE COMPARED WITH OUTPUT SOURCE
AT SELECTED POINTS

105

INPUT DATA

UNIT CIRCLE

INTERFACE AT $PI/4.00$

DIFFUSION COEFFICIENT = .25

Table XVII c

DOMAIN 1

SIGMA/D = 1.00000

SOURCE = 1.00000

DOMAIN 2

SIGMA/D = 1.00000

SOURCE = 0.00000

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
USING THE SPECIAL CASE

RADIUS	ANGLE	INPUT SOURCE	OUTPUT SOURCE
0.000	0.00X PI	1.00	.076923
.125	0.00X PI	1.00	.242746
.250	0.00X PI	1.00	.420818
.500	0.00X PI	1.00	.467306
.750	0.00X PI	1.00	.418354
1.000	0.00X PI	1.00	.454122
0.000	.13X PI	1.00	.076923
.125	.13X PI	1.00	.219695
.250	.13X PI	1.00	.366199
.500	.13X PI	1.00	.463943
.750	.13X PI	1.00	.413584
1.000	.13X PI	1.00	.456132
0.000	.25X PI	.50	.076923
.125	.25X PI	.50	.160329
.250	.25X PI	.50	.211438
.500	.25X PI	.50	.222950
.750	.25X PI	.50	.214963
1.000	.25X PI	.50	.227154
0.000	.33X PI	0.00	.076923
.125	.33X PI	0.00	.146417
.250	.33X PI	0.00	.180733
.500	.33X PI	0.00	.144926
.750	.33X PI	0.00	.149841
1.000	.33X PI	0.00	.149598
0.000	.50X PI	0.00	.076923
.125	.50X PI	0.00	.108015
.250	.50X PI	0.00	.141403
.500	.50X PI	0.00	.150120
.750	.50X PI	0.00	.140941
1.000	.50X PI	0.00	.147648
0.000	.67X PI	0.00	.076923
.125	.67X PI	0.00	.054679
.250	.67X PI	0.00	.065607
.500	.67X PI	0.00	.138124
.750	.67X PI	0.00	.116944
1.000	.67X PI	0.00	.131925
0.000	.75X PI	0.00	.076923
.125	.75X PI	0.00	.024794
.250	.75X PI	0.00	-.007149
.500	.75X PI	0.00	-.014344
.750	.75X PI	0.00	-.009352
1.000	.75X PI	0.00	-.016971
0.000	1.00X PI	0.00	.076923
.125	1.00X PI	0.00	-.026716
.250	1.00X PI	0.00	-.138011
.500	1.00X PI	0.00	-.167066
.750	1.00X PI	0.00	-.136471
1.000	1.00X PI	0.00	-.158826

SERIES EXPANSION METHOD

$S(2) = 0.$
 $S(1) = 1.0000$
 $C(2) = 1.0000$
 $C(1) = 1.0000$
 $D = .2500$
 $ALPH = 45$

Figure 19

Table XVIII a

EXPANSION COEFFICIENTS FOR THE DIFFUSION EQUATION

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/4.00$
 DIFFUSION COEFFICIENT = 4.00

DOMAIN 1		DOMAIN 2	
SIGMA/D =	.06250	SIGMA/D =	16.00000
SOURCE =	.06250	SOURCE =	0.00000

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
 EACH HAVING 3 RADIAL EIGENVALUES
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

A(1, 1) =	.01211207258
A(2, 1) =	-.02822837518
A(2, 2) =	-.00302224284
A(2, 3) =	-.00110819745
A(3, 1) =	.00694588385
A(3, 2) =	.00131152780
A(3, 3) =	.00041705911
A(4, 1) =	-.00300477170
A(4, 2) =	-.00069769171
A(4, 3) =	-.00023552053

VALUES OF SELECTED POINTS

INPUT DATA

Table XVIII b

UNIT CIRCLE INTERFACE AT $\pi/4.00$
 DIFFUSION COEFFICIENT = 4.00

DOMAIN 1	DOMAIN 2
SIGMA/D = .06	SIGMA/D = 16.00
SOURCE = .06	SOURCE = 0.00

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
 EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

ANGLE	RADIUS	POINT VALUE
0.000 X π	0.000	.01211207
0.000 X π	.250	.01558561
0.000 X π	.500	.01705304
0.000 X π	1.000	.01798192
.250 X π	0.000	.01211207
.250 X π	.250	.01474325
.250 X π	.500	.01553193
.250 X π	1.000	.01597093
.333 X π	0.000	.01211207
.333 X π	.250	.01278626
.333 X π	.500	.01198493
.333 X π	1.000	.01103235
.500 X π	0.000	.01211207
.500 X π	.250	.00950692
.500 X π	.500	.00840634
.500 X π	1.000	.00770969
.667 X π	0.000	.01211207
.667 X π	.250	.00682700
.667 X π	.500	.00584440
.667 X π	1.000	.00563831
1.000 X π	0.000	.01211207
1.000 X π	.250	.00342823
1.000 X π	.500	-.00024036
1.000 X π	1.000	-.00256254

INPUT SOURCE COMPARED WITH OUTPUT SOURCE
AT SELECTED POINTS

INPUT DATA

UNIT CIRCLE

INTERFACE AT $PI/4.00$

DIFFUSION COEFFICIENT = 4.00

Table XVIII c

DOMAIN 1

SIGMA/D = .06250

SOURCE = .06250

DOMAIN 2

SIGMA/D = 16.00000

SOURCE = 0.00000

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
USING 8 EQUALLY SPACED COLLOCATION POINTS.

RADIUS	ANGLE	INPUT SOURCE	OUTPUT SOURCE
0.000	0.00X PI	.06	.000757
.125	0.00X PI	.06	.028958
.250	0.00X PI	.06	.042304
.500	0.00X PI	.06	.025117
.750	0.00X PI	.06	.010831
1.000	0.00X PI	.06	.016780
0.000	.13X PI	.06	.000757
.125	.13X PI	.06	.026882
.250	.13X PI	.06	.037436
.500	.13X PI	.06	.028120
.750	.13X PI	.06	.014859
1.000	.13X PI	.06	.020511
0.000	.25X PI	.03	.000757
.125	.25X PI	.03	.021625
.250	.25X PI	.03	.023856
.500	.25X PI	.03	.008169
.750	.25X PI	.03	.004359
1.000	.25X PI	.03	.007736
0.000	.33X PI	0.00	.193793
.125	.33X PI	0.00	.207725
.250	.33X PI	0.00	.190909
.500	.33X PI	0.00	.116678
.750	.33X PI	0.00	.134146
1.000	.33X PI	0.00	.124206
0.000	.50X PI	0.00	.193793
.125	.50X PI	0.00	.147453
.250	.50X PI	0.00	.121113
.500	.50X PI	0.00	.116463
.750	.50X PI	0.00	.119620
1.000	.50X PI	0.00	.111613
0.000	.67X PI	0.00	.193793
.125	.67X PI	0.00	.097330
.250	.67X PI	0.00	.073891
.500	.67X PI	0.00	.131698
.750	.67X PI	0.00	.118714
1.000	.67X PI	0.00	.114905
0.000	.75X PI	0.00	.193793
.125	.75X PI	0.00	.074936
.250	.75X PI	0.00	.031209
.500	.75X PI	0.00	.039004
.750	.75X PI	0.00	.036478
1.000	.75X PI	0.00	.022594
0.000	1.00X PI	0.00	.193793
.125	1.00X PI	0.00	.039326
.250	1.00X PI	0.00	-.048474
.500	1.00X PI	0.00	-.063973
.750	1.00X PI	0.00	-.053452
1.000	1.00X PI	0.00	-.080142

SERIES EXPANSION METHOD

$S(2) = .0$
 $S(1) = .0625$
 $C(2) = 16.0000$
 $C(1) = .0625$
 $D = 4.0000$
 $ALPH = 45$

Figure 20

EXPANSION COEFFICIENTS FOR THE DIFFUSION EQUATION

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/4$ 4.00

DIFFUSION COEFFICIENT = .25

DOMAIN 1

SIGMA/D = 16.00000

SOURCE = 16.00000

DOMAIN 2

SIGMA/D = .06250

SOURCE = 0.00000

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
EACH HAVING 3 RADIAL EIGENVALUES
USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

A(1, 1) = 1.34055573217

A(2, 1) = .37778266162

A(2, 2) = .03152091544

A(2, 3) = .00728732274

A(3, 1) = -.05077352688

A(3, 2) = -.00954520575

A(3, 3) = -.00276663944

A(4, 1) = .02327827952

A(4, 2) = .00503430535

A(4, 3) = .00163585377

VALUES OF SELECTED POINTS

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/4$ 4.00
 DIFFUSION COEFFICIENT = .25

DOMAIN 1 DOMAIN 2
 SIGMA/D = 16.00 SIGMA/D = .05
 SOURCE = 16.00 SOURCE = 0.00

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
 EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

ANGLE	RADIUS	POINT VALUE
0.000 X π	0.000	1.34055573
0.000 X π	.250	1.25598086
0.000 X π	.500	1.18651729
0.000 X π	1.000	1.13763718
.250 X π	0.000	1.34055573
.250 X π	.250	1.29438505
.250 X π	.500	1.26021829
.250 X π	1.000	1.23470046
.333 X π	0.000	1.34055573
.333 X π	.250	1.30186093
.333 X π	.500	1.27515539
.333 X π	1.000	1.25649410
.500 X π	0.000	1.34055573
.500 X π	.250	1.32469794
.500 X π	.500	1.31167352
.500 X π	1.000	1.30250850
.667 X π	0.000	1.34055573
.667 X π	.250	1.35430575
.667 X π	.500	1.36039030
.667 X π	1.000	1.36334506
1.000 X π	0.000	1.34055573
1.000 X π	.250	1.39341502
1.000 X π	.500	1.43682976
1.000 X π	1.000	1.46737983

INPUT SOURCE COMPARED WITH OUTPUT SOURCE
AT SELECTED POINTS

INPUT DATA

UNIT CIRCLE

INTERFACE AT PI/ 4.00

Table XIX c

DIFFUSION COEFFICIENT = .25

DOMAIN 1

SIGMA/D = 16.00000

SOURCE = 16.00000

DOMAIN 2

SIGMA/D = .06250

SOURCE = 0.00000

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
USING 8 EQUALLY SPACED COLLOCATION POINTS.

RADIUS	ANGLE	INPUT SOURCE	OUTPUT SOURCE
0.000	0.00X PI	16.00	21.448892
.125	0.00X PI	16.00	20.225394
.250	0.00X PI	16.00	18.912260
.500	0.00X PI	16.00	17.932152
.750	0.00X PI	16.00	17.790058
1.000	0.00X PI	16.00	17.537765
0.000	.13X PI	16.00	21.448892
.125	.13X PI	16.00	20.376825
.250	.13X PI	16.00	19.247330
.500	.13X PI	16.00	18.093588
.750	.13X PI	16.00	17.936579
1.000	.13X PI	16.00	17.670118
0.000	.25X PI	8.00	21.448892
.125	.25X PI	8.00	20.780795
.250	.25X PI	8.00	20.208465
.500	.25X PI	8.00	19.666441
.750	.25X PI	8.00	19.468735
1.000	.25X PI	8.00	19.327812
0.000	.33X PI	0.00	.083785
.125	.33X PI	0.00	-.176680
.250	.33X PI	0.00	-.308346
.500	.33X PI	0.00	-.129164
.750	.33X PI	0.00	-.089564
1.000	.33X PI	0.00	-.101749
0.000	.50X PI	0.00	.083785
.125	.50X PI	0.00	-.029117
.250	.50X PI	0.00	-.139100
.500	.50X PI	0.00	-.115294
.750	.50X PI	0.00	-.034689
1.000	.50X PI	0.00	-.043174
0.000	.67X PI	0.00	.083785
.125	.67X PI	0.00	.170619
.250	.67X PI	0.00	.145711
.500	.67X PI	0.00	-.117801
.750	.67X PI	0.00	-.040172
1.000	.67X PI	0.00	-.052167
0.000	.75X PI	0.00	.083785
.125	.75X PI	0.00	.278700
.250	.75X PI	0.00	.399148
.500	.75X PI	0.00	.397580
.750	.75X PI	0.00	.334796
1.000	.75X PI	0.00	.355041
0.000	1.00X PI	0.00	.083785
.125	1.00X PI	0.00	.460123
.250	1.00X PI	0.00	.826734
.500	1.00X PI	0.00	.747379
.750	1.00X PI	0.00	.478697
1.000	1.00X PI	0.00	.506980

SERIES EXPANSION METHOD

$S(2) = 0.$
 $S(1) = 16.0000$
 $C(2) = .0625$
 $C(1) = 16.0000$
 $D = .2500$
 $ALPH = 45$

CENTER PROFILE

INTERFACE PROFILE

ANGULAR PROFILE

Figure 21.

EXPANSION COEFFICIENTS FOR THE DIFFUSION EQUATION

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/4.00$
DIFFUSION COEFFICIENT = 4.00

DOMAIN 1

SIGMA/D = 16.00000

SOURCE = 16.00000

DOMAIN 2

SIGMA/D = .06250

SOURCE = 0.00000

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
EACH HAVING 3 RADIAL EIGENVALUES
USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

A(1, 1) = 1.33878645693

A(2, 1) = .53089997937

A(2, 2) = .01396687591

A(2, 3) = .03317484398

A(3, 1) = -.11787117651

A(3, 2) = -.02192470572

A(3, 3) = -.00636596250

A(4, 1) = .06081557462

A(4, 2) = .01299087487

A(4, 3) = .00429598686

VALUES OF SELECTED POINTS

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/4.00$
 DIFFUSION COEFFICIENT = 4.00

DOMAIN 1	DOMAIN 2
SIGMA/D = 16.00	SIGMA/D = .05
SOURCE = 15.00	SOURCE = 0.00

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
 EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

ANGLE	RADIUS	POINT VALUE
0.000 X π	0.000	1.33878646
0.000 X π	.250	1.27963662
0.000 X π	.500	1.25376315
0.000 X π	1.000	1.22452758
.250 X π	0.000	1.33878646
.250 X π	.250	1.29381162
.250 X π	.500	1.27909100
.250 X π	1.000	1.25964301
.333 X π	0.000	1.33878646
.333 X π	.250	1.32715587
.333 X π	.500	1.34093412
.333 X π	1.000	1.35468173
.500 X π	0.000	1.33878646
.500 X π	.250	1.38314884
.500 X π	.500	1.40255394
.500 X π	1.000	1.42448062
.667 X π	0.000	1.33878646
.667 X π	.250	1.42912434
.667 X π	.500	1.44780499
.667 X π	1.000	1.47405562
1.000 X π	0.000	1.33878646
1.000 X π	.250	1.48666106
1.000 X π	.500	1.55134473
1.000 X π	1.000	1.62443365

INPUT SOURCE COMPARED WITH OUTPUT SOURCE
AT SELECTED POINTS

117

INPUT DATA

UNIT CIRCLE

INTERFACE AT PI/ 4.00

DIFFUSION COEFFICIENT = 4.00

Table XX c

DOMAIN 1

SIGMA/D = 16.00000

SOURCE = 16.00000

DOMAIN 2

SIGMA/D = .06250

SOURCE = 0.00000

THIS APPROXIMATION USES 4 ANGULAR EIGENVALUES
EACH HAVING 3 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
USING 8 EQUALLY SPACED COLLOCATION POINTS.

RADIUS	ANGLE	INPUT SOURCE	OUTPUT SOURCE
0.000	0.00X PI	16.00	21.420583
.125	0.00X PI	16.00	20.312198
.250	0.00X PI	16.00	19.775489
.500	0.00X PI	16.00	19.866946
.750	0.00X PI	16.00	19.621995
1.000	0.00X PI	16.00	19.149509
0.000	.13X PI	16.00	21.420583
.125	.13X PI	16.00	20.380081
.250	.13X PI	16.00	19.909388
.500	.13X PI	16.00	19.859726
.750	.13X PI	16.00	19.620088
1.000	.13X PI	16.00	19.147961
0.000	.25X PI	8.00	21.420583
.125	.25X PI	8.00	20.562730
.250	.25X PI	8.00	20.294984
.500	.25X PI	8.00	20.458269
.750	.25X PI	8.00	20.194438
1.000	.25X PI	8.00	19.824280
0.000	.33X PI	0.00	.083674
.125	.33X PI	0.00	-.016581
.250	.33X PI	0.00	.298949
.500	.33X PI	0.00	1.438526
.750	.33X PI	0.00	.980351
1.000	.33X PI	0.00	.975078
0.000	.50X PI	0.00	.083674
.125	.50X PI	0.00	.482196
.250	.50X PI	0.00	.610469
.500	.50X PI	0.00	.232608
.750	.50X PI	0.00	.192585
1.000	.50X PI	0.00	.421229
0.000	.67X PI	0.00	.083674
.125	.67X PI	0.00	.885537
.250	.67X PI	0.00	.722625
.500	.67X PI	0.00	-.849726
.750	.67X PI	0.00	-.429195
1.000	.67X PI	0.00	.023938
0.000	.75X PI	0.00	.083674
.125	.75X PI	0.00	1.081595
.250	.75X PI	0.00	1.105706
.500	.75X PI	0.00	.110970
.750	.75X PI	0.00	.288097
1.000	.75X PI	0.00	.921061
0.000	1.00X PI	0.00	.083674
.125	1.00X PI	0.00	1.412081
.250	1.00X PI	0.00	1.839658
.500	1.00X PI	0.00	.580121
.750	1.00X PI	0.00	.446708
1.000	1.00X PI	0.00	1.208857

SERIES EXPANSION METHOD

$S(2) = 0.$
 $S(1) = 16.0000$
 $C(2) = .0625$
 $C(1) = 16.0000$
 $D = 4.0000$
 $ALPH = 45$

Figure 22

Table XXI a

EXPANSION COEFFICIENTS FOR THE DIFFUSION EQUATION

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/3.00$
 DIFFUSION COEFFICIENT = 2.00

DOMAIN 1		DOMAIN 2	
SIGMA/D =	.02000	SIGMA/D =	.10000
SOURCE =	1.00000	SOURCE =	0.00000

THIS APPROXIMATION USES 5 ANGULAR EIGENVALUES
 EACH HAVING 5 RADIAL EIGENVALUES
 USING 6 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

A(1, 1) =	8.10141214792
A(2, 1) =	-1.32789385202
A(2, 2) =	-.04454769089
A(2, 3) =	-.00958420983
A(2, 4) =	-.00335678129
A(2, 5) =	-.00110690512
A(3, 1) =	-.23644353953
A(3, 2) =	-.02088075855
A(3, 3) =	-.00575428419
A(3, 4) =	-.00236849632
A(3, 5) =	-.00086374913
A(4, 1) =	.05772418017
A(4, 2) =	.00909502752
A(4, 3) =	.00310206284
A(4, 4) =	.00150037691
A(4, 5) =	.00061577202
A(5, 1) =	.03155085798
A(5, 2) =	.00606105600
A(5, 3) =	.00224983154
A(5, 4) =	.00116640993
A(5, 5) =	.00050369397

Table XXI b

VALUES OF SELECTED POINTS

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/3.00$
 DIFFUSION COEFFICIENT = 2.00

DOMAIN 1	DOMAIN 2
SIGMA/D = .02	SIGMA/D = .10
SOURCE = 1.00	SOURCE = 0.00

THIS APPROXIMATION USES 5 ANGULAR EIGENVALUES
 EACH HAVING 5 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

ANGLE	RADIUS	POINT VALUE
0.000 X π	0.000	8.10141215
0.000 X π	.250	8.22942588
0.000 X π	.500	8.31672420
0.000 X π	1.000	8.38571463
.250 X π	0.000	8.10141215
.250 X π	.250	8.19294516
.250 X π	.500	8.25382931
.250 X π	1.000	8.30344959
.333 X π	0.000	8.10141215
.333 X π	.250	8.16687057
.333 X π	.500	8.20492153
.333 X π	1.000	8.23204496
.500 X π	0.000	8.10141215
.500 X π	.250	8.05748866
.500 X π	.500	8.01214630
.500 X π	1.000	7.96492015
.667 X π	0.000	8.10141215
.667 X π	.250	7.97340114
.667 X π	.500	7.88610400
.667 X π	1.000	7.81711414
1.000 X π	0.000	8.10141215
1.000 X π	.250	7.90503691
1.000 X π	.500	7.79088407
1.000 X π	1.000	7.70951379

INPUT SOURCE COMPARED WITH OUTPUT SOURCE
AT SELECTED POINTS

121

INPUT DATA

UNIT CIRCLE

INTERFACE AT $P_1/3.00$

DIFFUSION COEFFICIENT = 2.00

Table XXI c

DOMAIN 1

SIGMA/D = .02000

SOURCE = 1.00000

DOMAIN 2

SIGMA/D = .10000

SOURCE = 0.00000

THIS APPROXIMATION USES 5 ANGULAR EIGENVALUES
EACH HAVING 5 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
USING 6 EQUALLY SPACED COLLOCATION POINTS.

RADIUS	ANGLE	INPUT SOURCE	OUTPUT SOURCE
0.000	0.00X P_1	1.00	.162028
.125	0.00X P_1	1.00	.839491
.250	0.00X P_1	1.00	1.041486
.500	0.00X P_1	1.00	.855672
.750	0.00X P_1	1.00	.862733
1.000	0.00X P_1	1.00	.872364
0.000	.13X P_1	1.00	.162028
.125	.13X P_1	1.00	.781260
.250	.13X P_1	1.00	1.083549
.500	.13X P_1	1.00	1.053526
.750	.13X P_1	1.00	1.070623
1.000	.13X P_1	1.00	1.072828
0.000	.25X P_1	1.00	.162028
.125	.25X P_1	1.00	.597486
.250	.25X P_1	1.00	.844351
.500	.25X P_1	1.00	.901011
.750	.25X P_1	1.00	.917508
1.000	.25X P_1	1.00	.915318
0.000	.33X P_1	1.00	.162028
.125	.33X P_1	1.00	.422455
.250	.33X P_1	1.00	.439038
.500	.33X P_1	1.00	.403352
.750	.33X P_1	1.00	.402759
1.000	.33X P_1	1.00	.404636
0.000	.50X P_1	0.00	.810141
.125	.50X P_1	0.00	.429874
.250	.50X P_1	0.00	-.109432
.500	.50X P_1	0.00	-.172636
.750	.50X P_1	0.00	-.218529
1.000	.50X P_1	0.00	-.217395
0.000	.67X P_1	0.00	.810141
.125	.67X P_1	0.00	.127104
.250	.67X P_1	0.00	-.079558
.500	.67X P_1	0.00	.099272
.750	.67X P_1	0.00	.088051
1.000	.67X P_1	0.00	.077061
0.000	.75X P_1	0.00	.810141
.125	.75X P_1	0.00	.067456
.250	.75X P_1	0.00	-.071252
.500	.75X P_1	0.00	-.011719
.750	.75X P_1	0.00	-.014247
1.000	.75X P_1	0.00	-.019807
0.000	1.00X P_1	0.00	.810141
.125	1.00X P_1	0.00	.019823
.250	1.00X P_1	0.00	-.036598
.500	1.00X P_1	0.00	.061326
.750	1.00X P_1	0.00	.058140
1.000	1.00X P_1	0.00	.050966

EXPANSION COEFFICIENTS FOR THE DIFFUSION EQUATION

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/4$ 4.00

DIFFUSION COEFFICIENT = .50

DOMAIN 1

SIGMA/D = .02000

SOURCE = 1.00000

DOMAIN 2

SIGMA/D = .10000

SOURCE = 0.00000

THIS APPROXIMATION USES 5 ANGULAR EIGENVALUES
 EACH HAVING 5 RADIAL EIGENVALUES
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

A(1, 1) = 2.06762254698

A(2, 1) = -.27961706199

A(2, 2) = -.01201953112

A(2, 3) = -.00272712575

A(2, 4) = -.00098832871

A(2, 5) = -.00033301701

A(3, 1) = .03585776513

A(3, 2) = .00435218439

A(3, 3) = .00133964713

A(3, 4) = .00059818926

A(3, 5) = .00023122523

A(4, 1) = -.00972063895

A(4, 2) = -.00187461386

A(4, 3) = -.00069716622

A(4, 4) = -.00036193042

A(4, 5) = -.00015644032

A(5, 1) = .00479942316

A(5, 2) = .00112779902

A(5, 3) = .00046006011

A(5, 4) = .00025730377

A(5, 5) = .00011430968

Table XXII b

VALUES OF SELECTED POINTS

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/4.00$
 DIFFUSION COEFFICIENT = .50

DOMAIN 1 DOMAIN 2
 SIGMA/D = .02 SIGMA/D = .10
 SOURCE = 1.00 SOURCE = 0.00

THIS APPROXIMATION USES 5 ANGULAR EIGENVALUES
 EACH HAVING 5 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

ANGLE	RADIUS	POINT VALUE
0.000 X π	0.000	2.06762255
0.000 X π	.250	2.12355522
0.000 X π	.500	2.17004696
0.000 X π	1.000	2.20982643
.250 X π	0.000	2.06762255
.250 X π	.250	2.10144894
.250 X π	.500	2.12686765
.250 X π	1.000	2.14667386
.333 X π	0.000	2.06762255
.333 X π	.250	2.09376459
.333 X π	.500	2.11228038
.333 X π	1.000	2.12552658
.500 X π	0.000	2.06762255
.500 X π	.250	2.07461413
.500 X π	.500	2.08042560
.500 X π	1.000	2.08539803
.667 X π	0.000	2.06762255
.667 X π	.250	2.05289817
.667 X π	.500	2.04380789
.667 X π	1.000	2.03873247
1.000 X π	0.000	2.06762255
1.000 X π	.250	2.02567304
1.000 X π	.500	1.99080423
1.000 X π	1.000	1.96096964

INPUT SOURCE COMPARED WITH OUTPUT SOURCE
AT SELECTED POINTS

124 INPUT DATA
UNIT CIRCLE INTERFACE AT PI/ 4.00
DIFFUSION COEFFICIENT = .50

DOMAIN 1	DOMAIN 2
SIGMA/D = .02000	SIGMA/D = .10000
SOURCE = 1.00000	SOURCE = 0.00000

Table XXII c

THIS APPROXIMATION USES 5 ANGULAR EIGENVALUES
EACH HAVING 5 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
USING 8 EQUALLY SPACED COLLOCATION POINTS.

RADIUS	ANGLE	INPUT SOURCE	OUTPUT SOURCE
0.000	0.00X PI	1.00	.041352
.125	0.00X PI	1.00	.425383
.250	0.00X PI	1.00	.634947
.500	0.00X PI	1.00	.479754
.750	0.00X PI	1.00	.498938
1.000	0.00X PI	1.00	.506566
0.000	.13X PI	1.00	.041352
.125	.13X PI	1.00	.374350
.250	.13X PI	1.00	.587204
.500	.13X PI	1.00	.645202
.750	.13X PI	1.00	.653626
1.000	.13X PI	1.00	.651476
0.000	.25X PI	.50	.041352
.125	.25X PI	.50	.241680
.250	.25X PI	.50	.276182
.500	.25X PI	.50	.236379
.750	.25X PI	.50	.238359
1.000	.25X PI	.50	.240289
0.000	.33X PI	0.00	.206762
.125	.33X PI	0.00	.355437
.250	.33X PI	0.00	.330105
.500	.33X PI	0.00	.275017
.750	.33X PI	0.00	.271674
1.000	.33X PI	0.00	.274681
0.000	.50X PI	0.00	.206762
.125	.50X PI	0.00	.255039
.250	.50X PI	0.00	.281521
.500	.50X PI	0.00	.262587
.750	.50X PI	0.00	.265281
1.000	.50X PI	0.00	.266336
0.000	.68X PI	0.00	.206762
.125	.68X PI	0.00	.124949
.250	.68X PI	0.00	.189246
.500	.68X PI	0.00	.268565
.750	.68X PI	0.00	.272788
1.000	.68X PI	0.00	.268966
0.000	.75X PI	0.00	.206762
.125	.75X PI	0.00	.055487
.250	.75X PI	0.00	.028611
.500	.75X PI	0.00	.056938
.750	.75X PI	0.00	.054555
1.000	.75X PI	0.00	.052817
0.000	1.00X PI	0.00	.206762
.125	1.00X PI	0.00	-.082898
.250	1.00X PI	0.00	-.241790
.500	1.00X PI	0.00	-.128184
.750	1.00X PI	0.00	-.144353
1.000	1.00X PI	0.00	-.150680

Table XXIII a

EXPANSION COEFFICIENTS FOR THE DIFFUSION EQUATION

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/4$ 4.00
 DIFFUSION COEFFICIENT = 2.00

DOMAIN 1

SIGMA/D = .02000
 SOURCE = 1.00000

DOMAIN 2

SIGMA/D = .10000
 SOURCE = 0.00000

THIS APPROXIMATION USES 5 ANGULAR EIGENVALUES
 EACH HAVING 5 RADIAL EIGENVALUES
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

A(1, 1) = 7.59180463216

A(2, 1) = -.85219937884

A(2, 2) = -.02782377416

A(2, 3) = -.00595176856

A(2, 4) = -.00207721070

A(2, 5) = -.00068322087

A(3, 1) = .06046873911

A(3, 2) = .00781944168

A(3, 3) = .00246443155

A(3, 4) = .00112083339

A(3, 5) = .00043928817

A(4, 1) = -.02122697153

A(4, 2) = -.00396777673

A(4, 3) = -.00145510531

A(4, 4) = -.00074668592

A(4, 5) = -.00032006461

A(5, 1) = .00882887347

A(5, 2) = .00211210143

A(5, 3) = .00086938364

A(5, 4) = .00048946127

A(5, 5) = .00021667421

VALUES OF SELECTED POINTS

INPUT DATA

UNIT CIRCLE INTERFACE AT $\pi/4.00$
 DIFFUSION COEFFICIENT = 2.00

DOMAIN 1	DOMAIN 2
SIGMA/D = .02	SIGMA/D = .10
SOURCE = 1.00	SOURCE = 0.00

THIS APPROXIMATION USES 5 ANGULAR EIGENVALUES
 EACH HAVING 5 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
 USING 8 EQUALLY SPACED WEIGHTED COLLOCATION POINTS

ANGLE	RADIUS	POINT VALUE
0.000 X π	0.000	7.59180463
0.000 X π	.250	7.71413734
0.000 X π	.500	7.79272715
0.000 X π	1.000	7.85422417
.250 X π	0.000	7.59180463
.250 X π	.250	7.68298506
.250 X π	.500	7.73827084
.250 X π	1.000	7.77900914
.333 X π	0.000	7.59180463
.333 X π	.250	7.64260119
.333 X π	.500	7.66831253
.333 X π	1.000	7.68191805
.500 X π	0.000	7.59180463
.500 X π	.250	7.56122145
.500 X π	.500	7.54157400
.500 X π	1.000	7.52619975
.667 X π	0.000	7.59180463
.667 X π	.250	7.48697982
.667 X π	.500	7.42681769
.667 X π	1.000	7.38611987
1.000 X π	0.000	7.59180463
1.000 X π	.250	7.40830556
1.000 X π	.500	7.29042086
1.000 X π	1.000	7.19817533

INPUT SOURCE COMPARED WITH OUTPUT SOURCE
AT SELECTED POINTS

127

INPUT DATA
UNIT CIRCLE INTERFACE AT PI/ 4.00
DIFFUSION COEFFICIENT = 2.00

Table XXIII c

DOMAIN 1	DOMAIN 2
SIGMA/D = .02000	SIGMA/D = .10000
SOURCE = 1.00000	SOURCE = 0.00000

THIS APPROXIMATION USES 5 ANGULAR EIGENVALUES
EACH HAVING 5 RADIAL EIGENVALUES.

THE EXPANSION COEFFICIENTS WERE CALCULATED BY
USING 6 EQUALLY SPACED COLLOCATION POINTS.

RADIUS	ANGLE	INPUT SOURCE	OUTPUT SOURCE
0.000	0.00X PI	1.00	.151836
.125	0.00X PI	1.00	.731997
.250	0.00X PI	1.00	.949688
.500	0.00X PI	1.00	.774833
.750	0.00X PI	1.00	.794435
1.000	0.00X PI	1.00	.803026
0.000	.13X PI	1.00	.151836
.125	.13X PI	1.00	.680809
.250	.13X PI	1.00	.905293
.500	.13X PI	1.00	.966170
.750	.13X PI	1.00	.972567
1.000	.13X PI	1.00	.970590
0.000	.25X PI	.50	.151836
.125	.25X PI	.50	.546418
.250	.25X PI	.50	.598562
.500	.25X PI	.50	.532532
.750	.25X PI	.50	.535445
1.000	.25X PI	.50	.538632
0.000	.33X PI	0.00	.759180
.125	.33X PI	0.00	.948674
.250	.33X PI	0.00	.758498
.500	.33X PI	0.00	.558692
.750	.33X PI	0.00	.545246
1.000	.33X PI	0.00	.554671
0.000	.50X PI	0.00	.759180
.125	.50X PI	0.00	.612770
.250	.50X PI	0.00	.557271
.500	.50X PI	0.00	.599413
.750	.50X PI	0.00	.593586
1.000	.50X PI	0.00	.591135
0.000	.67X PI	0.00	.759180
.125	.67X PI	0.00	.321697
.250	.67X PI	0.00	.414433
.500	.67X PI	0.00	.572238
.750	.67X PI	0.00	.582088
1.000	.67X PI	0.00	.573577
0.000	.75X PI	0.00	.759180
.125	.75X PI	0.00	.161079
.250	.75X PI	0.00	.078150
.500	.75X PI	0.00	.170561
.750	.75X PI	0.00	.162477
1.000	.75X PI	0.00	.156523
0.000	1.00X PI	0.00	.759180
.125	1.00X PI	0.00	-.119280
.250	1.00X PI	0.00	-.452277
.500	1.00X PI	0.00	-.199425
.750	1.00X PI	0.00	-.234388
1.000	1.00X PI	0.00	-.249094

6. Conclusions

In this report we have analyzed the Neumann problem for the neutron diffusion equation over a circular disc with two segment regions characterized by regionwise constant diffusion and absorption coefficients and constant sources (see Figure 1). The purpose of this analysis is to obtain information on the necessity of constructing numerical methods that take the vertex singularities into account. Parallel to this investigation, but reported elsewhere, calculations were done with finite difference methods over triangular-quadrilateral networks and with finite element methods over triangular networks. As a first step, the problem was reformulated as a pair of coupled partial differential equations, and brought into dimensionless form. Thus the original problem which contains eight parameters was reduced to four parameters (c_1, c_2, d, α). This dimensional analysis shows clearly that in numerical calculations the important quantities are not the absolute mesh size but the magnitude of the coefficient c_1 (Equation 14), and as was shown subsequently, the value of d .

In order to obtain an analytic result, the solution was assumed to have the form $u = \sum \psi_\ell(\theta) v_\ell(r)$. For zero absorption coefficients, this method leads in a straightforward procedure to an eigenvalue problem for the functions $\psi_\ell(\theta)$ (Equation (20)), and subsequently to an eigenvalue problem for the function $v_\ell(r)$ (Equation (21)). It was then shown, that the method of separation of variables also yields a unique solution of the same form if the diffusion lengths (see (2) and (14)) are equal in both regions. The solution is given by (23), (33), (35) and (36). The Fourier coefficients a_{ij} occurring in these equations are easily bounded (42). Mathematically, the set of base functions

$\{u_{ij}\}$ (Equation (22)) is complete in L_2 . Hence, one may assume this same set as the base functions for the general problem. While this is true mathematically, it is an unsatisfactory choice from a computational point of view, as has been verified by the parameter study of Section 5.

Indeed, the analytic investigation of Section 4 shows that the set of base functions for the general problem is

$$\{\Psi_\ell\} \times \{J_{\nu_k}(\mu_{kj}r)\}$$

and not the previous set given by (22),

$$\{\Psi_\ell J_{\nu_\ell}(\mu_{\ell j}r)\}$$

where in both cases Ψ_ℓ is given by (23) and (20) and the $J_{\nu_\ell}(x)$ are Bessel functions of the first kind.

Referring to the general solution (85), we sum over ℓ and obtain with

$$G_{ik}(\theta) = \sum_{\ell=0}^{\infty} c_{\ell ki} \Psi_\ell(\theta)$$

the general form for the solution

$$u = \sum_{i,k=0}^{\infty} G_{ik} J_{\nu_k}(\mu_{ki}r)$$

Thus, with our present knowledge, we conclude that any improved finite difference method or finite element method should be based on the following assumptions:

In the radial direction from the vertex, the solution has the form

$$\begin{aligned} & a_0 + a_{10} r^{\nu_0} + a_{20} r^{\nu_1} + \dots \\ & + a_{11} r^{\nu_0+2} + a_{21} r^{\nu_1+2} + \dots \\ & + \dots \end{aligned}$$

while in the non-radial directions, piecewise polynomial approximation is sufficient (we must account for the discontinuity $\partial u / \partial \theta$ at the interface $\theta = \alpha$). At this point, the reader may review the values $v_0, v_1, \dots$ for $d = 1/2$ and $d = 2$ by consulting Figures 2 to 10.

At the present time there is a desire to develop high-order approximation codes both for the diffusion and transport equation with the expectation of using a very coarse network. We hope it is now generally accepted that the accuracy of the numerical methods depends *not* primarily on the technique (say finite differences vs. finite elements) but on the underlying interpolation space. In Appendix B we have developed an error analysis for the interpolation and least squares approximation method. Combining (B8) to (B11), we see that for interpolation,

$$v \equiv r^v \sum_{k=0}^{\infty} b_k r^k = r^{v-\alpha} \sum_{k=0}^n b'_k r^k + E(r)$$

where the error

$$E(r) = h^v \sum_{k=0}^{\infty} b_k F(\alpha+k, x) x^{v-\alpha}$$

and h is a reference parameter and $x = r/h$. If the interpolation points are at $x = ih (i = 0, 1, \dots, n)$, the dimensionless error functions $F(\alpha, x)$ are plotted in Figures 23 to 25 for quadratic and cubic interpolation of

$$x^\alpha = \sum_{k=1}^n a_k x^k + F \quad (n = 2, 3)$$

when α is between 0.5 and 4. The above expression for $E(r)$ clearly indicates that for $h \ll 1$,

$$E = O(h^\alpha)$$

and thus asymptotically, very little is gained by using higher order

approximations. The situation however becomes rather complex when $h \sim 1$. Clearly, if $\alpha = 1$, or $\alpha = 2$, then in Figures 23 and 24 $F(\alpha; x) = 0$, similarly $F = Y = 0$ for $\alpha = 3$ in Figure 25. However, the more α deviates from these integer values, the larger the Chebyshev norm of the error function. Note the errors for $\alpha = 3$ in Figure 23 and for $\alpha = 4$ in Figure 25. Thus, using collocation methods (to use an example) the introduction of broken powers $r_k^{\nu+2j}$ in the base functions promises high rewards.

Figures 26 to 33 represent the dimensionless error functions (B21)

$$Y = F(\alpha, 0; x) = x^\alpha - \sum_{k=0}^n a_k x^k$$

for least squares approximations of order $n = 1$ to $n = 5$ of x^α over the unit interval. We conclude from (B23) that the Chebyshev error behaves again as

$$E = O(h^\nu) \quad \text{for} \quad h \ll 1$$

but when $h \sim 1$, the behavior is quite different. The computed extreme errors for $h = 1$ have been tabulated in Table XXV. It is seen again, that a close approximation of the base functions to the analytic behavior of the true function is preferable to a high order approximation. On the other hand high order least squares approximations will always yield better results than low order approximations (disregarding computational difficulties).

We return to Figures 26 to 33 and notice that the zero points of the error functions fall in rather small intervals. As explained in Appendix B, and tabulated in Table XXIV, they can be calculated with relative ease. We can therefore conclude that collocation methods with points in these intervals will produce rather superior results. As

expected, the function x^α is very poorly defined at $x = 0$.

To our knowledge, the problem discussed in this paper, the analytical behavior of the solution near the vertex in rigorous form, as well as the observation of the "zero-intervals" for least squares approximations of x^α have never been published. In addition we believe that the procedure using the Householder method on rectangular blocks in a Block-Gauss-Seidel iteration is a novel approach. It falls between the interpolation and discrete least squares approximation method.

By giving the explicit dimensionless error functions F in Appendix B we have advanced from the traditional soft mathematical approach to hard mathematics and large intervals. There are still many questions unanswered. E.g. would it be profitable to seek Chebyshev approximations and "zero-intervals" for x^α ?

Our approach to establish the analytical behavior is mathematically very interesting, but computationally unsatisfactory. In Appendix A we showed that the constructive method may be very unstable. Indeed we have observed in our calculations that the Fourier coefficients decreased very satisfactorily in general, but suddenly a coefficient became unexpectedly large. In addition, there is always the possibility that logarithmic terms have to be introduced according to the construction given in Appendix A. For this reason, other approaches are being studied to obtain a formulation of the analytic behavior of the solution to our problem near the vertex.

In summary, we have developed a mechanism that permits the creation of physically and mathematically meaningful benchmark problems for testing diffusion codes at interior (and exterior) corners.

Appendix B

If a function $v(r)$ has the expansion

$$v = r^{\nu} \sum_{k=0}^{\infty} b_k r^k \quad (\nu > 0) \quad (B1)$$

and is approximated by

$$g = r^{\nu'} \sum_{k=0}^n b'_k r^k \quad (\nu \geq \nu' > 0) \quad (B2)$$

then the error is defined by

$$E = v - g \quad (B3)$$

Let g approximate v by collocation at the points r_i ($i = 0, 1, \dots, n$).

Define

$$E(\alpha; n; r) = r^{\alpha} - \sum_{k=0}^n a_k^{(\alpha)} r^k \quad (B4)$$

The coefficients $a_k^{(\alpha)}$ are determined from the collocation points r_i by solving the system of linear equations

$$\sum_{k=0}^n a_k^{(\alpha)} r_i^k = r_i^{\alpha} \quad (i = 0, 1, \dots, n) \quad (B5)$$

Obviously, the coefficients b'_k are determined by solving

$$\sum_{k=0}^n b'_k r_i^k = \sum_{k=0}^{\infty} b_k r_i^{\nu - \nu' + k} \quad (i = 0, 1, \dots, n) \quad (B6)$$

From the last two equations follows

$$b'_k = \sum_{\ell=0}^{\infty} a_k^{(\nu - \nu' + \ell)} b_{\ell} \quad (B7)$$

and from (B3) follows

$$E = v - g = r^{\nu'} \sum_{k=0}^{\infty} b_k E(\nu - \nu' + k; n; r) \quad (B8)$$

Clearly, $E = 0$ at $r = r_i$. Hence, given (B1), v' , and (B4) for a set of collocation points $\{r_i\}$, we can analyze the behavior of the approximation g of (B2).

We bring (B4) in dimensionless form. Let

$$r = xh \quad (B9)$$

where h is a reference length. Then

$$\begin{aligned} E(\alpha; r) &= r^\alpha - \sum_{k=0}^n a_k r^k \\ &= h^\alpha (x^\alpha - \sum_{k=0}^n c_k x^k) = h^\alpha F(\alpha, x) \end{aligned}$$

We have for the collocation points $x_i = r_i/h$ ($i = 0, 1, \dots, n$)

$$E(\alpha; r) = h^\alpha F(\alpha, x) \quad (B10)$$

$$F(\alpha, x) = x^\alpha - \sum_{k=0}^n c_k x^k \quad (B11)$$

$$c_k = \frac{\begin{vmatrix} 1 & x_0^1 & \dots & x_0^\alpha & \dots & x_0^n \\ 1 & x_1^1 & \dots & x_1^\alpha & \dots & x_1^n \\ \vdots & \vdots & & \vdots & & \vdots \\ 1 & x_n^1 & \dots & x_n^\alpha & \dots & x_n^n \end{vmatrix}}{\begin{vmatrix} 1 & x_0^1 & \dots & x_0^k & \dots & x_0^n \\ 1 & x_1^1 & \dots & x_1^k & \dots & x_1^n \\ \vdots & \vdots & & \vdots & & \vdots \\ 1 & x_n^1 & \dots & x_n^k & \dots & x_n^n \end{vmatrix}} \quad (B12)$$

$$a_k = h^{\alpha-k} c_k \quad (B13)$$

Figure 23 represents the dimensionless error function $F(\alpha, x)$ for the collocation points $x = 0, 1, 2$. Note that $F(1; x) = F(2; x) = 0$, and $|F(3; x)| < 0.4$. Figures 24 and 25 show the error curve $Y = F(\alpha, x)$

for the cubic interpolation at $x = 0, 1, 2, 3$ for α between 0.5 and

4. Note that for equidistant interpolation through the origin,

$c_0 = 0$, and in general

$$c_k = \frac{\begin{vmatrix} 1 & 1^2 & \dots & 1^\alpha & \dots & 1^n \\ \vdots & \vdots & & \vdots & & \vdots \\ n & n^2 & & n^\alpha & & n^n \end{vmatrix}}{\begin{vmatrix} 1 & 1^2 & \dots & 1^k & \dots & 1^n \\ \vdots & \vdots & & \vdots & & \vdots \\ n & n^2 & \dots & n^k & \dots & n^n \end{vmatrix}} \quad (\text{B14})$$

which shows immediately that $c_k = 1$ for $\alpha = k$, $c_k = 0$ for α an integer $\neq k$.

We consider now the least squares approximation of (B1) over the interval $[0, h]$. Hence we solve the normal equation for the b'_k of (B2),

$$\sum_{k=0}^n b'_k \int_0^h r^{2v'+k+i} dr = \int_0^h v(r) r^{v'+i} dr \quad (i = 0, 1, \dots, n)$$

or

$$\sum_{k=0}^n \frac{h^{v'+k}}{2v'+k+i+1} b'_k = \sum_{\ell=0}^{\infty} b_\ell \frac{h^{v+\ell}}{v+v'+\ell+i+1} \quad (\text{B15})$$

Similar to the collocation method approach we define an error function

$$E(\alpha, \beta; n, r) = r^\alpha - r^\beta \sum_{k=0}^n a_k r^k \quad (\text{B16})$$

for which the normal equation is

$$\sum_{k=0}^n \frac{h^{\beta+k}}{2\beta+k+i+1} a_k = \frac{h^\alpha}{\alpha+\beta+i+1} \quad (i = 0, 1, \dots, n) \quad (\text{B17})$$

We let (corresponding to (B13))

$$a_k = h^{\alpha-\beta-k} c_k = a_k^{(\alpha, \beta)} \quad (\text{B18})$$

where by (B17)

$$c_k = \frac{\begin{vmatrix} \frac{1}{2\beta+1} & \cdots & \frac{1}{\alpha+\beta+1} & \cdots & \frac{1}{2\beta+n+1} \\ \vdots & & \vdots & & \vdots \\ \frac{1}{2\beta+n+1} & \cdots & \frac{1}{\alpha+\beta+n+1} & \cdots & \frac{1}{2\beta+2n+1} \end{vmatrix}}{\begin{vmatrix} \frac{1}{2\beta+1} & \cdots & \frac{1}{2\beta+k+1} & \cdots & \frac{1}{2\beta+n+1} \\ \vdots & & \vdots & & \vdots \\ \frac{1}{2\beta+n+1} & \cdots & \frac{1}{2\beta+k+n+1} & \cdots & \frac{1}{2\beta+2n+1} \end{vmatrix}} \quad (\text{B19})$$

is again independent of h as is (B14). Now let $r = xh$ as in (B9). Then from (B16) follows

$$E(\alpha, \beta; n, r) = h^\alpha F(\alpha, \beta; x) \quad (\text{B20})$$

$$F(\alpha, \beta; x) = x^\alpha - x^\beta \sum_{k=0}^n c_k x^k \quad (\text{B21})$$

where (B21) is independent of h as is (B11). Hence, corresponding to (B7), we have for the least squares approximation

$$b'_k = \sum_{\ell=0}^{\infty} a_k^{(\nu+\ell, \nu')} b_\ell \quad (\text{B22})$$

and for the least squares error function (B3)

$$\begin{aligned} E = v - g &= \sum_{\ell=0}^{\infty} b_\ell E(\nu+\ell, \nu') \\ &= h^\nu \sum_{\ell=0}^{\infty} b_\ell h^\ell F(\nu+\ell, \nu'; x) \end{aligned} \quad (\text{B23})$$

In practical applications, one lets usually $\nu' = 0$. Therefore, we have plotted in Figures 26 to 33 the dimensionless error functions

$Y = F(\alpha, 0; x)$ for values of α between 0.5 and 4 (inclusive), and for linear, quadratic, cubic, quartic and quintic least squares approxima-

tions. In addition we calculated the zeros of the error curve which happen to coincide with the zeros of the Jacobi polynomials $P_{n+1}^{0, \alpha}(2x-1)$

where we used the definition given by Abramowitz and Stegun (1965, p. 774, Eqs. 22.2.1/2). If $\alpha = n + 1$, these zeros x_i are the zeros of the Legendre polynomial $P_{n+1}(2x-1)$. All zeros are tabulated in Table XXIV. Finally in Table XXV we summarized the extreme values for the dimensionless error functions $F(\alpha, 0; x)$, together with the corresponding abscissae. These tables and the figures given considerable quantitative insight into the problem of least squares approximation.

Figure 24

Error functions $F(\alpha; x)$ for cubic interpolation of x^α

Figure 25

Error functions $F(\alpha; x)$ for cubic interpolation of x^α

Figure 26

Error functions $F(\alpha, 0; x)$ for linear least squares approximation of x^α

Figure 27

Error functions $F(\alpha, 0; x)$ for quadratic least squares
 approximations of x^α

Figure 28

Error function $F(1/2, 0; x)$ for cubic least squares
approximation of $x^{1/2}$.

Figure 29
 Error function $F(\alpha, 0; x)$ for cubic least squares
 approximation of x^α

Figure 30.

Error function $F(1/2, 0; x)$ for quartic least squares
approximation of $x^{1/2}$.

Figure 31

Error functions $F(\alpha, 0; x)$ for quartic least squares approximations of x^α .

Figure 32

Error function $F(1/2, 0; x)$ for quintic least squares
approximations of $x^{1/2}$.

Figure 33
 Error functions $F(\alpha, 0; x)$ for quintic least squares
 approximations of x^α .

Table XXIV

Zeros x_i of the dimensionless error functions $F(\alpha, 0; n, x)$ of Eq. (B21)
for different orders of least squares approximations of x^α .

Abcissae x_i

α	linear (n=1)		quadratic (n=2)			cubic (n=3)			
1/2	.1487	.7471	.07009	.4322	.8657	.04025	.2673	.6229	.9187
3/4	.1621	.7556	.07668	.4413	.8684	.04408	.2740	.6273	.9197
7/8	.1682	.7596	.07970	.4456	.8697	.04583	.2771	.6294	.9203
1	-	-	-	-	-	-	-	-	-
1.5	.1945	.7770	.09255	.4648	.8756	.05328	.2913	.6394	.9227
2	G .2113	.7887	-	-	-	-	-	-	-
2.5	.2253	.7988	.10727	.4896	.8838	.06172	.3097	.6531	.9261
3	.2370	.8071	G .11270	.5000	.8873	-	-	-	-
3.5	.2469	.8156	.11720	.5093	.8905	.06733	.3240	.6648	.9292
4	.2553	.8227	.12094	.5177	.8935	G .06943	.3300	.6700	.9306
G:	.2113	.7887	.11270	.5000	.8873	.06943	.3300	.6700	.9306

α	quartic (n=4)					quintic (n=5)				
1/2	.0260	.1788	.4465	.7381	.9460	.0185	.1272	.3297	.5787	.8097
3/4	.0285	.1836	.4507	.7403	.9465	.0199	.1308	.3333	.5812	.8109
7/8	.0296	.1859	.4527	.7414	.9467	.0207	.1324	.3350	.5824	.8114
1	-	-	-	-	-	-	-	-	-	-
1.5	.0345	.1961	.4621	.7465	.9487	.0241	.1400	.3429	.5882	.8142
2	-	-	-	-	-	-	-	-	-	-
2.5	.0399	.2092	.4753	.7540	.9495	.0279	.1496	.3540	.5965	.8184
3	-	-	-	-	-	-	-	-	-	-
3.5	.0435	.2194	.4864	.7606	.9510	.0304	.1571	.3632	.6040	.8223
4	-	-	-	-	-	-	-	-	-	-
G:	.0470	.2308	.5000	.7692	.9531	.0338	.1694	.3807	.6193	.8306

G. Gaussian Abcissae x_i : $P_{n+1}(2x_i-1) = 0$ (Abramowitz and Stegun (1965) p. 921)

Table XXV

Extreme values of the dimensionless error functions $F(\alpha, 0; n, x)$ of Eq. (B21) for different orders of polynomial least squares approximations of x^α .

Extreme values (all occurring either at $x_i = 0$ or $x_i = 1$)

α	n=1	x_i	n=2	x_i	n=3	x_i	n=4	x_i	n=5	x_i
1/2	-.2667	0	-.1714	0	-.1270	0	-.1010	0	-.0839	0
3/4	-.1039	0	-.0519	0	-.0328	0	-.0232	0	-.0175	0
7/8	-.0464	0	-.0202	0	-.0117	0	-.0078	0	-.0056	0
1	0	-	0	-	0	-	0	-	0	-
1.5	+.1945	0	+.0190	0	+.0069	0	+.0033	0	+.0019	0
2	+.1667	0; 1	0	-	0	-	0	-	0	-
2.5	+.7988	1	-.0260	0	-.0027	0	-.0007	0	-.0002	0
3	+.3000	1	-.0500	0; 1	0	-	0	-	0	-
3.5	+.3536	1	-.0816	1	+.0062	0	+.0005	0	+.0001	0
4	+.4000	1	-.1143	1	+.0143	0; 1	0	-	0	-

7. Operating Instructions for Computer Programs

This package of programs is divided into three main sections which are:

- I. EIGEN
- II. a. EQSP
b. BLKIT
- III. a. PTS
b. RESI
c. PRØFI

The first section consists of the program EIGEN. EIGEN computes the eigenvalues and eigenfunctions to the problem (16) and (10) to (15) for a given interface angle (α). α is defined in this program as $\alpha = (\pi/\text{ALPH})$, where ALPH is a rational number greater than 2. The second section uses the punched output from the first to calculate the expansion coefficients, a_{ij} . The third section uses the punched output from the second section to examine the results of the series.

We use the octal format for our punched output to avoid any round off error. This is accomplished by expressing the memory's word in octal form (20 octal digits for a 60 bit word in the CDC 6400).

(Eigenvalues)

Input:

Card 1: ALPH D Format (2F12.5)
Card 2: NN MM Format (2I5)

π/ALPH is the interface angle where ALPH is greater than 2 and is rational

D is the diffusions coefficient.

NN is number of angular eigenvalues desired.

MM is number of associated radial eigenvalues desired with each angular eigenvalues.

Output:

A table of eigenvalues and the relative error is printed. The punched output is in the following form.

Card 1: Cover Card "Eigenvalues for the following Input"

Card 2: ALPH D Format (2F12.5)

Card 3: NN MM Format (2I5)

Card 4: v_i Format (Ø20)

Card 5: μ_{ij} $j=1, \dots, MM$ Format (4020)

Program EQSP

(Equation Special)

This program calculates the expansion coefficients, a_{ij} , directly for the special case when $\sigma_1/D_1 = \sigma_2/D_2$ by using (35) and (36).

Input:

Card 1: JK LK Format (2I5)
 Card 2: ALPH D S1 S2 C1 C2 Format (6F12.5)
 Card 3-N: Punched output from EIGEN with the first two cards removed.

JK is the number of angular eigenvalues used for the approximation.
 LK is the number of radial eigenvalues used with each angular eigenvalue.
 PI/ALPH is the interface angle.
 D is the diffusion coefficient ratio.
 S1, S2 are the respective source terms (S_i/D_i).
 C1, C2 are the absorption coefficients (σ_i/D_i).

Output:

A printed table of the input and the expansion coefficients is printed. The punched output is of the following form:

Card 1: ALPH D S1 S2 C1 C2 Format (6F12.5)
 Card 2: JK LK 0 (zero) Format (3I5)
 Card 3-N: a_{ij} v_i μ_{ij} Format (3D20)

Note: Coefficients $a_{1,2} \dots a_{1,LK}$ have been omitted since they were found to be very close to zero.

Program BLKIT

(Block Iteration)

This program calculates the expansion coefficients, a_{ij} , by using a least square fit of an over determined system with Gaussian block iteration. Formulae (103) and (104) are used to generate a system which is solved by the iterative process described in the text.

Input:

Card 1: JK MK LK Format (3I5)
 Card 2: ALPH D S1 S2 C1 C2 Format (6F12.5)
 Card 3-N: Punched output from EIGEN with the first two cards removed.

JK is the number of angular eigenvalues to be used in this approximation.
 MK is the number of radial eigenvalues to be used with each angular eigenvalue in this approximation.
 LK is the number of equally spaced collocation points throughout the radius to be used in generating an overdetermined system.
 PI/ALPH is the interface angle.
 D is the diffusion coefficient ratio.
 S1, S2 are the source terms (S_i/D_i).
 C1, C2 are the absorption coefficients (σ_i/D_i).

Output:

A table of the input and the expansion coefficients is printed. The punched output is of the following form:

Card 1: ALPH D S1 S2 C1 C2 Format (6F12.5)
 Card 2: JK LK 0 (zero) Format (3I5)
 Card 3-N: a_{ij} v_i μ_{ij} Format (3020)

Note: Coefficients $a_{1,2\dots a_{1,MK}}$ have been omitted since they have found to be very close to zero.

Program RESI

(Residues)

This program tests the accuracy of the series by comparing the output source computed by the series with the input source at selected points.

Input:

Card 1:	NA	NR	Format (2I5)
Cards 2-N:	AN(I)	I=1,...,NA	Format (6F12.5)
Cards N+1-M:	RR(I)	I=1,...,NR	Format (6F12.5)
Cards M+1-L:	Punched output from either EQSP or BLKIT.		

NA is the number of angles desired.

NR is the number of radial points along each angle.

AN(I)* π are the angles.

RR(I) are the radial points along each angle.

Output:

A printed table of the input parameters and the input and output source is printed for each point.

Program PTS

(Points)

This program calculates the value of the flux at selected points across the region. This is done by evaluating the series calculated by either EQSP or BLKIT.

Input:

Card 1:	NA	NR	Format (2I5)
Cards 2-N:	AN(I)	I=1, NA	Format (6F12.5)
Cards N+1-M:	RR(I)	I=1, NR	Format (6FP2.5)
Cards M+1- :	The punched output from either BLKIT or EQSP.		

where:

NA	is the number of angles desired.
NR	is the number of radial points along each angle
AN(I)* π	are the angles.
RR(I)	are the radial points along each angle.

Program PROF1

(Profile)

This program plots profiles of the neutron flux across the axis $(1,0)-(1-\pi)$; $(1,\alpha)-(0,0)-(1,\pi-\alpha)$, and for $(1,\theta)$ where $0 \leq \theta \leq \pi$.

Input:

Card 1: NI NC NA Format (3I5)

Card 2-N: Punched output from either EQSP or BLKIT.

NI is the number of divisions across each radius of the interface profile.

NC is the number of division across each radius of the center profile.

NA is the number of divisions across the angular profile.

Output:

A plot of the center profile, interface profile and the angular profile.

Appendix C

```

PROGRAM EIGEN(INPUT,OUTPUT,PUNCH)
CC  THIS PROGRAM FINDS THE EIGENVALUES FOR THE DIFFUSION EQUATION
C    REIV= EIGENVALUES(RADIUS)
C    SAA,SAB = STARTING POINTS FOR ANG. EIGENVALUES ITERATION
C    NN= NO OF EIGENVALUES(ANGLE)
C    MM= NO OF EIGENVALUES(RADIUS)
C    AEIV =EIGEN VALUES (ANGLE)
C    SRA,SKB LIKEWISE FOR RADIUS
C    TEST =TERMINATION VALUE
    DIMENSION AEIV(15),REIV(15,15),STOR(101)
    TEST=0.00000000005
    PI=3.141592653589793
    PUNCH, 5
5   FORMAT(28EIGENVALUES FOR THIS INPUT)
    READ 25,ALPH,D
    PUNCH 25,ALPH,D
25  FORMAT(6F12.8)
    READ 35,NN,MM
    PUNCH 35,N,I,MM
35  FORMAT(2I5)
C    ALPH MUST BE GREATER THAN .2
    TH=PI/ALPH
    CP=ALPH
    SAA=0.13
    SAB=0.083
    KK=0
    CXN=0.0
    KJZ=-1
    PRINT 30
30  FORMAT(1H1,///,25X,* EIGENVALUE TABLE*)
    PRINT 32,ALPH
32  FORMAT(/,25X,* INTERFACE ANGLE AT PI/*,F5.2)
    PRINT 31,D
31  FORMAT( 26X,*THE DIFFUSION COEFFICIENT IS *,F5.3 )
C    THIS CODE FINDS ANGULAR EIGENVALUES USING RECURRENCE RELATION
    DO 100 I=1,NN
    KJZ=KJZ+1
    IF(KJZ.GT. 2) CXN =CXN+1.0
    IF(KJZ.GT. 2) KJZ=1
43  SA=SAA
    SB=SAB
    IF(I.EQ. 1)SA=0.0
    IF(I.LE.3) GO TO 48
    SA=AEIV(KJZ+1)+CP*CXN
    SB=SA+.28
48  TP=SB*(PI TH)
    FB= TAN(TP)+D *TAN(SB*TH)
C    TO INCREASE ITERATIONS EXPAND DO LOOP
    DO 50 IA=1,19
    TP=SA*(PI TH)

```

```

FA= TAN(TP)+D *TAN(SA*TH)
IF(ABS(FA). LE. TEST) GO TO 68
DSA=SA-SB
DFA=FA-FB
SB=SA
SA=FB-(DSA/DFA)*FA
50 FB=FA
63 SAA=SAA+0.175
SAB=SAB+0.175
KK=KK+1
C IF A PROPER VALUE CANNOT BE FOUND IN A SEGMENT PROGRAM ABORTS
C CHANGE HERE FOR MORE ATTEMPS
IF(KK. GE. 15) GO TO 303
GO TO 43
68 IF(I.EQ. 1) GO TO 66
TP=SA-AEIV(I-1)
IF(ABS(TP). LE. TEST) GO TO 63
KK=0
66 PRINT 69,I,SA,FA
69 FORMAT(/,25X,* NU(*,I2,*) = *,3X,F17.11,5X,*F(NU) = *,E12.5)
AEIV(I)=SA
SAA=SA+0.20
SAB=SA+0.175
C
C THIS CODE FINDS RADIAL EIGENVAUES ASSOCIATED WITH EACH ANGULAR VALUE
C USES MCMAHON SERIES AND IMPROVES WITH THE SECANT METHOD
DO 100 J=1,MM
IA=0
C START SERIES FOR DERIVATIVE AS DEFINED IN SP. PAPER
C BP= LETA PRIME FROM PAPER
C SRD= ROOT DESIRED
C SERIES COUNT STARTS WITH ZERO
C SPECIAL CASE FOR ZERO
IF(AEIV(I). LE. TEST. AND. J.EQ.1) GO TO 71
SRD=J-1
BP=0.25*PI*((2.0*AEIV(I))+(4.0*SRD)+1.0)
EBP=8.0*BP
C SM= MIN PAPER
SM=4.0*(AEIV(I)**2)
SMS=SM*SM
SMC=SM*SMS
PS=BP-((SM 3.0)/(8.0*BP))
TP=(4.0*(7.0*SMS+82.0*SM-9.0))/(3.0*(EBP**3))
PS=PS-TP
TP=(32.0*(83.*SMC+2075.*SMS-3039*SM+3537.))/(15.*
1 (EBP**5))
PS=PS-TP
TPB=105.*(EBP**7)
TP=(64.*(6949.*(SM**4)+296492.*SMC-1248002.*SMS+
1 7414380.*SM-5853627.))/TPB
PS=PS-TP
TPB=945.*(EBP**9)
TP=(128.*(70197.*(SM**5)+ 4535387.*SMC*SM 38051230.*
1 SMC+527973862.*SMS-2491515495.*SM+2014126479.))/TPB
TPB=14395*(EBP**11)

```

160

TP=(256.*(5592657.*(SM**6) 507585210.*(SM**5)-
1 7121376977.*(SM**4)+177311093036.*SMC-2079338304417*
2 SMS+9969910286202.*SM-8057531232063.)))/TPB

RA=PS-TP

C TEST TO SEE IF CLOSE ENOUGH

IP=AEIV(I)

TP=IP

FRAC=AEIV(I)-TP

C ROUND OFF HERE ON INTERGERS

TPX=1.0-TEST

IF(FRAC. GE. TPX) IP=IP+1

IF(FRAC. GE. TPX) FRAC=0.0

IL=IP+1

C STOR(1)= J(0),RA))

C STOR(IP+1)=J(AEIV,X)

CALL BESLJ(RA,FRAC,IL, STOR,IE)

IF(IE. GT. 0) GO TO 305

FRA=(AEIV(I)/RA)*STOR(IP+1)-STOR(IP+2)

GO TO 72

C SPECIAL CASE FOR ZERO VALUE

71 RA=0.0

FRA=0.0

72 IF(ABS(FRA). LE. TEST) GO TO 83

RB=RA

FRB=FRA

RA=RB-0.005

C

C START ITERATION FOR RADIUS EIGENVALUES

C FOR MORE ITERATIONS INCREASE DO-LOOP

DO 80 IA=1,19

CALL BESLJ(RA,FRAC,IL, STOR,IE)

IF(IE. GT. 0) GO TO 305

FRA=(AEIV(I)/RA)*STOR(IP+1)-STOR(IP+2)

IF(ABS(FRA).LE. TEST) GO TO 83

DR=RA-RB

DF=FRA-FRB

RB=RA

RA=RB-(DR/DF)*FRA

IF(RA. LE. 0.0) RA= ABS(RA)

80 FRB=FRA

C NO CONVERGENCE

GO TO 303

83 REIV(I,J)=RA

PRINT 86,I,J,RA,FRA

86 FORMAT(25X,* MU(*,I2,*,*,I2,*) = *,F17.11 ,4X,* F(MU) = *,E12.5)

100 CONTINUE

PRINT 137

137 FORMAT(1H1,/,*, COEFFICIENTS FOR TRANSPORT EQATION*)

PRINT 138,ALPH

138 FORMAT(/,*, INTERFACE ANGLE OF PI/*,F4.2)

139 FORMAT(/,*, DIFFUSION COEFFICIENT OF *,F6.2)

PRINT 139, D

PRINT 144,S1,S2

144 FORMAT(/,*, SOURCE TERMS S1= *,E12.5,* S2= *,E12.5)

PRINT 142,C1,C2

```
142 FORMAT(/,* SIGMA 1 = *,E12.5,* SIGMA 2 = *,E12.5)
    PRINT 110,NN,MM
110 FORMAT(/,3X,I3,* ANGULAR MODES EACH HAVING*,I3,*RADIAL MODES*)
    PRINT 190
190 FORMAT(/,5X,* EIGEN VALUE TABLE* )
    DO 194 I=1,NN
    PRINT 195,I, AEIV(I)
    PUNCH 200, AEIV(I)
    PUNCH 200,(REIV(I,J),J=1,MM)
194 PRINT 196,(REIV(I,J),J=1,MM)
195 FORMAT(/,*,* MODE NO. *,I3,* ANGULAR EIGENVALUE *,E12.5)
196 FORMAT(* ASSOCIATED RADIAL VALUES *,6(E12.5,2X))
200 FORMAT(4020)
    GO TO 400
303 PRINT 304
304 FORMAT(* COULDN-T FIND ALL EIGENVALUES*)
    GO TO 400
305 PRINT 306
306 FORMAT(* TROUBLE WITH BESSEL FUNCTION ROUTINE*)
400 CONTINUE
    END
```

```

PROGRAM BLK.IT(INPUT,OUTPUT,PUNCH)
CC  USEING IMPROVED BLOCK ITERATION
C   REIV= EIGENVALUES(RADIUS)
C   NN= NO OF EIGENVALUES(ANGLE)
C   MM= NO OF EIGENVALUES(RADIUS)
C   AEIV =EIGEN VALUES (ANGLE)
C   PI/ALPH = INTERFACE ANGLE
C   D= DIFFUSION RATIO      D1/D2
C   TEST =TERMINATION VALUE
C   JK AND MK ARE THE NUMBER OF AVAILABLE  ANGULAR AND RADIAL EIGENVALUES
C   USED IN THIS PROGRAM
C   LK = NO OF COOLOCATION POINTS
C   S1,S2 ARE THE SOURCE TERMS
C   C1, C2 = SIGMA(I)/D(I)
      DIMENSION STOR(101)
      COMMON/ZAP/AK(112,57),CON(112),AA(57) ,JK,MK,LK
      COMMON AEIV(12),REIV(12,12),ALA(12),ALB(12)
      COMMON PI,TH,TEST,D
      TEST=0.000000000005
      PI=3.141592653589793
      READ 35,JK,MK,LK
      IF(JK. GT. NN. OR. MK. GT. MM) GO TO 803
      PUNCH 10
10  FORMAT(42HEIGENVALUES AND COEFFICIENTS FOR PROBLEM )
      READ 25,ALPH,D,S1,S2,C1,C2
      PUNCH 25,ALPH,D,S1,S2,C1,C2
25  FORMAT(6F12.8)
      READ 35,NN,MM
      PUNCH 35,NN,MM
      PUNCH 35,JK,MK,LK
35  FORMAT(3I5)
      TH=PI/ALPH
      PMT=PI-TH
      DO 30 I=1,NN
      READ 33,AEIV(I)
      READ 33,(REIV(I,J),J=1,MM)
33  FORMAT(4020)
      ALA(I)=COS(AEIV(I)*PMT)
30  ALB(I)=COS(AEIV(I)*TH)
C   GENERATE MATRIX
      LROW=0
      TLK=LK
      DO 150 JJ=1,JK
      DO 150 LL=1,LK
      LCOL=0
      LROW=LROW+1
      TP=LL
      RR=TP/TLK
C   RR=RADIUS BEING USED
      CALL SIVECT(TP,S1,S2,JJ)

```

```

CON(LROW)=TP*RR
DO 150 II=1,JK
DO 150 KK=1,MK
IF(II.EQ. 1 .AND. KK. GT. 1) GO TO 150
LCOL=LCOL+1
CALL JFUNCT(FJ,RR,II,KK,IE)
IF(IE. GT.0) GO TO 805
CALL SIGMA(SIG,C1,C2,II,JJ)
IF(II.EQ. JJ) GO TO 105
AK(LROW,LCOL)=FJ*SIG *RR
GO TO 150
105 CALL ANGNORM(ANF,II)
AK(LROW,LCOL)=FJ*(SIG+(REIV(II,KK)**2)*ANF) *RR
150 CONTINUE
C
C SOLVE SYSTEM TO GET AA COEFFICIENTS
C SET ALL COEFF TO 0
LP=(JK-1)*IK+1
DO 190 I=1,LP
190 AA(I)=0.0
DO 250 I=1,JK
CALL LSBK(I,I)
IF(I.EQ. 1) GO TO 225
LQ=I-1
DO 210 L=1,LQ
CALL LSBK(L,I)
210 CONTINUE
225 IP=0
PRINT 263
263 FORMAT(1H )
DO 250 II=1,JK
DO 250 JJ=1,MK
IF(II.EQ. 1 .AND. JJ.GT. 1) GO TO 250
IP=IP+1
PRINT 235,II,JJ,AA(IP)
235 FORMAT( * A(*,I2,*,*,I2,*)= *,E15.8)
250 CONTINUE
C PURIFYING SOLUTION BY JACOBIAN ITERATION
DO 275 I=1,6
TA=AA(1)
TB=AA(2)
DO 260 L=1,JK
260 CALL LSBK(L,JK)
TAC=AA(1) TA
TBC=AA(2)-TB
PRINT 263
IP=0
DO 270 II=1,JK
DO 270 JJ=1,MK
IF(II.EQ. 1 .AND. JJ. GT. 1) GO TO 270
IP=IP+1
PRINT 235,II,JJ, AA(IP)
270 CONTINUE
IF(ABS(TAC) .LE. 0.000001 .AND. ABS(TBC) .LE. 0.000001) GO TO 299
275 CONTINUE

```

```

299 PRINT 336
336 FORMAT(1H1,///,25X,* EXPANSION COEFFICIENTS FOR THE DIFFUSION *,
1 *EQUATION*)
PRINT 337, ALPH
337 FORMAT( /,25X,*INPUT DATA*,//,25X,*UNIT CIRCLE*,10X,
1 *INTERFACE AT PI/* ,F5.2)
PRINT 338,D
338 FORMAT( 25X,*DIFFUSION COEFFICIENT = *,F5.2)
PRINT 339
339 FORMAT( /,25X,* DOMAIN 1*,20X,*DOMAIN 2*)
PRINT 340,C1, C2
340 FORMAT( 25X,* SIGMA/D = *,F9.5,5X,*SIGMA/D = *,F9.5)
PRINT 341, S1,S2
341 FORMAT( 25X,* SOURCE = *,F9.5,5X,* SOURCE = *,F9.5)
PRINT 345,JK
345 FORMAT(//,25X, *THIS APPROXIMATION USES *,I2,
1* ANGULAR EIGENVALUES*)
PRINT 346,MK ,LK
346 FORMAT(25X, *EACH HAVING *,I2,* RADIAL EIGENVALUES *,/,
1 25X,*USING *,I2,* EQUALLY SPACED WEIGHTED COLLOCATION POINTS*)
IP=0
DO 380 II=1,JK
PRINT 381
381 FORMAT(1H )
DO 380 JJ=1,MK
IF(II.EQ. 1. AND .JJ. GT. 1) GO TO 380
IP=IP+1
PRINT 335,II,JJ,AA(IP)
335 FORMAT(25X,*A(*,I2,*,*,I2,*) = *,F17.11)
PUNCH 379, AA(IP),AEIV(II),REIV(II,JJ)
379 FORMAT(3020)
380 CONTINUE
GO TO 900
C ERROR MESSAGES
803 PRINT 804
804 FORMAT(* REQUESTED NO OF EIGENVALUES NOT AVAILABLE*)
GO TO 900
805 PRINT 806
806 FORMAT(* TROUBLE WITH BESSEL-S FCTN*)
900 CONTINUE
END
SUBROUTINE SIVECT(SL,S1,S2,I)
C SL=S(L)
COMMON AEIV(12),REIV(12,12),ALA(12),ALB(12)
COMMON PI,TH,TEST,D
IF(AEIV(I). LE. TEST) GO TO 10
TP=AEIV(I)*TH
TPX=AEIV(I)*(PI-TH)
SL=(ALA(I)*S1*SIN(TP))*D/ AEIV(I)
SL=SL+(ALB(I)*S2*SIN(TPX))/ AEIV(I)
RETURN
10 SL=S1*D+S2*(PI-TH)
RETURN
END
SUBROUTINE ANGNORM(ANF, II)

```

```

C   ANF=ANGLE NORMALIZATION FACTOR
      COMMON AEIV(12),REIV(12,12),ALA(12),ALB(12)
      COMMON PI,TH,TEST,D
      TPX=PI-TH
      IF(AEIV(I).LE.TEST) GO TO 15
      TP=2*AEIV(I)
      AN1=(ALA(I)**2)*0.5 *(TH+(SIN(TP*TH)/TP))
      AN2=(ALB(I)**2)*0.5 *(TPX ( SIN(TP*TPX)/( TP)))
      ANF=D*AN1+AN2
      RETURN
15  ANF=D*TH+TPX
      RETURN
END
SUBROUTINE JFUNCT(FR,RR,I,J,IE)
C   THIS IS OUR BESSEL FUNCTION
C   IE= 1 MEANS ERROE
      DIMENSION STOR(100)
      COMMON AEIV(12),REIV(12,12),ALA(12),ALB(12)
      COMMON PI,TH,TEST,D
      IE=0
      IP=AEIV(I)
      TP=IP
      FRAC=AEIV(I)-TP
      TPX=1.0-TEST
      IF( FRAC .GE. TPX) IP=IP+1
      IF( FRAC .GE. TPX) FRAC= 0.0
      XX=RR*REIV(I,J)
C   TEST FOR REIV=0
      IF(XX.LE.TEST) GO TO 10
      CALL BESLJ(XX,FRAC,IP,STOR,IE)
      FR=STOR(IP-1)
      RETURN
10  FR=0.0
      IF(AEIV(I).LE.TEST) FR=1.0
      RETURN
END
SUBROUTINE SIGMA(PSI,C1,C2,I,II)
C   INTEGRAL OF PSI*PSI*C(TH) = PSI
      COMMON AEIV(12),REIV(12,12),ALA(12),ALB(12)
      COMMON PI,TH,TEST,D
      IF(I.EQ. II) GO TO 20
      TP=AEIV(II)-AEIV(I)
      TPX=AEIV(II)+AEIV(I)
      THM=PI-TH
      PSI= C1*0.5 *D *ALA(I)*ALA(II)*((SIN(TP*TH)/TP)+(SIN(TPX*TH)
1 /TPX))
      PSI=PSI+ (C2 *0.5*ALB(I)*ALB(II)*((SIN(TP*THM)/TP)+
1 (SIN(TPX*THM)/TPX)) )
      RETURN
20  TPX=PI-TH
      IF(AEIV(I).LE.TEST) GO TO 30
      TP=2*AEIV(I)
      AN1=(ALA(I)**2)*0.5 *(TH+(SIN(TP*TH)/TP))
      AN2=(ALB(I)**2)*0.5 *(TPX ( SIN(TP*TPX)/( TP)))
      PSI=D*C1*AN1+C2*AN2

```



```

      RETURN
C     PSI= INTEGRAL OF ANG FUN.**2 *C(TH)
30    PSI=D*CI*TH+TPX*C2
      RETURN
      END
      SUBROUTINE LSBK(I,LQ)
C     THIS SUBROUTINE DOES A LEAST SQUARE ON AN INCREASING
C     OVERTERMINED SYSTEM
C     TO CHANGE THE SIZE OF THE ARRAYS  SET ZA(K1,K2)
C     WHERE K1 = NUMBER OF ROWS IN AK AND K2 = (K1+3*MM)/MM
C     ALSO SET KZ =K1
      COMMON/ZAP/AK(112,57),CON(112),AA(57),JK,MK,LK
      DIMENSION ZA(112,25)
      KZ=112
      LZ=1
      MA=MK
      IF(I.EQ. 1)MA=1
      LC=(JK-1)*MK+1
      IS=(I-2)*MA+1
      ID=(I-1)*MA+1
      LR=LQ*LK
C     IS =START OF COLUMN      ID= END OF COLIMN
      DO 15 I=1,LR
      SUM=0.0
      DO 10 L=1,LC
      IF(L. GT. IS. AND. L. LE. ID) GO TO 10
      SUM=SUM+AA(L)*AK(I,L)
10    CONTINUE
15    ZA(I,MA+1)=CON(I)-SUM
      DO 20 I=1,LR
      DO 20 J=1,MA
20    ZA(I,J)=AK(I,IS+J)
      CALL HLSQ(ZA,KZ,LR,MA,1,IER,1.E-8)
      DO 30 I=1,MA
30    AA(IS+I)=ZA(I,MA+1)
      I=LZ
      RETURN
      END

```

Flow Chart for Block Iteration
BLKIT Generation of Matrix

Figure 34

Block Iteration
Iteration Driver

Figure 35

LSBK
Driver for Householder Routine

Figure 36.

Note: The routine use for the Householder Method is HLSQ No. 221 in Vogelback Library.

```

      1
PROGRAM EQSP(INPUT,OUTPUT,PUNCH)
C   THIS PROGRAM CALCULATES THE SPECIAL CASE
C   WHERE  $\text{SIGMA}(1)/D = \text{SIGMA}(2)/D$ 
  DIMENSION STOR(101),AA(64)
  COMMON AEIV(12),REIV(12,12),ALA(12),ALB(12)
  COMMON PI,TH,TEST ,D
C   NN= NO OF EIGENVALUES(ANGLE)
C   MM= NO OF EIGENVALUES(RADIUS)
C   AEIV =EIGEN VALUES (ANGLE)
C   REIV= EIGENVALUES(RADIUS)
  PI=3.141592653589793
C   TEST =TERMINATION VALUE
  TEST=0.000000000005
  READ 35,JK,LK
  READ 25,ALPH,D,S1,S2,C1,C2
  PUNCH 25, ALPH,D,S1,S2,C1,C2
25  FORMAT(6F12.8)
  READ 35,NN,MM
  PUNCH35,NN,MM
  I=0
  PUNCH 35,JK,LK ,I
35  FORMAT(3I5)
  TH=PI/ALPH
  PMT=PI-TH
  DO 40 I=1,NN
  READ 43,AEIV(I)
  READ 43,(REIV(I,J),J=1,MM)
43  FORMAT(4020)
  ALA(I)=COS(PMT*AEIV(I))
40  ALB(I)=COS(TH*AEIV(I))
C   HEADING
  PRINT136
136  FORMAT(1H1,/,25X,* EXPANSION COEFFICIENTS FOR THE DIFFUSION *,
1    *EQUATION*)
  PRINT 137, ALPH
137  FORMAT( /,25X,*INPUT DATA*,/,25X,*UNIT CIRCLE*,10X,
1    *INTERFACE AT PI/*,F5.2)
  PRINT 138,D
138  FORMAT( 25X,*DIFFUSION COEFFICIENT = *,F5.2)
  PRINT 139
139  FORMAT( /,25X,* DOMAIN 1*,12X,*DOMAIN 2*)
  PRINT 140,C1,C2
140  FORMAT( 25X,* SIGMA/D = *,F5.2,5X,*SIGMA/D = *,F5.2)
  PRINT 141, S1,S2
141  FORMAT( 25X,* SOURCE = *,F5.2,5X,* SOURCE = *,F5.2)
  PRINT 145,JK
145  FORMAT(/,25X, *THIS APPROXIMATION USES *,I2,
1    * ANGULAR EIGENVALUES*)
  PRINT 146,LK
146  FORMAT(25X, *EACH HAVING *,I2,* RADIAL EIGENVALUES.*)

```

```

171
147 PRINT 147
147 FORMAT(/,25X,*THE EXPANSION COEFFICIENTS WERE CALCULATED BY *,/,
1 25X,*USING THE SPECIAL CASE*)
IP=0
DO 150 II=1,JK
PRINT 325
325 FORMAT(1H )
DO 150 JJ=1,LK
IF(II.EQ. 1 .AND. JJ.GT.1) GO TO 150
IP=IP+1
CALL SIVECT(TP,S1,S2,II)
CALL IJR(FJ,II,JJ,IE)
IF(IE. GT.0) GO TO 805
CALL ANGNORM(AG, II)
CALL JNOR(XJ,II,JJ,IE)
IF(IE. GT. 0) GO TO 805
TPX = (AG*XJ*((REIV(II,JJ)**2)+C1) )
TPA=TP*FJ
AA(IP)=TPA/TPX
PUNCH 148,AA(IP),AEIV(II),REIV(II,JJ)
148 FORMAT(3020)
PRINT 149,II,JJ, AA(IP)
149 FORMAT(25X,*A(*,I2,**,I2,*)= *,F17.11)
150 CONTINUE
GO TO 900
805 PRINT 806
806 FORMAT(* TROUBLE WITH BESSEL-S FCTN*)
900 CONTINUE
END
SUBROUTINE SIVECT(SL,S1,S2,I)
C SL=S(L)
COMMON AEIV(12),REIV(12,12),ALA(12),ALB(12)
COMMON PI,TH,TEST ,D
IF(AEIV(I). LE. TEST) GO TO 10
TP=AEIV(I)*TH
TPX=AEIV(I)*(PI-TH)
SL=(ALA(I)*S1*SIN(TP))*D/ AEIV(I)
SL=SL+(ALB(I)*S2*SIN(TPX))/ AEIV(I)
RETURN
10 SL=S1*D*TH+S2*(PI-TH)
RETURN
END
SUBROUTINE ANGNORM(ANF, I)
C ANF=ANGLE NORMALIZATION FACTOR
COMMON AEIV(12),REIV(12,12),ALA(12),ALB(12)
COMMON PI,TH,TEST ,D
TPX=PI-TH
IF(AEIV(I). LE. TEST) GO TO 15
TP=2*AEIV(I)
AN1=(ALA(I)**2)*0.5 *(TH+(SIN(TP*TH)/TP))
AN2=(ALB(I)**2)*0.5 *(TPX ( SIN(TP*TPX)/( TP)))
ANF=AN1*D AN2
RETURN
15 ANF=TH*D+TPX
RETURN

```

```

END
SUBROUTINE IJR(FR,I,J,IE)
C   COMPUTES INTERGRAL OF J(MU)*R
C   IE= 1 MEANS ERROE
DIMENSION STOR(101)
COMMON AEIV(12),REIV(12,12),ALA(12),ALB(12)
COMMON PI,TH,TEST ,D
IE=0
IF(REIV(I,J) .LE. TEST) GO TO 40
IP=AEIV(I)
TP=IP
FRAC=AEIV(I)-TP
TPX=1.0-TEST
IF( FRAC .GE. TPX) IP=IP+1
IF( FRAC .GE. TPX) FRAC= 0.0
IX=IP+45
CALL BESLJ(REIV(I,J),FRAC,IX,STOR,IE)
TXX=TEST/(AEIV(I)+2.0)
PS=0.0
BOT=1.0
AA=(AEIV(I)+4.0)/2.0
DO 20 IL=1,45
TL=IL
NO=IP+IL
IF(IL. GT. 1)BOT=(AA+TL-2.0)*GAM
GAM=BOT
ILL=IL-1
TP=((REIV(I,J)/2.0)**ILL)*STOR(NO))/BOT
PS=PS+TP
IF(ABS(TP). LE. TXX) GO TO 30
IF(IL.LT. 45) GO TO 20
PRINT 10,TP
10 FORMAT(* TROUBLE WITH POWER SERIES LAST TERM*,E12.5)
IE=1
20 CONTINUE
30 FR=PS/(AEIV(I)+2.0)
RETURN
40 FR=0.5
RETURN
END
SUBROUTINE JNOR(XJ,I,J,IE)
C   COMPUTES INTERGRAL OF J*J
C   DIMENSION STOR(101)
COMMON AEIV(12),REIV(12,12),ALA(12),ALB(12)
COMMON PI,TH,TEST ,D
IE=C
IF(REIV(I,J) .LE. TEST) GO TO 40
IP=AEIV(I)
TP=IP
FRAC=AEIV(I)-TP
TPX=1.0-TEST
IF( FRAC .GE. TPX) IP=IP+1
IF( FRAC .GE. TPX) FRAC= 0.0
CALL BESLJ(REIV(I,J),FRAC,IP,STOR,IE)
TP=.5*(1.0 - ((AEIV(I)/REIV(I,J))**2))

```

```
XJ=TP*(STOR(IP+1)**2)
RETURN
40 XJ=0.0
IF(AEIV(1).LE.TEST) XJ=0.5
RETURN
END
```



```

      1
      PROGRAM RESI(INPUT,OUTPUT)
C     THIS PROGRAM CALCULATES THE FOLLOWING PROBLEM USING THE
C     EIGENVALUES AND SERIES EXPANSION COEFFICIENTS TO COMPUTE
C     U(R,THETA)
C     (DEL**2)*U(R,THETA)+(C(I)/D(I))*U(R,THETA )= S(I)
C     WHERE S(I) IS THE COMPUTED RESIDUES TO THIS PROBLEM
C     THE INPUT IS THE PUNCHED OUTPUT FROM A COEFFICIENT PROGRAM
C     REMOVE TITLE CARD AND PLACE THE NO OF POINTS AND LOCATION
C     OF THESE POINTS IN FRONT OF THE PUNCHED DATA
      DIMENSION AN(15),RR(15)
      COMMON AA(12,12),TH,PI,ALA(12),ALB(12),JK,MK,AEIV(12),REIV(12,12)
      1 ,C1,C2
      PI=3.141592653589793
      READ 35,NA,NR
      READ 25,(AN(I),I=1,NA)
      READ 25,(RR(I),I=1,NR)
C     PUNCHED OUTPUT FROM COEFFICIENT PROGRAM
      READ 25,ALPH,D,S1,S2,C1,C2
      25 FORMAT(6F12.5)
      TH=PI/ALPH
      READ 35,JK,MK,LK
      35 FORMAT(3I5)
      TP=PI-TH
      DO 34 I=1,JK
      DO 34 J=1,MK
      IF(I.EQ.1 .AND. J.GT. 1) GO TO 34
      READ 45,AA(I,J),AEIV(I),REIV(I,J)
      45 FORMAT(3020)
      ALA(I)= COS(AEIV(I)*TP)
      ALB(I)= COS(AEIV(I)*TH)
      34 CONTINUE
      PRINT 36
      36 FORMAT(1H1,///,25X,* INPUT SOURCE COMPARED WITH OUTPUT SOURCE*,/
      1 ,25X,* AT SELECTED POINTS*)
      PRINT 37,ALPH
      37 FORMAT( /,25X,*INPUT DATA*, /,25X,*UNIT CIRCLE*,10X,
      1 *INTERFACE AT PI/*,F5.2)
      PRINT 38,D
      38 FORM/T( 25X,*DIFFUSION COEFFICIENT = *,F5.2)
      PRINT 39
      39 FORMAT( /,25X,* DOMAIN 1*,20X,*DOMAIN 2*)
      PRINT 40,C1,C2
      40 FORMAT( 25X,* SIGMA/D = *,F9.5,5X,*SIGMA/D = *,F9.5)
      PRINT 41, S1,S2
      41 FORMAT( 25X,* SOURCE = *,F9.5,5X,* SOURCE = *,F9.5)
      PRINT 42,JK
      42 FORMAT(///,25X,* THIS APPROXIMATION USES *,I2,
      1* ANGULAR EIGENVALUES*)
      PRINT 46, MK
      46 FORMAT(25X,* EACH HAVING *,I2,* RADIAL EIGENVALUES.*)

```

```

      IF(LK.LT. 1) GO TO 48
      PRINT 47,LK
47  FORMAT(/,25X,*THE EXPANSION COEFFICIENTS WERE CALCULATED BY *,/,
      1 25X,*USING *,I2, * EQUALLY SPACED COLLOCATION POINTS.*,/)
      GO TO 50
48  PRINT 49
49  FORMAT(/,25X,*THE EXPANSION COEFFICIENTS WERE CALCULATED BY *,/,
      1 25X,*USEING THE SPECIAL CASE*,/)
50  PRINT 99
99  FORMAT( 25X,*RADIUS*,6X,*ANGLE*,7X,* INPUT*,5X,*OUTPUT*,/ ,49X,
      1 *SOURCE*,5X,*SOURCE*,/)
      DO 80 I=1,NA
      PRINT 90
90  FORMAT(1H )
      DO 80 IR=1,NR
      ACT=PI*AN(I)
      CALL DFX(RES,ACT,RR(IR))
      SX=(S1+S2)/2.0
      IF(ACT.LT. TH) SX=S1
      IF(ACT.GT. TH) SX=S2
      PRINT 60,RR(IR),AN(I),SX,RES
60  FORMAT(25X,F6.3,5X,F5.2,*X PI*,4X,F5.2,5X,F10.6)
80  CONTINUE
      END
      SUBROUTINE DFX( SUM,ANG,RR)
      DIMENSION STOR(101)
      COMMON AA(12,12),TH,PI,ALA(12),ALB(12),NN,MM,AEIV(12),REIV(12,12)
      1 ,C1,C2
      SUM=0.0
      DO 15 I=1,NN
      TP=AEIV(I)*ANG
      TPX=AEIV(I)*(PI-ANG)
      CHA=1.0
      CHB=1.0
      IF(ANG.LE. TH)CHB=0.0
      IF(ANG.GT. TH)CHA=0.0
      SIG=CHA*C1 CHB*C2
      PSI=ALA(I)*CHA*COS(TP)+ALB(I)*CHB*COS(TPX)
      IP=AEIV(I)
      TP=IP
      FRAC=AEIV(I)-TP
      IF(FRAC.GE. 0.999995) IP=IP+1
      IF(FRAC.GE. 0.999995) FRAC=0.0
      DO 15 K=1,MM
      IF(I.EQ. 1 .AND. K.GT.1) GO TO 15
      XX=REIV(I,K)*RR
      IF(XX.LE. 0.000005) GO TO 13
      CALL BESLJ(XX,FRAC,IP,STOR,IE)
      GO TO 14
13  STOR(IP+1)=0.0
      IF(AEIV(I).LE.0.000005) STOR(IP+1)=1.0
14  SUM=SUM+PSI*AA(I,K)*STOR(IP+1)*((REIV(I,K)**2)+SIG)
15  CONTINUE
      RETURN
      END

```

```

1
PROGRAM PTS(INPUT,OUTPUT)
DIMENSION AN(12),RR(12)
COMMON AA(12,12),TH,PI,ALA(12),ALB(12),JK,MK,AEIV(12),REIV(12,12)
C AEIV = EIGEN VALUES (ANGLE)
C REIV = EIGENVALUES(RADIUS)
PI=3.141592653589793
READ 35,NA,NR
READ 25,( AN(I),I=1,NA)
READ 25,( RR(I),I=1,NR)
READ 25,ALPH,D,S1,S2,C1,C2
25 FORMAT(6F12.8)
READ 35,JK,MK,LK
TH=PI/ALPH
35 FORMAT(3I5)
PMT=PI-TH
DO 40 I=1,JK
DO 40 J=1,MK
IF(I.EQ. 1 .AND. J.GT.1) GO TO 40
READ 43, AA(I,J),AEIV(I),REIV(I,J)
43 FORMAT(3020)
ALA(I)=COS(PMT*AEIV(I))
ALB(I)=COS(TH*AEIV(I))
40 CONTINUE
C HEADING
PRINT136
136 FORMAT(1H1,///,25X,* VALUES OF SELECTED POINTS*,/)
PRINT 137, ALPH
137 FORMAT( /,25X,*INPUT DATA*,//,25X,*UNIT CIRCLE*,10X,
1 *INTERFACE AT PI/*,F5.2)
PRINT 138,D
138 FORMAT( 25X,*DIFFUSION COEFFICIENT = *,F5.2)
PRINT 139
139 FORMAT( /,25X,* DOMAIN 1*,12X,*DOMAIN 2*)
PRINT 140,C1,C2
140 FORMAT( 25X,* SIGMA/D = *,F5.2,5X,*SIGMA/D = *,F5.2)
PRINT 141, S1,S2
141 FORMAT( 25X,* SOURCE = *,F5.2,5X,* SOURCE = *,F5.2)
PRINT 145,JK
145 FORMAT(//,25X, *THIS APPROXIMATION USES *,I2,
1* ANGULAR EIGENVALUES*)
PRINT 146,MK
146 FORMAT(25X, *EACH HAVING *,I2,* RADIAL EIGENVALUES.*)
IF(IK.LT. 1) GO TO 149
PRINT 147,LK
147 FORMAT(/,25X,*THE EXPANSION COEFFICIENTS WERE CALCULATED BY *,/,
1 25X,*USING *,I2,* EQUALLY SPACED WEIGHTED COLLOCATION POINTS*,/)
GO TO 151
149 PRINT 150
150 FORMAT(/,25X,*THE EXPANSION COEFFICIENTS WERE CALCULATED BY *,/,
1 25X,*USING THE SPECIAL CASE*,/)

```

```

151 PRINT 148
148 FORMAT(25X,*ANGLE*,12X,*RADIUS*,11X,*POINT VALUE*,/)
    DO 160 IA=1,NA
      ANG=AN(IA)*PI
      PRINT 154
154 FORMAT(1H )
      DO 160 IR=1,NR
        CALL FUN(VAL,ANG,RR(IR))
        PRINT 155, AN(IA),RR(IR),VAL
155 FORMAT(25X,F6.3,* X PI*,5X,F6.3,8X,F15.8)
160 CONTINUE
      END
      SUBROUTINE FUN (SUM, ANG,RR)
C      THIS SUBROUTINE CALCULATES THE SERIES AT A CERTAIN POINT
      COMMON AA(12,12),TH,PI,ALA(12),ALB(12),JK,MK,AEIV(12),REIV(12,12)
      DIMENSION STOR(101)
      PMT=PI-ANG
      SUM=0.0
      DO 15 I=1,JK
        IF(ANG.LE.TH)PSI=ALA(I)*COS(AEIV(I)*ANG)
        IF(ANG.GT.TH)PSI=ALB(I)*COS(AEIV(I)*PMT)
        IP=AEIV(I)
        TP=IP
        FRAC=AEIV(I)-TP
        IF(FRAC.GE. 0.999995) IP=IP+1
        IF(FRAC.GE. 0.999995) FRAC=0.0
        DO 15 K=1,MK
          IF(I.EQ. 1 .AND. K.GT.1) GO TO 15
          XX=REIV(I,K)*RR
          IF(XX.LE. 0.000005) GO TO 13
          CALL BESLJ(XX,FRAC,IP,STOR,IE)
          GO TO 14
13      STOR(IP+1)=0.0
          IF(AEIV(I).LE.0.000005) STOR(IP+1)=1.0
14      SUM=SUM+PSI*AA(I,K)*STOR(IP+1)
15      CONTINUE
      RETURN
      END

```

```

      1
PROGRAM PROF1(INPUT,OUTPUT, TAPE 99)
DIMENSION AY(22),AX(11),GY(36),GX(36),SS(2),SC(2),IPTO(10),NOO(9),
1 NOT(9),NOTH(10),F(42,1),ASX(3),ASY(6),Z(42)
COMMON AA(12,12),TH,PI,ALA(12),ALB(12),JK,MK,AEIV(12),REIV(12,12)
DATA NOO/1,4,7,12,17,23,29,35,40/
DATA NOT/13,14,15,16,17,22,27,32,31/
DATA NOTH/1,2,8,13,19,25,31,37,41,40/
DATA IPTO/36,2,4,6,8,10,12,14,16,18/
DATA ASX/1.,1.3,1.7/
DATA ASY/3.07,2.88,2.7,2.3,2.12,1.93/
C SCZZ= UPPER LIMIT OF X AXIS
SCZZ=0.02
SCZ1=SCZZ/2.0
CALL NAMPLT
READ 35 ,NIP,NCP,NAP
35 FORMAT(3I5)
READ 35,NPO
PI=3.141592653589793
DO 250 IXX=1,NPO
READ 25,ALPH,D,S1,S2,C1,C2
25 FORMAT(6F12.5)
TH=PI/ALPH
READ 35,JK,MK,LK
TP=PI-TH
DO 40 I=1,JK
DO 40 J=1,MK
IF(I.EQ.1.AND.J.GT.1) GO TO 40
READ 45,AA(I,J),AEIV(I),REIV(I,J)
45 FORMAT(30Z0)
ALA(I)=COS(AEIV(I)*TP)
ALB(I)=COS(AEIV(I)*TH)
40 CONTINUE
PRINT 38,ALPH
38 FORMAT(/,* INTERFACE ANGLE IS PI/*,F5.2)
PRINT 39,D
39 FORMAT(/,* DIFFUSION COEFFICIENT IS *,F5.2)
PRINT 44,S1,S2
44 FORMAT(/,* SOURCE TERMS S1= *,E12.5,* S2= *,E12.5)
PRINT 42,C1,C2
42 FORMAT(/,* SIGMA 1 = *,E12.5,* SIGMA 2 = *,E12.5)
PRINT 333
333 FORMAT(* COEFFICIENTS USING COLLOCATION METHOD*,/,
1 * USING IMPROVED BLOCK ITERATION*,/)
PRINT 110,JK,MK
110 FORMAT(/,3X,I3,* ANGULAR MODES EACH HAVING*,I3,*RADIAL MODES*)
PRINT 112,JK,LK
112 FORMAT(/,* USING *,I3,* DIFFERENT EQUATIONS EVALUATED AT*,I3,
1 * DIFFERENT POINTS*)
DO 111 I=1,20
111 AY(I)=I*.5

```

```

DO 100 I=1,9
100  AX(I)=I
    CALL SCALE(AY,10.,20,1)
    CALL SCALE(AX,10., 9,1)
    CALL SYMBOL(0.,3.5,.15,23HSERIES EXPANSION METHOD,90.0,23)
    CALL SYMBOL(1.3,6.5,.15,11HS(2) = 0. ,90.,11)
    CALL SYMBOL(1.6,6.5,.15,6HS(1) =,90.,6)
    CALL NUMBER(1.6,7.5,.15,S1 ,90.,4HF9.4)
    CALL SYMBOL(1.9,6.5,.15,6HC(2) =,90.,6)
    CALL NUMBER(1.9,7.5,.15, C2,90.,4HF9.4)
    CALL SYMBOL(2.2,6.5,.15,6HC(1) =,90.,6)
    CALL NUMBER(2.2,7.5,.15, C1,90.,4HF9.4)
    CALL SYMBOL(2.5,6.5,.15,6HD =,90.,6)
    CALL NUMBER(2.5,7.5,.15, D,90.,4HF9.4)
    CALL SYMBOL(2.8,6.5,.15,10HALPH = ,90.,10)
    TP=(90.0/ALPH)*2.0
    IANG=TP
    CALL NUMBER(2.8,7.70,.15,IAN,90.0,2HI2)
    CALL SYMBOL(7.5,1.1,.10,14HCENTER PROFILE,90.,14)
    CALL PLOT(7.,4.,3)
    CALL PLOT(7.,0.,2)
    CALL PLOT(5.,0.,2)
    CALL SYMBOL(7.5,6.1,.10,17HINTERFACE PROFILE,90.,17)
    CALL PLOT(7.,9.,3)
    CALL PLOT(7.,5.,2)
    CALL PLOT(5.,5.,2)
    CALL SYMBOL(11.5,1.1,.10,15HANGULAR PROFILE,90.,15)
    CALL PLOT(11.,6.,3)
    CALL PLOT(11.,0.,2)
    CALL PLOT(9.,0.,2)
    CALL NUMBER(5.,-.25,.1,SCZZ,90.0,4HF5.2)
    CALL NUMBER(6.,-.25,.1,SCZ1,90.0,4HF5.2)
    CALL NUMBER(5.,4.75,.1,SCZZ,90.0,4HF5.2)
    CALL NUMBER(6.,4.75,.1,SCZ1,90.0,4HF5.2)
    CALL NUMBER(10.,-.25,.1,SCZ1,90.0,4HF5.2)
    CALL NUMBER(9.0,-.25 ,.1,SCZZ,90.0,4HF5.2)
    XO=0.
    DO 116 I=1,9
    IO=NOO(I)
116  CALL NUMBER(7.15,XO,.10,IO,90.0,2HI2)
    XO=XO+.5
    DO 117 I=1,9
    XO=XO+.5
    IT=NOT(I)
117  CALL NUMBER(7.15,XO,.10,IT,90.,2HI2)
    XO=0.
    DO 118 I=1,10,3
    ITH=NOTH(I)
118  CALL NUMBER(11.15,XO,.10,ITH,90.,2HI2)
    XO=XO+2.
    CALL NUMBER(.5,2.5,.08,42,90.,2HI2)
    CALL PLOT(2.,1.,3)
    CALL PLOT(2.,4.,2)
    DO 113 I=1,36
    THET=I*3.1415926/18. +3.1415926/2.

```

```

      FY=1.5*COS(THET)
      FX=1.5*SIN(THET)
      GY(I)=2.+FY
      GX(I)=2.5+FX
113  CALL PLOT(GY(I),GX(I),2)
      CALL PLOT(GY(6),GX(6),3)
      CALL PLOT(GY(24),GX(24),2)
      CALL PLOT(GY(12),GX(12),3)
      CALL PLOT(2.,2.5,2)
      DO 114 I=1,10
        IJJ=NOTH(I)
        IJ=IPTO(I)
114  CALL NUMBER(GY(IJ),GX(IJ),.10,IJJ,90.,2HI2)
      XC=4.
      CALL PLOT(2.,4.,3)
      DO 115 I=1,9
        KJ=NOO(I)
115  CALL NUMBER( 2.,XC,.10,KJ,90.,2HI2)
      XC=XC-.375
      DO 119 I=1,3
        L1=4-I
        L2=I+1
        L3=I+3
        L4=I+5
        CALL NUMBER(ASX(I),ASY(I),.10,NOT(L2),90.,2HI2)
119  CALL NUMBER(ASX(L1),ASY(L3),.10,NOT(L4),90.,2HI2)
      C  SX=SCALER
      C  FIX 30 LARGEST =2.0
      SX=2.0/SCZZ
      CALL PLOT(7.0,0.0,-6)
      TP =NIP
      PTSZ=1.0/TP
      STSZ=2.0/TP
      PY=-STSZ
      RR=1.0+PTSZ
      ANG=0.0
      NI=2*NIP+1
      DO 211 I=1,NI
        RR=RR-PTSZ
        RA=RR
        IF(RR.LT. 0.0) ANG=PI
        IF(RR.LT. 0.0) RA=-RR
        PY=PY+STSZ
        CALL FUN(PX,ANG,RA)
        PX=-PX*SX
        CALL PLOT(PX,PY,2)
211  CONTINUE
      CALL PLOT(0.0,5.0,-6)
      TP=NCP
      PTSZ=1.0/TP
      STSZ=2.0/TP
      PY=-STSZ
      RR=1.0+PTSZ
      ANG=TH
      NI=2*NCP+1

```

```

DO 212 I=1,NI
RR=RR-PTSZ
RA=RR
IF(RR.LT. 0.0)RA=-RR
IF(RR.LT. 0.0) ANG=PI-TH
CALL FUN(PX,ANG,RA)
PY=PY+STSZ
PX=-PX*SX
CALL PLOT(PX,PY,2)
212 CONTINUE
CALL PLOT(4.0,-5.0,-6)
TP=NAP
PTSZ=6.0/TP
RR=1.0
PY=-PTSZ
NI=NAP+1
DO 213 I=1,NI
TX=I-1
ANG=(TX/TP)*PI
PY=PY+PTSZ
CALL FUN(PX,ANG,RA)
PX=-PX*SX
213 CALL PLOT(PX,PY,2)
CALL PLOT(3.0,0.0,-6)
250 CONTINUE
CALL ENDPLT
END
C SUBROUTINE FUN (SUM, ANG,RR)
THIS SUBROUTINE CALCULATES THE SERIES AT A CERTAIN POINT
COMMON AA(12,12),TH,PI,ALA(12),ALB(12),JK,MK,AEIV(12),REIV(12,12)
DIMENSION STOR(101)
PMT=PI-ANG
SUM=0.0
DO 15 I=1,JK
IF(ANG.LE. TH)PSI=ALA(I)*COS(AEIV(I)*ANG)
IF(ANG.GT. TH)PSI=ALB(I)*COS(AEIV(I)*PMT)
IP=AEIV(I)
TP=IP
FRAC=AEIV(I)-TP
IF(FRAC.GE. 0.999995) IP=IP+1
IF(FRAC.GE. 0.999995) FRAC=0.0
DO 15 K=1,MK
IF(I.EQ. 1 .AND. K.GT.1) GO TO 15
XX=REIV(I,K)*RR
IF(XX.LE. 0.000005) GO TO 13
CALL BESLJ(XX,FRAC,IP,STOR,IE)
GO TO 14
13 STOR(IP+1)=0.0
IF(AEIV(I).LE.0.000005) STOR(IP+1)=1.0
14 SUM=SUM+PSI*AA(I,K)*STOR(IP+1)
15 CONTINUE
RETURN
END

```


References

- Abramowitz, Milton and Stegun, Irene, A Handbook of Mathematical Functions, Dover Publications, Inc., New York, N.Y., 1965
- Abu-Shumays, I.K. and Bareiss, E.H., "Adjoining Appropriate Singular Elements to Transport Theory Computations," J. Math. Anal. Appl. (1974), Northwestern University, Appl. Math. Rep. 73-2 (1973)
- Babuska, I. and Kellog, R.B., "Mathematical and Computational Problems in Reactor Calculations," U. Maryland Tech. Note BN-760 (1973)
- Bareiss, E.H., "Finite Elements in Neutron Transport Theory," Northwestern University Report COO-2280-2 (1973)
- Bateman, Harry and staff, Higher Transcendental Functions, Vol. 2, McGraw-Hill Book Co., Inc., New York, N.Y., 1953
- Bickley, W.G. et al., British Association for the Advancement of Science Mathematical Tables, Vol. 10, University Press, Cambridge, 1953
- Birkhoff, G., "Angular Singularities of Elliptic Problems," J. Approx. Th. 6, 215-230, 1972
- Boffi, V.C. and Molinari, V.G., "Exact Solutions to the Three Dimensional Transport Equations for Mono Energetic Neutrons," C.N.E.N. Report RT/FI(68) 43 (1968)
- Davison, B., Neutron Transport Theory, Oxford at the Clarendon Press, 1957
- Derstine, K.L., A Contribution to the Mechanization of Error Analysis in Neutron Transport, Master Thesis, Northwestern, 1973
- Glasstone, Samuel and Lund, E.D., Milton, C., The Elements of Nuclear Reactor Theory, D. Van Nostrand Company, Inc., London, 1952
- Jahnke-Emde-Lösch, Tables of Higher Functions, McGraw-Hill Book Co., Inc., 1960
- Luke, Yudell L., Integrals of Bessel Functions, McGraw-Hill Book Co., Inc., New York, N.Y., 1962
- Luke, Yudell L., The Special Functions and Their Approximations, Vol. 1, Vol. 2, Academic Press, Inc., New York, N.Y., 1969
- McLachlan, Norman W., Bessel Functions for Engineers, Oxford Press, London, 1934

Strang, Gilbert and Fix, G.J., An Analysis of the Finite Element Method, Prentice-Hall, Englewood, N.J., 1973

Watson, G.M., A Treatise on the Theory of Bessel Functions, Cambridge at the University Press, London, 1966

U.S. ATOMIC ENERGY COMMISSION

UNIVERSITY-TYPE CONTRACTOR'S RECOMMENDATION FOR
DISPOSITION OF SCIENTIFIC AND TECHNICAL DOCUMENT

(See Instructions on Reverse Side)

1. AEC REPORT NO. C00 - 2280 - 17	2. TITLE The Interior Corner Problem for the Neutron Diffusion Equation
--	---

3. TYPE OF DOCUMENT (Check one):

☒ a. Scientific and technical report

☐ b. Conference paper

Title of conference _____

Date of conference _____

Exact location of conference _____

Sponsoring organization _____

☐ c. JOURNAL ARTICLE Submitted to _____

☐ d. Other (Specify) _____

4. RECOMMENDED ANNOUNCEMENT AND DISTRIBUTION (Check one):

☒ a. AEC's normal announcement and distribution procedures may be followed.

☐ b. Make available only within AEC and to AEC contractors and other U.S. Government agencies and their contractors.

5. REASON FOR RECOMMENDED RESTRICTIONS:

6. SUBMITTED BY: NAME AND POSITION (Please print or type)

Erwin H. Bareiss
Professor and Principal Investigator

Organization

Departments of Computer Sciences and Engineering Sciences
The Technological Institute
Northwestern University
Evanston, Illinois 60201

Signature

Erwin H. Bareiss

Date

March 1975

FOR AEC USE ONLY

7. AEC CONTRACT ADMINISTRATOR'S COMMENTS, IF ANY, ON ABOVE ANNOUNCEMENT AND DISTRIBUTION RECOMMENDATION:

PATENT CLEARANCE:

☐ a. AEC patent clearance has been granted by responsible AEC patent group.

☐ b. Report has been sent to responsible AEC patent group for clearance.

☐ c. Patent clearance not required.