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Determination of Thermal Conductivity from a Surface Temperature Measurement: Theory for the 3ω Method

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1. Introduction

In order to measure thermal conductivity of a material, a so-called 3ω method [1-5] is sometimes used. The key theoretical framework for the 3ω method is the determination of a steady-state time-harmonic temperature variation of the solid, whose thermal conductivity is to be measured, when the solid is heated by a time-harmonic heat source. Even though there have been a number of papers published on the 3ω method [2-5], none of them have provided an exact closed form solution for the steady-state time-harmonic temperature variation of the solid in a form which can lead to an exact closed form expression for the temperature variation amplitude and the phase shift. In this paper, an exact closed form solution is developed for a 1D heat conduction problem with a time-harmonic heat flux. This is a mathematical model for a thermal conductivity measurement for a thin film.

2. Problem definition

As a mathematical model for a thermal conductivity measurement, let us consider a 1D heat conduction problem as shown below.

Fig. 1. Schematic of 1D heat conduction problem

In a thermal conductivity experiment using the 3ω method, the solid is heated up by an electric AC heater attached at the surface, and the solid temperature rises. But eventually the solid temperature becomes steady state. The heat generated by the AC heater is modeled as a time-harmonic heat flux at $x = 0$. By measuring the resulting time-harmonic temperature in the solid, the thermal conductivity of the solid can be determined. The 1D heat conduction problem shown above is very similar to the heat conduction problem associated with a photo-acoustic measurement [6] for obtaining the thermal conductivity. The main difference is that, in the photo-acoustic measurement, usually the heat conduction problem is defined for a multilayer system. The extension to a multilayer system for our problem is relatively straightforward. For our system, the governing equation for the heat conduction in the solid is given by

$$\frac{1}{\alpha_s} \frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} \quad 0 \leq x \leq l \quad (1)$$

where α_s is the thermal diffusivity of the solid given by

$$\alpha_s = \frac{k}{\rho c_p} \quad (2)$$

Here ρ is the density, and c_p is the specific heat of the material. In Eq. (1), ϕ is a temperature difference, and is defined as

$$\phi = T_s - T_0 \quad (3)$$

where T_s is a temperature in the solid, and T_0 is an ambient temperature. The boundary conditions are given by

$$\begin{aligned} -k \frac{\partial \phi}{\partial x}(0, t) &= q(1 - \cos \omega t) \quad (q > 0) \\ \phi(l, t) &= 0 \end{aligned} \quad (4)$$

where q is the heat flux in the positive x -direction (into the solid). Since the heat is put into the solid in our problem, q is positive. The solution to (1) and (4) can be expressed as

$$\phi(x, t) = \text{Re}[\phi_1(x, t)] - \frac{q}{k}(x - l) \quad (5)$$

where $\text{Re} [\]$ denotes the real part, and $\phi_1(x, t)$ is the solution to the following equations.

$$\frac{1}{\alpha_s} \frac{\partial \phi_1}{\partial t} = \frac{\partial^2 \phi_1}{\partial x^2} \quad 0 \leq x \leq l \quad (6)$$

$$-k \frac{\partial \phi_1}{\partial x}(0, t) = -q e^{i\omega t} \quad (7)$$

$$\phi_1(l, t) = 0 \quad (8)$$

Since our goal is to obtain the solution as $t \rightarrow \infty$, initial conditions are not needed. Eqs. (6), (7) and (8) define the boundary value problem for our 1 D heat conduction problem.

3. Mathematical formulation

Let us set

$$\phi_1(x, t) = \Phi_1(x) e^{i\omega t} \quad (9)$$

Here it should be noted that $\phi_1(x, t)$ as well as $\Phi_1(x)$ are both complex functions. Substituting (9) into (6), we obtain

$$\frac{d^2 \Phi_1}{dx^2} - \frac{i\omega}{\alpha_s} \Phi_1 = 0 \quad (10)$$

Substituting (9) into (7) and (8), we obtain

$$\begin{aligned} -k \frac{d\Phi_1}{dx}(0) &= -q \\ \Phi_1(l) &= 0 \end{aligned} \quad (11)$$

Solving (10), we obtain

$$\Phi_1(x) = S_1 e^{\lambda x} + S_2 e^{-\lambda x} \quad (12)$$

where S_1 and S_2 are the unknown constants to be determined from the boundary conditions (11), and

$$\lambda = \sqrt{\frac{i\omega}{\alpha_s}} = \frac{1+i}{\sqrt{2}} \sqrt{\frac{\omega}{\alpha_s}} = \sqrt{\frac{\omega}{2\alpha_s}} (1+i) \quad (13)$$

Substituting (12) into (11), and solving the resulting linear algebraic equation, we obtain

$$S_1 = \frac{q}{k\lambda} \frac{e^{-\lambda l}}{D}, \quad S_2 = -\frac{q}{k\lambda} \frac{e^{\lambda l}}{D} \quad (14)$$

where

$$D = e^{\lambda l} + e^{-\lambda l} \quad (15)$$

Substituting (14) into (12), we have

$$\Phi_1(x) = \frac{q}{k\lambda D} [e^{\lambda(x-l)} - e^{-\lambda(x-l)}] \quad (16)$$

From (13) and (15), we have

$$\frac{1}{\lambda} = \frac{1}{\sqrt{\frac{\omega}{2\alpha_s}}(1+i)} = \frac{1-i}{2} \sqrt{\frac{2\alpha_s}{\omega}} = \sqrt{\frac{\alpha_s}{2\omega}}(1-i) \quad (17-1)$$

$$\begin{aligned} \frac{1}{D} &= \frac{1}{e^{z(1+i)} + e^{-z(1+i)}} = \frac{1}{e^z(\cos z + i \sin z) + e^{-z}(\cos z - i \sin z)} \\ &= \frac{1}{\cos z(e^z + e^{-z}) + i \sin z(e^z - e^{-z})} = \frac{\cos z(e^z + e^{-z}) - i \sin z(e^z - e^{-z})}{e^{2z} + e^{-2z} + 2 \cos 2z} \\ &= \frac{\cos z \cosh z - i \sin z \sinh z}{\cos 2z + \cosh 2z} \end{aligned} \quad (17-2)$$

where z is a non-dimensional parameter defined by

$$z = \sqrt{\frac{\omega}{2\alpha_s}} l \quad (18)$$

Substituting (17) into (16), we obtain

$$\begin{aligned} \Phi_1(x) &= \frac{q}{k} \sqrt{\frac{\alpha_s}{2\omega}}(1-i) \frac{1}{D} [e^{-\lambda l(1-\frac{x}{l})} - e^{\lambda l(1-\frac{x}{l})}] \\ &= \frac{q}{k} \sqrt{\frac{\alpha_s}{2\omega}}(1-i) \frac{1}{D} [e^{-z(1-\frac{x}{l})(1+i)} - e^{z(1-\frac{x}{l})(1+i)}] \equiv \frac{q}{k} \sqrt{\frac{\alpha_s}{2\omega}} \frac{P}{a} \end{aligned} \quad (19)$$

where

$$a = \cos 2z + \cosh 2z \quad (20)$$

$$P = (1-i)(\cos z \cosh z - i \sin z \sinh z) \left[e^{-z(1-\frac{x}{l})(1+i)} - e^{z(1-\frac{x}{l})(1+i)} \right]$$

Let us set

$$y = z(1 - \frac{x}{l}) \quad (21)$$

From (20-2), we have

$$P = -2(1-i)(P_1 + iP_2) = -2\sqrt{2}e^{-i\frac{\pi}{4}}\sqrt{P_1^2 + P_2^2}(\cos \delta + i \sin \delta) \quad (22)$$

where

$$P_1 = \cos z \cosh z \cos y \sinh y + \sin z \sinh z \sin y \cosh y \quad (23)$$

$$P_2 = \cos z \cosh z \sin y \cosh y - \sin z \sinh z \cos y \sinh y$$

$$\cos \delta = \frac{P_1}{\sqrt{P_1^2 + P_2^2}}, \quad \sin \delta = \frac{P_2}{\sqrt{P_1^2 + P_2^2}} \quad (24)$$

From (23), we have

$$P_1^2 + P_2^2 = \frac{1}{4}(\cosh 2z + \cos 2z)(\cosh 2y - \cos 2y) \quad (25)$$

Substituting (25) into (22), we obtain

$$P = -\sqrt{2}\sqrt{(\cosh 2z + \cos 2z)(\cosh 2y - \cos 2y)}e^{i(\delta - \frac{\pi}{4})} \quad (26)$$

Substitution of (26) into (19) yields

$$\Phi_1(x, t) = -\frac{q}{k} \sqrt{\frac{\alpha_s}{\omega}} \sqrt{\frac{\cosh 2y - \cos 2y}{\cosh 2z + \cos 2z}} e^{i(\delta - \frac{\pi}{4})} \quad (27)$$

where (20-1) has been used. From (27), (9) and (5), we finally obtain

$$\phi(x, t) = -\frac{q}{k} \sqrt{\frac{\alpha_s}{\omega}} \sqrt{\frac{\cosh 2y - \cos 2y}{\cosh 2z + \cos 2z}} \cos(\omega t - \beta) + \frac{ql}{k} (1 - \frac{x}{l}) \quad (28)$$

where

$$\beta = \frac{\pi}{4} - \delta \quad (29)$$

From (23) and (24), we have

$$\begin{aligned} \cos \delta &= \frac{2(\cos z \cosh z \cos y \sinh y + \sin z \sinh z \sin y \cosh y)}{\sqrt{(\cosh 2z + \cos 2z)(\cosh 2y - \cos 2y)}} \\ \sin \delta &= \frac{2(\cos z \cosh z \sin y \cosh y - \sin z \sinh z \cos y \sinh y)}{\sqrt{(\cosh 2z + \cos 2z)(\cosh 2y - \cos 2y)}} \end{aligned} \quad (30)$$

Eq. (28) is the solution to our original problem. Eq. (28) can be rewritten as

$$\phi(x, t) = -C \cos(\omega t - \beta) + \frac{ql}{k} \left(1 - \frac{x}{l}\right) \quad (31)$$

where the amplitude C (> 0) is defined by

$$C = \frac{q}{k} \sqrt{\frac{\alpha_s}{\omega}} \sqrt{\frac{\cosh 2y - \cos 2y}{\cosh 2z + \cos 2z}} \quad (32)$$

Eq. (21) can be rewritten as

$$y = mz, \quad \text{where} \quad m = 1 - \frac{x}{l} \quad (33)$$

Since $0 \leq x \leq l$, we have from (33)

$$0 \leq m \leq 1 \quad (34)$$

From (18), we have

$$\sqrt{\frac{\alpha_s}{\omega}} = \frac{l}{\sqrt{2}z} \quad (35)$$

Substituting (33) and (35) into (32), we have

$$C = \frac{ql}{\sqrt{2}k} \frac{1}{z} \sqrt{\frac{\cosh 2mz - \cos 2mz}{\cosh 2z + \cos 2z}} \quad (36)$$

From (35) and (36), we have

$$\lim_{\omega \rightarrow \infty} C = \lim_{z \rightarrow \infty} C = 0 \quad (37)$$

Therefore, the temperature difference amplitude C goes to zero as the angular frequency ω goes to infinity everywhere. However, the amplitude drop would be more significant in the inside of the material ($0 < x \leq l$, $0 \leq m < 1$) than at the surface ($x = 0$, $m = 1$) because of the negative exponential term which comes out of cosh function inside the square root in Eq. (36). For the phase shift angle, β , the high frequency limit does not exist in the inside of the material. However, at the surface ($m = 1$), there is a nice high frequency limit as will be shown in the following.

From (28), we have

$$\phi(0, t) = -\frac{q}{k} \sqrt{\frac{\alpha_s}{\omega}} \sqrt{\frac{\cosh 2z - \cos 2z}{\cosh 2z + \cos 2z}} \cos(\omega t - \beta) + \frac{ql}{k} \quad (38)$$

where β is defined by (29), and

$$\begin{aligned} \cos \delta &= \frac{\sinh 2z}{\sqrt{\cosh^2 2z - \cos^2 2z}} \\ \sin \delta &= \frac{\sin 2z}{\sqrt{\cosh^2 2z - \cos^2 2z}} \end{aligned} \quad \text{at } x = 0 \quad (39)$$

From (39), we obtain

$$\tan \delta = \frac{\sin 2z}{\sinh 2z} \quad \text{at } x = 0 \quad (40)$$

It can be shown from (29) and (39) that $0 \leq \beta < \pi/2$. From (29), (38) and (39), we also obtain

$$\begin{aligned} \lim_{z \rightarrow 0} \delta(z) &= \frac{\pi}{4} & \lim_{z \rightarrow 0} \beta(z) &= 0 & \text{at } x = 0 \\ \lim_{z \rightarrow \infty} \delta(z) &= 0, & \lim_{z \rightarrow \infty} \beta(z) &= \frac{\pi}{4} & \text{at } x = 0 \end{aligned} \quad (41)$$

$$\lim_{\omega \rightarrow \infty} \phi(0, t) = \frac{q}{k_s} \sqrt{\frac{\alpha_s}{\omega}} \cos(\omega t - \frac{\pi}{4}) + \frac{ql}{k_s}$$

The above expression (41-3) gives the asymptotic limit of the surface temperature difference as the angular frequency $\omega \rightarrow \infty$. It can be also shown from (28) that after some algebraic manipulations

$$\lim_{\omega \rightarrow 0} \phi(x, t) = 0 \quad (42)$$

which is consistent with the definition of the boundary value problem defined by (1) and (4), when $\omega = 0$.

4. Numerical results and discussion

Let us first rewrite (38) as

$$\phi(0, t) = -F \cos(\omega t - \beta) + \frac{ql}{k} \quad (43)$$

where the temperature difference amplitude F (> 0) is given by

$$F = \frac{ql}{\sqrt{2k}} \frac{1}{z} \sqrt{\frac{\cosh 2z - \cos 2z}{\cosh 2z + \cos 2z}} \quad (44)$$

The phase shift angle β is defined by (29), and δ in (29) is defined by (39) when $x = 0$. z is defined by (18), which is repeated below for completeness.

$$z = \sqrt{\frac{\omega}{2\alpha_s}} l \quad (25)$$

From (44), a non-dimensional temperature difference amplitude $f(z)$ can be defined as

$$f(z) \equiv \frac{F}{\frac{ql}{\sqrt{2k}}} = \frac{1}{z} \sqrt{\frac{\cosh 2z - \cos 2z}{\cosh 2z + \cos 2z}} \quad (45)$$

Since we already know that $0 \leq \beta < \pi/2$ from the previous section, the phase shift angle β can be uniquely determined by

$$\beta = \frac{\pi}{4} - \tan^{-1} \left[\frac{\sin 2z}{\sinh 2z} \right] \quad (0 \leq \beta < \pi/2) \quad (46)$$

The numerical results for the non-dimensional temperature difference amplitude $f(z)$ and the phase shift angle β as a function of z are shown in Fig.2, and Fig. 3, respectively.

Fig.2. Non-dimensional temperature difference amplitude $f(z)$

Fig.3. Phase shift angle β

It should be noted from (45), (46), and Figs. 2 and 3 that

$$\lim_{z \rightarrow 0} f(z) = \sqrt{2} \quad (47)$$

$$\lim_{z \rightarrow \infty} f(z) = 0$$

$$\lim_{z \rightarrow 0} \beta = 0 \quad (48)$$

$$\lim_{z \rightarrow \infty} \beta = \frac{\pi}{4}$$

It can be also shown from (61) that

$$\text{Max}_{z>0} \beta(z) = 0.813266 \quad \text{at } z = 1.9633 \quad (49)$$

Therefore, the experimentally useful range of z for the phase shift angle measurement is $0 < z < z_{\max} (=1.9633)$.

5. Determination of the thermal conductivity and the thermal diffusivity from a surface temperature measurement

Suppose that we obtain F and β (and therefore δ) from a measurement for a given heat flux q , a given angular frequency ω , and a given length l of the specimen. Then the thermal conductivity k and the thermal diffusivity α_s can be determined as follows.

1. Solve (40) for z knowing the value of δ . Since this is a transcendental equation, it has to be solved numerically.
2. Once z is determined, the thermal diffusivity α_s can be determined from (18).
3. Once z is determined, the thermal conductivity k can be determined from (44) using the value of F .

From (2), we have

$$\rho c_p = \frac{k}{\alpha_s} \quad (50)$$

Thus the heat capacity ρc_p can be also obtained. If we know the density ρ , we can thus determine the specific heat c_p .

On the other hand, if we already know the heat capacity ρc_p , then we can determine the thermal conductivity k from (50), knowing the value of α_s after the step 2 above. The thermal conductivity thus determined can be compared to the thermal conductivity obtained from the step 3 above, and therefore, it can be used as a consistency check of the experimental data.

5. Conclusion

This paper solves a heat conduction problem associated with a 3ω method for obtaining the thermal conductivity of a solid. The thermal conductivity measurement is modeled as a 1 D heat conduction problem, and the AC current heat source is modeled as a time-harmonic heat flux at the boundary of the solid. An exact closed form solution is obtained for this boundary value problem, which could be used with experimental data obtained from the 3ω method to determine the thermal conductivity of a solid. Also this exact solution may be used as a bench mark solution for a numerical solution, or other approximate solutions.

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