**GENERALIZED ERROR ANALYSIS FOR
CONVENTIONAL AND REMOTE REFERENCE
MAGNETOTELLURICS**

by

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ABSTRACT

An error analysis which applies to both conventional and remote reference magnetotelluric impedance and tipper estimates is developed based on the assumption that noise in the field measurements is governed by a complex normal distribution. Under the assumed model of noise it is shown that the theoretical expressions for the variances and covariances derived recently by Gamble et al (1979b) specifically for remote reference estimates apply to conventional estimates as well. However, calculations are biased if the impedance or tipper functions are biased. The impedance and tipper functions are calculated as ratios of two random functions of noisy field measurements. The expressions for the variances and covariances account for noise in both the numerator and denominator of the estimates. They are useful provided the probability that the magnitude of the random error in the denominator exceeds the magnitude of its expected value is small.

Expressions for the bias errors of the impedance and tipper functions are obtained in order to assess the relative contributions of random and bias errors to the mean squared error of the estimates. The relative magnitude of both random and bias errors depends on the noise level and on the values of the sample coherencies between various pairs of the field measurements used to compute a particular estimate.

INTRODUCTION

In the frequency domain, the equations governing the magnetotelluric (MT) relations between the signal components (subscript s) of the electric and magnetic fields at the earth's surface are

$$E_{si} = Z_{ix}H_{sx} + Z_{iy}H_{sy}, \quad i = x \text{ or } y, \quad (1)$$

and
$$H_{sz} = T_{zx}H_{sx} + T_{zy}H_{sy}. \quad (2)$$

The tensor impedances, Z_{ix} and Z_{iy} , and the tipper functions, T_{zx} and T_{zy} , are functions of frequency and conductivity structure. It is assumed implicitly that the impedances and tippers are the earth's response to a primary plane wave source field which is refracted to propagate vertically into the earth at the air-earth interface.

Equations (1) and (2) each can be written in the general form

$$O_{si} = G_{ix}I_{sx} + G_{iy}I_{sy}, \quad i = x, y, \text{ or } z, \quad (3)$$

where, from the viewpoint of linear system theory, G_{ix} and G_{iy} are transfer functions of a dual input, single output linear system through which the inputs, I_{sx} and I_{sy} , are related deterministically to the output, O_{si} . The transfer functions are estimated from noisy field measurements, I_x , I_y , and O_i .

The transfer functions are usually calculated from noisy field data as the least squares estimates (subscript m)

$$G_{mij} = W_{mij}/V_m, \quad (4)$$

where

$$W_{mix} = \overline{O_i A^*} \overline{I_y B^*} - \overline{I_y A^*} \overline{O_i B^*},$$

$$W_{miy} = \overline{I_x A^*} \overline{O_i B^*} - \overline{O_i A^*} \overline{I_x B^*},$$

and

$$V_m = \overline{I_x A^*} \overline{I_y B^*} - \overline{I_y A^*} \overline{I_x B^*}.$$

In equation (4), A and B are two reference fields whose signal components, A_s and B_s , are related linearly to the input signals, I_{sx} and I_{sy} , i.e., $[A_s \ B_s]^T = \underline{M}[I_{sx} \ I_{sy}]^T$, where $\underline{M}$ is a nonsingular transfer matrix. The quantities XY^* are the averages, $\overline{XY^*} = N^{-1} \sum_{k=1}^N X_k Y_k^*$, where X_k and Y_k are harmonics of discrete Fourier transforms of the field data within a narrow bandwidth or from different data sets, and * indicates complex conjugate. These averaged crossproducts are actually spectral estimates, which we assume are resolved sufficiently so that G_{mij} gives G_{ij} with negligible error when I_x , I_y , and O_i are noise-free. Hereafter, the subscripts ij in G_{mij} and W_{mij} are suppressed except where required to identify particular estimates.

We distinguish between conventional and remote reference estimates. Conventional estimates are calculated by using particular pairs of I_x , I_y , and O_i as local references (e.g. Sims et al, 1971). Alternatively, remote reference estimates are calculated by using references obtained simultaneously with I_x , I_y , and O_i , but at a different site (Goubau et al, 1978; Gamble et al, 1979a).

The estimates G_m are nonlinear functions of the field measurements. They contain random or bias errors when the base field measurements I_x , I_y , or O_i are noisy. We require some means of estimating confidence intervals and bias errors of the estimates in order to specify the errors adequately.

It is generally recognized that correlations between the noise components of the references, A or B, and those of I_x , I_y , or O_i introduce bias errors. In particular, autopower noise terms in conventional estimates are recognized as primary sources of bias errors (e.g. Swift, 1967; Sims et al, 1971; Pedersen, 1982). However, additional $O(N^{-1})$ bias errors which exist due to the nonlinearity of the estimates have not been examined systematically.

Random error analyses have been presented by several authors. Bentley (1973) attempted to compute variances and confidence intervals for conventional estimates obtained using either the electric or magnetic fields as references, but was not careful to distinguish clearly between signals and noises in his analysis. Reddy et al (1976) and Pedersen (1982) have computed confidence intervals for G_{mix} and G_{miy} based on the principal axis decomposition of the least squares covariance matrix and the distribution of the residual squared errors. However, these intervals strictly apply only when the magnetic fields are noise free and are used as the reference field pair. Finally, Gamble et al (1979b) have presented a useful random error analysis, but have derived their results specifically for remote reference estimates.

It is possible to show that the expressions for the variances and covariances of the G_m obtained by Gamble et al apply to conventional

estimates as well, provided certain restrictions on the statistical behavior of the noise components of the field measurements are met. In this paper, we develop an analysis of random and bias errors of the G_m which is valid for both conventional and remote reference estimates, and examine the relative contributions of random and bias errors to the mean squared errors of the estimates. Specification of probability distributions for functions of the G_m of primary interest in MT, which are required to compute confidence intervals, is deferred to a subsequent paper.

In the next section, restrictions on the statistical behavior of the noises in the field measurements are outlined under which a general error analysis can be developed. In subsequent sections, derivations are presented for the variances and bias errors of the G_m after careful specification of the noise terms in W_m and V_m which cause random or bias errors in the ratio, $G_m = W_m/V_m$. The range of validity of these expressions is explored with regard to the noise levels in W_m and V_m . Then relative contributions of random and bias errors to the mean squared error of the estimates are examined as functions of the ordinary coherencies between the references, A and B, and between I_x , I_y , and O_i , providing insight into the problem of estimation.

THE NOISE MODEL

Our starting point is the model of signals and noises in MT field measurements presented in Stodt (1982a). Recapitulating, the field measurements consist of signals and additive random noises, e.g., $I_x = I_{sx} + i_x$, where lower case letters represent noise components. By definition, the signal components (subscript s) satisfy equation (3) while the noise components do not. We assume the signals and noises are statistically independent. The signals are treated as parameters when analyzing errors in estimates of the transfer functions by defining statistical expectation with respect to the conditional distribution of the noises, rather than with respect to the joint distribution of signals and noises. Such treatment is consistent with the assumption that signal components are related deterministically to each other through the transfer functions, G_{ix} and G_{iy} . In our case, the conditional distribution of the noises is the joint distribution of the noise terms only, because the signals and noises are statistically independent.

Subsequent derivations require that we impose the additional restriction that the noise components of the field measurements are realizations from a stationary Gaussian random process. The real and imaginary parts of the Fourier transformed noise components are then governed by a joint Gaussian distribution. In addition, subsequent analyses are simplified considerably if we assume that the covariance

matrix has a special form which permits the 2n-dimensional distribution of real and imaginary parts to be written compactly as an n-dimensional distribution of complex noise terms.

We outline briefly the conditions under which the complex distribution is a valid representation. Let $\underline{Z} = \underline{Z}_s + \underline{z}$ be an n-dimensional complex vector of Fourier transformed field measurements, $Z_i = Z_{si} + z_i$, $i = 1, 2, \dots, n$. Treating the signal components, $\underline{Z}_s$, as parameters, we are interested in the conditional distribution of the noises, $\underline{z} = \underline{Z} - \underline{Z}_s$. Define the 2n-dimensional random vector, $\underline{w}^T = [\underline{x}^T, \underline{y}^T]$. Then the joint Gaussian distribution of the real and imaginary parts of the noise components is defined by

$$g(\underline{w}) = \frac{1}{(2\pi)^n (\det \underline{M})^{1/2}} \exp[-(\underline{w}^T \underline{M}^{-1} \underline{w})/2] , \quad (5)$$

where $\underline{M}$ is the symmetric 2n x 2n covariance matrix

$$\underline{M} = \begin{matrix} \text{Cov}(x_i, x_j) & \text{Cov}(x_i, y_j) \\ \text{Cov}(x_j, y_i) & \text{Cov}(y_i, y_j) \end{matrix} \quad i, j = 1, 2, \dots, n .$$

For example, one entry of $\underline{M}$ is

$$\text{Cov}(x_i, y_j) = E[x_i y_j] - E[x_i] E[y_j] ,$$

where $E[\cdot]$ denotes statistical expectation. The diagonal elements of $\underline{M}$ are the variances of the real and imaginary components of the complex vector $\underline{z}$. If $\underline{M}$ has the special structure

$$\text{Cov}(x_i, y_j) = -\text{Cov}(x_j, y_i) , \quad (6)$$

and

$$\text{Cov}(x_i, x_j) = \text{Cov}(y_i, y_j) \quad , \quad (7)$$

then $g(\underline{w})$ can be written compactly as an n -dimensional distribution of the complex vector $\underline{z}$ (Miller, 1974),

$$g(\underline{z}) = \frac{1}{\pi^n \det \underline{R}} \exp [-\underline{z}^* \underline{R}^{-1} \underline{z}] .$$

We can also write

$$g(\underline{z}) = \frac{1}{\pi^n \det \underline{R}} \exp [-(\underline{z}^* - \underline{z}_s^*)^T \underline{R}^{-1} (\underline{z} - \underline{z}_s)], \quad (8)$$

where the signal components, z_{si} , are the expected values of the Z_i when expectation is defined with respect to the conditional distribution of the noises. The matrix $\underline{R}$ is the Hermitian $n \times n$ covariance matrix,

$$\underline{R} = [\text{Cov}(Z_i, Z_j^*)] \quad i, j = 1, 2, \dots, n ,$$

where $\text{Cov}(Z_i, Z_j^*) = E[Z_i Z_j^*] - E[Z_i] E[Z_j^*]. \quad (9)$

The diagonal elements of $\underline{R}$ are the variances of the Z_i , i.e.

$$R_{ii} = \text{Var } Z_i = E[|Z_i|^2] - |E[Z_i]|^2 . \quad (10)$$

By definition, $\text{Var } Z_i$ is the sum of $\text{Var } X_i$ and $\text{Var } Y_i$. Note that with regard to the real and imaginary parts of the individual variables Z_i , equation (6) requires that $\text{Cov}(X_i, Y_i)$ be zero, while equation (7) requires that $\text{Var } X_i$ and $\text{Var } Y_i$ are equal. However, the Gaussian random vectors $\underline{X}$ and $\underline{Y}$ are statistically independent if and only if $\text{Cov}(X_i, Y_j) = 0$ for all i and j , and in this special case it can be shown that the covariance matrix $\underline{R}$ becomes real and symmetric.

Equations (6) and (7) can be written compactly as

$$\text{Cov}(z_i, z_j) = \text{Cov}(Z_i, Z_j) = 0 \quad (11)$$

for all values of i and j .

We close this section with an expansion for the fourth order moment, $E[Z_1 Z_2^* Z_3 Z_4^*]$, which is used in subsequent derivations. Miller (1974, p. 85) calculates the moment generating function for $g(\underline{Z})$ and shows in particular the result,

$$\begin{aligned} E[Z_1 Z_2^* Z_3 Z_4^*] = \\ E[Z_1 Z_2^*] E[Z_3 Z_4^*] + E[Z_1 Z_4^*] E[Z_3 Z_2^*] - Z_{s1}^* Z_{s2}^* Z_{s3}^* Z_{s4}^* . \end{aligned} \quad (12)$$

Equation (12) is valid only if equation (11) is satisfied so that $\text{Cov}(Z_1, Z_3)$ and $\text{Cov}(Z_2^*, Z_4^*)$ are zero.

BIAS, VARIANCE, AND MEAN SQUARED ERROR
OF COMPLEX RANDOM FUNCTIONS

When noise is present in the field measurements, the estimates G_m are random functions obtained from the ratios W_m/V_m . The W_m and V_m are in turn random functions having the form

$$F_m = F(\underline{Z}) = \overline{Z_1 Z_3^*} \overline{Z_2 Z_4^*} - \overline{Z_2 Z_3^*} \overline{Z_1 Z_4^*},$$

where equation (4) leads to the identifications

$$W_{mix} = F(O_i, I_y, A, B),$$

$$W_{miy} = F(I_x, O_i, A, B),$$

$$V_m = F(I_x, I_y, A, B).$$

In general, the errors in a random function F_m are specified by calculating the finite moments $E[F_m^r F_m^{*s}]$ and the probability distribution of F_m . Bias errors and variances are calculated from the first and second moments, when they exist. If the true value, $F = F(\underline{Z}_S)$, is a nonlinear function of the elements of $\underline{Z}_S$, it is possible that the expected value of F_m differs from the true value, F . Then the bias of F_m is defined by

$$\text{Bias } F_m = E[F_m - F] = F_b - F, \quad (13)$$

where F_b is the expected value of F_m and the subscript b indicates the

possible presence of bias error. However if F_m is distributed symmetrically about the true value F , then the expected value, F_b , and the true value, F , are equal and F_m is unbiased. The variance is defined from the first and second moments of F_m as

$$\text{Var } F_m = E[|F_m - F_b|^2] = E[|F_m|^2] - |F_b|^2, \quad (14)$$

while the mean squared error of F_m is given by

$$\text{MSE} = E[|F_m - F|^2] = \text{Var } F_m + |\text{Bias } F_m|^2. \quad (15)$$

The MSE and variance of F_m are equal only if F_m is unbiased. Note that equations (13) through (15) are defined only when the appropriate moments of F_m exist.

The variance of a sum of complex random variables is required subsequently. Let $F_m = \sum_i a_i F_{mi}$, where the a_i are complex constants. Then

$$\text{Var } F_m = \sum_i |a_i|^2 \text{Var } F_{mi} + \sum_{\substack{i,j \\ i \neq j}} a_i a_j^* \text{Cov}(F_{mi}, F_{mj}^*) \quad (16)$$

when equation (14) is expanded.

SPECIFICATION OF ERRORS IN G_{mij}

Approximate expressions for the variances and bias errors of the G_m can be derived and their accuracy assessed once the errors in G_m are specified in suitable form. We can write the expression

$$G_{mij} = \frac{W_{mij}}{V_m} = \frac{W_{ij} + \Delta W_{ij}}{V + \Delta V}, \quad (17)$$

where the true value, G_{ij} , is given by the ratio, W_{ij}/V , and ΔW_{ij} and ΔV are noise terms which introduce random or bias errors in G_{mij} . The quantities W_{ij} , ΔW_{ij} , V , and ΔV are obtained explicitly as functions of the signal and noise components of the field measurements in Appendix A.

Equation (17) can be rearranged to express the error in G_{mij} in terms of the errors in W_{mij} and V_m , i.e.,

$$\Delta G_{ij} = \frac{\Delta W_{ij} - G_{ij} \Delta V}{V_m}, \quad (18)$$

where $\Delta G_{ij} = G_{mij} - G_{ij}$ is the total error in G_{mij} . Note in particular that $\Delta W_{ij} - G_{ij} \Delta V$ is zero when I_x , I_y , and O_i are noise free, as discussed in Appendix A. Alternatively, Gamble et al (1979b) define the base field measurement error,

$$N_i = O_i - G_{ix} I_x - G_{iy} I_y, \quad (19)$$

and derive the expression

$$\Delta G_{ij} = \frac{\overline{N_i R_j^*}}{V_m}, \quad (20)$$

where

$$R_x = A \overline{I_y^* B} - B \overline{I_y^* A} ,$$

and (21)

$$R_y = B \overline{I_x^* A} - A \overline{I_x^* B} .$$

Comparing equations (18) and (20) shows that $\overline{N_i R_j^*} = \Delta W_{ij} - G_{ij} \Delta V$.

If V_m is noisy or if W_m and V_m are biased due to correlations between base and reference field noises, then equations (17) through (21) can be rearranged to separate random and bias errors. Equation (17) can be rewritten as

$$G_{mij} = \frac{W_{bij} + w_{ij}}{V_b + v} , \quad (22)$$

where W_b and V_b are the expected values of W_m and V_m , and w and v are the random errors in W_m and V_m . Defining the random error, $g = G_m - G_b$, where G_b is the expected value of G_m , then we find

$$\begin{aligned} g_{ij} &= \frac{W_{mij} - G_{bij} V_m}{V_m} \\ &= \frac{W_{bij} + w_{ij} - G_{bij} (V_b + v)}{V_m} . \end{aligned} \quad (23)$$

Note that when V_m is noisy, G_b does not equal the ratio W_b/V_b due to the nonlinearity of the relation, $G_m = W_m/V_m$. Equation (20) can also be redefined to account for bias errors in the G_{mij} by redefining the base field measurement error, N_i , i.e.

$$N_{bi} = O_i - G_{bix} I_x - G_{biy} I_y . \quad (24)$$

Multiplying through by references A^* and B^* in turn, averaging, and

rewriting in matrix notation gives

$$[N_b R] = [OR] - G_b [IR], \quad (25)$$

where $[N_b R]$, $[OR]$, and $[IR]$ are spectral matrices, e.g.

$$[IR] = \begin{array}{cc} \overline{I_x^A}^* & \overline{I_x^B}^* \\ \overline{I_y^A}^* & \overline{I_y^B}^* \end{array} .$$

Multiplying equation (25) through by $[IR]^{-1}$ and recognizing that $[G_{mij}] = [OR][IR]^{-1}$, we find

$$[g_{ij}] = [N_b R][IR]^{-1} ,$$

where the individual elements g_{ij} can be written as

$$g_{ij} = \frac{\overline{N_{bi} R_j}^*}{V_m} , \quad (26)$$

and the R_j are defined by equation (21).

VARIANCE EXPRESSIONS

The tensor estimates G_m are calculated as the ratios, W_m/V_m . When the probability is zero that $|v| > |V_b|$, it can be shown (Stodt, 1982c) that the variance of the ratio is given exactly by

$$\text{Var } G_m = \frac{A - B}{E[|V_m|^2]}, \quad (27)$$

where

$$A = \text{Var } W_m + |G_b|^2 \text{Var } V_m - 2\text{Re}[G_b^* \text{Cov}(W_m, V_m^*)],$$

$$B = \text{Cov}(|g|^2, |V_m|^2) - |\text{Cov}(g, V_m)|^2,$$

and

$$E[|V_m|^2] = |V_b|^2 + \text{Var } V_m.$$

In practice, equation (27) is a good approximation when the probability is small that $|v|$ exceeds $|V_b|$. In this case, B is small compared to A and can be neglected. Equation (27) is interesting because it shows explicitly the contributions of the individual noise terms to $\text{Var } G_m$, but individual expressions for the variances and covariances of W_m and V_m are cumbersome to derive and calculate.

Gamble et al (1979b) derive variance expressions for remote reference estimates which are less cumbersome to calculate. They ignore random errors in the denominator of equation (20) and calculate the approximation,

$$\text{Var } G_{mij} \approx \frac{\text{Var } \overline{N_i R_j^*}}{|V_m|^2} \approx \frac{E[|N_i|^2]E[|R_j|^2]}{N|V_m|^2}. \quad (28)$$

The nature of their approximation becomes clear when we recall that $\overline{N_i R_j^*} = \Delta W_{ij} - G_{ij} \Delta V$. Calculating the variance of the difference from equation (16) shows that equation (28) gives essentially the same approximation as that given by neglecting B and replacing the expected value $E[|V_m|^2]$ with the measured value $|V_m|^2$ in equation (27). Gamble et al calculate $\text{Var } \overline{N_i R_j^*}$ under the specific requirement that base field noises are uncorrelated with reference field noises.

The following derivation shows that equation (28) is valid even when the base and reference field noises correlate. Recognizing that $\text{Var } \overline{N_i R_j^*}$ is the variance of a sum and applying equations (14) and (16) gives

$$\begin{aligned} \text{Var } G_{mij} &\approx N^{-2} |V_m|^{-2} \sum_{k=1}^N \sum_{l=1}^N \\ &\{ E[N_{ik} R_{jk}^* N_{il} R_{jl}^*] - E[N_{ik} R_{jk}^*] E[N_{il} R_{jl}^*] \}. \end{aligned} \quad (29)$$

Note that N_{ik} and N_{il} are normally distributed by assumption, but that R_{jk} and R_{jl} are normally distributed only if the crosspowers in equations (21) can be treated as constants. The expected value and variance of the crossproduct, $\overline{XY^*}$, are derived in Appendix B. From those expressions, the crossproduct can be written as

$$\overline{XY^*} = E[\overline{XY^*}] + O(N^{-1/2}),$$

where

$$E[\overline{XY^*}] = \overline{X_S Y_S^*} + E[\overline{xy^*}]$$

and $O(N^{-1/2})$ represents an error term which depends on the noise to signal powers of X and Y , and decreases like $N^{-1/2}$. If we replace the crosspowers in R_{jk} and R_{jl} with their expected values and expand the fourth order moment in equation (29) using equation (12), it can be shown that an $O(N^{-1})$ error is introduced into the resulting expression for $\text{Var } G_{mij}$. With this approximation, equation (29) reduces to

$$\begin{aligned}\text{Var } G_{mij} &\approx N^{-2} |V_m|^{-2} \sum_{k=1}^N \sum_{l=1}^N E[N_{ik} N_{il}^*] E[R_{jk}^* R_{jl}] \\ &= N^{-2} |V_m|^{-2} \sum_{k=1}^N E[|N_{ik}|^2] E[|R_{jk}|^2] \\ &= N^{-1} |V_m|^{-2} E[|N_i|^2] E[|R_j|^2].\end{aligned}\quad (30)$$

The successive reductions in equation (30) are performed by assuming N_{ik} and N_{il} are statistically independent so that $E[N_{ik} N_{il}^*]$ is zero when $k \neq l$, and by assuming the noises are stationary so that $E[|N_{ik}|^2]$ can be replaced by $E[|N_i|^2]$ and taken outside the summation.

In practice, $E[|N_i|^2]$ can be estimated by replacing G_{ix} and G_{iy} with G_{mix} and G_{miy} in equation (19) and computing the average, $\overline{|N_{mi}|^2}$. If G_{mix} and G_{miy} are unbiased, $\overline{|N_{mi}|^2}$ is an asymptotically unbiased estimate of $E[|N_i|^2]$. The expected value $E[|R_j|^2]$ is also replaced by the measured value $\overline{|R_j|^2}$ in equation (30). These substitutions introduce additional $O(N^{-1})$ errors into the variance calculations.

Equation (30) applies to conventional estimates as well as to remote reference estimates, since no assumptions concerning noise correlations were involved in the derivation. However, conventional estimates are usually biased so that persistent bias errors are introduced into the estimate $\overline{|N_{mi}|^2}$ of $E[|N_i|^2]$.

An alternate expression for $\text{Var } G_m$ which utilizes the biased values G_b can be obtained by starting with equation (26). A derivation similar to that which produced equation (30) leads to the expression

$$\text{Var } G_{mij} = N^{-1} |V_m|^{-2} \{ E[|N_{bi}|^2] E[|R_j|^2] - |E[N_{bi}]|^2 |E[R_j]|^2 \}, \quad (31)$$

where

$$E[N_{bi}] = 0_{si} - G_{bix} I_{sx} - G_{biy} I_{sy},$$

$$E[R_x] = A_s E[I_y^* B] - B_s E[I_x^* A] + O(N^{-1}),$$

and

$$E[R_y] = B_s E[I_x^* A] - A_s E[I_y^* B] + O(N^{-1}).$$

Calculation of equation (31) requires estimates of the autopowers and crosspowers of the signal components of the measured fields. If estimates of the signal autopowers are not available, then comparing equations (30) and (31) shows that variances calculated from equation (30) by replacing $E[|N_i|^2]$ with $|N_{mi}|^2$ are always biased high if G_{mix} or G_{miy} are biased in any manner.

The variances of various nonlinear functions of the G_m which are useful in MT analysis are also required. Approximate expressions can be obtained by propagation of errors (Gamble et al, 1979b; Stodt, 1982c). The variance expressions for functions of more than one element G_{mij} contain the terms $\text{Cov}(G_{mij}, G_{mkl}^*)$ where $i \neq k$ or $j \neq l$. The covariances are derived in the same manner as the variances, with the results

$$\text{Cov}(G_{mij}, G_{mkl}^*) \approx N^{-1} |V_m|^{-2} E[N_i N_k^*] \overline{E[R_j^* R_l]} \quad (32)$$

or

$$\begin{aligned} \text{Cov}(G_{mij}, G_{mkl}^*) \approx N^{-1} |V_m|^{-2} \{ & E[N_{bi} N_{bk}^*] \overline{E[R_j^* R_l^*]} \\ & - \overline{E[N_{bi}] E[N_{bk}^*]} \overline{E[R_j] E[R_l^*]} \}. \end{aligned} \quad (33)$$

Equations (32) and (33) are respectively the counterparts of equations (30) and (31).

BIAS ERRORS

By definition, the expected value of a product of two complex random variables A and B is given by

$$E[AB] = E[A]E[B] + \text{Cov}(A,B).$$

By analogy, the expected value G_b of the ratio $G_m = W_m/V_m$ can be written as

$$G_b = W_b E[V_m^{-1}] + \text{Cov}(W_m, V_m^{-1}).$$

The distribution of V_m can generally be approximated by equation (8) (Stodt, 1982c). Under this assumption, the expected value of V_m^{-1} is given by (Miller, 1975 p. 93)

$$E[V_m^{-1}] = V_b^{-1} [1 - \exp(-|V_b|^2 / \text{Var } V_m)].$$

Noting that $\text{Cov}(W_m, V_m^{-1})$ can be rewritten as $V_b^{-1} \text{Cov}(w, V_b/V_m)$, we can write

$$G_b = \frac{W_b}{V_b} [1 - \exp(-|V_b|^2 / \text{Var } V_m) + \text{Cov}(w/W_b, V_b/V_m)], \quad (34)$$

and the bias error in G_m is the difference, $G_b - G$. Note that G_m is biased if the percentage bias errors in W_m and V_m are different, if V_m is noisy, or if $\text{Cov}(w, V_b/V_m)$ is nonzero.

The bias errors in W_m and V_m are calculated from the expected values $E[\Delta W]$ and $E[\Delta V]$ in Appendix A. The bias error in V_m is given

by

$$\begin{aligned} \text{Bias } V_m = & \{E[i_x a^*] \overline{I_{sy} B_s^*} + E[i_y b^*] \overline{I_{sx} A_s^*} + E[i_x a^*] E[i_y b^*] \\ & - E[i_x b^*] \overline{I_{sy} A_s^*} - E[i_y a^*] \overline{I_{sx} B_s^*} - E[i_x b^*] E[i_y a^*]\} [1 - N^{-1}]. \end{aligned} \quad (35)$$

The bias errors in W_{mij} can be deduced from Bias V_m . Bias W_{mix} is deduced by replacing I_{sx} and i_x with O_{si} and o_i , while Bias W_{miy} is deduced by replacing I_{sy} and i_y with O_{si} and o_i , in equation (35).

If remote reference measurements are made and the base and reference field noises are uncorrelated, then equation (35) shows that W_m and V_m are unbiased. Thus, from equation (34), G_m is unbiased if V_m is noise free. If conventional measurements are made, then equation (35) can be simplified once a reference field pair is chosen. For example, if O_x and O_y are chosen as A and B and if input and output noises are uncorrelated, V_m is unbiased. The bias errors in W_{mix} and W_{miy} are given by

$$\text{Bias } W_{mix} = \{E[o_i o_x^*] \overline{I_{sy} O_{sy}^*} - E[o_i o_y^*] \overline{I_{sy} O_{sx}^*}\} [1 - N^{-1}]$$

and

$$\text{Bias } W_{miy} = \{E[o_i o_y^*] \overline{I_{sx} O_{sx}^*} - E[o_i o_x^*] \overline{I_{sx} O_{sy}^*}\} [1 - N^{-1}].$$

RELATIVE CONTRIBUTIONS OF RANDOM AND BIAS ERRORS

When I_x and I_y are used as references, it is possible to express the variances of the G_{mij} in terms of sample ordinary and multiple coherence functions. In this section, we define the coherence estimates and then examine the behavior of the variances after rewriting them as functions of the sample coherencies. We also examine the relative contributions of random and bias errors to the mean squared error of the estimates.

The ordinary coherence measures the frequency domain correlation between two field components X and Y and is estimated from measured data as

$$c_{XY}^2 = \frac{|\overline{XY^*}|^2}{P_X P_Y} \quad (36)$$

where, e.g., $P_X = \overline{XX^*}$. The multiple coherence measures the simultaneous correlation of a given field component to more than one other field component. In particular, the multiple coherence defined by (e.g. Bendat and Piersol, 1971)

$$c_{I_x I_y - O_i}^2 = \frac{|\overline{I_x O_i^*}|^2 P_{I_y} + |\overline{I_y O_i^*}|^2 P_{I_x} - 2\text{Re}[\overline{I_x O_i^*} \overline{I_y O_i^*} \overline{I_x I_y^*}]}{P_{O_i} P_{I_x} P_{I_y} [1 - c_{I_x I_y}^2]} \quad (37)$$

measures the linear relationship of O_i to I_x and I_y , and is unity when all three measurements are noise free. If only O_i is noisy, then

$C_{I_{sx}I_{sy}-0_i}^2$ estimates the ratio $P_{0_{si}}/P_{0_i}$, while additional noise in I_x or I_y further decreases the estimate. Both the ordinary and multiple coherence estimates satisfy the inequality $0 \leq C^2 \leq 1$. Stodt (1982a) discusses the relationship between the sample coherencies C^2 and their expected values, γ^2 . In the sequel it is important to note that it is the sample coherencies, not their expected values, which are the relevant parameters.

Three ordinary coherencies, $C_{I_x I_y}^2$, $C_{I_x 0_i}^2$, and $C_{I_y 0_i}^2$ can be defined from the field measurements, I_x , I_y , and 0_i . The following comments apply to the coherencies between the signal components and provide insight into their behavior with regard to G_{ix} and G_{iy} . In general, the values of $C_{I_{sx}0_{si}}^2$ and $C_{I_{sy}0_{si}}^2$ depend on the relative magnitudes of G_{ix} and G_{iy} , and on the input coherence $C_{I_{sx}I_{sy}}^2$. In the particular case when G_{ix} is zero, $C_{I_{sy}0_{si}}^2$ is unity and $C_{I_{sx}0_{si}}^2$ is identical to $C_{I_{sx}I_{sy}}^2$. Similarly, if G_{iy} is zero, $C_{I_{sx}0_{si}}^2$ is unity and $C_{I_{sy}0_{si}}^2$ is identical to $C_{I_{sx}I_{sy}}^2$. If the input coherence is zero, then from equation (37) we have

$$C_{I_{sx}0_{si}}^2 + C_{I_{sy}0_{si}}^2 = 1, \quad (38)$$

and the relative values of $C_{I_{sx}0_{si}}^2$ and $C_{I_{sy}0_{si}}^2$ reflect the relative values of the squared magnitudes of G_{ix} and G_{iy} . Alternatively, as the input coherence approaches unity, $C_{I_{sx}0_{si}}^2$ and $C_{I_{sy}0_{si}}^2$ approach unity regardless of the values of G_{ix} and G_{iy} .

From the defining equations (21), (36), and (37), it is straightforward to show that $\overline{|R_j|}^2$ can be written as

$$\overline{|R_{x,y}|^2} = P_{I_{y,x}} P_A P_B [1 - C_{AB}^2] C_{AB-I_{y,x}}^2. \quad (39)$$

When the references are chosen as I_x and I_y it can be shown that the multiple coherencies $C_{AB-I_{y,x}}^2$ are unity, and that $|V_m|^2$ is given by

$$|V_m|^2 = P_{I_x}^2 P_{I_y}^2 [1 - C_{I_x I_y}^2]^2.$$

Substituting these results into equation (28) gives

$$\text{Var } G_{mij} \approx \frac{\overline{|N_{mi}|^2} / P_{I_j}}{N[1 - C_{I_x I_y}^2]}. \quad (40)$$

Equation (40) shows how the variances of the G_{mij} are inflated as the sample input coherence $C_{I_x I_y}^2$ approaches unity.

The different behavior of the percentage errors in G_{mix} and G_{miy} can be examined from the expression for the variance of $\ln |G_{mij}|^2$, where $\ln(\cdot)$ is the natural logarithm. Using error propagation, we obtain the result

$$\text{Var } \ln |G_{mij}|^2 \approx \frac{2 \text{Var } G_{mij}}{N |G_{mij}|^2} \approx \frac{2 \overline{|N_{mi}|^2} \overline{|R_j|^2}}{N |W_{mij}|^2}. \quad (41)$$

When the references are I_x and I_y , it can be shown that

$$|W_{mix,y}|^2 = P_{I_{x,y}} P_{I_{y,x}}^2 P_{0_i} [C_{I_x I_y - 0_i}^2 - C_{0_i I_{y,x}}^2] [1 - C_{I_x I_y}^2], \quad (42)$$

and substituting equations (39) and (42) into equation (41) gives the result

$$\text{Var } \ln |G_{mij}|^2 \approx \frac{2 \overline{|N_{mi}|^2} / P_{0_i}}{N [C_{I_x I_y - 0_i}^2 - C_{0_i I_k}^2]}, \quad (43)$$

where the subscript k in $C_{0_i I_k}^2$ is x when the subscript j in G_{mij} is y , and vice versa. Equation (43) shows how the percentage errors in the squared magnitudes of G_{mix} and G_{miy} differ depending on the relative values of the sample coherencies $C_{I_x 0_i}^2$ and $C_{I_y 0_i}^2$. We can write the ratio

$$\frac{\text{Var } \ln |G_{mix}|^2}{\text{Var } \ln |G_{miy}|^2} = \frac{(1 - X)}{(1 - Y)} = \frac{1 + R - RS}{1 + R - S}, \quad (44)$$

where

$$X = C_{I_x 0_i}^2 / C_{I_x I_y - 0_i}^2,$$

$$Y = C_{I_y 0_i}^2 / C_{I_x I_y - 0_i}^2,$$

and we have defined the parameters $R = X/Y$ and $S = X + Y$. Figure 1 is a plot of equation (44) versus the ratio R for various values of the sum S . In the special case when only 0_i is noisy, X and Y reduce to $C_{I_{sx} 0_{si}}^2$ and $C_{I_{sy} 0_{si}}^2$ respectively, and their sum satisfies the inequality, $1 \leq S \leq 2$. From the previous discussion of the behavior of $C_{I_{sx} 0_{si}}^2$, $C_{I_{sy} 0_{si}}^2$, and $C_{I_{sx} I_{sy}}^2$, it is apparent that the particular value of the sum S depends on the input coherence; as $C_{I_{sx} I_{sy}}^2$ approaches unity, S approaches its upper limit. Figure 1 shows that as the input coherence increases, the relative levels of random noise in G_{mix} and G_{miy} become sensitive to small differences in the values of the ordinary coherencies, $C_{I_{sx} 0_{si}}^2$ and $C_{I_{sy} 0_{si}}^2$, between the signal components of the base field measurements.

We now examine the general case where the references A and B are

not chosen as the input measurements I_x and I_y . The signal components of any reference pair are related to the input signals through a transfer matrix $\underline{M}$, i.e. $[A_s \ B_s]^T = \underline{M} \underline{I}_s$. Define the vector, $\underline{I}_r = \underline{M}^{-1}[A \ B]^T$, where the subscript r is used to distinguish between the vectors $\underline{I} = \underline{I}_s + \underline{i}$ and $\underline{I}_r = \underline{I}_s + \underline{r}$, and $\underline{r}$ is the noise vector defined by $\underline{r} = \underline{M}^{-1}[a \ b]^T$. With these definitions it can be shown that

$$W_{mij}(A, B) = \det \underline{M}^* W_{mij}(I_{rx}, I_{ry}),$$

$$V_m(A, B) = \det \underline{M}^* V_m(I_{rx}, I_{ry}),$$

and

$$R_j(A, B) = \det \underline{M}^* R_j(I_{rx}, I_{ry}), \quad (45)$$

where the notation (A, B) is used to indicate computation with a particular reference pair and $\det \underline{M}^*$ is the determinant of $\underline{M}^*$.

The behavior of $\text{Var } G_m$ when the coherence $C_{A_s B_s}^2$ between the reference signals and the coherence $C_{I_{sx} I_{sy}}^2$ between the input signals are different is of interest. If the references are noise-free, $\underline{I}_r$ and $\underline{I}_s$ are identical and equations (45) together with equation (28) show that $\text{Var } G_m$ is independent of $C_{A_s B_s}^2$. However, when $C_{A_s B_s}^2$ is much larger than $C_{I_{sx} I_{sy}}^2$ the condition number of $\underline{M}$ is large, and if the references are noisy it is likely that $||\underline{r}||/||\underline{I}_s|| > ||[a \ b]||/||[A_s \ B_s]||$ where $||\cdot||$ is an appropriate vector norm (e.g. Forsythe and Moler, 1967, p. 20). Such noise amplification in $\underline{I}_r$ inflates $\text{Var } G_{mij}$.

The relative contributions of random and bias errors to the mean

squared error of the estimates are of interest. From equations (28) and (34) we obtain the approximation,

$$\frac{|\text{Bias } G_{mij}|^2}{\text{Var } G_{mij}} \approx \frac{N|A + B|^2}{|\overline{N_{mi}}|^2 |\overline{R_j}|^2}, \quad (46)$$

where

$$A = \text{Bias } W_{mij} - G_{ij} \text{Bias } V_m$$

and

$$B = \text{Cov}(W_{mij}, V_b/V_m) - W_{bij} \exp(-|V_b|^2/\text{Var } V_m).$$

Equation (35) shows that the quantity A represents a persistent bias error. In contrast, B usually decreases as N increases. As expected, equation (46) shows that when A is nonzero the proportion of bias to random error in G_m increases as N increases. Pedersen (1982) has shown similar behavior by estimating the random and bias errors for the scalar estimate, $G_{mij} = \overline{O_i I_j^*}/P_{I_j}$, as functions of N and the noise to signal powers of O_i and I_j . With regard to tensor estimates, note from equation (39) that $|\overline{R_j}|^2$ decreases as the coherence C_{AB}^2 between the references increases. Thus equation (46) shows additionally that the proportion of bias to random error in the tensor estimates increases or as the coherence C_{AB}^2 between the references increases.

When base and reference field noises are uncorrelated, W_m and V_m are unbiased. However, nonlinear functions of W_m and V_m may contain $O(N^{-1})$ bias errors which become significant as the random errors in W_m or V_m increase. Equation (34) shows how G_m is biased when V_m is noisy, due to the nonlinearity of the ratio W_m/V_m . Since apparent resistivities are calculated from the squared magnitude of G_m , it is

also of interest to examine the bias errors in $|W_m|^2$ and $|V_m|^2$ due to nonlinearity. We assume that W_m and V_m are normally distributed because of the central limit theorem. When the distribution of a complex random variable F_m is given by equation (8), it can be shown (Miller, 1974, p. 85) that

$$E[|F_m|^{2n}] = n!(\text{Var } F_m)^n L_n(-|F|^2/\text{Var } F_m)$$

where F is the expected value $E[F_m]$ and $L_n(\cdot)$ is the Laguerre polynomial. Using this result with the defining equations (13) and (14) gives

$$\text{Bias } |F_m|^2 = \text{Var } F_m,$$

and

$$\text{Var } |F_m|^2 = 2|F|^2 \text{Var } F_m + (\text{Var } F_m)^2.$$

The ratio $\text{Bias } |F_m|^2 / (\text{Var } |F_m|^2)^{1/2}$ is plotted versus the ratio $(\text{Var } F_m)^{1/2} / |F|$ in Figure 2.

CONCLUSIONS

We have shown that when the noise in the Fourier transformed field measurements is governed by a complex normal distribution, the variance and covariance expressions derived by Gamble et al (1979b) apply to conventional as well as to remote reference estimates. The expressions, given by equations (28) and (32), account for errors in both W_m and V_m and are useful provided the probability is small that $|v|$ exceeds $|V_b|$. However, if the G_{mij} are biased, equations (31) and (33) show that the variances calculated by replacing $E[|N_i|^2]$ and $E[N_i N_j^*]$ with the estimates $\overline{|N_{mi}|^2}$ and $\overline{N_{mi} N_{mj}^*}$ are biased. In particular, equation (31) shows that when G_m is biased, variances calculated by replacing $E[|N_i|^2]$ with $\overline{|N_{mi}|^2}$ are biased high.

The random error analysis shows that no significant inherent reduction in random error can be expected by using remote references in place of local references. However, in the ideal situation where base and reference field noises are completely uncorrelated, the expected values of W_m and V_m are the true values, W and V . Thus, remote references provide a means to reduce or eliminate bias errors in W_m and V_m which often are largely responsible for the observed scatter in conventional MT estimates.

Both random and bias errors in the G_{mij} are influenced by the values of the sample ordinary coherencies between various pairs of the field measurements. The coherence C_{AB}^2 between the references and the

input coherence $C_{I_x I_y}^2$ both act primarily to amplify random or bias errors as either value approaches unity. In particular, if the coherence between the references is much larger than the input coherence then bias errors may be inflated to a greater degree than are random errors. When the coherencies $C_{I_x 0_i}^2$ and $C_{I_y 0_i}^2$ between the input and output measurements are different, the percentage errors in G_{mix} and G_{miy} are different. For a given value of their ratio, $C_{I_x 0_i}^2 / C_{I_y 0_i}^2$, the difference in percentage errors in G_{mix} and G_{miy} increases as the sum $X + Y$ increases, where X and Y are defined in equation (44).

The analysis presented in this paper elucidates the behavior of the errors in impedance and tipper estimates calculated from noisy field data. An understanding of the effects of noise on the estimates should aid in optimizing data acquisition and reduction procedures.

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LIST OF FIGURES

1. Plot of the ratio $\text{Var } \ln|G_{mix}|^2 / \text{Var } \ln|G_{miy}|^2$ v.s. the ratio X/Y for various values of the sum $S = X + Y$, where $X = C_{I_x 0_i}^2 / C_{I_x I_y - 0_i}^2$ and $Y = C_{I_y 0_i}^2 / C_{I_x I_y - 0_i}^2$. If only 0_i is noisy, X and Y reduce to $C_{I_{sx} 0_{si}}^2$ and $C_{I_{sy} 0_{si}}^2$ respectively and S satisfies the inequality $1 \leq S \leq 2$, where the particular value depends on the input coherence, $C_{I_{sx} I_{sy}}^2$.
2. Plot of Bias $|F_m|^2 / (\text{Var } |F_m|^2)^{1/2} = A / (A^2 + 2)^{1/2}$ v.s. the ratio $A = (\text{Var } F_m)^{1/2} / |F|$ when the distribution of F_m is given by equation (8). Note that the bias error in $|F_m|^2$ due to nonlinearity is always less than the standard deviation of $|F_m|^2$.

FIGURE 1

FIGURE 2

APPENDIX A
SPECIFICATION OF ERRORS IN W_m , V_m , and G_m

In this appendix, W_m , V_m , and G_m are separated into signal and noise contributions and the bias errors in W_m and V_m are derived.

We first rewrite W_{mix} , W_{miy} , and V_m to separate the base fields, I_x , I_y , and O_i from the reference fields A and B. For example, consider V_m which can be written as the summation,

$$\begin{aligned} V_m &= N^{-2} \sum_{k=1}^N \sum_{l=1}^N [I_{xk} A_k^* I_{yl} B_l^* - I_{yk} A_k^* I_{xl} B_l^*] \\ &= N^{-2} \sum_k \sum_l D_{kl} A_k^* B_l^* , \end{aligned} \quad (A-1)$$

where

$$D_{kl} = [I_{xk} I_{yl} - I_{yk} I_{xl}] . \quad (A-2)$$

Recognizing that $D_{kl} = -D_{lk}$ and that $D_{kk} = 0$, equation (A-1) reduces to the result,

$$V_m = N^{-2} \sum_{k=1}^{N-1} \sum_{l=k+1}^N D_{kl} R_{kl}^* , \quad (A-3)$$

where

$$R_{kl} = [A_k B_l - B_k A_l] .$$

Similarly, W_{mix} and W_{miy} can be written as

$$W_{mij} = N^{-2} \sum_{k=1}^{N-1} \sum_{l=k+1}^N N_{ijkl} R_{kl}^* , \quad (A-4)$$

where

$$N_{ixkl} = [0_{ik} I_{yl} - I_{yk} 0_{il}] ,$$

and

$$N_{iykl} = [I_{xk} 0_{il} - 0_{ik} I_{xl}] .$$

Thus, the transfer functions G_m can be written as

$$G_{mij} = \frac{W_{mij}}{V_m} = \frac{N^{-2} \sum_{p=1}^P N_{ijp} R_p^*}{N^{-2} \sum_{p=1}^P D_p R_p^*} , \quad (A-5)$$

where p is a summation index over the $P = N(N - 1)/2$ combinations of the subscripts $k = 1, 2, \dots, N - 1$ and $l = k + 1, k + 2, \dots, N$. Note that the individual terms N_{ijp}/D_p are estimates of G_{ij} which would be obtained from the l th and k th data sets. Thus all possible pairings of the N sets of harmonics which could be used to estimate G_{ij} separately are utilized in equation (A-5).

When the measured fields are noisy, N_{ijp} , D_p , and R_p can be written as sums of signal and noise components;

$$N_{ijp} = N_{sijp} + N_{nijp} ,$$

$$D_p = D_{sp} + D_{np} , \quad (A-6)$$

and

$$R_p = R_{sp} + R_{np} .$$

Explicit expressions for the signal and noise terms are deduced easily, e.g., from equation (A-2) we obtain

$$D_{np} = [i_{xk} I_{yl} - I_{yk} i_{xl}] + [I_{s xk} i_{yl} - i_{yk} I_{s xl}] \quad (A-7)$$

and

$$D_{sp} = [I_{sxk} I_{sy} - I_{syk} I_{sx}] . \quad (A-8)$$

The discussion just presented indicates that equation (A-5) can be written as

$$G_{mij} = \frac{W_{ij} + \Delta W_{ij}}{V + \Delta V} ,$$

where $G_{ij} = W_{ij}/V$, and

$$\Delta W_{ij} = \sum_{p=1}^P [N_{nijp} R_p^* + N_{sijp} R_{np}^*] \quad (A-9)$$

and

$$\Delta V = \sum_{p=1}^P [D_{np} R_p^* + D_{sp} R_{np}^*] \quad (A-10)$$

are noise terms. The errors in G_m can be expressed as a function of the errors in W_m and V_m , i.e.,

$$\Delta G_{ij} = \frac{\Delta W_{ij} - G_{ij} \Delta V}{V_m} , \quad (A-11)$$

where $G_m = G + \Delta G$. Note that when I_x , I_y , and O_i are noise-free, N_{nijp} and D_{np} are zero. Then, in view of the equality, $N_{sijp} = G_{ij} D_{sp}$, it is apparent that ΔG_{ij} is zero. Thus, as Goubau et al (1978) point out, correct impedances are obtained when the base field measurements are noise-free even if the references are two separate random sequences with no correlation to the base fields. However, the sums $\sum_p N_{sijp} R_{np}^*$ and $\sum_p D_{sp} R_{np}^*$ approach zero as N becomes large since the expected value of R_{np} is zero. Consequently, for the estimates G_m to be stable when the base field measurements are noisy, the references A and B must have signal components A_s and B_s such that $[A_s \ B_s]^T = \underline{M} \underline{I}_s$,

where $\underline{M}$ is a nonsingular transfer matrix and the input signals I_{sx} and I_{sy} are not perfectly coherent.

The bias errors in W_m and V_m are calculated from the expected values of ΔW and ΔV . For example, from equation (A-10) we obtain

$$E[\Delta V] = E[\sum_p D_{np} R_{np}^*],$$

where $E[D_{np} R_{sp}^*]$ and $E[D_{sp} R_{np}^*]$ are zero in view of equation (11). The noise terms D_{np} and R_{np} are given by

$$D_{np} = [i_{xk} I_{sy1} - I_{syk} i_{x1}] + [I_{sxx} i_{y1} - i_{yk} I_{sx1}] + [i_{xk} i_{y1} - i_{yk} i_{x1}]$$

and

$$R_{np} = [a_k B_{s1} - B_{sk} a_1] + [A_{sk} b_1 - b_k A_{s1}] + [a_k b_1 - b_k a_1].$$

Making the identifications

$$D_{np} = \Delta D_1 + \Delta D_2 + \Delta D_3$$

and

$$R_{np} = \Delta R_1 + \Delta R_2 + \Delta R_3$$

and noting that the expected values of $\Delta D_3 \Delta R_1^*$, $\Delta D_3 \Delta R_2^*$, $\Delta D_1 \Delta R_3^*$, and $\Delta D_2 \Delta R_3^*$ are zero in view of equation (12), we obtain the result

$$E[D_{np} R_{np}^*] = E[(\Delta D_1 + \Delta D_2) (\Delta R_1 + \Delta R_2)^* + \Delta D_3 \Delta R_3^*].$$

The individual terms are evaluated by recognizing that they each can be written in the same form as equation (A-1), e.g.

$$E[\Delta D_1 \Delta R_1^*] = N^{-2} E[\sum_{k=1}^N \sum_{l=1}^N (i_{xk} a_k^* I_{sy1} B_{s1}^* - I_{syk} a_k^* i_{x1} B_{s1}^*)].$$

Carrying out the expectation operation gives

$$\begin{aligned} E[\Delta D_1 \Delta R_1^*] &= \overline{E[i_x a^*]} \overline{I_{sy} B_s^*} - N^{-2} \sum_{k=1}^N E[i_{xk} a_k^*] I_{syk} B_{sk}^* \\ &= \overline{E[i_x a^*]} \overline{I_{sy} B_s^*} [1 - N^{-1}] , \end{aligned}$$

where the last equality holds if the noises are stationary. When all terms are calculated, the final result is given by equation (35) in the text.

APPENDIX B
EXPRESSIONS FOR $\overline{XY^*}$ and $\text{VAR } \overline{XY^*}$

In this appendix we calculate the expected value and variance of the averaged crossproduct, $\overline{XY^*} = \sum_{k=1}^N X_k Y_k^*$, where $X_k = X_{sk} + x_k$, $Y_k = Y_{sk} + y_k$, and expectation is defined with respect to the conditional distribution of the noises, x_k and y_k .

The expected value can be written down immediately as

$$\begin{aligned} \overline{XY^*} &= \sum_{k=1}^N E[X_{sk} Y_{sk}^* + x_k Y_{sk}^* + X_{sk} y_k^* + x_k y_k^*] \\ &= \overline{X_s Y_s^*} + E[\overline{xy^*}] , \end{aligned} \quad (\text{B-1})$$

where the last equality holds if signals and noises are independent so that $E[x_k Y_{sk}^*]$ and $E[X_{sk} y_k^*]$ are zero.

The variance is calculated from equation (14), i.e.,

$$\begin{aligned} \text{Var } \overline{XY^*} &= E[|\overline{XY^*}|^2] - |\overline{XY^*}|^2 \\ &= N^{-2} \sum_{k=1}^N \sum_{l=1}^N \{ E[X_k Y_k^* X_l^* Y_l] - E[X_k Y_k^*] E[X_l^* Y_l] \} \\ &= N^{-2} \sum_k \{ E[X_k Y_k^* X_k^* Y_k] - E[X_k Y_k^*] E[X_k^* Y_k] \} , \end{aligned}$$

where the last equality requires that the individual harmonics are statistically independent when $k \neq l$. If the joint distribution of X_k and Y_k is given by equation (8), the fourth order moment can be expanded using equation (12) to give

$$\overline{\text{Var } XY^*} = N^{-2} \sum_k \{ E[X_k X_k^*] E[Y_k Y_k^*] - X_{sk} X_{sk}^* Y_{sk} Y_{sk}^* \}.$$

From equation (B-1), we deduce the results,

$$E[X_k X_k^*] = X_{sk} X_{sk}^* + E[x_k x_k^*]$$

and

$$E[Y_k Y_k^*] = Y_{sk} Y_{sk}^* + E[y_k y_k^*].$$

Assuming the noises are stationary, we can write

$$\overline{\text{Var } XY^*} = N^{-1} [P_x \overline{Y_s Y_s^*} + P_y \overline{X_s X_s^*} + P_x P_y], \quad (\text{B-2})$$

where $P_x = E[x_k x_k^*]$ and $P_y = E[y_k y_k^*]$ are the expected values of the periodogram estimates of the stationary noise powers. Equation (B-2) can be written compactly as

$$\overline{\text{Var } XY^*} = N^{-1} [P_X P_Y - P_{X_S} P_{Y_S}], \quad (\text{B-3})$$

where, e.g., we have defined $E[\overline{XX^*}] = P_X = P_{X_S} + P_x$ and $P_{X_S} = \overline{X_S X_S^*}$.