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by

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Abstract

This paper treats the vibration of a two-span beam. The beam is modeled as a Bernoulli-Euler beam. The boundary conditions are clamped-pinned-pinned. An exact closed-form solution is obtained for this problem. Even though there has been an enormous amount of work on beam vibration, most of the studies are conducted on a single-span beam. It is believed that the solution obtained in this paper is new. The method of solution developed in this paper can be easily extended to the vibration of a multi-span beam as well as a two-span beam with other types of boundary conditions. This closed-form solution can be used as a benchmark solution for a two-span beam vibration.

1. Introduction

There has been an enormous amount of study on the vibration of beams in the literature. However, most of these studies are conducted on a single-span beam [1-4]. Studies on multi-span beam vibration [5-9] focus on either numerical approximate solutions or natural frequencies. There are some exact analyses of forced vibration [6,7]. However, they are for a special case of equal span and uniform cross section for the entire span. To the best of the author's knowledge, none of the studies in the past treats an exact solution for a forced vibration of a general two-span beam. In this report, we have obtained an exact closed-form solution for two-span beam vibration, and examined the numerical results of this solution.

In the study of a beam impact problem, the solution to two-span beam vibration constitutes a key part of the investigation. In Section 2, the statement of the problem is given. The mathematical formulation is given in Section 3, the determination of eigenfunctions is given in Sections 4 and 5, applications to the vibration of a beam with a base motion are given in Sections 6 and 7, and finally the numerical results and discussion are given in Section 8.

2. Problem statement

The schematic of a two-span beam is shown in Fig. 1. The left end is clamped, and the mid point and the right end are simply supported. Our objective is to determine the dynamic behavior of this two-span beam. The beam is modeled as a Bernoulli-Euler beam. The governing equation for the Bernoulli-Euler beam in each span is given by

$$E_1 I_1 \frac{\partial^4 y_1}{\partial x^4} + \rho_1 A_1 \frac{\partial^2 y_1}{\partial t^2} = q_1(x, t) \quad (0 < x < l_1)$$

$$E_2 I_2 \frac{\partial^4 y_2}{\partial x^4} + \rho_2 A_2 \frac{\partial^2 y_2}{\partial t^2} = q_2(x, t) \quad (0 < x < l_2)$$
(1)

Fig. 1 Schematic of the two-span beam vibration with coordinates system

where the positive direction of the spatial coordinate (x) is defined in the direction to the right for the left span ($0 < x < l_1$), and it is defined in the direction to the left for the right span ($0 < x < l_2$). This notational convention is adopted here to simplify the algebra.

The boundary conditions are given by

$$y_1(0, t) = 0 \quad y_1'(0, t) = 0$$

$$y_1(l_1, t) = 0 \quad y_2(l_2, t) = 0 \quad y_1'(l_1, t) = -y_2'(l_2, t) \quad E_1 I_1 y_1''(l_1, t) = E_2 I_2 y_2''(l_2, t) \quad (2)$$

$$y_2(0, t) = 0 \quad y_2''(0, t) = 0$$

The initial conditions are given by

$$\begin{aligned}
y_1(x, 0) &= f_1(x), & \frac{\partial y_1}{\partial t}(x, 0) &= g_1(x) \\
y_2(x, 0) &= f_2(x), & \frac{\partial y_2}{\partial t}(x, 0) &= g_2(x)
\end{aligned} \tag{3}$$

Our goal is to obtain the solution for (1) together with (2) and (3). In this report, a method is developed specifically to solve the boundary value problem defined by (1)-(3). However, this method can be applied to any other two-span beam vibration with boundary conditions different from (2) as long as the appropriate eigenfunctions needed in the analysis are used.

3. Mathematical formulation

In order to obtain the solution to (1), (2) and (3), let us first consider a set of homogeneous equations, which is derived from (1).

$$\begin{aligned}
E_1 I_1 \frac{\partial^4 y_1^H}{\partial x^4} + \rho_1 A_1 \frac{\partial^2 y_1^H}{\partial t^2} &= 0 & (0 < x < l_1) \\
E_2 I_2 \frac{\partial^4 y_2^H}{\partial x^4} + \rho_2 A_2 \frac{\partial^2 y_2^H}{\partial t^2} &= 0 & (0 < x < l_2)
\end{aligned} \tag{4}$$

Let us seek a solution of the following form.

$$\begin{aligned}
y_1^H(x, t) &= Y_1(x) e^{i\omega t} \\
y_2^H(x, t) &= Y_2(x) e^{i\omega t}
\end{aligned} \tag{5}$$

By substituting (5) into (4), we obtain

$$\begin{aligned}
\frac{d^4 Y_1}{dx^4} - \frac{\omega^2}{a_1^2} Y_1 &= 0, & 0 \leq x \leq l_1, & \quad a_1^2 = \frac{E_1 I_1}{\rho_1 A_1} \\
\frac{d^4 Y_2}{dx^4} - \frac{\omega^2}{a_2^2} Y_2 &= 0, & 0 \leq x \leq l_2, & \quad a_2^2 = \frac{E_2 I_2}{\rho_2 A_2}
\end{aligned} \tag{6}$$

Even though both Y_1 and Y_2 are developed for the homogeneous equations (4), they are useful for constructing the solution to the original inhomogeneous equations (1), as will be seen later. The boundary conditions for Y_1 and Y_2 are given by

$$Y_1(0) = 0 \quad Y_1'(0) = 0$$

$$Y_1(l_1) = 0 \quad Y_2(l_2) = 0 \quad Y_1'(l_1) = -Y_2'(l_2) \quad E_1 I_1 Y_1''(l_1) = E_2 I_2 Y_2''(l_2) \quad (7)$$

$$Y_2(0) = 0 \quad Y_2''(0) = 0$$

Eqs. (6) and (7) define an eigenvalue problem with eigenfunctions Y_{1n} and Y_{2n} and eigenvalues ω_n . The eigenvalues ω_n are determined from the boundary conditions (7). There are an infinite number of solutions to the above equations. It can be shown that, among the solutions, the following orthogonality condition holds (see Appendix A).

$$\rho_1 A_1 \int_0^{l_1} Y_{1m} Y_{1n} dx + \rho_2 A_2 \int_0^{l_2} Y_{2m} Y_{2n} dx = P_n \delta_{mn} \quad (n \text{ not summed}) \quad (8)$$

where

$$P_n = \rho_1 A_1 \int_0^{l_1} Y_{1n}^2 dx + \rho_2 A_2 \int_0^{l_2} Y_{2n}^2 dx \quad (9)$$

The orthogonality relation (8) is the key equation to enable the analysis of two-span beam. Let us expand the solution to (1) as

$$\begin{aligned} y_1(x, t) &= \sum_{n=1}^{\infty} Y_{1n}(x) \phi_n(t) \quad (0 < x < l_1) \\ y_2(x, t) &= \sum_{n=1}^{\infty} Y_{2n}(x) \phi_n(t) \quad (0 < x < l_2) \end{aligned} \quad (10)$$

Substituting (10) into (1), we have

$$\begin{aligned} E_1 I_1 \sum_{n=1}^{\infty} \frac{d^4 Y_{1n}}{dx^4} \phi_n + \rho_1 A_1 \sum_{n=1}^{\infty} Y_{1n} \frac{d^2 \phi_n}{dt^2} &= q_1(x, t) \\ E_2 I_2 \sum_{n=1}^{\infty} \frac{d^4 Y_{2n}}{dx^4} \phi_n + \rho_2 A_2 \sum_{n=1}^{\infty} Y_{2n} \frac{d^2 \phi_n}{dt^2} &= q_2(x, t) \end{aligned} \quad (11)$$

Let us rewrite (6) for the n-th eigenfunction as

$$\begin{aligned} \frac{d^4 Y_{1n}}{dx^4} - \beta_{1n}^4 Y_{1n} &= 0, \quad \beta_{1n}^2 = \frac{\omega_n}{a_1} \quad (0 < x < l_1) \\ \frac{d^4 Y_{2n}}{dx^4} - \beta_{2n}^4 Y_{2n} &= 0, \quad \beta_{2n}^2 = \frac{\omega_n}{a_2} \quad (0 < x < l_2) \end{aligned} \quad (12)$$

From (11) and (12), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} Y_{1n} \left[\frac{d^2 \phi_n}{dt^2} + \omega_n^2 \phi_n \right] &= \frac{q_1(x, t)}{\rho_1 A_1} \\ \sum_{n=1}^{\infty} Y_{2n} \left[\frac{d^2 \phi_n}{dt^2} + \omega_n^2 \phi_n \right] &= \frac{q_2(x, t)}{\rho_2 A_2}\end{aligned}\tag{13}$$

From (13), we have

$$\begin{aligned}\rho_1 A_1 \int_0^{l_1} \sum_{n=1}^{\infty} Y_{1m} Y_{1n} \left[\frac{d^2 \phi_n}{dt^2} + \omega_n^2 \phi_n \right] dx + \rho_2 A_2 \int_0^{l_2} \sum_{n=1}^{\infty} Y_{2m} Y_{2n} \left[\frac{d^2 \phi_n}{dt^2} + \omega_n^2 \phi_n \right] dx \\ = \int_0^{l_1} Y_{1m}(x) q_1(x, t) dx + \int_0^{l_2} Y_{2m}(x) q_2(x, t) dx\end{aligned}\tag{14}$$

By using the orthogonality condition (8) in (14), and renaming the index m to n after the algebraic manipulations, we obtain

$$\frac{d^2 \phi_n}{dt^2} + \omega_n^2 \phi_n = h_n(t)\tag{15}$$

where

$$h_n(t) = \frac{1}{P_n} \left[\int_0^{l_1} Y_{1n}(x) q_1(x, t) dx + \int_0^{l_2} Y_{2n}(x) q_2(x, t) dx \right]\tag{16}$$

and P_n is defined in (9). It can be easily seen from (10) and (7) that $y_1(x, t)$ and $y_2(x, t)$ satisfy the boundary conditions (2). Substituting (10) into (3), we have

$$\begin{aligned}\sum_{n=1}^{\infty} Y_{1n}(x) \phi_n(0) &= f_1(x) & \sum_{n=1}^{\infty} Y_{1n}(x) \frac{d\phi_n}{dt}(0) &= g_1(x) \\ \sum_{n=1}^{\infty} Y_{2n}(x) \phi_n(0) &= f_2(x) & \sum_{n=1}^{\infty} Y_{2n}(x) \frac{d\phi_n}{dt}(0) &= g_2(x)\end{aligned}\tag{17}$$

From the orthogonality condition (8) and (17), we obtain

$$\begin{aligned}\phi_n(0) &= \frac{1}{P_n} \left[\rho_1 A_1 \int_0^{l_1} Y_{1n}(x) f_1(x) dx + \rho_2 A_2 \int_0^{l_2} Y_{2n}(x) f_2(x) dx \right] \\ \frac{d\phi_n}{dt}(0) &= \frac{1}{P_n} \left[\rho_1 A_1 \int_0^{l_1} Y_{1n}(x) g_1(x) dx + \rho_2 A_2 \int_0^{l_2} Y_{2n}(x) g_2(x) dx \right]\end{aligned}\tag{18}$$

It is seen from (15) and (18) that, to determine $\phi(t)$, we need to solve the following initial value problem.

$$\frac{d^2 \phi_n}{dt^2} + \omega_n^2 \phi_n = h_n(t) \quad (19)$$

$$\phi_n(0) = c_n \quad (20)$$

$$\frac{d\phi_n}{dt}(0) = d_n$$

where

$$c_n = \frac{1}{P_n} \left[\rho_1 A_1 \int_0^{l_1} Y_{1n}(x) f_1(x) dx + \rho_2 A_2 \int_0^{l_2} Y_{2n}(x) f_2(x) dx \right] \quad (21)$$

$$d_n = \frac{1}{P_n} \left[\rho_1 A_1 \int_0^{l_1} Y_{1n}(x) g_1(x) dx + \rho_2 A_2 \int_0^{l_2} Y_{2n}(x) g_2(x) dx \right]$$

The solution to (19) with (20) is given by

$$\phi_n(t) = \frac{d_n}{\omega_n} \sin \omega_n t + c_n \cos \omega_n t + \frac{1}{\omega_n} \int_0^t h(u) \sin \omega_n(t-u) du \quad (22)$$

Therefore, the solution to (1) together with (2) and (3) is given from (10) as

$$y_1(x, t) = \sum_{n=1}^{\infty} Y_{1n}(x) \left[\frac{d_n}{\omega_n} \sin \omega_n t + c_n \cos \omega_n t + \frac{1}{\omega_n} \int_0^t h(u) \sin \omega_n(t-u) du \right] \quad (0 < x < l_1) \quad (23)$$

$$y_2(x, t) = \sum_{n=1}^{\infty} Y_{2n}(x) \left[\frac{d_n}{\omega_n} \sin \omega_n t + c_n \cos \omega_n t + \frac{1}{\omega_n} \int_0^t h(u) \sin \omega_n(t-u) du \right] \quad (0 < x < l_2)$$

The bending moment $M(x, t)$ can be obtained from (23) as

$$M_1(x, t) = E_1 I_1 y_1''(x, t)$$

$$= E_1 I_1 \sum_{n=1}^{\infty} \frac{d^2 Y_{1n}}{dx^2}(x) \left[\frac{d_n}{\omega_n} \sin \omega_n t + c_n \cos \omega_n t + \frac{1}{\omega_n} \int_0^t h(u) \sin \omega_n(t-u) du \right] \quad (0 < x < l_1) \quad (24-1)$$

$$\begin{aligned}
M_2(x, t) &= E_2 I_2 y_2''(x, t) \\
&= E_2 I_2 \sum_{n=1}^{\infty} \frac{d^2 Y_{2n}}{dx^2}(x) \left[\frac{d_n}{\omega_n} \sin \omega_n t + c_n \cos \omega_n t + \frac{1}{\omega_n} \int_0^t h(u) \sin \omega_n (t-u) du \right] \quad (0 < x < l_2)
\end{aligned} \tag{24-2}$$

4. Determination of the eigenfunctions

The eigenfunctions Y_1 and Y_2 are defined by (12) and (7). The solution to (12) and (7) is given by

$$\begin{aligned}
Y_{1n}(x) &= \sinh \beta_{1n} x - \sin \beta_{1n} x - \gamma_{1n} (\cosh \beta_{1n} x - \cos \beta_{1n} x) \\
Y_{2n}(x) &= \gamma_{3n} (\gamma_{2n} \sinh \beta_{2n} x - \sin \beta_{2n} x)
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
\gamma_{1n} &= \frac{\sinh \beta_{1n} l_1 - \sin \beta_{1n} l_1}{\cosh \beta_{1n} l_1 - \cos \beta_{1n} l_1} \\
\gamma_{2n} &= \frac{\sin \beta_{2n} l_2}{\sinh \beta_{2n} l_2} \\
\gamma_{3n} &= -\frac{\beta_{1n} [\cosh \beta_{1n} l_1 - \cos \beta_{1n} l_1 - \gamma_{1n} (\sinh \beta_{1n} l_1 + \sin \beta_{1n} l_1)]}{\beta_{2n} [\gamma_{2n} \cosh \beta_{2n} l_2 - \cos \beta_{2n} l_2]} \\
\beta_{1n}^2 &= \frac{\omega_n}{a_1} \quad a_1^2 = \frac{E_1 I_1}{\rho_1 A_1} \\
\beta_{2n}^2 &= \frac{\omega_n}{a_2} \quad a_2^2 = \frac{E_2 I_2}{\rho_2 A_2}
\end{aligned} \tag{26}$$

and ω_n is the n-th root of the following characteristic equation.

$$\begin{aligned}
&2E_2 I_2 \beta_{2n} (1 - \cos \beta_{1n} l_1 \cosh \beta_{1n} l_1) \sin \beta_{2n} l_2 \sinh \beta_{2n} l_2 \\
&+ E_1 I_1 \beta_{1n} (\sin \beta_{1n} l_1 \cosh \beta_{1n} l_1 - \cos \beta_{1n} l_1 \sinh \beta_{1n} l_1) (\sin \beta_{2n} l_2 \cosh \beta_{2n} l_2 - \cos \beta_{2n} l_2 \sinh \beta_{2n} l_2) = 0
\end{aligned} \tag{28}$$

Since (28) is a transcendental equation, it has to be solved numerically.

5. Determination of the eigenfunctions when $E_1 I_1 = E_2 I_2$ and $\rho_1 A_1 = \rho_2 A_2$

Let us now consider a special case when $E_1 I_1 = E_2 I_2 = EI$ and $\rho_1 A_1 = \rho_2 A_2 = \rho A$. Then we have

$$\beta_{1n}^2 = \beta_{2n}^2 = \frac{\omega_n^2}{a^2} (\equiv \beta_n^2) \quad (29)$$

$$a_1^2 = a_2^2 = \frac{EI}{\rho A} (\equiv a^2)$$

From (25)-(28), the eigenfunctions are given by

$$\begin{aligned} Y_{1n}(x) &= \sinh \beta_n x - \sin \beta_n x - \gamma_{1n} (\cosh \beta_n x - \cos \beta_n x) \\ Y_{2n}(x) &= \gamma_{3n} (\gamma_{2n} \sinh \beta_n x - \sin \beta_n x) \end{aligned} \quad (30)$$

where

$$\begin{aligned} \gamma_{1n} &= \frac{\sinh \beta_n l_1 - \sin \beta_n l_1}{\cosh \beta_n l_1 - \cos \beta_n l_1} \\ \gamma_{2n} &= \frac{\sin \beta_n l_2}{\sinh \beta_n l_2} \\ \gamma_{3n} &= -\frac{\cosh \beta_n l_1 - \cos \beta_n l_1 - \gamma_{1n} (\sinh \beta_n l_1 + \sin \beta_n l_1)}{\gamma_{2n} \cosh \beta_n l_2 - \cos \beta_n l_2} \end{aligned} \quad (31)$$

and β_n is defined as the n-th root of the following characteristic equation.

$$\begin{aligned} &2(1 - \cos \beta l_1 \cosh \beta l_1) \sin \beta l_2 \sinh \beta l_2 \\ &+ (\sin \beta l_1 \cosh \beta l_1 - \cos \beta l_1 \sinh \beta l_1) (\sin \beta l_2 \cosh \beta l_2 - \cos \beta l_2 \sinh \beta l_2) = 0 \end{aligned} \quad (32)$$

Since (32) is also a transcendental equation, it has to be solved numerically.

6. Application to the vibration of a beam with a base motion

Let us consider the case where there is a base motion to the beam but no other applied load. The governing equation for the Bernoulli-Euler beam in each span is given by

$$\begin{aligned}
E_1 I_1 \frac{\partial^4 y_{1tot}}{\partial x^4} + \rho_1 A_1 \frac{\partial^2 y_{1tot}}{\partial t^2} &= 0 & (0 < x < l_1) \\
E_2 I_2 \frac{\partial^4 y_{2tot}}{\partial x^4} + \rho_2 A_2 \frac{\partial^2 y_{2tot}}{\partial t^2} &= 0 & (0 < x < l_2)
\end{aligned} \tag{33}$$

where y_{1tot} and y_{2tot} are the total displacement of each beam, and they are given by

$$\begin{aligned}
y_{1tot}(x, t) &= y_1(x, t) + \left(1 - \frac{x}{l_{tot}}\right) y_{B1}(t) + \frac{x}{l_{tot}} y_{B2}(t) & (0 < x < l_1) \\
y_{2tot}(x, t) &= y_2(x, t) + \left(1 - \frac{x}{l_{tot}}\right) y_{B2}(t) + \frac{x}{l_{tot}} y_{B1}(t) & (0 < x < l_2)
\end{aligned} \tag{34}$$

where

$$l_{tot} = l_1 + l_2 \tag{35}$$

Here $y_{B1}(t)$ and $y_{B2}(t)$ are a given displacement of the left base, and of the right base, respectively. Also in (34), as before, the spatial coordinate (x) is defined in the direction to the right for the left span ($0 < x < l_1$), and it is defined in the direction to the left for the right span ($0 < x < l_2$). Substituting (34) into (33), we obtain

$$\begin{aligned}
E_1 I_1 \frac{\partial^4 y_1}{\partial x^4} + \rho_1 A_1 \frac{\partial^2 y_1}{\partial t^2} &= q_{1B}(t) & (0 < x < l_1) \\
E_2 I_2 \frac{\partial^4 y_2}{\partial x^4} + \rho_2 A_2 \frac{\partial^2 y_2}{\partial t^2} &= q_{2B}(t) & (0 < x < l_2)
\end{aligned} \tag{36}$$

where

$$\begin{aligned}
q_{1B}(t) &= -\rho_1 A_1 \left[\left(1 - \frac{x}{l_{tot}}\right) \frac{\partial^2 y_{B1}}{\partial t^2} + \frac{x}{l_{tot}} \frac{\partial^2 y_{B2}}{\partial t^2} \right] & (0 < x < l_1) \\
q_{2B}(t) &= -\rho_2 A_2 \left[\left(1 - \frac{x}{l_{tot}}\right) \frac{\partial^2 y_{B2}}{\partial t^2} + \frac{x}{l_{tot}} \frac{\partial^2 y_{B1}}{\partial t^2} \right] & (0 < x < l_2)
\end{aligned} \tag{37}$$

The boundary conditions for the relative displacements y_1 and y_2 are given by

$$y_1(0, t) = 0 \quad y_1'(0, t) = 0$$

$$y_1(l_1, t) = 0 \quad y_2(l_2, t) = 0 \quad y_1'(l_1, t) = -y_2'(l_2, t) \quad E_1 I_1 y_1''(l_1, t) = E_2 I_2 y_2''(l_2, t) \quad (38)$$

$$y_2(0, t) = 0 \quad y_2''(0, t) = 0$$

The initial conditions are given by

$$\begin{aligned} y_1(x, 0) &= f_1(x), & \frac{\partial y_1}{\partial t}(x, 0) &= g_1(x) \\ y_2(x, 0) &= f_2(x), & \frac{\partial y_2}{\partial t}(x, 0) &= g_2(x) \end{aligned} \quad (39)$$

Therefore, the vibration of a beam with a base motion is reduced to a special case of the beam vibration which was considered in the previous sections.

7. Vibration of a beam with a harmonic base motion

Let us consider the case where

$$\begin{aligned} E_1 I_1 &= E_2 I_2 = EI \\ \rho_1 A_1 &= \rho_2 A_2 = \rho A \\ y_{B1}(t) &= y_{B2}(t) = y_B(t) \end{aligned} \quad (40)$$

Then we have

$$q_1(x, t) = q_2(x, t) = q(t) \quad (41)$$

where

$$q(t) = -\rho A \frac{\partial^2 y_B}{\partial t^2} \quad (42)$$

Let us set the initial conditions as

$$\begin{aligned} y_1(x, 0) &= 0, & \frac{\partial y_1}{\partial t}(x, 0) &= 0 \\ y_2(x, 0) &= 0, & \frac{\partial y_2}{\partial t}(x, 0) &= 0 \end{aligned} \quad (43)$$

From (23), the solution is given by

$$y_1(x, t) = \sum_{n=1}^{\infty} \frac{Y_{1n}(x)}{\omega_n} \int_0^t h(u) \sin \omega_n(t-u) du \quad (0 < x < l_1) \quad (44)$$

$$y_2(x, t) = \sum_{n=1}^{\infty} \frac{Y_{2n}(x)}{\omega_n} \int_0^t h(u) \sin \omega_n(t-u) du \quad (0 < x < l_2)$$

Substituting (42) into (16), we have

$$h_n(t) = -\frac{\rho A}{P_n} \frac{\partial^2 y_B}{\partial t^2} \left[\int_0^{l_1} Y_{1n}(x) dx + \int_0^{l_2} Y_{2n}(x) dx \right] \quad (45)$$

Substituting (40) into (9), we have

$$P_n = \rho A \left[\int_0^{l_1} Y_{1n}^2 dx + \int_0^{l_2} Y_{2n}^2 dx \right] \quad (46)$$

Substituting (46) into (45), we obtain

$$h_n(t) = -L_n \frac{\partial^2 y_B}{\partial t^2} \quad (47)$$

where

$$L_n = \frac{\int_0^{l_1} Y_{1n}(x) dx + \int_0^{l_2} Y_{2n}(x) dx}{\int_0^{l_1} Y_{1n}^2 dx + \int_0^{l_2} Y_{2n}^2 dx} \quad (48)$$

Substituting (47) into (44), we obtain

$$y_1(x, t) = -\sum_{n=1}^{\infty} \frac{L_n}{\omega_n} Y_{1n}(x) \psi_n(t) \quad (0 < x < l_1) \quad (49)$$

$$y_2(x, t) = -\sum_{n=1}^{\infty} \frac{L_n}{\omega_n} Y_{2n}(x) \psi_n(t) \quad (0 < x < l_2)$$

where

$$\psi_n(t) = \int_0^t \frac{\partial^2 y_B}{\partial u^2} \sin \omega_n(t-u) du \quad (50)$$

Eq. (49) is the solution for the vibration of a beam with a general base motion when $E_1 I_1 = E_2 I_2 = EI$ and $\rho_1 A_1 = \rho_2 A_2 = \rho A$, and the initial conditions are all zero. When the base motion is harmonic, we can set

$$\frac{\partial^2 y_B}{\partial u^2} = \alpha_B \sin \omega_B u \quad (51)$$

where α_B is the base acceleration. Substituting (51) into (50), and performing the integration, we obtain

$$\psi_n(t) = \frac{\alpha_B}{\omega_n^2 - \omega_B^2} (\omega_n \sin \omega_B t - \omega_B \sin \omega_n t) \quad (52)$$

Substituting (52) into (49), we finally obtain

$$\begin{aligned} y_1(x, t) &= -\alpha_B \sum_{n=1}^{\infty} \frac{L_n}{\omega_n^2 - \omega_B^2} Y_{1n}(x) \left(\sin \omega_B t - \frac{\omega_B}{\omega_n} \sin \omega_n t \right) \quad (0 < x < l_1) \\ y_2(x, t) &= -\alpha_B \sum_{n=1}^{\infty} \frac{L_n}{\omega_n^2 - \omega_B^2} Y_{2n}(x) \left(\sin \omega_B t - \frac{\omega_B}{\omega_n} \sin \omega_n t \right) \quad (0 < x < l_2) \end{aligned} \quad (53)$$

A non-dimensional displacement can be defined from (53) as

$$\begin{aligned} y_1^*(x, t) &= \frac{y_1(x, t)}{l} = -\frac{\alpha_B}{l} \sum_{n=1}^{\infty} \frac{L_n}{\omega_n^2 - \omega_B^2} Y_{1n}(x) \left(\sin \omega_B t - \frac{\omega_B}{\omega_n} \sin \omega_n t \right) \quad (0 < x < l_1) \\ y_2^*(x, t) &= \frac{y_2(x, t)}{l} = -\frac{\alpha_B}{l} \sum_{n=1}^{\infty} \frac{L_n}{\omega_n^2 - \omega_B^2} Y_{2n}(x) \left(\sin \omega_B t - \frac{\omega_B}{\omega_n} \sin \omega_n t \right) \quad (0 < x < l_2) \end{aligned} \quad (54)$$

where

$$l = \frac{l_1 + l_2}{2} \quad (55)$$

The bending moment is obtained from (53) as

$$\begin{aligned} M_1(x, t) &= -\alpha_B EI \sum_{n=1}^{\infty} \frac{L_n}{\omega_n^2 - \omega_B^2} Y_{1n}''(x) \left(\sin \omega_B t - \frac{\omega_B}{\omega_n} \sin \omega_n t \right) \quad (0 < x < l_1) \\ M_2(x, t) &= -\alpha_B EI \sum_{n=1}^{\infty} \frac{L_n}{\omega_n^2 - \omega_B^2} Y_{2n}''(x) \left(\sin \omega_B t - \frac{\omega_B}{\omega_n} \sin \omega_n t \right) \quad (0 < x < l_2) \end{aligned} \quad (56)$$

A non-dimensional bending moment can be defined from (56) as

$$M_1^*(x, t) = \frac{M_1(x, t)}{\frac{EI}{l}} = -\alpha_B l \sum_{n=1}^{\infty} \frac{L_n}{\omega_n^2 - \omega_B^2} Y_{1n}''(x) \left(\sin \omega_B t - \frac{\omega_B}{\omega_n} \sin \omega_n t \right) \quad (0 < x < l_1) \quad (57)$$

$$M_2^*(x, t) = \frac{M_2(x, t)}{\frac{EI}{l}} = -\alpha_B l \sum_{n=1}^{\infty} \frac{L_n}{\omega_n^2 - \omega_B^2} Y_{2n}''(x) \left(\sin \omega_B t - \frac{\omega_B}{\omega_n} \sin \omega_n t \right) \quad (0 < x < l_2)$$

8. Numerical results and discussion

Let us consider the vibration of a beam with a harmonic base motion with the following parameters.

$$E_1 I_1 = E_2 I_2 = EI, \quad \rho_1 A_1 = \rho_2 A_2 = \rho A$$

$$l_1 = l_2 = l = 1 \text{ (m)}$$

$$a = \sqrt{\frac{EI}{\rho A}} = 75 \text{ (m}^2/\text{sec)} \quad (58)$$

$$\omega_B = 100 \text{ (rad / sec)}$$

$$\alpha_B = 100 \text{ (m / sec}^2\text{)}$$

The value of “a” given above approximately corresponds to the steel bar of a square cross section of 0.1 (m) x 0.1 (m). The acceleration given above is about 10g. The time histories of the non-dimensional beam displacement at the mid-span (left span and right span) are shown in Figs. 2-3. The non-dimensional beam displacements over the entire span are shown in Figs. 4-8. Here it should be noted that in order to use the conventional spatial coordinate x continuously for both spans, the non-dimensional displacement is redefined as

$$y_1^{rev}(x, t) = y_1^*(x, t) \quad (0 < x < l_1) \quad (59)$$

$$y_2^{rev}(x, t) = y_2^*(l_{tot} - x, t) \quad (l_1 < x < l_{tot})$$

$$y^*(x, t) = \begin{cases} y_1^*(x, t) & 0 < x < l_1 \\ y_2^*(l_{tot} - x, t) & l_1 < x < l_{tot} \end{cases} \quad (60)$$

where

$$\ell_{\text{tot}} = \ell_1 + \ell_2 \quad (61)$$

A similar definition is made also for the non-dimensional bending moment. The time histories of the non-dimensional bending moment at four different locations are shown in Figs. 9-12. The non-dimensional bending moment over the entire span are shown in Figs. 13-20. It can be seen from the time histories that the period of the beam vibration corresponds approximately to the period of the harmonic excitation (i.e., $2\pi/100 = 0.0628$ sec). It is seen from Figs. 4-8 that the beam displacement is larger in the right span as expected. It is also seen from Figs. 13-20 that the magnitude of the bending moment at the mid-support is often the greatest.

Fig. 2 Time history of the non-dimensional beam displacement at $x = 0.5$ (in the left half of the beam)

Fig. 3 Time history of the non-dimensional beam displacement at $x = 1.5$ (in the right half of the beam)

Fig. 4 Non-dimensional beam displacement over the entire span ($0 < x < 2$) at $t = 0.01$

Fig. 5 Non-dimensional beam displacement over the entire span ($0 < x < 2$) at $t = 0.02$

Fig. 6 Non-dimensional beam displacement over the entire span ($0 < x < 2$) at $t = 0.03$

Fig. 7 Non-dimensional beam displacement over the entire span ($0 < x < 2$) at $t = 0.04$

Fig. 8 Non-dimensional beam displacement over the entire span ($0 < x < 2$) at $t = 0.05$

Fig. 9 Time history of the non-dimensional bending moment at $x = 0$ (the clamped end)

Fig. 10 Time history of the non-dimensional bending moment at $x = 0.5$ (left half)

Fig. 11 Time history of the non-dimensional bending moment at $x = 1$ (the mid-support)

Fig. 12 Time history of the non-dimensional bending moment at $x = 1.5$ (right half)

Fig. 13 Non-dimensional bending moment over the entire span ($0 < x < 2$) at $t = 0.01$

Fig. 14 Non-dimensional bending moment over the entire span ($0 < x < 2$) at $t = 0.02$

Fig. 15 Non-dimensional bending moment over the entire span ($0 < x < 2$) at $t = 0.03$

Fig. 16 Non-dimensional bending moment over the entire span ($0 < x < 2$) at $t = 0.032$

Fig. 17 Non-dimensional bending moment over the entire span ($0 < x < 2$) at $t = 0.035$

Fig. 18 Non-dimensional bending moment over the entire span ($0 < x < 2$) at $t = 0.04$

Fig. 19 Non-dimensional bending moment over the entire span ($0 < x < 2$) at $t = 0.0475$

Fig. 20 Non-dimensional bending moment over the entire span ($0 < x < 2$) at $t = 0.05$

9. Conclusion

An exact closed-form solution to a two-span beam vibration was obtained by an eigenfunction expansion. The solution seems to be new to the best of the author's knowledge. A general solution was constructed for a beam with a clamped support, a simple support and a simple support. As an application of the solution, the vibration of a beam with base motion was also solved, and numerical results were obtained for a particular case. It was found from the numerical results that the magnitude of the bending moment was often the greatest at the mid-support. The solution obtained in this report will be further used in the analysis of a beam impact problem.

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Appendix A. Proof of the orthogonality condition

In order to prove the orthogonality condition, let us consider the following eigenvalue problem.

$$\begin{aligned} L(u_1) - \lambda_1 u_1 &= 0, & 0 \leq x \leq l_1, \\ L(u_2) - \lambda_2 u_2 &= 0, & 0 \leq x \leq l_2, \end{aligned} \tag{A1}$$

where

$$L = \frac{d^4}{dx^4} \tag{A2}$$

$$\lambda_1 = \frac{\omega^2}{a_1^2} \quad a_1^2 = \frac{E_1 I_1}{\rho_1 A_1} \tag{A3}$$

$$\lambda_2 = \frac{\omega^2}{a_2^2} \quad a_2^2 = \frac{E_2 I_2}{\rho_2 A_2}$$

The boundary conditions for u_1 and u_2 are given by

$$\begin{aligned} u_1(0) &= 0 & u_1'(0) &= 0 \\ u_1(l_1) &= 0 & u_2(l_2) &= 0 & u_1'(l_1) &= -u_2'(l_2) & E_1 I_1 u_1''(l_1) &= E_2 I_2 u_2''(l_2) \\ u_2(0) &= 0 & u_2''(0) &= 0 \end{aligned} \tag{A4}$$

We will first prove the following.

$$E_1 I_1 \int_0^{l_1} [v_1 L(u_1) - u_1 L(v_1)] dx + E_2 I_2 \int_0^{l_2} [v_2 L(u_2) - u_2 L(v_2)] dx = 0 \tag{A5}$$

where (v_1, v_2) is a set of solutions for the following eigenvalue problem.

$$\begin{aligned} L(v_1) - \lambda_1' v_1 &= 0, & 0 \leq x \leq l_1, \\ L(v_2) - \lambda_2' v_2 &= 0, & 0 \leq x \leq l_2, \end{aligned} \tag{A6}$$

where

$$\lambda_1' = \frac{(\omega')^2}{a_1^2}, \quad \lambda_2' = \frac{(\omega')^2}{a_2^2} \quad (\text{A7})$$

The boundary conditions for v_1 and v_2 are given by

$$\begin{aligned} v_1(0) &= 0 & v_1'(0) &= 0 \\ v_1(l_1) &= 0 & v_2(l_2) &= 0 & v_1'(l_1) &= -v_2'(l_2) & E_1 I_1 v_1''(l_1) &= E_2 I_2 v_2''(l_2) \\ v_2(0) &= 0 & v_2''(0) &= 0 \end{aligned} \quad (\text{A8})$$

Using the integration by parts and the boundary conditions, we obtain

$$\begin{aligned} \int_0^{l_1} [v_1 L(u_1) - u_1 L(v_1)] dx &= -u_1''(l_1) v_1'(l_1) + u_1'(l_1) v_1''(l_1) \\ \int_0^{l_2} [v_2 L(u_2) - u_2 L(v_2)] dx &= -u_2''(l_2) v_2'(l_2) + u_2'(l_2) v_2''(l_2) \end{aligned} \quad (\text{A9})$$

Substituting (A9) into (A5), we have

$$\begin{aligned} & E_1 I_1 \int_0^{l_1} [v_1 L(u_1) - u_1 L(v_1)] dx + E_2 I_2 \int_0^{l_2} [v_2 L(u_2) - u_2 L(v_2)] dx \\ &= E_1 I_1 [-u_1''(l_1) v_1'(l_1) + u_1'(l_1) v_1''(l_1)] + E_2 I_2 [-u_2''(l_2) v_2'(l_2) + u_2'(l_2) v_2''(l_2)] \end{aligned} \quad (\text{A10})$$

By using the boundary conditions, the right hand side of (A10) can be shown to be 0. This proves (A5).

By using (A5) together with (A1), (A3) and (A6), (A7), we obtain

$$(\lambda_1 - \lambda_1') \left[E_1 I_1 a_2^2 \int_0^{l_1} u_1 v_1 dx + E_2 I_2 a_1^2 \int_0^{l_2} u_2 v_2 dx \right] = 0 \quad (\text{A11})$$

If $\lambda_1 \neq \lambda_1'$, we finally obtain from (A11)

$$\frac{E_1 I_1}{a_1^2} \int_0^{l_1} u_1 v_1 dx + \frac{E_2 I_2}{a_2^2} \int_0^{l_2} u_2 v_2 dx = 0 \quad (\text{A12})$$

This is the orthogonality condition. By using (A3), (A12) can also be written as

$$\rho_1 A_1 \int_0^{l_1} u_1 v_1 dx + \rho_2 A_2 \int_0^{l_2} u_2 v_2 dx = 0 \quad (\text{A13})$$

(A13) holds when the eigenvalues of u_1 and v_1 are different. If they are the same, we have

$$\rho_1 A_1 \int_0^{l_1} u_1 v_1 dx + \rho_2 A_2 \int_0^{l_2} u_2 v_2 dx = P \quad (\text{A14})$$

where

$$P = \rho_1 A_1 \int_0^{l_1} (u_1)^2 dx + \rho_2 A_2 \int_0^{l_2} (u_2)^2 dx \quad (\text{A15})$$

(A13) and (A14) are equivalent to the orthogonality condition (6).

Q.E.D.