Universality and $m_X$ cut effects in $B \to X_s \ell^+ \ell^-$

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The most precise comparison between theory and experiment for the $B \to X_s \ell^+ \ell^-$ rate is in the $q^2 < 6$ GeV$^2$ region. The hadronic uncertainties associated with an experimentally required cut on $m_X$ potentially spoil the extraction of short distance flavor-changing neutral current couplings. We compute the $m_X$ cut dependence of $d\Gamma(B \to X_s \ell^+ \ell^-)/dq^2$ using the $B \to X_s \gamma$ shape function, and show that the effect is universal for all short distance contributions in the limit $m_X^2 < m_B^2$. This universality is not spoiled by realistic values of the $m_X$ cut, nor by $\alpha_s$ corrections. Alternatively, normalizing the $B \to X_s \ell^+ \ell^-$ rate to $B \to X_s \ell\bar{\nu}$ with the same cuts removes the main uncertainties.

We find that the forward-backward asymmetry vanishes near $q_0^2 = 3$ GeV$^2$.

I. INTRODUCTION

In the standard model (SM) the flavor-changing neutral current process $B \to X_s \ell^+ \ell^-$ does not occur at tree level, and is a sensitive probe of new physics. Predicting its rate involves integrating out the $W$, $Z$, and $t$ at a scale of order $m_W$ by matching on to the Hamiltonian

$$H_W = \frac{G_F}{\sqrt{2}} V_{tb} V_{ts}^* \left\{ \sum_{i=1}^{6} C_i O_i + \frac{1}{4\pi^2} \sum_{i=7}^{10} C_i O_i \right\},$$

(1)
evolving to $m_b$, and computing matrix elements of $H_W$. Here $O_1 - O_6$ are four-quark operators and

$$O_7 = \bar{m}_b \bar{s} \alpha \mu \nu \epsilon P^\mu P^\nu b,$$

$$O_8 = \bar{m}_b \bar{s} \alpha \mu \nu \gamma G^{\mu\nu} b,$$

$$O_9 = e^2 (\bar{s} \gamma_\mu P_L b)(\bar{\ell} \gamma^\mu \ell),$$

$$O_{10} = e^2 (\bar{s} \gamma_\mu P_L b)(\bar{\ell} \gamma^\mu \gamma_5 \ell),$$

(2)

where $P_{L,R} = (1 \mp \gamma_5)/2$. The dilepton invariant mass spectrum, $q^2 = (p_{\ell^+} + p_{\ell^-})^2$, can be calculated in an operator product expansion (OPE), and the leading nonperturbative corrections are suppressed by $\Lambda^2_{QCD}/m_b^2$ \[3\]. The matching and anomalous dimension calculations for $C_i$ are known at next-to-next-to-leading log (NNLL) order \[4\], as are the largest perturbative QCD corrections to the matrix elements of $O_i$.

An important complication in $B \to X_s \ell^+ \ell^-$ compared to $B \to X_s \gamma$ is that the long distance contributions, $B \to J/\psi X_s$ and $\psi' X_s$ followed by $J/\psi, \psi' \to \ell^+ \ell^-$, are of order $m_b^2/m_X^2$ or $m_b^2/m_{\psi'}^2$, which is smaller than the short distance $m_X^2/m_b^2$. The low $q^2$ region, $q^2 < 6$ GeV$^2$, allows the most precise comparison with the SM, but requires a cut on the invariant mass of the hadronic final state, $m_X < m_{X_s}$. In the latest Belle analysis $m_X^{\text{cut}} = 2$ GeV \[5\], while Babar uses $m_X^{\text{cut}} = 1.8$ GeV \[6\]. These cuts are to remove backgrounds, and will likely be required for quite some time \[7\]. The high $q^2$ region is unaffected by the $m_X$ cut, but the rate is lower, and calculating it involves an expansion in $\Lambda^2_{QCD}/(m_b^2 - q^2)$.

In this letter we investigate the effects of the $m_X$ cut predictions for $B \to X_s \ell^+ \ell^-$ decay in the low $q^2$ region. This was previously studied in the Fermi-motion model in Ref. \[1\]. For $(m_X^{\text{cut}})^2 = \mathcal{O}(\Lambda_{QCD} m_b)$, the local OPE breaks down, and is replaced by an OPE involving nonlocal operators, whose matrix elements are $b$ quark distribution functions in the $B$ meson. We define

$$\Gamma_{ij}^\text{cut} = \int_{q_i^2}^{q_f^2} dq^2 \int_0^{m_{X_s}^{\text{cut}}} dm_X \text{Re}(c_i c_j^*) \frac{d\Gamma(B \to X_s \ell^+ \ell^-)}{dq^2 dm_X} \quad (3)$$

$$= \bar{\eta}_{ij}(m_X^{\text{cut}}, q_1^2, q_2^2) \frac{\Gamma_0}{m_B^2} \int_{q_i^2}^{q_f^2} dq^2 \text{Re}(c_i c_j^*) \left( \frac{m_b^2 - q^2}{m_b^2} \right)^2 G_{ij},$$

where $ij = \{77, 99, 00, 79\}$ label contributions of time-ordered products $T(O_i^\dagger O_j)$.

(4)
II. $m_X$ CUT EFFECTS AT LEADING ORDER

For simplicity, consider the kinematics in the $B$ meson’s rest frame. Since $q = p_B - px$,

$$2m_BE_X = m_B^2 + m_X^2 - q^2.$$  \hspace{1cm} (5)

If $m_X^2 \ll m_B^2$ and $q^2$ is not near $m_B^2$, then $E_X = \mathcal{O}(m_B)$. Since $E_X \gg m_X$, $px$ is near the light-cone, with $p_X^\pm = E_X - |p_X| = \mathcal{O}(\Lambda_{QCD})$ and $p_X^\mp = E_X + |p_X| = \mathcal{O}(m_B)$. Of the variables symmetric in $p_+^+$ and $p_-^-$ ($p_X^\pm$, $E_X$, $q^2$, $m_X^2$), only two are independent, and we work with $q^2$ and $p_X^\mp$ or $m_X$. The phase space cuts are shown in Fig. 1.

For the $p_X^\mp \ll p_X^\pm$ region, factorization of the form $d\Gamma = HJ \otimes \hat{f}(0)$ has been proven for semileptonic and radiative $B$ decays \cite{13}, where $H$ contains perturbative physics at $\mu_0 \sim m_b$, $J$ at $\mu_1 \sim \Lambda_{QCD} m_b$, and $\hat{f}(0)/\omega$ is a universal nonperturbative shape function. This factorization also applies for $B \to X_s \ell^+ \ell^-$ with the same universal $\hat{f}(0)$, as long as $q^2$ is not parametrically small \cite{14}.

In the $q^2 < 6\mathrm{GeV}^2$ region, $[C_9^{\text{mix}}(\mu_0 = 4.8\mathrm{GeV})] = 4.52$ to better than 1%, and can be taken to be constant. We neglect $\alpha_s$ corrections in this section and find

$$\frac{d\Gamma}{dp_X^\pm dq^2} = \hat{f}(0)(p_X^\pm) \frac{\Gamma_0}{m_B^2} \left[ \frac{(m_B - p_X^\mp)^2 - q^2}{(m_B - p_X^\pm)^3} \right]$$

$$\times \left\{ [C_9^{\text{mix}}]^2 + C_{10}^2 \right\} \left[ 2q^2 + (m_B - p_X^\pm)^2 \right]$$

$$\times \left[ 2 + 4(m_B - p_X^\mp)^2 \right]$$

$$\times 12m_B \Re[C_7^{\text{mix}} C_9^{\text{mix}}](m_B - p_X^\pm),$$  \hspace{1cm} (6)

where $\hat{f}(0)/\omega$ has support in $\omega \in [0, \infty)$. As a function of $p_X^\pm$, the kinematic terms in Eq. (6) vary only on a scale $m_B$, while $\hat{f}(0)(p_X^\pm)$ varies on a scale $\Lambda_{QCD}$. Writing $m_B = m_b + \Lambda$ and expanding in $(p_X^\pm - \Lambda)/m_B$, decouples the $p_X^\pm$ and $q^2$ dependences in Eq. (6), and gives the local OPE prefactors, $(m_X^2 - q^2)^2 G_{ij}(q^2)$, in Eq. (6). For $\eta_{ij}(p_X^\pm, q_1^2, q_2^2)$ the $p_X^\pm$ integration is over a rectangle in Fig. 1 whose boundaries do not couple $p_X^\mp$ and $q^2$. Thus, $\eta = \int dp_X^\pm \hat{f}(0)(p_X^\pm)$, independent of $ij$ and $q_1^2, q_2^2$. While the $m_X$ cut retains more events than the $p_X^\pm$ cut, the latter may give theoretically cleaner constraints on short distance physics when statistical errors become small.

The effect of the $m_X$ cut is $q^2$ dependent, because the upper limit of the $p_X^\pm$ integration is $q^2$ dependent, as shown in Fig. 1. Including the full $p_X^\pm$ dependence in Eq. (5), the universality of $\eta_{ij}(m_X^\text{cut}, q_1^2, q_2^2)$ is maintained to better than 3% for $1\mathrm{GeV}^2 < q_1^2 < 2\mathrm{GeV}^2$, $5\mathrm{GeV}^2 < q_2^2 < 7\mathrm{GeV}^2$, and $m_X^\text{cut} \geq 1.7\mathrm{GeV}$, because the region where the $p_X^\pm$ and $q^2$ integration limits are coupled has a small effect on the $ij$ dependence. This is exhibited in Fig. 2, where the solid curves show $\eta_{ij}(m_X^\text{cut}, 1\mathrm{GeV}^2, 6\mathrm{GeV}^2)$ with the shape function set to model-1 of \cite{15} with $m_b^{18} = 4.68\mathrm{GeV}$ and $\lambda_1$ from \cite{16}. (Taking $q_1^2 = 1\mathrm{GeV}^2$ instead of $4m_b^2$ increases the sensitivity to $C_9^{ij}$, but one may be concerned by local duality/resonances near $q^2 = 1\mathrm{GeV}^2$. To estimate this uncertainty, assume the $\phi$ is just below the cut and $B(B \to X_s \phi) \sim 10 \times B(B \to X_s K^{(*)}\phi)$. Then $B \to X_s \phi \to X_s \ell^+ \ell^-$ is $\sim 2\%$ of the $X_s \ell^+ \ell^-$ rate.)

The local OPE results for $\eta_{ij}(m_X^\text{cut}, q_1^2, q_2^2)$ are obtained by replacing $\hat{f}(0)(p_X^\pm)$ by $\delta(\Lambda - p_X^\pm)$ in Eq. (6). Performing the $p_X^\pm$ integral sets $(m_B - p_X^\pm) = m_b$ and implies $m_X^2 > \Lambda(m_b - q^2/m_b).$ This makes the lower limit on $q^2$ equal $\max\{q_1^2, m_b(m_b - (m_X^\text{cut})^2/\Lambda)\}$, and so the $\eta_{ij}$'s depend on the shape of $d\Gamma_i$. In Fig. 2 the local OPE results are shown by dashed lines, and clearly $\eta_{77} \neq \eta_{99}$. However, the local OPE is not applicable for $p_X^\pm \sim \Lambda_{QCD}$.

The universality of $\eta_{ij}$ can be broken by $\alpha_s$ corrections in the hard and jet functions, or by renormalization group evolution, since these effects couple $p_X^\pm$ and $q^2$ and have been neglected so far. We consider these next.

III. CALCULATION AND RESULTS AT $\mathcal{O}(\alpha_s)$

A complication in calculating $B \to X_s \ell^+ \ell^-$ compared to $B \to X_u \ell \nu$ is that, in the evolution of the effective Hamiltonian down to $m_b$, $C_9(\mu)$ receives a $\ln(m_c^2/m_b^2)$
enhanced contribution from the mixing of $O_2$. Thus, formally, $C_9 \sim \mathcal{O}(1/\alpha_s)$, and conventionally one expands the amplitude in $\alpha_s$, treating $\alpha_s \ln(m_{W_2}^2/m_t^2) = \mathcal{O}(1)$. In the local OPE this is reasonable, since the nonperturbative corrections are small, and at next-to-leading-log (NLL) all dominant terms in the rate are included. However, in the shape function region nonperturbative effects are $\mathcal{O}(1)$ and only the rate is calculable. With the traditional counting the $C_9$ contribution to the rate would be needed to $\mathcal{O}(\alpha_s^2)$ before the $C_{10b}$ terms could be included.

This would be a bad way to organize the perturbative corrections (numerically $|C_9(m_b)| \approx |C_{10}|$). It can be circumvented by using a “split matching” procedure to decouple the perturbation series above and below the scale $m_b$. This allows us to consider the short distance coefficients $C_{\text{mix}}^{\text{mix}}$, $C_9^{\text{mix}}$, and $C_{10}$ as $\mathcal{O}(1)$ numbers when organizing the perturbation theory at $m_b^2$ and $m_b^2 \Lambda_{\text{QCD}}$.

The rate and the forward-backward asymmetry are

$$d^2T/dq^2dp_X^+ = \frac{\Gamma_0}{m_B^2} H(q^2, p_X^+) F^{(0)}(p_X^+, p^-),$$

$$d^2A_{FB}/dq^2dp_X^+ = \frac{\Gamma_0}{m_B^2} K(q^2, p_X^+) F^{(0)}(p_X^+, p^-),$$

where $p^- = m_b - q^2/(m_b - p_X^+)$. The hard functions $H$ and $K$ were computed in Ref. 13 using SCET \cite{18} and split matching, which factorizes the dependence on scales above and below $m_b$ as $H_1(\mu, H_2(\mu))$. Here, to the order one is working at, $H_1$ is $\mu$ independent, the $\mu$ dependence in $H_2$ and $F^{(0)}$ cancels, and $F^{(0)}$ is $\mu_1$ independent. The shape function model is specified at $\mu_\Lambda$. The convolution of jet and shape functions at NLL including $\mathcal{O}(\alpha_s)$ corrections is

$$F^{(0)}(p_X^+, p^-) = U_H(p^+, \mu_1, \mu_b) \left( f^{(0)}(p_X^+, \mu_1) + \frac{\alpha_s(\mu_1)}{4\pi} C_F \left[ 2 \ln \frac{p_X^+ p^-}{\mu_1^2} - 3 \ln \frac{p_X^+ p^-}{m_b^2} + 7 - \pi^2 \right] f^{(0)}(p_X^+, \mu_1) \right) + \int_0^1 dz \left[ 4 \ln \frac{p_X^+ p^-}{\mu_1^2} - d \left[ f^{(0)}(p_X^+ (1 - z), \mu_1) - f^{(0)}(p_X^+, \mu_1) \right] \right],$$

$$f^{(0)}(\omega, \mu_\Lambda) = \frac{e^{V_1(\mu_1, \mu_2)}}{1 + \eta} \left( \frac{\omega}{\mu_\Lambda} \right)^\eta \int_0^1 d\omega \left[ \omega (1 - 1/\eta), \mu_\Lambda \right],$$

where $U_H$ was computed in Ref. 13, the one-loop jet function in Ref. 20, 21, and the shape function evolution up to $\mu_1$ in Refs. 18, 21 (for earlier calculations, see Refs. 13, 22). The $H$ and $K$ are

$$H(q^2, p_X^+) = \frac{[1 - \hat{p}_X^+)^2 - \hat{q}^2]^2}{(1 - \hat{p}_X^+)^3} \left\{ [C_9^{\text{mix}}(s, \mu_0)^2 + C_{10}^{\text{mix}}(s, \mu_0) + (1 - \hat{p}_X^+)^2 \Omega_2^2(s, \mu_0)] \right\} + 4[C_7^{\text{mix}}(\mu_0)^2 \Omega_2^2(s, \mu_0)^2 + 2(1 - \hat{p}_X^+)^2 \Omega_2^2(s, \mu_0)^2 + 2Re\left[C_7^{\text{mix}}(\mu_0)^2 \right] \right\} \left[ 1 - \hat{p}_X^+ \right] \Omega_E(s, \mu_0),$$

$$K(q^2, p_X^+) = \frac{3q^2(1 - \hat{p}_X^+)^2 - \hat{q}^2)^2}{(1 - \hat{p}_X^+)^3} \Omega_A(s, \mu_0) \Re\left[C_9^{\text{mix}}(s, \mu_0) \Omega_B(s, \mu_0) + \frac{2(1 - \hat{p}_X^+)^2}{\hat{q}^2} C_7^{\text{mix}}(\mu_0) \Omega_D(s, \mu_0) \right],$$

where $s = q^2/m_b^2$, $\hat{q}^2 = q^2/m_B^2$, $\hat{p}_X^+ = p_X^+/m_B$, and

$$\Omega_A = 1 + \frac{\alpha_s}{\pi} \omega^V_a(s, \mu_b), \quad \Omega_C = 1 + \frac{\alpha_s}{\pi} \omega^T_a(s, \mu_b),$$

$$\Omega_B = 1 + \frac{\alpha_s}{\pi} \left[ \omega^V_a(s, \mu_b) + \frac{(1 - \hat{p}_X^+)^2 - \hat{q}^2}{2(1 - \hat{p}_X^+)^2} \omega^V_a(s, \mu_b) + \omega^V_a(s, \mu_b) \right],$$

$$\Omega_D = 1 + \frac{\alpha_s}{\pi} \left[ \omega^T_a(s, \mu_b) - \omega^T_a(s, \mu_b) \right],$$

$$\Omega_E = (2 \Omega_A \Omega_D + \Omega_B \Omega_C)/3.$$  

Here $\alpha_s = \alpha_s(\mu_b)$ and $\omega^{V,T}$ are defined in Ref. 13.

In Fig. 8 we plot $\eta_0(m_{W_2}^2, 1 \text{ GeV}^2, 6 \text{ GeV}^2)$, including the $\alpha_s$ corrections. For each $\hat{f}^{(0)}$, the deviations of the $\eta_j$s from being universal is still below 3%. We use five different models for the shape function, constructed to obey the known constraints on its moments \cite{24}. The orange, green and purple (medium, light, dark) curves correspond to $m_{W_2}^{1S} = 4.68 \text{ GeV}$, 4.63 GeV, and 4.73 GeV, respectively, using the central values $\mu_0 = \mu_b = 4.8 \text{ GeV}$ and $\mu_1 = 2.5 \text{ GeV}$. For $m_{W_2}^{1C} = 2 \text{ GeV}$, varying $\mu_b$ in the range $3.5 \text{ GeV} < \mu_b < 7.5 \text{ GeV}$ changes $\eta_00$ by $\pm 6\%$. We find a $\pm 5\%$ variation for $2 \text{ GeV} < \mu_b < 3 \text{ GeV}$. The curves with slightly lower [higher] values of $\eta_00$ at large $m_{W_2}^{1C}$ correspond to $\mu_1 = 1.5 \text{ GeV} [2 \text{ GeV}]$.

The $\mu_b$ dependence of the rate is similar to that in the local OPE, and will be reduced by including the known NNLL corrections \cite{25,26,27}. We did not study it here.

Using the $c_i$s at NLL, for $1 \text{ GeV}^2 < \hat{q}^2 < 6 \text{ GeV}^2$ and $m_{W_2}^{1C} = 1.8$ and 2.0 GeV, we obtain $\Gamma^{\text{cut}} \tau_B = (1.20 \pm 0.15) \times 10^{-6}$ and $(1.48 \pm 0.14) \times 10^{-6}$, respectively.

The largest uncertainty in the rate and the largest source of universality breaking in the $\eta_j$s are due to sub-
leading shape functions, which affect the rate by about 5% for $m^\text{cut} = 2$ GeV and by about 10% for $m^\text{cut} = 1.8$ GeV [23].

If the $m^\text{cut}$ dependence were not universal, it would modify the zero of the forward-backward asymmetry, $A_{FB}(q^2_0) = 0$. For $m^\text{cut} = 2$ GeV we find at NLL $\Delta q^2_0 \approx -0.04$ GeV$^2$, much below the higher order uncertainties [7]. However, we obtain $q^2_0 = 2.8$ GeV$^2$, lower than earlier results [3]. In the local OPE limit we get $q^2_0 = 2m_b[\overline{m}_b(\mu)C^\text{eff}(\mu)]/\overline{\text{Re}}[\overline{C}^\text{eff}(q^2_0)]$). Here $m_b$ can be taken to be $m_b^\text{pole}$ or expanded about $m_b^S$, but to ensure that the $\mu$ dependence cancels at the order we are working, we cannot reexpand $\overline{m}_b(\mu)$ in terms of $m_b^\text{pole}$.

In conclusion, we pointed out that the experimentally used upper cut on $m^\text{cut}$ makes the observed $B \to X_s\ell^+\ell^-$ rate in the low $q^2$ region sensitive to the shape function. In this region there is an OPE only for the decay rate and not for the amplitude, necessitating a reorganization of the usual perturbation expansion. Since one can use the shape function measured in other processes, the sensitivity to new physics is not reduced. We found that the $q^2$s for the different operators’ contributions are universal to a good approximation. The theoretical uncertainties are reduced by raising the $m^\text{cut}$. Another possibility is to keep $m^\text{cut} < m_D$ and measure with the same cuts

$$ R = \Gamma^\text{cut}(B \to X_s\ell^+\ell^-)/\Gamma^\text{cut}(B \to X_u\ell\bar{\nu}), $$

since the effect of $m^\text{cut}$, as well as the $m_b$ dependence, are drastically reduced in this ratio. These results also apply for $B \to X_d\ell^+\ell^-$, which may be studied at a higher luminosity $B$ factory. Subleading $\Lambda_{\text{QCD}}/m_b$ as well as NNLL corrections to the rate and the forward-backward asymmetry will be studied in a separate publication [23].

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