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MODES IN TOKAMAKS

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Nonlinear Coupling of Kink Modes in Tokamaks

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ABSTRACT

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The $m = 2$, $n = 1$ kink mode is shown to be capable of destabilizing the $m = 1$, $n = 1$ internal kink. A nonlinear Lagrangian theory is developed for the coupling of modes of different pitch, and it is applied to the interaction of these modes. The coupling to the $m = 2$ mode provides sufficient additional destabilization to the internal mode to permit it to account even quantitatively (where it had failed when considered by itself) for many of the features of the disruptive instability.

1. INTRODUCTION

One of the most intriguing aspects of the tokamak discharge has been the occurrence of the "disruptive instability" which is accompanied by negative spikes of the gap voltage. [1, 2] Several attempts have been made to explain these phenomena; none completely successful.

Within the framework of a cylindrical geometry analysis, MHD and resistive instabilities with azimuthal mode numbers $m \geq 2$ have been shown inadequate to explain the observed effects. [3] The nonlinear evolution of such instabilities has been studied using a Friedrichs type of analysis [4] according to which a given linear mode generates harmonics which in turn interact with and modify the stability of the mode. An analysis of highly nonlinear states has also been given. [5]

The $m = 1$ internal kink mode [6] is the only MHD mode which possesses the correct features: it predicts a negative voltage spike and a major radius contraction of the plasma column. [7] However, it quantitatively falls very short of explaining the experimental observations.

The occurrence of the "disruptive instability" is always observed to be accompanied by the appearance of $m = 2$ and sometimes of even higher azimuthal number magnetic fluctuations. [1] This fact along with the qualitatively correct nonlinear behavior of the $m = 1$ internal kink immediately suggests that the interaction of such fluctuations may play an important role in understanding the nature of the negative voltage spikes of the

tokamak discharge. In particular, we are led to examine the nonlinear coupling of the $m = 2, n = 1$ kink mode with the $m = 1, n = 1$ internal mode.

The basic geometry used here is that of a cylindrical plasma column of radius $r = a$, which can either be infinite in length or finite with periodic boundary conditions at its ends. As shown in Fig. 1, we consider the case where a conducting wall is present near the plasma at $r = b$. The longitudinal current profile $j_z(r)$ that we assume is such as to enable us to study the internal mode:

$$j_z(r) = \begin{cases} j_{z0} = \text{const.} & 0 < r < c \\ 0 & c < r < a \end{cases}$$

The interval $c < r < a$ is assumed to be filled with zero pressure conducting plasma. A vacuum exists between this plasma and the wall, $a < r < b$. In what is to follow, we call the above intervals, regions I, II, and III respectively. The entire configuration is immersed in a strong longitudinal magnetic field $B_z^{(0)}$. The longitudinal current described above generates the azimuthal field $B_\theta^{(0)}$. The equilibrium plasma pressure within region I is assumed to be parabolic. (This is a realistic but not an essential assumption here.)

The complexity of this problem forces us to confine ourselves to a nonlinear equilibrium analysis of the behavior of the two modes rather than a study of the dynamics of the nonlinear

coupling. Thus, we consider the case of a marginally stable $m = 2$, $n = 1$ mode only. The $m = 1$ internal mode is stable to lowest order, but in reality it is growing with a very small growth rate given in Ref. [7].

Figure 2 shows the usual linear kink growth rate (for our current profile) plotted against the value q_a of the safety factor q evaluated at the plasma boundary. It is seen that there are two cases which we can study in general: (a) the marginal stability point for $m = 2$ at $q_a = 2$; (b) the marginal point for the same mode at $q_c = 1 + (c/b)^4$. Case (a) is the more interesting one experimentally. However, it is the case which is difficult to treat analytically. One then is led to study a special situation (Fig. 2) similar to that first studied in Ref. [3]. The geometry is specified so that the two marginal points are close together. Then we can work near marginal point (a). It is easy to see from Fig. 2 that to make the $m = 2$ mode completely marginal, we have to have:

$$\frac{a^2}{c^2} \left(1 + \left(\frac{c}{b} \right)^4 \right) = 2 \quad (1)$$

We must also note that the range for instability for the $m = 1$ internal kink as calculated in Ref. [7] is rather large once q falls below one somewhere inside the plasma.

2. THE ANALYTICAL MODEL

In this work, the equilibrium is defined to consist of a sum of small amplitude perturbations imposed on a zeroth order configuration of cylindrical symmetry. This configuration was described in Section 1. The perturbations are described by the form of the Lagrangian displacement vector $\underline{\xi}(\underline{r}_0)$. The subscript zero refers to the zeroth order equilibrium.

Following the perturbation, the resulting position of the plasma element is given as:

$$\underline{r} = \underline{r}_0 + \underline{\xi}(\underline{r}_0) \quad (2)$$

The total magnetic field at this point is then [8]:

$$\underline{B}(\underline{r}) = \underline{B}^{(0)}(\underline{r}_0) \cdot |\underline{I} + \underline{\nabla}_0 \underline{\xi}(\underline{r}_0)|/J$$

Here $\underline{B}^{(0)}(\underline{r}_0)$ is the zeroth order equilibrium field, $\underline{I}$ is the unit dyadic, and J is the Jacobian determinant of the $\underline{r}_0$ to $\underline{r}$ transformation.

Considering only the case of an incompressible plasma, (the presence of a strong $B_z^{(0)}$ justifies this), we set $J = 1$ from here on. We then expand $\underline{B}(\underline{r})$ in a Taylor series about $\underline{r}_0$:

$$\begin{aligned} \underline{B}(\underline{r}_0 + \underline{\xi}) = & \underline{B}(\underline{r}_0) + \underline{\xi} \cdot \underline{\nabla}_0 \underline{B}(\underline{r}_0) + 1/2 \underline{\xi} \underline{\xi} : \underline{\nabla} \underline{\nabla} \underline{B}(\underline{r}_0) \\ & + 1/3 \underline{\xi} \underline{\xi} \underline{\xi} : \underline{\nabla} \underline{\nabla} \underline{\nabla} \underline{B}(\underline{r}_0) + O(|\underline{\xi}|^4) \end{aligned} \quad (4)$$

Introducing a small parameter ϵ which is proportional to the amplitude of $\underline{\xi}$, $\epsilon \sim |\underline{\xi}|/a$ with $\epsilon \ll 1$, we visualize perturbation expansions for the fields $\underline{\xi}$ and $\underline{B}$ of the form:

$$\underline{\xi} = \epsilon \underline{\xi}^{(1)} + \epsilon^2 \underline{\xi}^{(2)} + \epsilon^3 \underline{\xi}^{(3)} + O(\epsilon^4) \quad (5)$$

$$\underline{B}(\underline{r}_0) = \underline{B}^{(0)}(\underline{r}_0) + \epsilon \underline{B}^{(1)}(\underline{r}_0) + \epsilon^2 \underline{B}^{(2)}(\underline{r}_0) + \epsilon^3 \underline{B}^{(3)}(\underline{r}_0) + O(\epsilon^4) \quad (6)$$

Combining Eqs. (3, 4, 5, 6), and dropping the subscript zero on $\underline{r}$ except where necessary for clarity, we obtain order by order in ϵ :

$$\begin{aligned} \underline{B}^{(1)} &= \underline{B}^{(0)} \cdot \underline{\nabla} \underline{\xi}^{(1)} - \underline{\xi}^{(1)} \cdot \underline{\nabla} \underline{B}^{(0)} \\ \underline{B}^{(2)} &= \underline{B}^{(0)} \cdot \underline{\nabla} \underline{\xi}^{(2)} - \underline{\xi}^{(2)} \cdot \underline{\nabla} \underline{B}^{(0)} - \underline{\xi}^{(1)} \cdot \underline{\nabla} \underline{B}^{(1)} - 1/2 \underline{\xi} \underline{\xi} : \underline{\nabla} \underline{\nabla} \underline{B}^{(0)} \\ \underline{B}^{(3)} &= \underline{B}^{(0)} \cdot \underline{\nabla} \underline{\xi}^{(3)} - \underline{\xi}^{(3)} \cdot \underline{\nabla} \underline{B}^{(0)} - \underline{\xi}^{(2)} \cdot \underline{\nabla} \underline{B}^{(1)} - \underline{\xi}^{(1)} \cdot \underline{\nabla} \underline{B}^{(2)} \\ &\quad - 1/2 \underline{\xi}^{(1)} \underline{\xi}^{(1)} : \underline{\nabla} \underline{\nabla} \underline{B}^{(1)} - 1/2 \underline{\xi}^{(2)} \underline{\xi}^{(1)} : \underline{\nabla} \underline{\nabla} \underline{B}^{(0)} \\ &\quad - 1/2 \underline{\xi}^{(1)} \underline{\xi}^{(2)} : \underline{\nabla} \underline{\nabla} \underline{B}^{(0)} - 1/3 \underline{\xi}^{(1)} \underline{\xi}^{(1)} \underline{\xi}^{(1)} : \underline{\nabla} \underline{\nabla} \underline{\nabla} \underline{B}^{(0)} \end{aligned} \quad (7)$$

Considering the component in the $\underline{r}_0$ direction only, we have:

$$B_r^{(1)} = \underline{B}^{(0)} \cdot \underline{\nabla} \xi_r^{(1)} \quad (8)$$

$$B_r^{(2)} = \underline{B}^{(0)} \cdot \underline{\nabla} \xi_r^{(2)} - \xi_r^{(1)} \cdot \underline{\nabla} B_r^{(1)} + \xi_\theta^{(1)} B_\theta^{(1)} / r - \left(\xi_r^{(1)} \xi_\theta^{(1)} / r \right) \left(B_\theta^{(0)} / r - B_\theta'^{(0)} \right) \quad (9)$$

$$\begin{aligned} B_r^{(3)} = & \underline{B}^{(0)} \cdot \underline{\nabla} \xi_r^{(3)} - \xi_r^{(2)} \cdot \underline{\nabla} B_r^{(1)} + \xi_\theta^{(2)} B_\theta^{(1)} / r - \xi_r^{(1)} \cdot \underline{\nabla} B_r^{(2)} \\ & + \xi_\theta^{(1)} B_\theta^{(2)} / r - \xi_r^{(1)} \xi_\theta^{(2)} / r \left(B_\theta^{(0)} / r - B_\theta'^{(0)} \right) \\ & - \xi_r^{(2)} \xi_\theta^{(1)} / r \left(B_\theta^{(0)} / r - B_\theta'^{(0)} \right) \end{aligned} \quad (10)$$

Here the prime denotes differentiation with respect to r_0 .

Our basic assumption in this work will be:

$$\xi^{(1)}(r_0) = \varepsilon \xi_1(r_0) \cos \tau_1 + \varepsilon \xi_2(r_0) \cos \tau_2$$

$$\equiv \sum_m \varepsilon \xi_m \cos \tau_m, \quad m = 1, 2 \quad (11)$$

with $\tau_m = m\theta + kz$, where k is the wavenumber in the z -direction.

As we mentioned before, we are limiting ourselves to considering an incompressible plasma. If such is the case, setting the Jacobian of the transformation between the $\underline{r}_0$ and $\underline{r}$ coordinates equal to unity implies

$$\exp\{\text{Tr} \ln [\underline{I} + \underline{\nabla}_0 \underline{\xi}(\underline{r}_0)]\} = 1. \quad (12)$$

Expanding Eq. (12), we obtain

$$\underline{\nabla} \cdot \underline{\xi} + 1/2 (\underline{\nabla} \cdot \underline{\xi})^2 - 1/2 (\underline{\nabla} \underline{\xi} : \underline{\nabla} \underline{\xi}) + O(\varepsilon^3) = 0 . \quad (13)$$

From this the incompressibility constraint order by order in ε is given by:

$$\underline{\nabla} \cdot \underline{\xi}^{(1)} = 0 , \quad \xi_{\theta m}^{(1)} = - \frac{1}{m} \left(r \xi_{rm}^{(1)} \right)' \sin \tau_m^- \quad (14)$$

$$\underline{\nabla} \cdot \underline{\xi}^{(2)} + 1/2 (\underline{\nabla} \cdot \underline{\xi}^{(1)})^2 - 1/2 (\underline{\nabla} \underline{\xi}^{(1)} : \underline{\nabla} \underline{\xi}^{(1)}) = 0 ,$$

or

$$\begin{aligned} \xi_{\theta ij}^{(2)} = & - \frac{1}{(i+j)} \left[r \xi_{rij}^{(2)} \right]' + \frac{1}{(i+j)} \sum_{k,l=1}^2 \left(\xi_{\theta k}^{'(1)} \dot{\xi}_{rl}^{(1)} \right. \\ & \left. - \xi_{\theta k}^{'(1)} \xi_{\theta l}^{(1)} - \xi_{rk}^{(1)} \xi_{rl}^{'(1)} - \xi_{rk}^{'(1)} \dot{\xi}_{\theta l}^{(1)} \right) \end{aligned} \quad (15)$$

where the dot denotes differentiation with respect to θ . In the above expressions, second order quantities bear a double subscript ij . When $i=j$, we would refer to the second harmonics of the two fundamentals given in Eq. (11) each with cyclic behavior dependent on $2 \tau_m$. When $i \neq j$, we refer to the beat waves generated by the same fundamentals; the 12 subscript refers to azimuthal dependence on $\tau_1 + \tau_2$, the 21 subscript on $\tau_2 - \tau_1$. In Eq. (15), the + sign is to be used for the 12 subscript, and the - sign for the 21 subscript. Note that for $i \neq j$ and $\xi_1 = \text{const.}$ the second term vanishes.

Using Eq. (8) to calculate the magnetic field $B_r^{(1)}$ in terms of our assumed Lagrangian displacement, we obtain:

$$B_{rm}^{(1)} = - F_m \xi_m(r) \sin \tau_m, \quad F_m \equiv m B_\theta^{(0)}/r + k B_z^{(0)}. \quad (16)$$

Combining Eqs. (16) and (9) gives to next order:

$$B_{r12}^{(2)} = - F_{12} \xi_{12}^{(2)} + \frac{3}{4} \xi_1 \xi_2' F_2 + \frac{\xi_1 \xi_2}{4r} \left(6B_\theta' - 12 \frac{B_\theta}{r} - 3k B_z \right)$$

$$B_{r21}^{(2)} = - F_{21} \xi_{21}^{(2)} + \frac{1}{4} \xi_1 \xi_2' F_2 + \frac{\xi_1 \xi_2}{4r} \left(3k B_z + 2B_\theta' + 4 \frac{B_\theta}{r} \right). \quad (17)$$

Here $F_{12} \equiv 3B_\theta^{(0)}/r + 2k B_z^{(0)}$

$$F_{21} \equiv B_\theta^{(0)}/r$$

and we have suppressed all angular dependence.

The effect of the generation of second harmonics on the stability problem was studied in Ref. [3]. We will concentrate here on the effect of the generation of beats; we are then interested only in the components for which $i \neq j$. This is possible because the second harmonic of each of the fundamental modes of our problem does not interact with the other mode to yield a third order contribution to the first mode. To $O(\epsilon^3)$, we have omitted all terms cubic in $\xi^{(1)}$ and we rewrite Eq. (10) in the form

$$\begin{aligned}
 B_{rm}^{(3)} = & B^{(0)} \cdot \nabla \xi_m^{(3)} + \sum_{i,j,k=1}^2 \left[- \xi_{ij}^{(2)} \cdot \nabla B_{rk}^{(1)} + \xi_{\theta ij}^{(2)} B_{\theta k}^{(1)} / r \right. \\
 & - \xi_i^{(1)} \cdot \nabla B_{rik}^{(2)} + \xi_{\theta i}^{(1)} B_{\theta jk}^{(2)} / r - \xi_{ri}^{(1)} \left(\xi_{\theta jk}^{(2)} / r \right) \left(B_{\theta}^{(0)} / r - B_{\theta}'^{(0)} \right) \\
 & \left. - \xi_{rij}^{(2)} \left(\xi_{\theta k}^{(1)} / r \right) \left(B_{\theta}^{(0)} / r - B_{\theta}'^{(0)} \right) \right] . \quad (18)
 \end{aligned}$$

In particular, we can simplify further ignoring terms in the sum containing factors of the type $\xi_{ii}^{(2)}$ and we write Eq. (18) as

$$B_{rm}^{(3)} = - F_m \xi_m^{(3)} + \Delta B_m \quad (19)$$

where ΔB_m symbolizes the 3rd order nonlinear part of $B_{rm}^{(3)}$. The expressions ΔB_m in Eq. (19), because of their excessive length, are given in Appendix A. Note that the expected three indices on $B_r^{(3)}$ have been suppressed to one; we are only interested in that part of $B_r^{(3)}$ which has the same kind of angular dependence as our fundamentals.

Finally, we have

$$\nabla \cdot \underline{B} = 0 \quad (20)$$

Equation (20) enables us to calculate the azimuthal component of the perturbed magnetic field. To each order, we have:

$$\left. \begin{aligned} B_{\theta m}^{(1)} &= \frac{1}{m} [r B_{rm}^{(1)}]' \cos \tau_m \\ B_{\theta ij}^{(2)} &= \frac{1}{i+j} [r B_{rij}^{(2)}]' \cos \tau_{ij} \\ B_{\theta m}^{(3)} &= \frac{1}{m} [r B_{rm}^{(3)}]' \cos \tau_m - \frac{kr}{m} B_{zm}^{(2)} \cos \tau_m \end{aligned} \right\} \quad (21)$$

The quantity $B_{zm}^{(2)}$ is calculated in Appendix B.

3. EQUILIBRIUM

We start with the MHD equation of motion in its Eulerian form:

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B} \quad (22)$$

Here ρ is the plasma density, p the plasma pressure, $\mathbf{B}$ is the magnetic field, $\mathbf{j}$ the current density, and $\mathbf{v}$ the velocity of the plasma element. Operating on Eq. (22) with $\hat{\mathbf{e}}_z \cdot \nabla \times$, where $\hat{\mathbf{e}}_z$ is a unit vector in the direction of the strong longitudinal magnetic field, we obtain:

$$\left[\nabla \times \left(\rho \frac{d\mathbf{v}}{dt} \right) \right]_z = \left(\mathbf{B} \cdot \nabla \mathbf{j} - \mathbf{j} \cdot \nabla \mathbf{B} \right)_z \quad (23)$$

where Eq. (20) was applied, and also $\nabla \cdot \mathbf{j} = 0$.

Setting $d/dt = 0$, we confine our attention to studying equilibrium situations only. Equation (23) then reduces to:

$$\mathbf{B} \cdot \nabla j_z = \mathbf{j} \cdot \nabla B_z \quad (24)$$

To lowest order, Eq. (24) yields

$$\left(\frac{B_{\theta}^{(0)}}{r} \frac{\partial}{\partial \theta} + B_z^{(0)} \frac{\partial}{\partial z} \right) \frac{1}{r} \frac{\partial}{\partial r} \left(r B_{\theta}^{(0)} \right) = 0$$

which is satisfied by our particular choice of zeroth order equilibrium.

To first order in our expansion parameter ϵ , we have:

$$\left(\frac{B_{\theta}^{(0)}}{r} \frac{\partial}{\partial \theta} + B_z^{(0)} \frac{\partial}{\partial z} \right) \left(\frac{1}{r} \frac{\partial}{\partial r} r B_{\theta}^{(1)} - \frac{1}{r} \frac{\partial}{\partial \theta} B_r^{(1)} \right) = 0$$

Then for any r for which the first factor in the last equation is not equal to zero ($r \neq s_1, s_2$, the points at which $\underline{k} \cdot \underline{B}^{(0)} = 0$ for the $m = 1$ and the $m = 2$ modes respectively) we obtain using Eq. (21):

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \chi_m^{(1)} - \frac{m^2}{r^2} \chi_m^{(1)} = 0 \quad (25)$$

with

$$\chi_m^{(1)} \equiv r B_{rm}^{(1)}(r)$$

to $O(\epsilon^2)$.

Similarly we have

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \chi_{ij}^{(2)} - \frac{(i+j)^2}{r^2} \chi_{ij}^{(2)} = 0 \quad (26)$$

$$\chi_{ij}^{(2)} \equiv r B_{rij}^{(2)}(r) .$$

Here again the + sign holds only for $i < j$, and finally,

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \chi_m^{(3)} - \frac{m^2}{r^2} \chi_m^{(3)} = - \frac{1}{r} \frac{\partial}{\partial r} [k r^2 B_{zm}^{(2)}] - m j_z^{(0)} k B_{zm}^{(2)} / F_m .$$

$$\chi_m^{(3)} \equiv r B_{rm}^{(3)}(r) . \quad (27)$$

We remark that the last equation is an inhomogeneous one in the case of finite k . We have adopted the ordering

$$B_{\theta}^{(0)} / B_z^{(0)} \sim k a \sim \epsilon .$$

4. PERTURBED MAGNETIC FIELD SOLUTIONS

We now examine the solution to Eqs. (25), (26), (27) in the various regions of our model. Again, we refer to Fig. 1 and the assumptions outlined in Section 1. We choose the singular surface $r = s_{12}$ for the $m = 3$, $n = 2$ harmonic falling within the well-conducting plasma, so that this mode too is stable (if $s_{12} = a$, it is marginally stable).

Region I: $0 < r < c$

Applying the condition at $r = 0$ that χ_m should be finite, we obtain from Eqs. (16), (25) to order ε

$$\chi_1^I = \chi_{1c}^I r/c ; \chi_{1c}^I = \text{const}$$

$$\xi_1^I = - \frac{\chi_{1c}^I}{c F_1^I} = \xi_{1c} = \text{const}$$

$$\chi_2^I = \chi_{21c}^I r^2/c^2 ; \chi_{2c}^I = \text{const}$$

$$\xi_2^I = - \frac{\chi_{2c}^I}{c F_2^I} \frac{r}{c} = \xi_{2c} \frac{r}{c} ; \xi_{2c} = \text{const} \quad (28)$$

From Eqs. (26), (17):

$$\chi_{12}^I = \chi_{12c}^I r^3/c^3 ; \chi_{12c}^I = \text{const}$$

$$\xi_{12}^I = - \frac{\chi_{12c}^I}{c F_{12}^I} \frac{r^2}{c^2}$$

$$\chi_{21}^I = \chi_{21c}^I \frac{r}{c} ; \chi_{21c}^I = \text{const}$$

$$\xi_{21}^I = \left(2 + \frac{k c B_z}{B_{\theta c}} \right) \frac{\xi_{1c}^I \xi_{2c}^I}{c} - \frac{\chi_{21c}^I}{c F_{21}^I} = \xi_{21c} = \text{const} \quad (29)$$

From Eq. (27) to order ϵ^3

$$\begin{aligned}\chi_1^{(3)I} &= \delta\chi_1^I = \delta\chi_{1c}^I \frac{r}{c} + \delta\chi_{1inh}^I ; \delta\chi_{1c}^I = \text{const} \\ \chi_2^{(3)I} &= \delta\chi_2^I = \delta\chi_{2c}^I \frac{r^2}{c^2} + \delta\chi_{2inh}^I ; \delta\chi_{2c}^I = \text{const}\end{aligned}\quad (30)$$

where

$$\delta\chi_{minh} = \frac{1}{2m} \int^r dr' \left(\frac{r'^m}{r^m} - \frac{r^m}{r'^m} \right) r' \left[\frac{1}{r'} \frac{\partial}{\partial r'} (kr'^2 B_{zm}^{(2)}) + mj_{zo} k B_{zm}^{(2)} / F_m \right]$$

is the inhomogeneous part of the solution with (see Appendix B)

$$B_{zm}^{(2)} = \frac{kr}{m} \left(\frac{1}{m^2} \frac{\partial^2}{\partial \theta \partial r} [r B_{rm}^{(1)}] + j_{zo} \xi_m^{(1)} \right) .$$

Combining the last expression with Eq. (28) gives

$$B_{zm}^{I(2)} = \frac{kr}{m} \left(\frac{r}{c} \right)^{m-1} \left(-F_m^I + j_{zo} \right) \xi_{mc} \quad (31)$$

so that

$$\begin{aligned}\delta\chi_1^I &= \delta\chi_{1c}^I \frac{r}{c} - \frac{1}{8} k^2 r^3 \left(-F_1^I + j_{zo} \right) \xi_{1c} \left(3 + j_{zo} / F_1^I \right) \\ \delta\chi_2^I &= \delta\chi_{2c}^I \frac{r^2}{c^2} - \frac{1}{6} (k^2 r^4 / c) \left(-F_2^I + j_{zo} \right) \xi_{2c} \left(1 + j_{zo} / 2F_2^I \right)\end{aligned}\quad (32)$$

Using Eqs. (19), (27) in Eq. (32), we obtain:

$$\xi_{r1}^{(3)I} = \delta \xi_1^I = \frac{3}{8} \frac{k^2 r^2}{F_1^I} (-F_1^I + j_{zo}) (3 + j_{zo}/F_1^I) \xi_{1c} + \frac{\Delta B_1^I}{F_1^I} - \frac{\delta \chi_{1c}^I}{c F_1^I} \frac{r}{c} \quad (33)$$

$$\xi_{r2}^{(3)I} = \delta \xi_2^I = \frac{1}{6} \frac{k^2 r^3}{c F_2^I} (-F_2^I + j_{zo}) (1 + j_{zo}/2 F_2^I) \xi_{2c} + \frac{\Delta B_2^I}{F_2^I} - \frac{\delta \chi_{2c}^I}{c F_2^I} \frac{r^2}{c}$$

Regions II and III: $c < r < b$

Outside the current channel, special care has to be exercised because of the presence of singular surfaces. The $m = 1$, $n = 1$, and the $m = 3$, $n = 2$ modes possess such surfaces in region II. Their presence will cause discontinuities in the marginal field functions.

When the proper boundary conditions are applied (see Section E), the solutions to order ϵ are well known in this region [3]. For $r < s_1$, we have

$$\chi_1^{II} = -r F_1^{II} \xi_1^{II} = -r F_1^{II} \xi_{1c} = -k B_z (r - s_1^2/r) \xi_{1c}$$

$$\xi_1^{II}(r) = \xi_{1c} = \text{const} \quad (34)$$

For $r > s_1$, we have

$$\chi_1^{III} = \xi_1^{III} = \chi_1^{II} = \xi_1^{II} = 0$$

Similarly for the $m = 2$, $n = 1$ mode, when the corresponding singular surface $r = s_2$ is in the vacuum, we can write for the marginal case an expression valid throughout both regions II and III obeying the boundary condition at the conducting wall $\chi_2 = 0$ at $r = b$, so that

$$\chi_2^{II,III} = \chi_{2c}^{II} \frac{c^2/r^2 - c^2 r^2/b^4}{1 - c^4/b^4}; \quad \chi_{2c}^{II} = \text{const} \quad (35)$$

Using Eq. (16):

$$\xi_2^{II,III} = - \frac{\chi_{2c}^{II}}{kB_z (1 - c^4/b^4)} \frac{c^2/r^2 - c^2 r^2/b^4}{r(1 - s_2^2/r^2)} \quad (36)$$

with

$$\chi_{2c}^{II} = - \xi_{2c} kB_z (1 - 2s_1^2/c^2)$$

To order ε^2 , we solve Eq. (26) in the region II-1 where $c < r < s_1$ to obtain

$$\chi_{21}^{II-1} = \chi_{21c}^{II-1} \frac{c/r - cr/s_1^2}{1 - c^2/s_1^2} + \chi_{21s}^{II-1} \frac{r/s_1 - c^2/rs_1}{1 - c^2/s_1^2}; \quad \chi_{21c}^{II-1}, \chi_{21s}^{II-1} = \text{const} \quad (37a)$$

and in the region II-2, where $s_1 < r < b$

$$\chi_{21}^{II-2,III} = \chi_{21s}^{II-2,III} \frac{s_1/r - rs_1/b^2}{1 - s_1^2/b^2}; \quad \chi_{21s}^{II-2,III} = \text{const} \quad (37b)$$

Using Eq. (17), we then obtain

$$\begin{aligned} \xi_{21}^{II-1} = & - \frac{\chi_{21c}^{II-1}}{B_\theta} \frac{c/r - cr/s_1^2}{1 - c^2/s_1^2} - \frac{\chi_{21s}^{II-1}}{B_\theta} \frac{r/s_1 - c^2/rs_1}{1 - c^2/s_1^2} \\ & + \frac{1}{4} r \frac{\xi_{1c} \xi_2^{II-1}}{B_\theta} F_2 + \frac{1}{4} \frac{\xi_{1c} \xi_2^{II-1}}{B_\theta} \left(2B'_\theta + 4 \frac{B_\theta}{r} + 3kB_z \right) \end{aligned} \quad (38a)$$

and

$$\xi_{21}^{II-2,III} = - \frac{\chi_{21s}^{II-2,III}}{B_\theta} \frac{s_1/r - rs_1/b^2}{1 - s_1^2/b^2} \quad (38b)$$

Similarly, for the other beat harmonic, in the subintervals II-1 where $c < r < s_1$, and II-2 where $s_1 < r < s_{12}$

$$\chi_{12}^{II-1} = \chi_{12c}^{II-1} \frac{c^3/r^3 - (c/s_1)^6 r^3/c^3}{1 - (c/s_1)^6} + \chi_{12s}^{II-1} \frac{r^3/s_1^3 - (c/s_1)^6 s_1^3/r^3}{1 - (c/s_1)^6} \quad (39a)$$

with χ_{12c}^{II-1} , $\chi_{12s}^{II-1} = \text{const.}$, and

$$\chi_{12}^{II-2} = \frac{27}{19} \left[\chi_{12s}^{II-2} \left(\frac{s_1^3}{r^3} - \frac{8}{27} \frac{r^3}{s_1^3} \right) + \chi_{12c}^{II-2} \left(\frac{r^3}{s_{12}^3} - \frac{8}{27} \frac{s_{12}^3}{r^3} \right) \right] \quad (39b)$$

with χ_{12c}^{II-2} , $\chi_{12s}^{II-2} = \text{const.}$, noting that for our model $s_{12}^2/s_1^2 = 3/2$. In the regions II-3,III where $s_{12} < r < b$, we have

$$\chi_{12}^{II-3,III} = \chi_{12s}^{II-3,III} \frac{s_{12}^3/r^3 - (s_{12}/b)^6 r^3/s_{12}^3}{1 - (s_{12}/b)^6} \quad (40)$$

with $\chi_{12s}^{II-3,III} = \text{const.}$ when Eqs. (39), (40) are combined with

Eq. (17), we obtain:

$$\xi_{12}^{II-1} = - \frac{\chi_{12}^{II-1}(r)}{r F_{12}^{II-1}} + \frac{3}{4} \xi_{1c} \xi_2^{II-1} \frac{F_2^{II-1}}{F_{12}^{II-1}} \quad (41a)$$

$$+ \frac{1}{4} \frac{\xi_{1c} \xi_2^{II-1}}{r F_{12}^{II-1}} (6B'_\theta - 3kB_z - 12B_\theta/r)$$

$$\xi_{12}^{II-2} = - \frac{\chi_{12}^{II-2}(r)}{r F_{12}^{II-2}} \quad (41b)$$

and

$$\xi_{12}^{II-3,III} = - \frac{\chi_{12s}^{II-3,III}}{r F_{12}^{II-3,III}} \frac{s_{12}^3/r^3 - (s_{12}/b)^6 r^3/s_{12}^3}{1 - (s_{12}/b)^6} \quad (42)$$

Finally, we solve Eq. (27) in the region II-1, where $c < r < s_1$

$$\delta\chi_1^{II-1} = \delta\chi_{1c}^{II-1} \frac{c/r - cr/s_1^2}{1 - c^2/s_1^2} + \delta\chi_{1s}^{II-1} \frac{r/s_1 - c^2/s_1 r}{1 - c^2/s_{1inh}^2} + \delta\chi_1^{II-1} \quad (43)$$

where $\delta\chi_{1c}^{II-1}$, $\delta\chi_{1s}^{II-1} = \text{const.}$, and

$$\delta\chi_{1inh}^{II-1} = \frac{1}{2} k^3 c^2 B_z \hat{s}_1^2 r \left(\ln \frac{r}{c} - \frac{1}{2} \right) \xi_{1c}$$

In the region II-2 where $s_1 < r < s_{12}$, we have

$$\delta\chi_1^{II-2} = 3 \left[\delta\chi_{1s}^{II-2} \left(\frac{s_1}{r} - \frac{2}{3} \frac{r}{s_1} \right) + \delta\chi_{1s}^{II-2} \left(\frac{r}{s_{12}} - \frac{2}{3} \frac{s_{12}}{r} \right) \right] \quad (44a)$$

where $\delta\chi_{1s}^{II-2}$, $\delta\chi_{1s}^{II-2} = \text{const.}$ while in the regions II-3, III

where $s_{12} < r < b$

$$\delta\chi_1^{II-3, III} = \delta\chi_{1s}^{II-3, III} \frac{s_{12}/r - s_{12}r/b^2}{1 - s_{12}^2/b^2} \quad (44b)$$

where $\delta\chi_{1s}^{II-3, III} = \text{const.}$, and we have used $\delta\chi_{1inh}^{II-2} = \delta\chi_{1inh}^{II-2, III} = 0$ since $B_{z1}^{(2)}(r > s_1) = 0$. Similarly, in the region II-1 where $c < r < s_1$

$$\delta\chi_2^{II-1} = \delta\chi_{2c}^{II-1} \frac{c^2/r^2 - c^2r^2/s_1^4}{1 - c^4/s_1^4} + \delta\chi_{2s}^{II-1} \frac{r^2/s_1^2 - c^4/r^2s_1^2}{1 - c^4/s_1^4} + \delta\chi_{2inh}^{II, III} \quad (45a)$$

where $\delta\chi_{2c}^{II-1}$, $\delta\chi_{2s}^{II-1} = \text{const.}$, and

$$\delta\chi_{2inh}^{II, III} = -\frac{1}{6} \xi_{2c} (1 - 2s_1^2) k^3 c^3 B_z \frac{r^4/b^4}{1 - c^4/b^4}$$

In the regions II-2, III where $s_1 < r < b$, we have

$$\delta\chi_2^{II-2, III} = \delta\chi_{2s}^{II-2, III} \frac{s_1^2/r^2 - r^2s_1^2/b^4}{1 - s_1^4/b^4} + \delta\chi_{2b} \frac{r^2/b^2 - s_1^4/b^2r^2}{1 - s_1^4/b^4} + \delta\chi_{2inh}^{II, III} \quad (45b)$$

where $\delta\chi_{2s}^{II-2, III} = \text{const.}$, and

$$\delta\chi_{2b} = -\delta\chi_{2inh}(b) = \frac{1}{6} \xi_{2c} \frac{(1-2\hat{s}_1^2)}{1-s^4/b^4} k_c^3 B_z.$$

The corrections of order ϵ^3 to the Lagrangian displacement in the Nth region are then given respectively by Eq. (19):

$$\delta\xi_m^N = -\frac{\delta\chi_m^N(r)}{rF_m^N} + \frac{\Delta B_m^N}{F_m^N}. \quad (46)$$

Note (see Appendix A) that for $r > s_1$ $\Delta B_2^N = 0$.

5. BOUNDARY CONDITIONS

Boundary conditions are needed on the plasma vacuum interface at $r = a$, on the edge of the current channel at $r = c$, and at the $\underline{k} \cdot \underline{B} = 0$ surfaces which fall within conducting plasma. We have already used the boundary condition on the conducting wall at $r = b$ and the finiteness condition at $r = 0$. The further conditions that we invoke are continuity of the tangential electric field and total pressure balance at the perturbed current-channel-plasma and plasma-vacuum interfaces. At the singular surface of a given mode when embedded in conducting plasma, we demand, in addition, that the Lagrangian displacement ξ remain finite for this particular mode. These conditions are sufficient to determine all of the unknown constants of our problem. We next look at them in detail.

Continuity of the tangential electric field

In the absence of a sheet current on the perturbed interface, this condition takes on the form [9]:

$$[\hat{n} \cdot \xi] = 0 \quad (47)$$

where the double brackets denote the jump in the indicated quantity, and $\hat{n}$ is the outward normal to the perturbed interface under consideration. It is our first task to calculate $\hat{n}$ to the order appropriate to our problem, namely, to order ϵ^2 . Considering the vector diagram shown in Fig. 3, the final position of a displaced plasma element is related to its original one through Eq. (2). The surface function which describes the original unperturbed magnetic surface is given merely as:

$$\phi(r_o, \theta_o) = r_o - r_1 = 0 ; r_1 = \text{const.} \quad (48)$$

and the normal $\hat{n}^{(0)}$ to this surface is

$$\hat{n}^{(0)} = \nabla_o \phi = \hat{e}_r .$$

The vector relation, Eq. (2), can be written in general as:

$$r = g(r_o, \theta_o) ; \theta = h(r_o, \theta_o) \quad (49)$$

where g and h are arbitrary functions of r_o and θ_o . Assuming that Eq. (49) has an inverse, we can write:

$$r_o = G(r, \theta) , \theta_o = H(r, \theta) . \quad (50)$$

Preserving the label r_1 for each magnetic surface, we express Eq. (48) in terms of the perturbed coordinates r and θ :

$$\phi(r, \theta) = G(r, \theta) - r_1 = 0 . \quad (51)$$

The unit normal for the deformed surface given by Eq. (51) is then:

$$\hat{n} = \nabla\phi(r, \theta) / |\nabla\phi(r, \theta)| \quad (52)$$

where ∇ is the gradient operator in the new coordinates r, θ ; it is related to ∇_0 via the chain rule:

$$\nabla = \frac{dr_0}{dr} \cdot \nabla_0$$

Combining the last relation with Eq. (2), we obtain:

$$\nabla\phi(r, \theta) = (\underline{I} + \nabla_0 \xi)^{-1} \cdot \nabla\phi(r_0, \theta_0) = (\underline{I} + \nabla_0 \xi)^{-1} \cdot \hat{e}_{r_0} \quad (53)$$

Substituting Eq. (53) into Eq. (52), we obtain:

$$\hat{n}(r_0, \theta_0) = \frac{(\underline{I} + \nabla_0 \xi)^{-1} \cdot \hat{e}_{r_0}}{|(\underline{I} + \nabla_0 \xi)^{-1} \cdot \hat{e}_{r_0}|} \quad (54)$$

Expanding the tensor in Eq. (53) and dropping the subscript zero, we have

$$(\underline{I} + \nabla \xi)^{-1} = \underline{I} - \nabla \xi + \nabla \xi \cdot \nabla \xi - O(|\xi|^3)$$

and

$$\begin{aligned}
 (\underline{\underline{I}} + \underline{\underline{\nabla}} \underline{\underline{\xi}})^{-1} \cdot \hat{\underline{\underline{e}}}_r &= \hat{\underline{\underline{e}}}_r \left[1 - \xi_r' + (\xi_r')^2 + \frac{\xi_\theta'}{r} (\xi_r' - \xi_\theta) \right] + \hat{\underline{\underline{e}}}_\theta \left[\frac{(\xi_\theta - \dot{\xi}_r)}{r} \right. \\
 &\quad \left. + \frac{\dot{\xi}_r \xi_r'}{r} - \frac{\xi_\theta \xi_r'}{r} + \frac{\xi_r}{r^2} (\dot{\xi}_r - \xi_\theta) + \frac{\dot{\xi}_\theta}{r^2} (\xi_r - \xi_\theta) \right] + O(|\xi|^3) \quad (55)
 \end{aligned}$$

while

$$\begin{aligned}
 |(\underline{\underline{I}} + \underline{\underline{\nabla}} \underline{\underline{\xi}})^{-1} \cdot \hat{\underline{\underline{e}}}_r|^{-1} &= 1 + \xi_r' - \frac{1}{2} \left(\frac{\dot{\xi}_r^2}{r^2} + \frac{\xi_\theta^2}{r^2} \right) - \frac{1}{r} \left(\xi_\theta' \dot{\xi}_r - \xi_\theta' \xi_\theta - \frac{\dot{\xi}_r \xi_\theta}{r} \right) \\
 &\quad + O(|\xi|^3) \quad (56)
 \end{aligned}$$

Combining Eqs. (55, 56) gives the components of the normal at the perturbed interface that we require. We present them order by order:

$$\begin{aligned}
 \hat{\underline{\underline{n}}}^{(0)} &= \hat{\underline{\underline{e}}}_r, \\
 \hat{\underline{\underline{n}}}^{(1)} &= \hat{\underline{\underline{e}}}_\theta \left(\frac{\xi_\theta^{(1)}}{r} - \frac{\dot{\xi}_r^{(1)}}{r} \right), \\
 \hat{\underline{\underline{n}}}^{(2)} &= \hat{\underline{\underline{e}}}_r \left[\frac{\dot{\xi}_r^{(1)} \xi_\theta^{(1)}}{r^2} - \frac{1}{2} \left(\frac{\dot{\xi}_r^{(1)} \dot{\xi}_r^{(1)}}{r^2} + \frac{\xi_\theta^{(1)} \xi_\theta^{(1)}}{r^2} \right) \right] + \hat{\underline{\underline{e}}}_\theta \left[\frac{\xi_\theta^{(2)} - \dot{\xi}_r^{(2)}}{r} \right. \\
 &\quad \left. + \frac{\xi_r^{(1)}}{r^2} \left(\dot{\xi}_r^{(1)} - \xi_\theta^{(1)} \right) + \frac{\dot{\xi}_\theta^{(1)}}{r^2} \left(\xi_r^{(1)} - \xi_\theta^{(1)} \right) \right] \quad (57)
 \end{aligned}$$

As will be seen from Eq. (60), it is indeed sufficient to calculate the normal only to $O(|\xi|^2)$. To $O(\epsilon)$, we have from Eqs. (47, 57):

$$\left[\hat{\underline{\underline{n}}}^{(0)} \cdot \underline{\underline{\xi}}^{(1)} \right] = \left[\xi_r^{(1)} \right] = 0 \quad (58)$$

Similarly to $O(\xi^2)$:

$$\left[\hat{n}^{(0)} \cdot \underline{\xi}^{(2)} + \hat{n}^{(1)} \cdot \underline{\xi}^{(1)} \right] = 0$$

which at $r = c$, s_1 yields, respectively,

$$\left[\xi_r^{(2)} \right]_c = 0 \quad (59)$$

$$\left[\xi_{mn} \right]_{s_1} = \pm \frac{\xi_{1c}}{4s_1} (r\xi_2' - 3\xi_2) \Big|_{s_1}$$

where the sign convention of Eq. (19) holds. To $O(\xi^3)$ we have:

$$\left[\hat{n}^{(0)} \cdot \underline{\xi}^{(3)} + \hat{n}^{(1)} \cdot \underline{\xi}^{(2)} + \hat{n}^{(2)} \cdot \underline{\xi}^{(1)} \right] = 0$$

which gives

$$\left[\delta \xi_1 \right]_c = 0 \quad (60)$$

$$\left[\delta \xi_2 \right]_c = - \frac{\xi_{1c}}{2} \left(\frac{1}{3} \left[\xi_{12}' \right]_c - \left[\xi_{21}' \right]_c \right)$$

$$\left[\delta \xi_2 \right]_{s_1} = - \frac{1}{2} \xi_{1c} \left(\frac{1}{3} \xi_{12}' - \xi_{21}' \right) \Big|_{s_1} + \frac{4}{3} \frac{\xi_{1c}}{s_1} \xi_{12} \Big|_{s_1} \quad (61)$$

$$\left[\delta \xi_1 \right]_{s_{12}} = 0$$

We should note that in writing Eqs. (59), (60), we have for simplicity used $\xi_1 = \text{const.}$

Pressure balance across the perturbed interface

The general condition at any interface where a discontinuity might arise is written as:

$$\llbracket p + B^2/8\pi \rrbracket = 0 \quad (62)$$

which holds true for any reasonably smooth interface. To zeroth order, Eq. (62) gives

$$\llbracket p^{(0)} + \frac{1}{8\pi} (B_{z0}^2 + B_{\theta}^{(0)2}) \rrbracket = 0 \quad (63)$$

For an incompressible plasma at equilibrium, the pressure remains constant on any magnetic surface after the surface is perturbed; the perturbations of p at the perturbed interface as functions of the unperturbed coordinates then vanish, so that the perturbed part of Eq. (62) merely assumes the form:

$$\llbracket \tilde{B}^2 \rrbracket = 0 \quad (64)$$

where the tilde denotes the perturbed part of the quantity. To first order in the amplitude, one obtains:

$$\llbracket 2\mathbf{B}^{(0)} \cdot \mathbf{B}^{(1)} + \xi^{(1)} \cdot \nabla B^{(0)2} \rrbracket = 0$$

or

$$\llbracket 2B_{\theta}^{(0)} B_{\theta}^{(1)} + 2B_z^{(0)} \tilde{B}_z^{(2)} + \xi_r^{(1)} (B_{\theta}^{(0)2})' \rrbracket = 0 \quad (65)$$

where we assumed $\partial(B_z^{(0)})/\partial r = 0$. Combining Eq. (65) with Eqs. (16) and (B-5), we have:

$$\left[\xi_m^{(1)} \right]_c = 0 \quad (66)$$

To second order in the amplitude:

$$\left[2B_z^{(0)} \cdot B_z^{(2)} + B_z^{(1)} \cdot B_z^{(1)} + \xi_z^{(2)} \cdot \nabla B_z^{(0)^2} + 1/2 \xi_z^{(1)} \xi_z^{(1)} : \nabla \nabla B_z^{(0)^2} + \xi_z^{(1)} \cdot \nabla (2B_z^{(0)} \cdot B_z^{(1)}) \right] = 0 \quad (67)$$

In more detail, Eq. (67) is written as:

$$\begin{aligned} & \left[2B_\theta^{(0)} B_\theta^{(2)} + 2B_z^{(0)} B_z^{(3)} + B_y^{(1)} B_y^{(1)} + B_\theta^{(1)} B_\theta^{(1)} + 2B_\theta^{(0)} \xi_r^{(2)} B_\theta^{(0)} \right. \\ & + \xi_\theta^{(1)} \xi_\theta^{(1)} B_\theta^{(0)} B_\theta^{(0)'} / r + \xi_r^{(1)} \xi_r^{(1)} (B_\theta^{(0)} B_\theta^{(0)''} + B_\theta^{(0)'}{}^2) + 2\xi_r^{(1)} B_\theta^{(0)'} B_\theta^{(1)} \\ & + 2\xi_r^{(1)} B_\theta^{(0)} B_\theta^{(1)'} + 2\xi_r^{(1)} B_z^{(0)} B_z^{(2)'} + 2\xi_\theta^{(1)} B_\theta^{(0)} B_\theta^{(1)'} / r \\ & \left. + 2\xi_{\theta 1}^{(1)} B_z^{(0)} B_z^{(2)'} / r \right] = 0 \quad (68) \end{aligned}$$

Combining Eq. (68) with Eqs. (15), (16), (17), (21), (28), (34), (35), and (B-7) we obtain the jumps of the derivatives of the Lagrangian displacement of the beats at $r = c$:

$$c \left[\left[\Xi'_{21} \right] \right]_c = 6 \hat{s}_1^{-2} - 10 \hat{s}_1^{-4} - 6 \Xi_{21c} \quad (69a)$$

$$+ \frac{2c \xi_{2c}''}{\hat{\xi}_{2c} \hat{s}_1^4} \left(1 - \frac{3}{8} \hat{s}_1^{-2} - \frac{1}{2} \hat{s}_1^2 \right)$$

and

$$c \left[\left[\Xi'_{12} \right] \right]_c = \left[\frac{3}{2} \hat{s}_1^2 + \frac{1}{6} \frac{10 \hat{s}_1^4 - 3 \hat{s}_1^2}{2 - 3 \hat{s}_1^2} + 16 \Xi_{12c} \left(\hat{s}_1^4 + \frac{2 \hat{s}_1^4 - (4/3) \hat{s}_1^2}{2 - 3 \hat{s}_1^2} \right) \right] \quad (69b)$$

$$+ \frac{c \xi_{2c}''}{2 \hat{\xi}_{2c}} + \left(\frac{1}{4} \hat{s}_1^2 - \frac{1}{8} + \frac{1}{24} \frac{15 - 53 \hat{s}_1^2 + 46 \hat{s}_1^4}{2 - 3 \hat{s}_1^2} \right) \left/ \left(9 \hat{s}_1^4 - 6 \hat{s}_1^2 + \frac{18 \hat{s}_1^4 - 24 \hat{s}_1^2 + 8}{2 - 3 \hat{s}_1^2} \right) \right.$$

where the hat denotes a normalization to c , and we have defined:

$$\Xi_{mn} \equiv \hat{\xi}_{mn} / \hat{\xi}_{1c} \hat{\xi}_{2c}$$

and

$$c \xi_{2c}'' = - 2 \hat{\xi}_{2c} \frac{12 \hat{s}_1^4 / \hat{b}^4 - 4 \hat{s}_1^4 + 2 \hat{s}_1^2 / \hat{b}^4 + 6 \hat{s}_1^2 - 6}{(1 - \hat{b}^{-4}) (1 - 2 \hat{s}_1^2)^2}.$$

On the other hand, at $r = s_1$, one obtains:

$$c \left[\left[\Xi'_{21} \right] \right]_{s_1} = -3 \frac{1-\hat{s}_1^2}{\hat{b}^2-\hat{s}_1^2} \hat{\chi}_{21s}^{II-1} + \frac{1-2\hat{s}_1^2}{2\hat{s}_1(1-\hat{b}^{-4})} (9\hat{s}_1^{-4} - 19\hat{b}^{-4})$$

and

(70)

$$c \left[\left[\Xi'_{12} \right] \right]_{s_1} = - \frac{1-2\hat{s}_1^2}{2\hat{s}_1(1-\hat{b}^{-4})} (\hat{b}^{-4} + 2\hat{s}_1^{-4})$$

The hat on the χ constants implies normalization to

$-cB_{\theta c} \hat{\xi}_{1c} \hat{\xi}_{2c} = kcB_z \hat{s}_1^3 \hat{\xi}_{1c} \hat{\xi}_{2c}$; on the $\delta\chi_m$ constants and on the ΔB_m it will imply normalization to $kcB_z \hat{s}_1^2 \hat{\xi}_{1c} \hat{\xi}_{2c}$ for $m = 1$ and to $kcB_z \hat{s}_1^2 \hat{\xi}_{1c} \hat{\xi}_{2c}$ for $m = 2$.

Finally to order ϵ^3 , when again only the effect of the beats is considered (no cubic products of the linear fields included):

$$\begin{aligned} & \left[2B_{\theta}^{(0)} B_{\theta}^{(3)} + 2B_z^{(0)} B_z^{(4)} + 2B_r^{(1)} B_r^{(2)} \right. \\ & + 2B_{\theta}^{(1)} B_{\theta}^{(2)} + 2B_{\theta}^{(0)} \xi_r^{(3)} B_{\theta}'^{(0)} + 2\xi_r^{(1)} \xi_r^{(2)} B_{\theta}^{(0)} B_{\theta}''^{(0)} \\ & + 2B_{\theta}^{(0)} \xi_r^{(2)} B_{\theta}'^{(1)} + 2B_z^{(0)} \xi_r^{(2)} B_z'^{(2)} + 2\xi_r^{(2)} B_{\theta}'^{(0)} B_{\theta}^{(1)} \\ & + 2B_z^{(0)} B_z^{(2)} \xi_{\theta}^{(2)} / r + 2B_{\theta}^{(0)} \xi_{\theta}^{(2)} B_{\theta}^{(1)} / r \\ & \left. + 2\xi_r^{(1)} (\partial/\partial r) (B_{\theta}^{(0)} B_{\theta}^{(2)} + B_z^{(0)} B_z^{(3)}) + 2B_{\theta}^{(0)} \xi_{\theta}^{(1)} \xi_{\theta}^{(2)} B_{\theta}'^{(0)} / r \right] = 0 \end{aligned}$$

(71)

which is again successively applied at $r = c$, s_1 , s_{12} . At $r = c$, one obtains:

$$\llbracket \delta \xi'_m \rrbracket_c = \frac{m^2 Y_{mc}}{2\kappa^2 B_z^2 (1 - m\hat{s}_1^2)^2} - \frac{\llbracket \delta \hat{\xi}_m \rrbracket_c}{(1 - m\hat{s}_1^2)^2}.$$

Here Y_m is the sum of the terms in Eq. (71) which do not involve $\delta \xi_m$; because of their excessive length, their detailed form is not given in this work [see Ref. (10)]. Using Eqs. (61), (33), (46), the last equation becomes

$$\llbracket \delta \xi'_1 \rrbracket_c = \frac{Y_{1c}}{2\kappa^2 B_z^2 (1 - \hat{s}_1^2)^2} \quad (72a)$$

$$\begin{aligned} \llbracket \delta \xi'_2 \rrbracket_c &= \frac{2Y_{2c}}{\kappa^2 B_z^2 (1 - 2\hat{s}_1^2)^2} - \frac{\llbracket \delta \hat{\xi}_2 \rrbracket_c}{(1 - 2\hat{s}_1^2)^2} \\ &= \frac{2Y_{2c}}{\kappa^2 B_z^2 (1 - 2\hat{s}_1^2)^2} + \frac{\hat{\xi}_{1c}}{2(1 - 2\hat{s}_1^2)^2} \left(\frac{1}{3} \llbracket \xi'_{12} \rrbracket_c - \llbracket \xi'_{21} \rrbracket_c \right) \end{aligned} \quad (72b)$$

Here, $\kappa \equiv kc$ Eq. (71) yields for $m = 2$ at $r = s_1$:

$$\llbracket \delta \xi'_2 \rrbracket_{s_1} = \frac{2Y_{2s_1}}{\kappa^2 B_z^2 \hat{s}_1^2} - \frac{1}{\hat{s}_1} \llbracket \delta \hat{\xi}_2 \rrbracket_{s_1} \quad (73)$$

while at $r = s_{12}$, it yields for the $m = 1$ mode:

$$\llbracket \delta \xi'_1 \rrbracket_{s_{12}} = 3 \frac{Y_{1s_{12}}}{\hat{s}_{12}^2 \kappa^2 B_z^2} \quad (74)$$

One of the difficult parts of the determination of the Y's is the detailed calculation of $B_z^{(4)}$ which is given by Eq. (B-9).

6. THE NONLINEAR STABILITY CONDITIONS

In Section 4 we calculated those constants of the problem which were easily obtained from the condition that the perturbed magnetic field be finite at the origin and zero at the conducting wall. Here we calculate the remaining constants.

As we mentioned in Section 5, when a singular surface exists in well-conducting plasma, we will demand that the Lagrangian displacement remain finite as this surface is approached. For the $m = 3$, $n = 2$ beat, making use of Eqs. (41) and (42), this condition takes on the form:

$$\chi_{12s}^{II-2} = 0, \quad \chi_{12}^{II-2} = \frac{27}{19} \chi_{12s}^{II-2} \left(\frac{s_1^3}{r^3} - \frac{8}{27} \frac{r^3}{s_1^3} \right)$$

$$\xi_{12}^{II-2}(r) = - \frac{27}{19} \frac{\chi_{12s}^{II-2}}{\kappa B_z} \frac{s_1^3/r^3 - 8/27 r^3/s_1^3}{r(2 - 3s_1^2/r^2)}$$

$$\chi_{12s}^{II-3,III} = 0, \quad \chi_{12}^{II-3,III}(r) = 0, \quad \xi_{12}^{II-3,III}(r) = 0. \quad (75)$$

The same condition applied to third order yields values for

$$\delta \chi_{1s}^{II-1}, \quad \delta \chi_{1s}^{II-2} :$$

$$\delta \hat{\chi}_{1s}^{II-1} = \hat{s}_1 \Delta \hat{B}_1^{II-1} (r \rightarrow \hat{s}_1^-) + \frac{1}{2} \kappa \hat{\xi}_{2c}^{-2} \hat{s}_1 \left(\frac{1}{2} - \ln \hat{s}_1 \right) \quad (76)$$

$$\delta \hat{\chi}_{1s}^{II-2} = \hat{s}_1 \Delta \hat{B}_1^{II-2} (\hat{r} \rightarrow \hat{s}_1^+)$$

where Eqs. (43, 44) were used. Combining Eqs. (29), (38), (41), (59), and (75), we obtain the following relations involving second order constants at $r = c$:

$$\hat{\chi}_{21c}^{II-1} = \hat{\chi}_{21c}^I + 1$$

$$\hat{\chi}_{12c}^{II-1} = \hat{\chi}_{12c}^I + 3$$

or in terms of E_{mnc} :

$$\hat{\chi}_{21c}^I = E_{21c} - 2 + \hat{s}_1^{-2}$$

$$\hat{\chi}_{21c}^{II-1} = E_{21c} - 1 + \hat{s}_1^{-2}$$

$$\hat{\chi}_{12c}^I = E_{12c} (3 - 2\hat{s}_1^{-2})$$

$$\hat{\chi}_{12c}^{II-1} = E_{12c} (3 - 2\hat{s}_1^{-2}) + 3$$

(77)

At $r = s_1$, using Eqs. (36), (38), (41), and (59), we have:

$$\hat{\chi}_{21s}^{II-2,III} = \hat{\chi}_{21s}^{II-1} - \frac{1-2\hat{s}_1^2}{1-\hat{b}^{-4}} \frac{1}{2\hat{s}_1^5} [1 - (\hat{s}_1/\hat{b})^4] \quad (78)$$

$$\hat{\chi}_{12s}^{II-2} = \hat{\chi}_{12s}^{II-1} + \frac{1-2\hat{s}_1^2}{1-\hat{b}^{-4}} \frac{1}{2\hat{s}_1^5} [7 + (\hat{s}_1/\hat{b})^4] .$$

Differentiating Eqs. (29), (38), and (41) and using Eqs. (37)

and (39), we obtain at $r = c$:

$$\begin{aligned} c \left[\left[\Xi'_{21} \right] \right]_c &= -2 \frac{\hat{\chi}_{21s}^{II-1}}{\hat{s}_1 (1-\hat{s}_1^2)} + \frac{2}{1-\hat{s}_1^2} (\Xi_{21c} - 1 + \hat{s}_1^{-2}) \\ &\quad - \frac{1}{4} \frac{1-2\hat{s}_1^2}{\hat{s}_1^2} \frac{c \xi_{2c}''}{\xi_{2c}} - 2\hat{s}_1^{-2} \end{aligned}$$

and

(79)

$$\begin{aligned} c \left[\left[\Xi'_{12} \right] \right]_c &= \left(3 \frac{1+\hat{s}_1^{-6}}{1-\hat{s}_1^{-6}} + \frac{2+3\hat{s}_1^2}{2-3\hat{s}_1^2} \right) \left(\frac{3\hat{s}_1^2}{2-3\hat{s}_1^2} - \Xi_{12c} \right) \\ &\quad - \frac{6\hat{\chi}_{12s}^{II-1}}{\hat{s}_1^3 (1-\hat{s}_1^{-6})} + \frac{3}{4} c \xi_{2c}'' + \frac{1-2\hat{s}_1^2}{2-3\hat{s}_1^2} \\ &\quad - \frac{12\hat{s}_1^2}{(2-3\hat{s}_1^2)^2} - 2\Xi_{12c} . \end{aligned}$$

Differentiating Eqs. (38) and (41) and using again Eqs. (37)

and (39), we next obtain at $r = s_1$:

$$\begin{aligned}
 c \left[\left[\Xi'_{21} \right] \right]_{s_1} &= - 2 \hat{\chi}_{21s}^{II-1} \left(\frac{1}{\hat{b}^2 - \hat{s}_1^2} - \frac{1}{1 - \hat{s}_1^2} \right) \\
 &+ \frac{1}{\hat{s}_1^3} \frac{1 - 2\hat{s}_1^2}{\hat{b}^2 - \hat{s}_1^2} \frac{1 - (\hat{s}_1/\hat{b})^4}{1 - \hat{b}^{-4}} + \frac{5}{4} \hat{s}_1^{-2} \\
 &- \frac{1}{4} \frac{c \xi_{2s}''}{\xi_{2c}} + \frac{3}{4 \hat{s}_1} \frac{\xi_{2s}'}{\xi_{2c}} \\
 &- \frac{2 \hat{s}_1}{1 - \hat{s}_1^2} \left(\Xi_{21c} - 1 + \hat{s}_1^{-2} \right)
 \end{aligned}$$

and

(80)

$$\begin{aligned}
 c \left[\left[\Xi'_{12} \right] \right]_{s_1} &= - 3 \frac{\hat{\chi}_{12s}^{II-1}}{\hat{s}_1^2} \left(\frac{200}{57} + \frac{1 + \hat{s}_1^{-6}}{1 - \hat{s}_1^{-6}} \right) \\
 &- \frac{105}{38} \frac{1 - 2\hat{s}_1^2}{\hat{s}_1^5} \frac{7 + (\hat{s}_1/\hat{b})^4}{1 - \hat{b}^{-4}} + \frac{39}{4} \hat{s}_1^{-2} \\
 &- \frac{3}{4} \frac{c \xi_{2s}''}{\xi_{2c}} + \frac{9}{4 \hat{s}_1} \frac{\xi_{2s}'}{\xi_{2c}} \\
 &+ \frac{6}{\hat{s}_1^3} \frac{1}{1 - \hat{s}_1^{-6}} \left[(3 - 2\hat{s}_1^{-2}) \Xi_{12c} + 3 \right]
 \end{aligned}$$

Combining Eqs. (79, 80) with Eqs. (77, 78) respectively, one determines Ξ_{mnc} and $\hat{\chi}_{mn}^N$. The resulting expressions are long and they are not presented here. (10)

Proceeding to the third order, combining Eqs. (32), (33), (43), (45), (46), and (60), we obtain at $r = c$:

$$\begin{aligned}\hat{\delta\chi}_{1c}^{II-1} &= \hat{\delta\chi}_{1c}^I + \left[\hat{\Delta B}_1 \right]_c + \frac{\kappa^2}{8} \frac{\hat{\xi}_{2c}^{-2}}{\hat{s}_1^2} \frac{3-7\hat{s}_1^4}{1-\hat{s}_1^2} \\ \hat{\delta\chi}_{2c}^{II-1} &= \hat{\delta\chi}_{2c}^I + \left[\hat{\Delta B}_z \right]_c \\ &\quad + \frac{(1-2\hat{s}_1^2)}{2\hat{s}_1^2} \left(\frac{1}{3} \left[\hat{\Xi}'_{12} \right]_c - \left[\hat{\Xi}'_{21} \right]_c \right) \\ &\quad + \frac{\kappa^2}{6} \frac{\hat{\xi}_{1c}^{-2}}{\hat{s}_1^2} \left(\frac{1-2\hat{s}_1^2}{\hat{b}^4-1} + \frac{1-3\hat{s}_1^2}{1-2\hat{s}_1^2} \right)\end{aligned}\quad (81)$$

Applying Eq. (61) at $r = s_1$ gives

$$\begin{aligned}\hat{\delta\chi}_{2s}^{II-2} &= \hat{\delta\chi}_{2s}^{II-1} - \hat{s}_1 \hat{\Delta B}_{2s}^{II-1} - \frac{1}{\hat{s}_1} \left(- \hat{\chi}_{12c}^{II-1} \frac{\hat{s}_1^{-3}}{1-\hat{s}_1^{-6}} \right. \\ &\quad \left. + \hat{\chi}_{12s}^{II-1} \frac{\hat{s}_1^{-6}}{1-\hat{s}_1^{-6}} + \hat{\chi}_{21c}^{II-1} \frac{\hat{s}_1^{-1}}{1-\hat{s}_1^{-2}} - \hat{\chi}_{21s}^{II-1} \frac{1}{1-\hat{s}_1^{-2}} \right) \\ &\quad + \frac{1}{\hat{s}_1^4} \frac{1-2\hat{s}_1^2}{1-\hat{b}^{-4}} \left(3\hat{s}_1^{-2} + \hat{b}^{-4}\hat{s}_1^2 \right)\end{aligned}\quad (82)$$

and at $r = s_{12}$:

$$\hat{\delta\chi}_{1s}^{II-3,III} = \hat{\delta\chi}_{1s}^{II-2} + \hat{s}_{12} \left[\hat{\Delta B}_1 \right]_{s_{12}} \quad (83)$$

In what is to follow, we set $\delta\chi_{1c}^I = \delta\chi_{2c}^I = 0$. We note that the algebraic dependence of the homogeneous terms in Eqs. (37) for $\delta\chi_1^I(r)$ and $\delta\chi_2^I(r)$ is identical with that of $\chi_1^I(r)$ and $\chi_2^I(r)$ respectively, so the presence of any such terms would be superfluous.

Finally, when Eq. (74) is combined with Eqs. (44), (46), (76), and (83), an expression for $\delta\hat{\chi}_{1s}^{II-2}$ results:

$$\delta\hat{\chi}_{1s}^{II-2} = \frac{1}{2} \left(5 + \frac{1+\hat{s}_{12}^2/\hat{b}^2}{1-\hat{s}_{12}^2/\hat{b}^2} \right)^{-1} \cdot \left(\frac{3Y_{1s_{12}}}{\kappa \hat{s}_1^2 \hat{B}_z^2 \hat{\xi}_{1c}^2 \hat{\xi}_{2c}^2} + \right. \quad (84)$$

$$\left. \frac{12\hat{s}_1^2}{\hat{s}_{12}} \left[\hat{\Delta B}_1 \right] s_{12} + 4\sqrt{6} \hat{s}_1 \hat{\Delta B}_{1s_{12}}^{II-2} - 3\hat{s}_1^2 c \left[\hat{\Delta B}_1' \right] s_{12} \right) - \hat{s}_{12} \left[\hat{\Delta B}_1 \right] s.$$

At this point, we have evaluated all of the constants of our problem, in terms of the amplitudes of the two fundamentals and some zeroth order equilibrium parameters. We can, thus, plot the profiles of the ξ 's and the χ 's that we have calculated throughout the entire domain of our problem. This is done only schematically in Figs. 4, 5, and 6 for the first, second, and third order fields, respectively. The significant features of these figures are discussed in section 7.

There are two more conditions to be satisfied however; these are the third order pressure balance conditions at $r = c$ for the $m = 1$ and the $m = 2$ modes. For the $m = 1$, we combine Eqs. (33), (43), and (46) to obtain:

$$\begin{aligned} \frac{[\delta \xi_1]_c}{\xi_{1c} \xi_{2c}^2} &= \frac{2\hat{s}_1^3}{(1-\hat{s}_1^2)^2} \delta \chi_{1s}^{II-1} - \frac{1}{4} \left(\frac{\kappa}{\xi_{2c}} \right)^2 \frac{5\hat{s}_1^4 + 4\hat{s}_1^2 - 3}{(1-\hat{s}_1^2)^2} \\ &+ \frac{\hat{s}_{1c}^2}{(1-\hat{s}_1^2)} [\hat{\Delta B}_1] - \frac{2\hat{s}_1^4}{(1-\hat{s}_1^2)^2} \hat{\Delta B}_{1c}^{II-1}. \end{aligned} \quad (85)$$

When Eq. (85) is combined with Eq. (72a), a relation results involving $\hat{\xi}_{2c}$ and the equilibrium alone. This relation will be completely independent of the form of $\delta \xi_1$ for $r > s_1$, although $\delta \xi_1 \neq 0$ for $r > s_1$. This does not mean that there will be no influences on the stability coming from $r > s_1$, as $\delta \xi_1$ and ΔB_1 depend on ξ_{12} , ξ_{21} , and ξ_2 which in turn are sensitive to the behavior outside the singular surface at $r = s_1$. Continuing on for the $m = 2$ mode we combine at $r = s_1$ Eqs. (45), (46), (73), and (82) to obtain:

$$\begin{aligned} \delta \chi_{2s}^{II-1} &= \left(\frac{1 + (\hat{s}_1/\hat{b})^4}{1 - (\hat{s}_1/\hat{b})^4} + \frac{1 + \hat{s}_1^{-4}}{1 - \hat{s}_1^{-4}} \right)^{-1} \left[- \frac{Y_{2s_1}}{\kappa^2 B_z^2 \hat{s}_1^2 \xi_{1c}^2 \xi_{2c}^2} \right. \\ &+ \frac{\hat{s}_1}{2} (\lambda + \hat{s}_1 \hat{\Delta B}_{2s}^{II-1}) + 2\delta \chi_{2c}^{II-1} \frac{\hat{s}_1^{-2}}{1-\hat{s}_1^{-4}} + \frac{\lambda}{2} \left(3 - 2 \frac{1 + (\hat{s}_1/\hat{b})^4}{1 - (\hat{s}_1/\hat{b})^4} \right) \\ &\left. - \frac{\hat{s}_{1c}^2}{2} [\hat{\Delta B}_2]_{s_1} - 2\hat{s}_1 [\hat{\Delta B}_2]_{s_1} + 2\delta \chi_{2b} \frac{(\hat{s}_1/\hat{b})^2}{1 - (\hat{s}_1/\hat{b})^4} \right] \end{aligned} \quad (86)$$

with $\lambda \equiv \delta \chi_{2s}^{II-2} - \delta \chi_{2s}^{II-1}$ as given by Eq. (82) while at $r = c$, Eqs. (33), (45), and (46) are combined to yield:

$$\begin{aligned}
 \frac{[\delta \xi_2']_c}{\hat{\xi}_{1c}^2 \hat{\xi}_{2c}} = & - \frac{4 \hat{\chi}_{2s}^{II-1}}{(1-2\hat{s}_1^2)(1-\hat{s}_1^{-4})} + \frac{\hat{s}_1^2}{1-2\hat{s}_1^2} \left(\frac{1+2\hat{s}_1^2}{1-2\hat{s}_1^2} + 2 \frac{1+\hat{s}_1^{-4}}{1-\hat{s}_1^{-4}} \right) \hat{\chi}_{2c}^{II-1} \\
 & + \frac{\hat{s}_1^2}{1-2\hat{s}_1^2} \frac{[\Delta B_2']_c}{1-2\hat{s}_1^2} - \frac{4 \Delta B_{2c}^{II-1} \hat{s}_1^4}{(1-2\hat{s}_1^2)^2} \\
 & + \frac{\kappa^2 \hat{\xi}_{1c}^{-2}}{6(1-2\hat{s}_1^2)} \left(\frac{3-10\hat{s}_1^2}{\hat{b}^4-1} + \frac{3(1-3\hat{s}_1^2)}{1-2\hat{s}_1^2} \right) . \quad (87)
 \end{aligned}$$

Combining Eqs. (85) and (87) with Eqs. (72a) and (72b) respectively, information is obtained on the nature of the interaction of the two modes under discussion. There results two expressions of the form:

$$\kappa^2 f_1 \hat{\xi}_{1c} + g_1 \hat{\xi}_{2c}^2 \hat{\xi}_{1c} = 0 \quad (88)$$

$$\kappa^2 f_2 \hat{\xi}_{2c} + g_2 \hat{\xi}_{1c}^2 \hat{\xi}_{2c} = 0 \quad (89)$$

where f_1 , g_1 , and f_2 , g_2 are functions of the zeroth order equilibrium parameters only:

$$f_1 = \left[\frac{3}{4} - \hat{s}_1^2 - \hat{s}_1^4 \left(\frac{3}{4} + \ln \hat{s}_1 \right) \right] (1 - \hat{s}_1^2)^{-2}$$

$$g_1 = \left(2\hat{s}_1^2 \hat{\Delta B}_{1s_1}^{II-1} + \hat{s}_1^2 (1 - \hat{s}_1^2) c \left[\hat{\Delta B}_1' \right]_c - 2\hat{s}_1^4 \hat{\Delta B}_{1c}^{II-1} - \hat{Y}_{1c}/2 \right) (1 - \hat{s}_1^2)^{-2}$$

$$f_2 = \left(- \frac{4(1 - 2\hat{s}_1^2)}{(1 - \hat{s}_1^4)} \delta \chi_{2s\kappa}^{II-1} + \hat{s}_1^2 (1 - 2\hat{s}_1^2) u_1 \delta \chi_{2c\kappa}^{II-1} - 2\hat{Y}_{2c\kappa} + \frac{1}{6} (1 - 2\hat{s}_1^2) u_2 \right) (1 - 2\hat{s}_1^2)^{-2}$$

$$g_2 = \left[- 2\hat{Y}_{2co} - \frac{c}{2} \left(\frac{1}{3} \left[\hat{\Xi}_{12}' \right]_c - \left[\hat{\Xi}_{21}' \right]_c \right) - \frac{4(1 - 2\hat{s}_1^2)}{1 - \hat{s}_1^4} \delta \chi_{2so}^{II-1} + \hat{s}_1^2 (1 - 2\hat{s}_1^2) u_1 \delta \chi_{2co}^{II-1} - \hat{s}_1^2 (1 - 2\hat{s}_1^2) c \left[\hat{\Delta B}_2' \right]_c - 4\hat{s}_1^4 \hat{\Delta B}_{2c}^{II-1} \right] (1 - 2\hat{s}_1^2)^{-2}$$

where the hat on Y_m means normalization to $\kappa^2 B_{f1c}^{2\hat{\xi}} \hat{\xi}_{2c}^2$ for $m = 1$ to $\kappa^2 B_{z1c}^{2\hat{\xi}} \hat{\xi}_{2c}^2$ for the $m = 2$ the subscripts κ and o denoting the part which comes from κ^2 contributions and that which does not respectively. Also

$$u_1 \equiv \frac{1 + 2\hat{s}_1^2}{1 - 2\hat{s}_1^2} + 2 \frac{1 + \hat{s}_1^{-4}}{1 - \hat{s}_1^{-4}}$$

$$u_2 \equiv \frac{3 - 10\hat{s}_1^2}{\hat{b}^4 - 1} + 3 \frac{1 - 3\hat{s}_1^2}{1 - 2\hat{s}_1^2}$$

We next relate the nonlinear stability picture emerging from Eqs. (88) and (89) to the linear one. For the $m = 1$ mode, no special effort is required. We have seen that this mode is not truly at linear equilibrium, but that it is growing with a small growth rate proportional to κ^2 . The κ^2 terms in Eq. (88) are proportional to this growth rate. They are not exactly equivalent to the growth rate given in Ref. [7] because here, for simplicity, we are considering a slightly different third order correction to the lowest order eigenmode. In particular, we force $\delta\xi_1$ to be finite as $r \rightarrow s_1$, while in Ref. [7] a boundary layer problem was solved. One expects the two conditions to yield only slightly differing results (at least for the linear case) since the shape of the resulting eigenfunctions involved is not all that different. A boundary layer treatment at $r = s_1$ (and at $r = s_{12}$) would still be appropriate to our problem but it would pose many difficulties which the condition used here helps us avoid.

For the $m = 2$ mode, we follow the procedure given in Ref. [3]. We imagine our system very close to equilibrium but not exactly at it. Then $[\xi_2']_c \neq 0$; in particular, $[\xi_2']_c = 0(\gamma^2 \hat{\xi}_{2c})$ where γ is the linear growth rate of the $m = 2$. When close enough to equilibrium, we can always pick

γ such that $[\xi_2']_c \sim 0(\epsilon^3)$. Then combining Eqs. (87) and (72b) and adding the small growth correction, Eq. (89) is modified to:

$$-\frac{4}{q_c^2(1 - 2s_1^2)^2} (q_c - 1 - b^{-4}) + \kappa^2 f_2 + g_2 \hat{\xi}_{1c}^2 = 0 \quad (90)$$

It then suffices to study whether there exist real values for $\hat{\xi}_{1c}$ and $\hat{\xi}_{2c}$ which are solutions of the system of Eqs. (88) and (90). These equations can be considered as they are, or can be made more complete by including the self interaction terms treated in Ref. [3] as well. A nonlinear equilibrium exists for the combination of modes if both $\hat{\xi}_{1c}^2 > 0$ and $\hat{\xi}_{2c}^2 > 0$. It does not exist if either or both of these quantities are negative.

7. RESULTS AND DISCUSSION

The detailed forms of the long algebraic expressions appearing in this work were handled by means of the FORMAC compiler. The main part of this operation focused on deriving the pressure balance quantities, the Y's . Ordinary FORTRAN programs were next employed using these expressions to obtain an answer. The details of this procedure are given in Ref. (10).

Figures 7, 8, and 9 summarize the results obtained in this manner. Figure 7 shows plots of $\hat{\xi}_{1c}^2$ and $\hat{\xi}_{2c}^2$ as functions of q_a for $\kappa = -0.22$ and fixed positions of the wall and of the plasma-vacuum interface. These were obtained by solving Eqs. (88, 90) for a succession of equilibrium profiles. The profiles were always picked so that to $O(\epsilon^0)$, Eq. (1) was satisfied and the singular surface for the $m = 3$, $n = 2$ mode remained in well conducting plasma (see Section 4). One observes that for high enough values of q_a the system of the combined distortions that we are considering can go to a nonlinear equilibrium state. However, when q_a falls below ~ 1.65 this equilibrium ceases to exist. Its destruction comes about because the nonlinear contributions to the $m = 2$ linear growth rate which are proportional to $\hat{\xi}_{1c}^2$ become destabilizing. At the same time, the reciprocal contribution (proportional to $\hat{\xi}_{2c}^2$) to the $m = 1$ linear growth rate is seen to be stabilizing. As a result, $\hat{\xi}_{1c}^2$ becomes negative while $\hat{\xi}_{2c}^2$ remains positive. This is then interpreted to mean that an instability exists which goes at $q_a \sim 1.65$ at a growth rate at least as large as that of the linear $m = 2$ growth rate. The immediate consequence of this instability is the

destruction of a relatively large amplitude internal $m = 1$ kink mode nonlinearly at equilibrium.

It is seen from Fig. 7 that $\hat{\xi}_{2c}^2$ is much smaller than $\hat{\xi}_{1c}^2$. We also note that a narrow region around $q_a \sim 1.65$ is not very well described by our theory as $\hat{\xi}_{1c}$ ceases to be a small quantity there. Something that is not shown in Fig. 7 but which can be seen in the computer results is that even $\hat{\xi}_{2c}^2$ can go negative for $q_a \lesssim 1.5$. This would not be very interesting since instability was seen to set in at higher q_a .

To further support the interpretation of the results stated above, we consider what the nonlinear corrections (Fig. 6) have done physically to the original linear fields (Fig. 4). The $m = 1$ internal mode is made to extend all the way to the exterior of the plasma and, which is even more interesting, it can even have a singular surface in the vacuum (the $m = 2$ singular surface). On the contrary, the $m = 2$ mode, purely due to nonlinearity, acquires a $1/r$ type of singularity near the origin which will tend to make it energetically unfavorable. Figure 5 shows the behavior of the beats generated by the interaction. One could place the $m = 3$, $n = 2$ singular surface in the vacuum making the corresponding beat mode be felt all the way to the conducting wall. This was not done here because it would complicate matters, to say the least.

Figure 8 shows the effect of the conducting wall on this phenomenon. As the wall is brought closer to the plasma, the onset of the instability moves to lower values of q_a and smaller

amplitudes although the effect is not really very big. The critical q_a changes from a value of about 1.65 for the wall at infinity to 1.61 when $\hat{b} = 1.59$. Figure 9 shows the effect of varying κ (or the aspect ratio). It is also small. Larger κ makes $\hat{\xi}_{1c}$ smaller but $\hat{\xi}_{2c}$ larger. We should also point out that for the $\kappa \sim 0.4$ cases our small κ theory is not very valid.

Figure 7 may be taken as an approximation to a dynamical picture of a shrinking current channel given that the shrinking is slow enough for the plasma to be considered at equilibrium at every successive time instant. A real dynamical theory would undoubtedly be more accurate in describing these phenomena. It would also be much more difficult to develop. The theory given here is only to serve as a hint at what goes on during the phase of the discharge when the disruptive instability occurs. The introduction of dissipation into the model will also have some effect and it is probably more appropriate to treat the $m = 2$ mode as a tearing rather than a kink mode.

In conclusion, if, as it is suggested by the nonlinear phenomena under study here, a larger growth rate can be considered in calculating the flux changes, the voltage spikes, and the major plasma radius contraction as was done in Ref. (7), one could succeed in explaining even quantitatively the features of the disruptive instability. The streak photographs of the discharges stretching over the time interval before, during, and after a negative voltage spike occurs⁽²⁾ seem to indicate a small amplitude mild process taking place for most of the time which would be amenable to a, strictly speaking, quasilinear description as

given here. One observes the $m = 2$ or $m = 3$ oscillation going on right through and after the occurrence of the voltage spike in many of the situations. It is possible then that an energy exchange takes place between the internal mode and the external oscillations which usually results in a negative voltage spike and sometimes (when q is low enough) in a disruption. Whether it is really an $m = 2$ or an $m = 3$ mode interacting with the internal kink is not very important since the behavior of the two external modes are qualitatively the same. There is also a chance that the $m = 3$ oscillation which is observed may very well be one of the two harmonics which are generated during the interaction. The behavior shown in the pictures of Ref. (2) would also seem to contradict a theory of the disruptive instability as given in Ref. (5).

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APPENDIX A
EXPRESSIONS IN CONNECTION
WITH THE THIRD-ORDER CALCULATION

The nonlinear parts to the third-order radial magnetic field generated by the interaction of the beat waves with the fundamentals are obtained as: [10]

$$\begin{aligned}
 \Delta B_1 = & -\frac{17}{12} \xi_2' \xi_{12}' krB_z - \frac{1}{4} \xi_2' \xi_{21}' krB_z - \frac{5}{12} \xi_2' \xi_{12}' kB_z + \frac{1}{4} \xi_2' \xi_{21}' kB_z \\
 & - \frac{1}{4} \xi_2'' \xi_2 \xi_1 kB_z - \frac{3}{4} (\xi_2')^2 \xi_1 kB_z - \frac{3}{4} \xi_{12}' \xi_2 kB_z + \frac{3}{4} \xi_{21}' \xi_2 kB_z \\
 & - 2\xi_2' B_\theta' \xi_{12}' - \frac{3}{16} \xi_2' B_\theta' \xi_{21}' - \frac{25}{12} \xi_2' \xi_{12}' B_\theta - \frac{7}{16} \xi_2' \xi_{21}' B_\theta - \xi_2^2 \xi_1 B_\theta / r^3 \\
 & + \xi_2 \xi_{12} B_\theta / r^2 + \xi_2 \xi_{21} B_\theta / r^2 + 4\xi_2' \xi_2 \xi_1 B_\theta / r^2 + B_\theta' \xi_2^2 \xi_1 / r^2 \\
 & - \frac{1}{4} \xi_2 \xi_{12} kB_z / r + \frac{3}{4} \xi_2 \xi_{21} kB_z / r + \frac{3}{4} \xi_2' \xi_2 \xi_1 kB_z / r + \frac{7}{6} \xi_2' \xi_{12} B_\theta / r \\
 & - \frac{3}{2} B_\theta' \xi_2 \xi_{12} / r + \frac{1}{2} \xi_2' \xi_{21} B_\theta / r + \frac{5}{16} B_\theta' \xi_2 \xi_{21} / r - \frac{1}{2} (\xi_2'') \xi_2 \xi_1 B_\theta / r \\
 & - \frac{3}{2} (\xi_2')^2 \xi_1 B_\theta / r - \frac{5}{4} \xi_{12}' \xi_2 B_\theta / r + \frac{17}{16} \xi_{21}' \xi_2 B_\theta / r - \frac{5}{2} \xi_2' B_\theta' \xi_2 \xi_1 / r \\
 & - \frac{1}{2} B_\theta'' \xi_2^2 \xi_1 / r ,
 \end{aligned} \tag{A-1}$$

$$\begin{aligned}
 \Delta B_2 = & - 2\xi_{12}'\xi_1' kB_z - \frac{1}{2} \xi_2''\xi_1'^2 kB_z + 4\xi_2\xi_1'^2 B_\theta/r^3 + \frac{3}{2} \xi_2\xi_1'^2 kB_z/r^2 \\
 & + 3\xi_1\xi_{12}B_\theta/r^2 - \xi_{21}\xi_1B_\theta/r^2 - \xi_2'\xi_1'^2 B_\theta/r^2 - B_\theta'\xi_2\xi_1'^2/r^2 \\
 & - \frac{3}{2} \xi_2'\xi_1'^2 kB_z/r - 3B_\theta'\xi_1\xi_{12}/r + \frac{5}{8} B_\theta'\xi_1\xi_{21}/r - 3\xi_{12}'\xi_1B_\theta/r \\
 & + \frac{5}{8} \xi_{21}'\xi_1B_\theta/r - \xi_2''\xi_1'^2 B_\theta/r - B_\theta''\xi_2\xi_1'^2/r - 2\xi_2'B_\theta'\xi_1'^2/r . \quad (A-2)
 \end{aligned}$$

We immediately realize that for $r > s_1$,

$$\Delta B_2 = 0 \quad (A-3)$$

while

$$\begin{aligned}
 \Delta B_1^{II-2} = & - \frac{17}{12} \xi_2'\xi_{12}' krB_z - \frac{1}{4} \xi_2'\xi_{21}' krB_z - \frac{5}{12} \xi_2'\xi_{12}' kB_z + \frac{1}{4} \xi_2'\xi_{21}' kB_z \\
 & - \frac{3}{4} \xi_{12}'\xi_2' kB_z + \frac{3}{4} \xi_{21}'\xi_2' kB_z - 2\xi_2'B_\theta'\xi_{12}' - \frac{3}{16} \xi_2'B_\theta'\xi_{21}' \\
 & - \frac{25}{12} \xi_2'\xi_{12}' B_\theta - \frac{7}{16} \xi_2'\xi_{21}' B_\theta + \xi_2\xi_{12}B_\theta/r^2 + \xi_2\xi_{21}B_\theta/r^2 \\
 & - \frac{1}{4} \xi_2\xi_{12}' kB_z/r + \frac{3}{4} \xi_2\xi_{21}' kB_z/r + \frac{7}{6} \xi_2'\xi_{12}B_\theta/r - \frac{3}{2} B_\theta'\xi_2\xi_{12}/r \\
 & + \frac{1}{2} \xi_2'\xi_{21}B_\theta/r + \frac{5}{16} B_\theta'\xi_2\xi_{21}/r - \frac{5}{4} \xi_2\xi_{12}' B_\theta/r + \frac{17}{16} \xi_{21}'\xi_2B_\theta/r .
 \end{aligned}$$

Even for $r > s_{12}$, ΔB_1 remains nonzero.

APPENDIX B

THE EQUILIBRIUM EQUATION IN THE RADIAL DIRECTION:

THE LONGITUDINAL MAGNETIC FIELD PERTURBATION

Operating with $\hat{e}_r \cdot \nabla$ rather than $\hat{e}_z \cdot \nabla$ on Eq. (22) and setting $d/dt = 0$, we obtain:

$$\underline{B} \cdot \nabla j_r - j \cdot \nabla B_r = 0. \quad (B-1)$$

From Ampere's law:

$$j_r = \frac{1}{r} \frac{\partial}{\partial \theta} B_z - \frac{\partial}{\partial z} B_\theta,$$

$$j_\theta = \frac{\partial}{\partial z} B_r - \frac{\partial}{\partial r} B_z,$$

$$j_z = \frac{1}{r} \frac{\partial}{\partial r} (r B_\theta) - \frac{1}{r} \frac{\partial}{\partial \theta} B_r. \quad (B-2)$$

Writing out Eq. (B-1), we have

$$B_r j_r' + \left(\frac{B_\theta}{r} \frac{\partial}{\partial \theta} + B_z \frac{\partial}{\partial z} \right) j_r = j_r B_r' + \left(\frac{j_\theta}{r} \frac{\partial}{\partial \theta} + j_z \frac{\partial}{\partial z} \right) B_r \quad (B-3)$$

and to the linear order

$$\left(\frac{B_\theta^{(0)}}{r} \frac{\partial}{\partial \theta} + B_z^{(0)} \frac{\partial}{\partial z} \right) j_r^{(2)} = j_{z0} \frac{\partial}{\partial z} B_r^{(1)}. \quad (B-4)$$

Combining Eqs. (B-2, 4) yields to lowest order (greater than zeroth order):

$$B_{zm}^{(2)} = \frac{kr}{m} \left[\frac{1}{m^2} \frac{\partial^2}{\partial \theta \partial r} \left(r B_{rm}^{(1)} \right) + j_{zo} \xi_m^{(1)} \right] . \quad (B-5)$$

To next order in ϵ , Eq. (B-3) becomes

$$\begin{aligned} B_r^{(1)} j_r'^{(2)} + \frac{B_\theta^{(1)}}{r} \frac{\partial}{\partial \theta} j_r^{(2)} + \left(\frac{B_\theta^{(0)}}{r} \frac{\partial}{\partial \theta} + B_z^{(0)} \frac{\partial}{\partial z} \right) j_r^{(3)} \\ = j_r^{(2)} B_r'^{(1)} + \frac{j_\theta^{(2)}}{r} \frac{\partial}{\partial \theta} B_r^{(1)} + j_{zo} \frac{\partial B_r^{(2)}}{\partial z} . \end{aligned} \quad (B-6)$$

Combining Eqs. (B-2, 6) yields the third-order part to the longitudinal field perturbation, which now contains nonlinear terms in addition to the linear ones:

$$\begin{aligned} B_{zij}^{(3)} = - \frac{r}{(i \pm j) F_{ij}} \left[j_r^{(2)} B_r'^{(1)} + \frac{j_\theta^{(2)}}{r} \frac{\partial}{\partial \theta} B_r^{(1)} + j_{zo} \frac{\partial}{\partial z} B_r^{(2)} \right. \\ \left. - B_r^{(1)} j_r'^{(2)} - \frac{B_\theta^{(1)}}{r} \frac{\partial}{\partial \theta} j_r^{(2)} \right. \\ \left. + \left(\frac{B_\theta^{(0)}}{r} \frac{\partial}{\partial \theta} + B_z^{(0)} \frac{\partial}{\partial z} \right) \frac{\partial B_\theta^{(2)}}{\partial z} \right] \end{aligned} \quad (B-7)$$

where again we are interested only in $i \neq j$, and the conversion about the $(\pm)$ signs holds as for Eq. (15). Finally, to $O(\epsilon^4)$, we have:

$$\begin{aligned}
 & B_r^{(1)} j_r^{(3)} + B_r^{(2)} j_r^{(2)} + \frac{B_\theta^{(1)}}{r} \dot{j}_r^{(3)} + \frac{B_\theta^{(2)}}{r} \dot{j}_r^{(2)} + B_z^{(2)} \frac{\partial}{\partial z} j_r^{(2)} \\
 & + \left(\frac{B_\theta^{(0)}}{r} \frac{\partial}{\partial \theta} + B_z^{(0)} \frac{\partial}{\partial z} \right) j_r^{(4)} \\
 & = j_r^{(3)} B_r^{(1)} + j_r^{(2)} B_r^{(2)} + \frac{j_\theta^{(3)}}{r} \dot{B}_r^{(1)} + \frac{j_\theta^{(2)}}{r} \dot{B}_r^{(2)} + j_{z0} \frac{\partial}{\partial z} B_r^{(3)} \quad (B-8)
 \end{aligned}$$

so that combining Eqs. (B-2, 8) we finally obtain:

$$\begin{aligned}
 B_{zm}^{(4)} = - \frac{r}{mF_m} & \left[j_r^{(3)} B_r^{(1)} + j_r^{(2)} B_r^{(2)} + \frac{j_\theta^{(3)}}{r} \dot{B}_r^{(1)} + \frac{j_\theta^{(2)}}{r} \dot{B}_r^{(2)} \right. \\
 & + j_{z0} \frac{\partial}{\partial z} B_r^{(3)} - B_r^{(1)} j_r^{(3)} - B_r^{(2)} j_r^{(2)} \\
 & \left. - \frac{B_\theta^{(1)}}{r} \dot{j}_r^{(3)} - \frac{B_\theta^{(2)}}{r} \dot{j}_r^{(2)} + \left(\frac{B_\theta^{(0)}}{r} \frac{\partial}{\partial \theta} + B_z^{(0)} \frac{\partial}{\partial z} \right) \frac{\partial}{\partial z} B_\theta^{(3)} \right] \quad (B-9)
 \end{aligned}$$

where again only the terms with dependences like $\cos \tau_m$,
 $m = 1, 2$, are to be considered.

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Fig. 1. The model. The safety factor q as a function of distance from the origin.

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Fig. 2. Qualitative plot of MHD external and internal kink growth rates.

Fig. 3. The Lagrangian displacement $\underline{\xi}$.

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Fig. 4. Schematic diagram of the first order (linear) fields.

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Fig. 5. Schematic diagram of second order fields (beats). (Sign of jumps shown may not be accurate.)

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Fig. 6. Schematic diagram of third order fields. (Sign of jumps shown may not be accurate.)

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Fig. 7. The squares of the normalized linear amplitudes of the $m = 1$ and $m = 2$ modes as functions of q_a for fixed κ , $\hat{a}$, $\hat{b}$.

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Fig. 8. Effect of the position of the wall.

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Fig. 9. Effect of varying κ (or the aspect ratio).

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