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ANALYTIC SOLUTION OF THE STEADY-STATE WAVE
EQUATION IN THREE DIMENSIONS AND IN IRREGULARLY SHAPED
DOMAINS WITH AN APPLICATION TO A REACTOR WITH
PARTIALLY INSERTED CONTROL RODS

by

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E. J. Betinis

I. INTRODUCTION

Many perturbation theories for boundary-value problems have been developed in the past, but the vast majority have been concerned with the effect on eigenvalues and eigenfunctions when operators of the form $\nabla^2 + \Lambda$ have been perturbed by the addition of a small perturbing operator of the form ϵ while the domain of the operator remained unchanged. In this regard, the literature is very extensive (for example, see references 1-3).

A paper by Brillouin⁽⁴⁾ is concerned with the problem of finding the corresponding distorted domain for a perturbed operator.

Other works regarding the perturbations on the eigenvalues and eigenfunctions due to perturbations in the boundary shape may be found in reference 5. In this same vein, a recent paper⁽⁶⁾ concerns itself explicitly with this type of perturbation for second-order operators in one-dimensional geometry. Since boundary-value problems of the eigenvalue type for distorted regions are actually concerned with geometries in which the differential equation is not separable, it is pertinent to cite reference 7 in this regard.

In addition, many numerical techniques have been developed for two-dimensional problems with good success. But for three-dimensional problems involving irregularly shaped boundaries there has been very little successful work done. There is an IBM 704 nuclear code, called Trixy,⁽⁸⁾ written by a group at The General Electric Company, which employs numerical techniques for solving three-dimensional problems and is successful for certain geometries.

A method of successive approximations has been developed for solving the steady-state wave equation

$$(\nabla^2 + \Lambda) \phi = 0 \tag{1}$$

in an arbitrarily shaped n-dimensional domain with ϕ subject to homogeneous boundary conditions. For the most part, the Helmholtz operator $\nabla^2 + \Lambda$ is considered; however, extensions to more general operators are also treated.

In particular, the technique developed has been extensively applied in solving the one-group diffusion equation, which is the same as eq. (1), for a bare, homogeneous cylindrical thermal reactor with many partially or fully inserted control rods. These calculations were made on the IBM 704 and have been included in this paper.

In addition, formulas were derived for the solution of eq. (1) for irregularly shaped rectangular domains.

The solutions of eq. (1) and related equations are in the form of a Fourier series. In the method of successive approximations developed in this paper, the Fourier coefficients necessary to solve a given boundary-value problem of the eigenvalue type mentioned above have been derived. As a further consequence of the technique developed, perturbation formulas for the eigenvalues of eq. (1) and related equations for the above boundary-value problem have also been derived.

II. STATEMENT OF THE PROBLEM

The eigenvalue problem to be considered may be solved subject to a variety of homogeneous boundary conditions. In the treatment that follows the homogeneous Dirichlet boundary conditions will be of concern in the main, although utilization of the method to be developed for the solution of the problem subject to other boundary conditions follows quite trivially.

We wish to determine the eigenfunctions and eigenvalues of the self-adjoint partial differential equation

$$\nabla^2 \phi + \Lambda \phi = 0 \text{ in } G \quad (2)$$

subject to one of the following homogeneous boundary conditions:

$$\phi|_{\Gamma} = 0 \quad (\text{Dirichlet}) \quad (3a)$$

$$\vec{\nu} \cdot \vec{\nabla} \phi|_{\Gamma} = 0 \quad (\text{Neumann}) \quad (3b)$$

or

$$(\phi + K \vec{\nu} \cdot \vec{\nabla} \phi)|_{\Gamma} = 0 \quad (\text{Mixed}) \quad (3c)$$

where G is an arbitrarily shaped, n -dimensional domain with continuous boundary surface Γ , ∇^2 the Laplacian for this domain, $\vec{\nu} \cdot \vec{\nabla} \phi$ the normal derivative of ϕ , and K is an arbitrary constant. In the sequel, the discussion and applications will be confined to three-dimensional considerations for the sake of clarity because in all that follows the n -dimensional considerations are equally valid. Actually in the treatment to follow the above boundary conditions are replaced by boundary conditions which are very similar if not in many cases equivalent. In this regard see page 11, eq. (23).

III. RESTATEMENT OF THE PROBLEM

Let $\bar{D}$ be a closed non-null subset of a three-dimensional Euclidean space, E_3 , with boundary surface Σ such that the eigenfunctions and eigenvalues of the equation

$$\nabla^2 f_n + \lambda_n f_n = 0 \text{ in } \bar{D} \quad , \quad (4)$$

subject to the homogeneous Dirichlet boundary condition

$$f_n|_{\Sigma} = 0 \quad , \quad (5)$$

may be determined. The index n has been written in place of the indices m , n , and l for the sake of clarity and brevity here and in what is to follow. Further let $\bar{D}_i$ with $i = 1, 2, \dots, k$ be k closed proper non-null subsets with continuous boundary surfaces σ_i , and define the union of these subsets as

$$\bar{D}_1 = \bigcup_{i=1}^k \bar{D}_i \quad . \quad (6)$$

Define the union of the k surfaces of these proper subsets as

$$\Sigma_1 = \bigcup_{i=1}^k \sigma_i \quad . \quad (7)$$

If we now let

$$G = \bar{D} - \bar{D}_1 = \overline{D - D_1} \quad (8)$$

and

$$\Gamma = \Sigma \cup \Sigma_1 \quad , \quad (9)$$

the problem as stated in Sect. II may be restated as follows: we wish to determine the eigenvalues and eigenfunctions of the equation

$$\nabla^2 \phi_n + \Lambda_n \phi_n = 0 \text{ in } \overline{D - D_1} \quad (10)$$

subject to the boundary condition

$$\phi_n|_{\sigma_1} = \phi_n|_{\sigma_2} = \dots \phi_n|_{\sigma_k} = \phi_n|_{\Sigma} = 0 \quad . \quad (11)$$

The boundary condition in eq. (11) could have been replaced equally well by either of the boundary conditions in eq. (3b) or eq. (3c). Furthermore, as will later be pointed out, the treatment to follow could apply equally well

to problems in which the boundary condition in eq. (11) would take on the form

$$\phi_n \Big|_{\sigma_1} = \phi_n \Big|_{\sigma_2} = \cdots = \phi_n \Big|_{\sigma_k} = \vec{\nu} \cdot \vec{\nabla} \phi_n \Big|_{\Sigma} = 0 \quad , \quad (11a)$$

or, possibly, of the form

$$\phi_n + K \vec{\nu} \cdot \vec{\nabla} \phi_n \Big|_{\sigma_1} = \phi_n + K \vec{\nu} \cdot \vec{\nabla} \phi_n \Big|_{\sigma_2} = \cdots = \phi_n + K \vec{\nu} \cdot \vec{\nabla} \phi_n \Big|_{\sigma_k} = \phi_n \Big|_{\Sigma} = 0 \quad . \quad (11b)$$

IV. CONSIDERATION OF THE STEADY-STATE WAVE EQUATION AND STATEMENT OF THEOREM I

There exists a function $S(\text{sink})$ which has the following properties:

$$(i) \quad \nabla^2 \phi + \Lambda \phi = S \text{ in } \bar{D}_1 \quad (12)$$

$$(ii) \quad \nabla^2 \phi + \Lambda \phi = 0 \text{ in } \overline{D-D_1} \quad (13)$$

such that it makes ϕ satisfy the boundary condition

$$(iii) \quad \phi \Big|_{\sigma_1} = \phi \Big|_{\sigma_2} = \cdots = \phi \Big|_{\sigma_k} = \phi \Big|_{\Sigma} = 0$$

and such that

$$(iv) \quad 0 < |S| \leq K \quad (14)$$

where K is a finite constant, and ϕ is analytic in $\overline{D-D_1}$.

The satisfaction of eq. (13) along with the boundary condition (iii) will then yield the eigenfunction solution ϕ_n and corresponding eigenvalue Λ_n .

V. PROOF OF THEOREM I BY A METHOD OF SUCCESSIVE APPROXIMATIONS

We assert, at this point, that the solution of eq. (13), subject to the satisfaction of boundary condition IV. (iii), may be expressed as the orthonormal expansion

$$\phi = \sum_n^{\infty} a_n f_n \quad . \quad (15)$$

The f_n 's satisfy the orthonormality condition

$$\int_{\overline{D}} f_m f_n = \delta_{mn} \quad (16)$$

and, in addition, satisfy eq. (4) and (5). The integration in eq. (16) and in the following equations will be understood to be Riemannian and over the domains indicated by the integral sign. Thus, all integrations pertain to bounded functions over the domains considered. The main argument of the theory to be presented here is that the a_n 's in eq. (16) may be determined by a series of successive approximations in such a way that ϕ will satisfy the boundary condition IV. (iii). The choice of the f_n 's by virtue of eq. (5) make ϕ satisfy the same boundary condition as eq. (5) with the properly constructed a_n 's, making ϕ satisfy the remaining boundary conditions in IV. (iii).

Denote by $S^{(k)}$ the k^{th} approximation of the sink function S . Define the first approximation of S so as to have the following properties:

$$S^{(1)} = f_n \text{ in } \overline{D}_1 \quad (17)$$

$$= 0 \text{ in } \overline{D} - \overline{D}_1 \quad (18)$$

where f_n satisfies eq. (4) and boundary condition (5). Furthermore, assume that $S^{(1)}$ is continuous in the interior of $\overline{D}_1$, which we designate as D_1 , and so admits of the Fourier expansion

$$S^{(1)} = \sum_n^{\infty} b_n f_n \quad , \quad (19)$$

where, by virtue of eq. (16), (17), and (18),

$$b_n = \int_{\overline{D}_1} S^{(1)} f_n \quad . \quad (19a)$$

This choice of S thus satisfies eq. (12) and (13) in the respective interiors of the domains $\overline{D}_1$ and $\overline{D} - \overline{D}_1$.

In order to make valid the interchange of summation and differentiation which is to follow, we assume that the solution ϕ , the first partial derivative of ϕ , and the second partial derivative of ϕ are continuous in the interior of $\overline{D-D_1}$, which we designate as $D-D_1$. Later, we will show that this restriction may be removed, for the series solution derived by this method is actually uniformly convergent and uniformly continuous. In addition, it will be shown that the series representations of the first and second-order partial derivatives of ϕ are uniformly convergent and uniformly continuous.

Denote the k^{th} approximation of the eigenvalue Λ by $\Lambda^{(k)}$ and the k^{th} approximation of the corresponding eigenfunction by $\phi^{(k)}$. We now find a first approximation $\phi^{(1)}$ of ϕ as follows: substitute eq. (15) and (19) into eq. (12) to get, after interchanging summation and differentiation,

$$\sum_n^{\infty} a_n (\Lambda - \lambda_n) f_n = \sum_n^{\infty} b_n f_n \quad . \quad (20)$$

Here ϕ is defined over $\overline{D}$ so that by application of the orthonormality property of the f_n 's

$$a_n = \frac{b_n f_n}{\Lambda - \lambda_n} \quad (21)$$

and, therefore, that

$$\phi^{(1)} = \sum_n^{\infty} \frac{b_n f_n}{\Lambda - \lambda_n} \quad . \quad (22)$$

However, we seek a $\phi^{(1)}$ such that boundary condition IV. (iii) holds as well. In place of the boundary condition IV. (iii), we impose the boundary condition

$$\int_{\Sigma_1} \phi^{(1)} = \left(\int_{\sigma_1} + \int_{\sigma_2} + \dots + \int_{\sigma_k} \right) \phi^{(1)} = 0 \quad . \quad (23)$$

Because of the choice of the f_n 's, we have that

$$\int_{\Sigma} \phi = 0 \quad .$$

If the series expression for $\phi^{(1)}$ in eq. (22) were a "pure" Fourier series, one could interchange integration and summation because the integration of the eigenfunction would introduce the additional factor $1/n$ in b_n .⁽⁹⁾ Since we have introduced another additional factor, $\Lambda - \lambda_n$, we are allowed to interchange integrations and summations a fortiori.

We now obtain the first approximation $\Lambda^{(1)}$ of the eigenvalue Λ by integrating $\phi^{(1)}$ in eq. (22) over the surface Γ and, after interchanging integration and summation, we have that

$$\sum_n \frac{b_n C_n}{\Lambda^{(1)} - \lambda_n} = 0 \quad , \quad (25)$$

where

$$C_n = \int_{\Gamma} f_n \quad . \quad (26)$$

Note that $\Lambda^{(1)}$ could never equal λ_n . If $\Lambda^{(1)}$ equaled λ_n , then this would imply that one of the eigenfunctions ϕ_n equals one of the eigenfunctions f_n . This would only be possible if the subdomains $\bar{D}_i$ vanished, but these have been assumed to be non-null sets.

Previously, we have mentioned that the f_n 's are bounded so that, in particular,

$$|C_n| = \left| \int_{\Gamma} f_n \right| \leq \frac{M}{g(n)} \quad ,$$

where $g(n)$ is a linear function of n which varies with the eigenfunction integrated. Therefore, the series in eq. (25) would converge absolutely, since

$$|b_n| = \left| \int_{\bar{D}_1} S^{(1)} f_n \right| \leq \frac{KM}{g(n)}$$

and since

$$\begin{aligned} \left| \sum_n \frac{b_n C_n}{\Lambda^{(1)} - \lambda_n} \right| &\leq \sum_n \left| \frac{b_n C_n}{\Lambda^{(1)} - \lambda_n} \right| \leq KM^2 \sum_n \frac{1}{|\Lambda^{(1)} - \lambda_n| g^2(n)} \\ &= KM^2 \sum_n \frac{1}{|\lambda_k + \epsilon - \lambda_n| g^2(n)} = KM^2 \sum_n \frac{1}{|k^2 - n^2 + \epsilon| n^2} \\ &= \frac{KM^2}{\epsilon k^2} + KM^2 \sum_{n \neq k} \frac{1}{|k^2 - n^2 + \epsilon| n^2} < \frac{KM^2}{\epsilon k^2} + KM^2 \sum_{n \neq k} \frac{1}{n^p} \quad , \end{aligned}$$

where $\lambda_k + \epsilon$ represents the first approximation of the k^{th} perturbed eigenvalue $\Lambda_k^{(1)}$, λ_n is known to be proportional to n^2 , and $1 < p \leq 2 + \frac{\log[(k^2 + \epsilon) + n^2]}{\log n}$ with $k \neq n \neq 1$. Further, the series

$$\sum_n \frac{b_n C_n}{\Lambda^{(1)} - \lambda_n}$$

would have one term, namely, the term in which $k = n$ dominates by far the entire series; that is, a term of the form

$$\left| \frac{b_k C_k}{\epsilon k^2} \right| = \frac{KM^2}{\epsilon k^2}$$

would always in absolute value be the largest term in the series. Consequently, the series in eq. (25) would have to converge to a positive number, a negative number, or zero depending on how much this term is dominating the series (in other words, depending on the choice of ϵ or, what is the same thing, the choice of $\Lambda^{(1)}$).

The first approximation of the eigenfunction is thus given by

$$\phi^{(1)} = \sum_n \frac{b_n f_n}{\Lambda^{(1)} - \lambda_n} \quad (27)$$

The series in eq. (27) also would have one dominating term regardless of the value of $\vec{X}$ [$\vec{X}$ denotes the n -tuple $(x_1, x_2, \dots, x_n)$] $\in \Gamma$, for the f_n are bounded. Therefore, $\phi^{(1)}$ would not oscillate along the surface Γ , so that eq. (25) would not hold because of negative and positive terms cancelling each other. The integral boundary condition therefore assures that $\phi^{(1)}$ be zero along Γ as well as that the integral of $\phi^{(1)}$ be zero.

The choice of $S^{(1)}$ expressed in eq. (17) and (18) does not necessarily guarantee that $\phi^{(1)}$, which now satisfies the boundary conditions imposed, still satisfies eq. (13). The argument here is that S may be a more complicated function than a simple "step" type of inhomogeneity. However, the first approximation of S , namely, $S^{(1)}$, admits of a Fourier expansion, whereas S itself could not be obtained by a simple Fourier expansion if it were a function of the form

$$S = S(\vec{x})$$

instead of the form

$$S = F(x_1) F(x_2) \cdots F(x_n) \quad .$$

The ultimate form of S is not known initially and no assumption of its form is made initially. Assuming, instead, the form $S^{(1)}$ and getting $\phi^{(1)}$, we check to see if $S^{(1)}$ still allows $\phi^{(1)}$ to satisfy eq. (13). This step is accomplished by substituting $\phi^{(1)}$ into eq. (13) to obtain, after interchanging summation and differentiation, the equation

$$S^{(2)} = \sum_n \frac{b_n(\Lambda - \lambda_n)f_n}{\Lambda^{(1)} - \lambda_n} = 0 \text{ in } \overline{D-D_1} \quad . \quad (28)$$

From eq. (28), we may, after interchanging summation and integration, obtain for the determination of $\Lambda^{(2)}$, the condition

$$\sum_n \frac{b_n(\Lambda^{(2)} - \lambda_n)d_n}{\Lambda^{(1)} - \lambda_n} = 0 \quad , \quad (29)$$

where

$$d_n = \int_{\overline{D-D_1}} f_n \quad . \quad (30)$$

Knowing $\Lambda^{(2)}$, we have $S^{(2)}$ from eq. (28). Repeating the above procedure with $S^{(2)}$, we find that $\Lambda^{(3)}$ must satisfy

$$\sum_n \frac{b_n(\Lambda^{(2)} - \lambda_n)C_n}{(\Lambda^{(3)} - \lambda_n)(\Lambda^{(1)} - \lambda_n)} = 0 \quad , \quad (31)$$

and immediately we have that

$$\phi^{(2)} = \sum_n \frac{b_n(\Lambda^{(2)} - \lambda_n)f_n}{(\Lambda^{(3)} - \lambda_n)(\Lambda^{(1)} - \lambda_n)} \quad . \quad (32)$$

Proceeding in this manner, we obtain the following successive approximations for S , Λ , and ϕ :

$$\sum_n b_n f_n = S^{(1)} \quad (33a)$$

$$\sum_n \frac{b_n C_n}{\Lambda^{(1)} - \lambda_n} = 0 \quad (33b)$$

$$\sum_n^{\infty} \frac{b_n f_n}{\Lambda^{(1)} - \lambda_n} = \phi^{(1)} \quad (33c)$$

$$\sum_n^{\infty} \frac{b_n d_n (\Lambda^{(2)} - \lambda_n)}{\Lambda^{(1)} - \lambda_n} = 0 \quad (33d)$$

$$\sum_n^{\infty} \frac{b_n (\Lambda^{(2)} - \lambda_n) f_n}{\Lambda^{(1)} - \lambda_n} = S^{(2)} \quad (33e)$$

$$\sum_n^{\infty} \frac{b_n C_n (\Lambda^{(2)} - \lambda_n)}{(\Lambda^{(1)} - \lambda_n)(\Lambda^{(3)} - \lambda_n)} = 0 \quad (33f)$$

$$\sum_n^{\infty} \frac{b_n (\Lambda^{(2)} - \lambda_n) f_n}{(\Lambda^{(1)} - \lambda_n)(\Lambda^{(3)} - \lambda_n)} = \phi^{(2)} \quad (33g)$$

$$\sum_n^{\infty} \frac{b_n d_n (\Lambda^{(2)} - \lambda_n)(\Lambda^{(4)} - \lambda_n) \dots (\Lambda^{(2k+2)} - \lambda_n)}{(\Lambda^{(1)} - \lambda_n)(\Lambda^{(3)} - \lambda_n) \dots (\Lambda^{(2k+1)} - \lambda_n)} = 0 \quad (33h)$$

$$\sum_n^{\infty} \frac{b_n (\Lambda^{(2)} - \lambda_n)(\Lambda^{(4)} - \lambda_n) \dots (\Lambda^{(2k+2)} - \lambda_n) f_n}{(\Lambda^{(1)} - \lambda_n)(\Lambda^{(3)} - \lambda_n) \dots (\Lambda^{(2k+1)} - \lambda_n)} = S^{(k)} \quad (33i)$$

$$\sum_n^{\infty} \frac{b_n C_n (\Lambda^{(2)} - \lambda_n)(\Lambda^{(4)} - \lambda_n) \dots (\Lambda^{(2k+2)} - \lambda_n)}{(\Lambda^{(1)} - \lambda_n)(\Lambda^{(3)} - \lambda_n) \dots (\Lambda^{(2k+1)} - \lambda_n)(\Lambda^{(2k+3)} - \lambda_n)} = 0 \quad (33j)$$

$$\sum_n^{\infty} \frac{b_n (\Lambda^{(2)} - \lambda_n)(\Lambda^{(4)} - \lambda_n) \dots (\Lambda^{(2k+2)} - \lambda_n) f_n}{(\Lambda^{(1)} - \lambda_n)(\Lambda^{(3)} - \lambda_n) \dots (\Lambda^{(2k+1)} - \lambda_n)(\Lambda^{(2k+3)} - \lambda_n)} = \phi^{(k)} \quad (33k)$$

We have demonstrated the existence of the k approximations of the sink functions S , and by construction each approximation yielded the k^{th} approximation of the eigenfunction and eigenvalue.

By this method of successive approximations, we have also constructed the eigenfunction so as to be zero on the boundary surfaces, provided the resulting series actually converges uniformly in $D-D_1$ and, in particular, uniformly to zero on the boundary surfaces. In proving that

$$\lim_{k \rightarrow \infty} \phi^{(k)} = \phi \quad (34a)$$

$$\lim_{k \rightarrow \infty} \Lambda^{(k)} = \Lambda \quad (34b)$$

$$\lim_{k \rightarrow \infty} S^{(k)} = S \quad (34c)$$

and further that

$$\lim_{k \rightarrow \infty} \left[\nabla^2 \phi^{(k)} + \Lambda^{(k)} \phi^{(k)} \right] = \nabla^2 \phi + \Lambda \phi = \lim_{k \rightarrow \infty} S^{(k)} = S \quad , \quad (34d)$$

we also prove uniform convergence of the various series as is needed.

First, we quote the following theorem concerning the eigenvalue problem being treated here.

Theorem 1.⁽¹⁰⁾ Under the boundary condition $\phi|_{\Gamma} = 0$, the n^{th} eigenvalue for a domain G never exceeds the n^{th} eigenvalue for a subdomain G' .*

In order to prove eq. (34a), we consider ϕ to be the first eigenfunction designated as $\phi_1 \equiv \phi$ with the corresponding first eigenvalue $\Lambda_1 \equiv \Lambda$. The argument to follow applies to all other eigenfunctions as well.

From Theorem 1. we have the inequality

$$\lambda_1 < \Lambda < \lambda_2 \quad , \quad (35)$$

with $G = \overline{D}$ and $G' = \overline{D-D_1}$. Let ϵ_k be the k^{th} error in the k^{th} approximation of Λ . Further, write the k^{th} approximation of Λ in the form

$$\Lambda^{(k)} = \lambda_1 + K + \epsilon_k \quad . \quad (36)$$

First, we will show that eq. (34a) holds. From eq. (33k) we must show that

$$\lim_{k \rightarrow \infty} \sum_n \frac{b_n}{\Lambda^{(k)} - \lambda_n} \prod_{S=1}^k \left\{ \frac{\Lambda^{(2S)} - \lambda_n}{\Lambda^{(2S+1)} - \lambda_n} \right\} f_n = \Phi \quad (37)$$

and that $\Phi = \phi$. We must, therefore, show that the product

$$\lim_{k \rightarrow \infty} \left\{ \frac{1}{\Lambda^{(k)} - \lambda_n} \prod_{S=1}^k \left(\frac{\Lambda^{(2S)} - \lambda_n}{\Lambda^{(2S+1)} - \lambda_n} \right) \right\} = \lim_{k \rightarrow \infty} P_k = P \quad (38)$$

*In fact, it is always smaller when we are dealing with a proper subdomain.

yields a result which makes the series in eq. (37) convergent. Substituting eq. (36) into eq. (38), we have to show that

$$\frac{1}{\lambda_1 - \lambda_n + K + \epsilon_1} \prod_{k=1}^{\infty} \left(\frac{\lambda_1 - \lambda_n + K + \epsilon_{2k}}{\lambda_1 - \lambda_n + K + \epsilon_{2k+1}} \right) = P \quad (39)$$

yields a convergent infinite product which is a function of the behavior of the errors ϵ_k .

Assume, first, that each error term becomes progressively larger, so that

$$\lim_{k \rightarrow \infty} \epsilon_k = \infty \quad (40)$$

The product in eq. (39) would then take on the indeterminate form

$$\frac{1}{\lambda_1 - \lambda_n + K + \epsilon_1} \prod_{k=1}^{\infty} \left(\frac{\lambda_1 - \lambda_n + K + \epsilon_{2k}}{\lambda_1 - \lambda_n + K + \epsilon_{2k+1}} \right) = \frac{\infty}{\infty} \quad (41)$$

The values of the indeterminate form in eq. (41) could take on the following values depending on how the errors behave, namely,

$$\begin{array}{ll} \text{Case (i)} & P = A, \\ \text{Case (ii)} & P = 0, \end{array}$$

and

$$\text{Case (iii)} \quad P = \infty,$$

where A is a constant, independent of n and greater than 0. Case (i) yields, then, that

$$\lim_{k \rightarrow \infty} \phi^{(k)} = A \sum_{n=1}^{\infty} b_n f_n = AS^{(1)} \quad (42)$$

where $S^{(1)}$ is again the sink function in eq. (19). Equation (42) means that, given $\epsilon > 0$,

$$| \phi^{(k)} - AS^{(k)} | < \epsilon \quad (43)$$

for all $\vec{x}$ [$\vec{x}$ denotes the n -tuple $(x_1, x_2, \dots, x_n)$] and for $k > N$. But for any $\vec{x} \in \overline{D-D_1}$, $S^{(1)} = 0$; therefore, eq. (43) yields that

$$| \phi^{(k)} | < \epsilon \quad (44)$$

Equation (44) in turn implies that for large k

$$\nabla^2 \phi^{(k)} + \Lambda^{(k)} \phi^{(k)} = S^{(k)} \rightarrow 0 \quad (45)$$

and, in particular, for $\vec{x} \in \Sigma_1$, which contradicts eq. (14).

Because $P = 0$ in case (ii), we have, immediately from eq. (43), the inequality

$$|\phi^{(k)}| < \epsilon \quad (46)$$

By the same argument as presented for case (i), we have the same contradiction of eq. (14) by case (ii).

For case (iii), we have, from eq. (42), the inequality

$$|\phi^{(k)}| > M, \quad (47)$$

where M is a constant which, for increasing k , becomes arbitrarily large. Thus, for large k we would have that

$$\nabla^2 \phi^{(k)} + \Lambda^{(k)} \phi^{(k)} = S^{(k)} \rightarrow \infty, \quad (48)$$

which again contradicts eq. (14).

If the ϵ_k 's did not change with all k 's, we would have the situations corresponding to case (i) or case (ii). If the ϵ_k 's increased without bound, we would have case (iii).

Therefore, the ϵ_k 's must decrease with increasing k . So we must have that

$$\lim_{k \rightarrow \infty} \epsilon_k = 0, \quad (49)$$

and eq. (49) implies that all the ratios in eq. (41) tend to one, so that

$$\prod_{k=1}^{\infty} \left(\frac{\lambda_1 - \lambda_n + K + \epsilon_{2k}}{\lambda_1 - \lambda_n + K + \epsilon_{2k+1}} \right) = B_n \quad (50)$$

Therefore, we have that

$$\lim_{k \rightarrow \infty} \phi^{(k)} = \sum_n \frac{b_n B_n f_n}{\Lambda^{(1)} - \lambda_n} = \Phi \quad (51)$$

Since the ϵ_k 's tend to zero with increasing k , we see from eq. (50) that

$$\lim_{n \rightarrow \infty} B_n \leq 1.$$

The series expression for ϕ in eq. (51) will now be shown to converge uniformly. Since $|S|$ is bounded,

$$|b_n| = \left| \int_{\overline{D}_1} S^{(1)} f_n \right| \leq \int_{\overline{D}_1} |S^{(1)} f_n| < K ||\overline{D}_1||, \quad (51a)$$

where K is a fixed number ≤ 1 independent of n and $||\overline{D}_1||$ is the magnitude of $\overline{D}_1$. Consequently,

$$\left| \frac{b_n B_n f_n}{\Lambda^{(1)} - \lambda_n} \right| < \frac{M B_n}{\Lambda^{(1)} - \lambda_n},$$

where $M = K ||\overline{D}_1||$. The series of constant terms $B_n / \Lambda^{(1)} - \lambda_n$ converges since λ_n is of the form n^2 ; hence, according to the Weierstrass M-test, the series in eq. (51) converges uniformly with respect to $\vec{x}$. Also, the terms of this series are continuous with respect to $\vec{x}$, so that Φ is uniformly continuous.

The terms of the series obtained by integrating eq. (51) over Γ satisfy the inequality

$$\left| \frac{b_n B_n F_n}{n(\Lambda^{(1)} - \lambda_n)} \right| \leq \frac{K M' B_n}{n |\Lambda^{(1)} - \lambda_n|},$$

where K is defined in eq. (36) and M' is defined in the equation

$$\left| \int_{\Gamma} f_n \right| \leq \frac{1}{n} \int_{\Gamma} |F_n| = \frac{N ||\Gamma'||}{n} = \frac{M'}{n}.$$

Here $||\Gamma'||$ is the magnitude of the surface Γ' over which the integration is carried out after a change of variables $\vec{x} \rightarrow \vec{x}' = \vec{x}/n$. As in the above argument, this series is uniformly convergent. Since Φ is of the same form as $\phi^{(1)}$, then by the same argument that $\phi^{(1)}$ is zero on Γ , also Φ is zero on Γ .

If the series in eq. (51) is differentiated once and then again, the following inequalities are derived:

$$\left| \frac{n b_n B_n G_n}{\Lambda^{(1)} - \lambda_n} \right| \leq \frac{A' n B_n}{\Lambda^{(1)} - \lambda_n} \quad (51b)$$

and

$$\left| \frac{n^2 b_n B_n f_n}{\Lambda^{(1)} - \lambda_n} \right| \leq \frac{M n^2 B_n}{\Lambda^{(1)} - \lambda_n}, \quad (51c)$$

where

$$nA' = |\vec{\nabla} f_n| \leq n |G_n b_n|$$

and M is as previously defined. Since A' and M are constants, we test the terms involving n in eq. (51b) and (51c) for uniform convergence by the Cauchy Ratio Test and find that

$$A' \lim_{n \rightarrow \infty} \left| \frac{(n+1) B_{n+1} (\Lambda^{(1)} - \lambda_n)}{n B_n (\Lambda^{(1)} - \lambda_{n+1})} \right| = A' < 1 \quad (52)$$

and that

$$M \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2 B_{n+1} (\Lambda^{(1)} - \lambda_n)}{n^2 B_n (\Lambda^{(1)} - \lambda_{n+1})} \right| = M = K ||\vec{D}_1||, \quad (53)$$

respectively. Since $||\vec{D}_1||$ is less than one, then for eigenfunctions having a bound $K < 1$, the right-hand side of eq. (53) is less than one, so that once and twice-differentiated series are uniformly convergent and may be differentiated term by term.

Next, we prove that $\Lambda^{(k)}$ converges, i.e., that

$$\lim_{k \rightarrow \infty} \Lambda^{(k)} = \bar{\Lambda}. \quad (54)$$

Now, we know that

$$\Lambda^{(k)} = \lambda_1 + K + \epsilon_k \quad (55)$$

and, therefore, that

$$\Lambda^{(k+1)} - \Lambda^{(k)} = \epsilon_{k+1} - \epsilon_k. \quad (56)$$

From eq. (53), we have

$$|\Lambda^{(k+1)} - \Lambda^{(k)}| = |\epsilon_{k+1} - \epsilon_k| \quad (57)$$

which, for $\epsilon > 0$ and large k , gives

$$|\Lambda^{(k+1)} - \Lambda^{(k)}| < \epsilon. \quad (58)$$

Since the ϵ_k 's $\rightarrow 0$ with increasing k , we have as in the above arguments that, in addition,

$$\lim_{k \rightarrow \infty} S(k) = \bar{S} \quad (59)$$

Proceeding from the above arguments, we have then that

$$\nabla^2 \Phi + \bar{\Lambda} \Phi = \bar{S} \text{ in } \bar{D}_1 \quad (60)$$

$$= 0 \text{ in } \overline{D - D_1} \quad (61)$$

and that

$$\Phi|_{\Gamma} = 0. \quad (62)$$

Therefore, by the Cauchy-Kowalewski theorem, we have

$$\begin{aligned} \phi &= \Phi \\ \Lambda &= \bar{\Lambda} \end{aligned} \quad (63)$$

Q.E.D.

VI. PERTURBATION FORMULAS FOR THE EIGENVALUES AND EIGENFUNCTIONS

From eq. (33f) and (33k), one may easily obtain a perturbation-like formula for the eigenvalues. In particular, we may rearrange eq. (33d) to obtain an expression for the second-order approximation:

$$\Lambda^{(2)} = \sum_n \frac{b_n d_n \lambda_n}{\Lambda^{(1)} - \lambda_n} \bigg/ \sum_n \frac{b_n d_n}{\Lambda^{(1)} - \lambda_n} \quad (64)$$

In practice, if one, for example, were looking for the n^{th} eigenvalue of the sequence $\{\Lambda_n\}$, he would start at $\Lambda_n^{(1)}$, given by

$$\lambda_n < \Lambda_n^{(1)} < \lambda_{n+1} \quad (65)$$

and increment $\Lambda_n^{(1)}$ for $n=1$ until eq. (64) was satisfied within some desired error.

In using eq. (33f) to find the third approximation $\Lambda^{(3)}$, and knowing the values of $\Lambda^{(2)}$ and $\Lambda^{(1)}$, we replaced d_n by

$$\frac{d_n (\Lambda^{(2)} - \lambda_n)}{(\Lambda^{(1)} - \lambda_n)(\Lambda^{(3)} - \lambda_n)} \quad (66)$$

and proceed as in the above manner to obtain

$$\Lambda^{(4)} = \sum_n \frac{b_n d_n (\Lambda^{(2)} - \lambda_n) \lambda_n}{(\Lambda^{(1)} - \lambda_n)(\Lambda^{(3)} - \lambda_n)} \bigg/ \sum_n \frac{b_n d_n (\Lambda^{(2)} - \lambda_n)}{(\Lambda^{(1)} - \lambda_n)(\Lambda^{(3)} - \lambda_n)} \quad ; \quad (67)$$

that is, we increment

$$\Lambda^{(3)} = \lambda_1 + K + \epsilon_3 \quad (68)$$

until $\Lambda^{(4)}$ times the denominator of eq. (67) equals the numerator.

Continuing in this manner, we find that the even-order approximation of the eigenvalue Λ may be found from the equation

$$\Lambda^{(2k+2)} = \sum_n b_n d_n \frac{\prod_{s=1}^k (\Lambda^{(2s)} - \lambda_n) \lambda_n}{\prod_{s=0}^k (\Lambda^{(2s+1)} - \lambda_n)} \bigg/ \sum_n b_n d_n \frac{\prod_{s=1}^k (\Lambda^{(2s)} - \lambda_n)}{\prod_{s=0}^k (\Lambda^{(2s+1)} - \lambda_n)}$$

with the values of all the even-order approximations being found by use of the equations for odd-order approximations.

If one expands eq. (67), one gets a perturbation-like formula of the form

$$\Lambda^{(2k+2)} = \frac{b_n D_n^k}{\sigma^k} \left[\lambda_n + \sum_{j \neq n}^{\infty} \frac{b_j D_j^k \lambda_j}{b_n D_n^k} \right], \quad (70)$$

where

$$D_n^k = \frac{d_n \prod_{s=1}^k (\Lambda^{(2s)} - \lambda_n) \lambda_n}{\prod_{s=0}^k (\Lambda^{(2s+1)} - \lambda_n)}$$

and

$$\sigma^k = \frac{\sum_n^{\infty} b_n d_n \prod_{s=1}^k (\Lambda^{(2s)} - \lambda_n)}{\prod_{s=0}^k (\Lambda^{(2s+1)} - \lambda_n)}.$$

In a similar manner, a perturbation-like formula

$$\phi_n^k = a_n E_n^k \left[f_n + \sum_{j \neq n}^{\infty} \frac{a_j E_j^k f_j}{a_n E_n^k} \right], \quad (71)$$

where

$$E_n^k = \prod_{s=1}^k (\Lambda^{(2s)} - \lambda_n) / \prod_{s=0}^k (\Lambda^{(2s+1)} - \lambda_n)$$

may be derived for eigenfunctions.

VII. DETERMINATION OF THE FLUX AND BUCKLING FOR A HOMOGENEOUS REACTOR WITH PARTIALLY INSERTED CONTROL RODS

The method developed will now be applied to determine the flux and buckling for a finite, cylindrical, bare, homogeneous thermal reactor with partially inserted control rods. In the approximation given by diffusion theory,⁽¹¹⁾ we wish to solve eq. (1) for the first eigenfunction subject to homogeneous Dirichlet boundary conditions on the extrapolated boundaries of the reactor and on the effective boundaries of the control rods.⁽¹²⁾ We also wish to determine the first eigenvalue. The first eigenfunction and corresponding eigenvalue are known as the flux and buckling, respectively, in reactor parlance.

Figure 1 shows how the r - and z - axes were chosen, where H is the extrapolated height, R the extrapolated radius of the reactor, h_i is the effective length of insertion of the i^{th} control rod, and r_i its effective radius. Only two control rods are shown in Fig. 1 for the sake of clarity since the theory applies equally well for many rods. We denote by $\bar{D}$ the

Fig. 1 Cylindrical Reactor with
Partially Inserted Control
Rods

extrapolated volume of the reactor, and by $\bar{\Sigma}$ its surfaces. We denote by $\bar{D}_i$ the effective volume of the i^{th} control rod, and by σ_i its surface. With these definitions we wish to solve eq. (12) and (13) subject to boundary conditions IV (iii).

Equations (12) and (13) become, for the geometry under consideration,

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} + \Lambda \phi = S \text{ in } \bar{D}_1 \quad (72)$$

$$= 0 \text{ in } \bar{D} - \bar{D}_1 \quad (73)$$

The eigenfunctions f_n are taken to be the unnormalized eigenfunctions⁽¹³⁾

$$f_{mn\ell} = J_m\left(\frac{\beta_{mn}r}{R}\right) \left(a_{mn} \cos m\theta + \bar{a}_{mn} \sin m\theta\right) \sin \frac{\ell\pi z}{H} \quad , \quad (73a)$$

and the eigenvalues λ_n become the eigenvalues

$$\lambda_{mn\ell} = \left(\frac{\beta_{mn}}{R}\right)^2 + \left(\frac{\ell\pi}{H}\right)^2 \quad ,$$

where the β_{mn} are the n roots of the Bessel function of the first kind of order m .

In the reactor without any rods, the first eigenfunction (which is ≥ 0 in D) is then of the form

$$f_{011} = J_0\left(\frac{\beta_{01}r}{R}\right) \sin \frac{\pi z}{H} \quad .$$

Thus, we define the first approximation of the sink S_i to be

$$S_i^{(1)}(r, z, \theta) = J_0\left(\frac{\beta_{01}r}{R}\right) \sin \frac{\pi z}{H} \text{ in } \bar{D}_i \quad (74)$$

$$= 0 \text{ in } \overline{D - D_i} \quad (i = 1, 2, \dots, k) \quad (75)$$

Figure 2 is a view of the reactor in the plane $z = 0$ and shows how

Fig. 2

Control Rod Positions
in Plane $z=0$

the various parameters were chosen to describe the positions of the i^{th} and $i^{\text{th}} + 1$ control rods. The effective control rod heights are understood to be h_i and h_{i+1} , respectively. Figure 2 also shows how the variables r and θ are changed to ρ and ψ when the origin is moved from O to O' , the center of the i^{th} rod. This change of variables and the other angles shown are to be used in conjunction with the addition theorems for Bessel functions, which in turn are needed to carry out the various integrations. The general form for the addition theorem is⁽¹⁴⁾

$$J_m(\lambda r) e^{im\theta} = \sum_{k=-\infty}^{\infty} J_{m+k}(\lambda d_i) J_k(\lambda \rho) e^{ik\psi}, \quad (75a)$$

where λ is a constant.

Since more than one control rod is present, the interaction effects of these control rods must be considered. An arbitrary reference radius may be taken as one side of the various angles $\alpha_1, \alpha_2, \dots, \alpha_i, \alpha_{i+1}$, but once chosen must remain fixed. Thus eq. (75a) now takes on the form

$$J_m(\lambda r) e^{im(\theta - \alpha_i)} = \sum_{k=-\infty}^{\infty} J_{m+k}(\lambda d_i) J_k(\lambda \rho) e^{ik\psi}. \quad (76)$$

Multiplying both sides of eq. (76) by $e^{im\alpha_i}$ and taking the real and imaginary parts of both sides of eq. (76), we have

$$J_m(\lambda r) \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} = \sum_{k=-\infty}^{\infty} J_{m+k}(\lambda d_i) J_k(\lambda \rho) \begin{Bmatrix} \cos(k\psi + m\alpha_i) \\ \sin(k\psi + m\alpha_i) \end{Bmatrix}. \quad (77)$$

If r varies very little over D_i , we could replace r by d_i in eq. (74) to get for the i^{th} sink function:

$$\begin{aligned} S_i(z, \theta) &= J_0\left(\frac{\beta_{0i} d_i}{R}\right) \sin \frac{\pi z}{H} \text{ in } \bar{D}_i \\ &= 0 \text{ in } \overline{D - D_i} \quad (i = 1, 2, \dots, k) \end{aligned} \quad (78)$$

However, in what follows, we will not make the assumption that $\bar{D}_i$ is very small relative to $\bar{D}$.

The sink function S in eq. (68) is actually the sum of all the sink functions S_i , since eq. (68) is to be solved in the domain $\overline{D - D_1} = \bar{D} - \bigcup_{i=1}^k D_i$.

We will thus consider the sink S_i first and sum the effects of the sinks from $i = 1$ to k to get the total sink function, S . In so doing, the interactions of the rods will also appear implicitly because of the angles α_i appearing in the function S .

Using the functions $f_{mn\ell}$, we form the eigenfunction expansion

$$S_i^{(1)} = \sum_{m,n,\ell} \left\{ \begin{matrix} a_{mn} \\ \bar{a}_{mn} \end{matrix} \right\} b_{\ell} J_m \left(\frac{\beta_{mn} r}{R} \right) \sin \frac{\ell \pi z}{H} \left\{ \begin{matrix} \cos m\theta \\ \sin m\theta \end{matrix} \right\}, \quad (80)$$

where $\left\{ \begin{matrix} \\ \end{matrix} \right\}$ denote the linear combination expressed in eq. (73a). We now multiply eq. (80) by the linear combination

$$\left\{ \begin{matrix} a_{m'n'} \\ \bar{a}_{m'n'} \end{matrix} \right\} J_{m'} \left(\frac{\beta_{m'n'} r}{R} \right) \sin \frac{\ell' \pi z}{H} \left\{ \begin{matrix} \cos m'\theta \\ \sin m'\theta \end{matrix} \right\}$$

and integrate over D . By use of the orthogonality properties,

$$\int_0^R r J_m \left(\frac{\beta_{mn} r}{R} \right) J_{m'} \left(\frac{\beta_{m'n'} r}{R} \right) dr = \frac{R^2}{2} J_{m+1}^2(\beta_{mn}) \quad ; \quad n = n'$$

$$= 0 \quad ; \quad n \neq n'$$

$$\int_0^H \sin \frac{\ell \pi z}{H} \sin \frac{\ell' \pi z}{H} dz = H/2 \quad ; \quad \ell = \ell'$$

$$= 0 \quad ; \quad \ell \neq \ell'$$

$$\int_0^{2\pi} \cos m\theta \cos m'\theta d\theta = 2\pi \quad ; \quad m = m' = 0$$

$$= \pi \quad ; \quad m = m' \neq 0$$

$$= 0 \quad m \neq m'$$

and

$$\int_0^{2\pi} \sin m\theta \sin m'\theta d\theta = \pi \quad ; \quad m = m'$$

$$= 0 \quad ; \quad \text{otherwise.}$$

We have that

$$\left\{ \begin{matrix} a_{mn} \\ \bar{a}_{mn} \end{matrix} \right\} b_\ell = \frac{4}{\pi R^2 H J_{m+1}^2(\beta_{mn})} \int_0^R \int_0^H \int_0^{2\pi} S_i(r, z, \theta) J_m\left(\frac{\beta_{mn} r}{R}\right) \left\{ \begin{matrix} \cos m\theta \\ \sin m\theta \end{matrix} \right\} \sin \frac{\ell \pi z}{H} r dr dz d\theta \quad (81)$$

From eq. (74) and (75), $S_i \neq 0$ in D_i and $S_i = 0$ in $\overline{D-D_i}$ and by use of eq. (77), we have eq. (81) in the form

$$\left\{ \begin{matrix} a_{mn} \\ \bar{a}_{mn} \end{matrix} \right\} b_\ell = \frac{4}{\pi R^2 H J_{m+1}^2(\beta_{mn})} \int_0^{r_1} \int_0^{h_i} \int_0^{2\pi} \sum_{k, k' = -\infty}^{\infty} J_{m+k}\left(\frac{\beta_{mn} d_i}{R}\right) \times J_k\left(\frac{\beta_{mn} \rho}{R}\right) J_{k'}\left(\frac{\beta_{mn} \rho}{R}\right) \times \left\{ \begin{matrix} \cos(k\psi + m\alpha_i) \\ \sin(k\psi + m\alpha_i) \end{matrix} \right\} \times \left\{ \begin{matrix} E_{k'} \cos k' \psi + \bar{E}_{k'} \sin k' \psi \\ E_{k'} \cos k' \psi + \bar{E}_{k'} \sin k' \psi \end{matrix} \right\} \sin \frac{\pi z}{H} \sin \frac{\ell \pi z}{H} \rho d\rho dz d\psi. \quad (82)$$

If we first carry out the integrations with respect to ψ , we have integrals of the form

$$\int_0^{2\pi} (E_{k'} \cos k' \psi + \bar{E}_{k'} \sin k' \psi) \left\{ \begin{matrix} \cos(k\psi + m\alpha_i) \\ \sin(k\psi + m\alpha_i) \end{matrix} \right\} d\psi = \begin{cases} 2\pi \left\{ \begin{matrix} \cos m\alpha_i \\ \sin m\alpha_i \end{matrix} \right\} & k' = k = 0 \\ \pi \left\{ \begin{matrix} \cos m\alpha_i \\ \sin m\alpha_i \end{matrix} \right\} & k' = k \\ 0 & k' \neq k \end{cases}$$

where $E_{k'}$, being an arbitrary constant, is taken equal to one.

The integration with respect to ρ yields integrals of the form⁽¹⁵⁾

$$\begin{aligned}
\int_0^{r_i} J_k \left(\frac{\beta_{01} \rho}{R} \right) J_k \left(\frac{\beta_{mn} \rho}{R} \right) \rho d\rho &= \frac{r_i R}{\beta_{01}^2 - \beta_{mn}^2} \left\{ \beta_{01} J_k \left(\frac{\beta_{mn} r_i}{R} \right) J_{k+1} \left(\frac{\beta_{01} r_i}{R} \right) \right. \\
&\quad \left. - \beta_{mn} J_k \left(\frac{\beta_{01} r_i}{R} \right) J_{k+1} \left(\frac{\beta_{mn} r_i}{R} \right) \right\} \beta_{01} \neq \beta_{mn} \\
&= \frac{r_i^2}{2} \left(\left\{ J'_k \left(\frac{\beta_{01} r_i}{R} \right) \right\}^2 + \left[1 - \frac{k^2}{\left(\frac{\beta_{01} r_i}{R} \right)^2} \right] J_k^2 \left(\frac{\beta_{01} r_i}{R} \right) \right) \beta_{01} = \beta_{mn}
\end{aligned}$$

by use of the relation⁽¹⁵⁾

$$\left[J'_\nu(kz) \right]^2 = \left[\frac{\nu J_\nu(kz)}{kz} - J_{\nu+1}(kz) \right]^2,$$

the latter integration is equal to

$$\frac{r_i^2}{2} \left[J_k^2 \left(\frac{\beta_{01} r_i}{R} \right) - \frac{2kR}{\beta_{0n} r_i} J_k \left(\frac{\beta_{01} r_i}{R} \right) J_{k+1} \left(\frac{\beta_{0n} r_i}{R} \right) + J_{k+1}^2 \left(\frac{\beta_{01} r_i}{R} \right) \right].$$

Substituting these equations and the equation

$$\frac{2}{H} \int_0^{h_i} \sin \frac{\pi z}{H} \sin \frac{\ell \pi z}{H} dz = b_\ell,$$

where

$$\begin{aligned}
b_\ell &= \frac{2}{\pi} \left(\frac{\pi h_i}{H} - \frac{1}{4} \sin \frac{2\pi h_i}{H} \right), \quad \ell = 1 \\
&= \frac{2}{\pi} \left(\frac{1}{1 - \ell^2} \right) \left(\ell \sin \frac{\pi h_i}{H} \cos \frac{\ell \pi h_i}{H} - \sin \frac{\ell \pi h_i}{H} \cos \frac{\pi h_i}{H} \right), \quad \ell > 1,
\end{aligned}$$

into eq. (82), we have that

$$\begin{aligned}
\left\{ \frac{a_{mn}}{\bar{a}_{mn}} \right\} b_\ell &= \frac{4r_i}{R J_{m+1}^2(\beta_{mn})} \left[J_0 \left(\frac{\beta_{01} d_i}{R} \right) J_m \left(\frac{\beta_{mn} d_i}{R} \right) \Gamma_0 \right. \\
&\quad \left. + \sum_{k=1}^{\infty} \left\{ J_k \left(\frac{\beta_{01} d_i}{R} \right) X \left[J_{m+k} \left(\frac{\beta_{mn} d_i}{R} \right) \Gamma_k + (-1)^{2k-m} \right. \right. \right. \\
&\quad \left. \left. J_{k-m} \left(\frac{\beta_{mn} d_i}{R} \right) \Gamma_{-k} \right] \right\} \left[\begin{matrix} \cos m\alpha_i \\ \sin m\alpha_i \end{matrix} \right] b_\ell \right], \quad (83)
\end{aligned}$$

where

$$\Gamma_{-k} = \frac{r_i}{2R} \left[J_k^2 \left(\frac{\beta_{mn} r_i}{R} \right) + (-1)^{2k-1} \left(\frac{2kR}{\beta_{mn} r_i} \right) J_k \left(\frac{\beta_{mn} r_i}{R} \right) J_{k-1} \left(\frac{\beta_{mn} r_i}{R} \right) + J_{k-1}^2 \left(\frac{\beta_{mn} r_i}{R} \right) \right]$$

and

$$\Gamma_k = \frac{r_i}{2R} \left[J_k^2 \left(\frac{\beta_{mn} r_i}{R} \right) - \frac{2kR}{\beta_{mn} r_i} J_k \left(\frac{\beta_{mn} r_i}{R} \right) J_{k+1} \left(\frac{\beta_{mn} r_i}{R} \right) + J_{k+1}^2 \left(\frac{\beta_{mn} r_i}{R} \right) \right]$$

The quantity Γ_{-k} is formed from Γ_k by use of the relation

$$J_{-k}(z) = (-1)^k J_k(z) \quad .$$

In addition,

$$\begin{aligned} \Gamma_0 &= \frac{r_i}{2R} \left[J_1^2 \left(\frac{\beta_{0n} r_i}{R} \right) + J_0^2 \left(\frac{\beta_{0n} r_i}{R} \right) \right] \quad , \quad \beta_{01} = \beta_{mn} \\ &= \frac{2}{\beta_{01}^2 - \beta_{mn}^2} \left[\beta_{01} J_0 \left(\frac{\beta_{mn} r_i}{R} \right) J_1 \left(\frac{\beta_{01} r_i}{R} \right) - \beta_{mn} J_0 \left(\frac{\beta_{01} r_i}{R} \right) J_1 \left(\frac{\beta_{mn} r_i}{R} \right) \right] \quad , \\ \beta_{01} &\neq \beta_{mn} \quad . \end{aligned}$$

The final form of the first approximation of the total sink function is, therefore,

$$S^{(1)} = \sum_{i=1}^k S_i^{(1)} = \sum_{i=1}^k C_i \left\{ \frac{a_{mn}}{\bar{a}_{mn}} \right\} b_\ell J_m \left(\frac{\beta_{mn} r}{R} \right) \left\{ \frac{\cos m\theta}{\sin m\theta} \right\} \sin \frac{\ell \pi z}{H} \quad , \quad (84)$$

where

$$C_i = 4r_i/R \quad .$$

As was stated previously, we are seeking the first eigenfunction and first eigenvalue, so in what follows it is understood that this is the case.

Knowing the first approximation, $S^{(1)}$, of the sink, we find, from eq. (33c), the first approximation of the eigenfunction:

$$\phi^{(1)} = \sum_{i=1}^k C_i \sum_{m,n,\ell} \frac{\left\{ \frac{a_{mn}}{\bar{a}_{mn}} \right\} b_\ell J_m \left(\frac{\beta_{mn} r}{R} \right) \left\{ \frac{\cos m\theta}{\sin m\theta} \right\}}{\Lambda^{(1)}_{mn\ell}} \sin \frac{\ell \pi z}{H} \quad , \quad (85)$$

with the first approximation $\Lambda^{(1)}$ of the eigenvalue Λ to be determined by the use of eq. (33b). Integrations of $\phi^{(1)}$ over the surfaces of the control rods require integrating eq. (77) over these surfaces. These integrations would be zero unless $k = 0$, as can be seen by considering first the integration with respect to ψ .

In order to get the interaction of these control rods, we let the $i^{\text{th}} + 1$ control rod now become the j^{th} control rod, with $i \neq j$ unless $i = 1$, that is, unless one rod is present only. Equation (77) for this integration then becomes effectively

$$J_m \left(\frac{\beta_{mn} r}{R} \right) e^{im\theta} = J_m \left(\frac{\beta_{mn} \rho}{R} \right) J_0 \left(\frac{\beta_{mn} \rho}{R} \right) e^{im\alpha_i} \quad (86)$$

Substituting eq. (86) in eq. (85), we then form the integrals

$$\begin{aligned} \sum_{j=1}^k \int_{\sigma_j} \phi_1 = & \sum_{i=1}^k C_i \sum_{m,n,\ell} \frac{\left\{ \begin{smallmatrix} a_{mn} \\ \bar{a}_{mn} \end{smallmatrix} \right\} b_\ell}{\Lambda^{(1)-\lambda}_{mn\ell}} \sum_{j=1}^k \left\{ J_m \left(\frac{\beta_{mn} d_j}{R} \right) \begin{Bmatrix} \cos m\alpha_j \\ \sin m\alpha_j \end{Bmatrix} \right. \\ & \times \left[\int_0^{r_j} J_0 \left(\frac{\beta_{mn} \rho}{R} \right) \rho d\rho \sin \frac{\ell \pi h_j}{H} \int_0^{2\pi} d\psi + r_j J_0 \left(\frac{\beta_{mn} r_j}{R} \right) \right. \\ & \left. \left. \times \int_0^{h_j} \sin \frac{\ell \pi z}{H} dz \int_0^{2\pi} d\psi \right] \right\} \quad (87) \end{aligned}$$

Carrying out the integrations by use of the relation

$$\int_0^{r_i} \rho J_0 \left(\frac{\beta_{mn} \rho}{R} \right) \rho d\rho = \frac{R r_i}{\beta_{mn}} J_1 \left(\frac{\beta_{mn} r_i}{R} \right) \quad ,$$

we have that the first approximation, $\Lambda^{(1)}$, must satisfy the equation

$$\begin{aligned} 2\pi \sum_{i=1}^k C_i \sum_{m,n,\ell} \frac{\left\{ \begin{smallmatrix} a_{mn} \\ \bar{a}_{mn} \end{smallmatrix} \right\} b_\ell}{\Lambda^{(1)-\lambda}_{mn\ell}} \sum_{j=1}^k \left\{ r_j J_m \left(\frac{\beta_{mn} d_j}{R} \right) \left[\frac{R J_1 \left(\frac{\beta_{mn} r_j}{R} \right) \sin \frac{\ell \pi h_j}{H}}{\beta_{mn}} \right. \right. \\ \left. \left. + \frac{H}{\ell \pi} J_0 \left(\frac{\beta_{mn} r_j}{R} \right) \left(1 - \cos \frac{\ell \pi h_j}{H} \right) \begin{Bmatrix} \cos m\alpha_j \\ \sin m\alpha_j \end{Bmatrix} \right] \right\} = 0 \quad (88) \end{aligned}$$

Equation (88) implicitly shows the complicated interaction of the control rods, for it must be remembered that $\left\{ \begin{matrix} a_{mn} \\ \bar{a}_{mn} \end{matrix} \right\}$ are also functions of the control rod positions.

The second-order approximation, $\Lambda^{(2)}$, of the eigenvalue Λ is obtained by use of eq. (33d). Using eq. (86) and the fact that the $i^{\text{th}} + 1$ rod is designated by j , we form the integrals

$$\begin{aligned} \frac{\int}{D-D_1} \sum_{i=1}^k S_i^{(k)} = \sum_{i=1}^k \left\{ \sum_{m,n,\ell}^{\infty} \frac{\left\{ \begin{matrix} a_{mn} \\ \bar{a}_{mn} \end{matrix} \right\} b_{\ell} \left(\Lambda^{(2)} - \lambda_{mn\ell} \right)}{\Lambda^{(1)} - \lambda_{mn\ell}} \right. \\ \times \left[\int_0^R \int_0^H \int_0^{2\pi} J_m \left(\frac{\beta_{mn} r}{R} \right) \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} \sin \frac{\ell\pi z}{H} r dr dz d\theta \right. \\ \left. \left. - \sum_{j=1}^k J_m \left(\frac{\beta_{mn} d_j}{R} \right) \int_0^{r_j} J_0 \left(\frac{\beta_{mn} \rho}{R} \right) \begin{Bmatrix} \cos m\alpha_j \\ \sin m\alpha_j \end{Bmatrix} \rho d\rho dz d\theta \right] \right\} = 0. \quad (89) \end{aligned}$$

Unless $m = 0$, the first integration in eq. (89) is zero. Carrying out the integrations in eq. (89), we have that the second-order approximation, $\Lambda^{(2)}$, must satisfy the equation

$$\begin{aligned} \sum_{i=1}^k \left\{ C_i \sum_{m=0,n,\ell}^{0,\infty,\infty} \frac{\left\{ \begin{matrix} a_{mn} \\ \bar{a}_{mn} \end{matrix} \right\} b_{\ell} \left(\Lambda^{(2)} - \lambda_{mn\ell} \right)}{\Lambda^{(1)} - \lambda_{mn\ell}} \left[\frac{2HR^2}{\ell\beta_{0n}} J_1(\beta_{0n}) (1 - \cos \ell\pi) \right] \right. \\ \left. - \sum_{m,n,\ell}^{\infty} \left[\frac{\left\{ \begin{matrix} a_{mn} \\ \bar{a}_{mn} \end{matrix} \right\} b_{\ell} \left(\Lambda^{(2)} - \lambda_{mn\ell} \right)}{\Lambda^{(1)} - \lambda_{mn\ell}} \times \sum_{j=1}^k \frac{2HRr_j}{\ell\beta_{mn}} J_m \left(\frac{\beta_{mn} d_j}{R} \right) \right. \right. \\ \left. \left. J_1 \left(\frac{\beta_{mn} r_j}{R} \right) \left(1 - \cos \frac{\ell\pi h_j}{H} \right) \begin{Bmatrix} \cos m\alpha_j \\ \sin m\alpha_j \end{Bmatrix} \right] \right\} = 0. \quad (90) \end{aligned}$$

Knowing $\Lambda^{(2)}$, we may find $S^{(2)}$ by use of eq. (33e). Using eq. (33f) and (84) through (88), we may find $\Lambda^{(3)}$. Using eq. (33g), we have the second-order approximation of ϕ :

$$\phi^{(2)} = \sum_{i=1}^k C_i \sum_{m,n,\ell}^{\infty} \frac{\left\{ \frac{a_{mn}}{\bar{a}_{mn}} \right\}^b \ell \left(\Lambda^{(2)-\lambda}_{mn\ell} \right)}{\left(\Lambda^{(1)-\lambda}_{mn\ell} \Lambda^{(3)-\lambda}_{mn\ell} \right)} J_m \left(\frac{\beta_{mn} r}{R} \right) \left\{ \begin{array}{l} \cos m\theta \\ \sin m\theta \end{array} \right\} \sin \frac{\ell \pi z}{H} . \quad (91)$$

Since this is an eigenvalue problem, the solution may be multiplied by an arbitrary constant. It is now quite clear how this procedure may be continued in this application of the theory to obtain eq. (33h) through (33k).

As previously mentioned in Sect. II, the theory developed may be used equally well when the homogeneous boundary conditions imposed may be of the form shown in eq. (11a). In this case, the roots β_{mn} would have to be such that

$$\left. \frac{\partial \left[J_m \left(\frac{\beta_{mn} r}{R} \right) \right]}{\partial r} \right|_{\Sigma_r} = 0$$

and the integers ℓ would have to be such that

$$\left. \frac{\partial \left(\sin \frac{\ell \pi z}{H} \right)}{\partial z} \right|_{\Sigma_z} = 0$$

where Σ_r and Σ_z are those portions of the surface which are normal to the components ν_r and ν_z of the surface normal $\bar{\nu}$. The remaining boundary conditions in eq. (11a) may be satisfied by proceeding as already shown. Similar arguments may be applied to the other boundary-value problems mentioned in Sect. II.

VIII. THE SPECIAL CASE OF A FULLY INSERTED CENTRAL ROD AND A RESULTING IDENTITY

The derivation of the first eigenfunction of eq. (1) for the case of a central rod fully inserted in a finite, bare, homogeneous thermal reactor, subject to homogeneous Dirichlet boundary conditions on the extrapolated boundaries of the reactor and the effective boundaries of the control rod may be found in reference 18. The solution of this problem is

$$\Theta(r, z) = A \left[J_0 \left(\frac{\beta r}{R} \right) - \frac{J_0(\beta)}{Y_0(\beta)} Y_0 \left(\frac{\beta r}{R} \right) \right] \sin \frac{\pi z}{H} \quad , \quad (92)$$

where $Y_0 \left(\frac{\beta r}{R} \right)$ is the zero-order Bessel function of the second kind, and A is an arbitrary normalization constant.

Since this geometry admits of a separate solution, the first eigenvalue, Λ_1 , is of the form

$$\Lambda_1 = (\beta/R)^2 + (\pi/H)^2 \quad , \quad (93)$$

where $(\beta/R)^2$ is the radial component and $(\pi/H)^2$ is the vertical component of the eigenvalue. For the case of the rod radius $r_1 \ll R$, the reactor radius, β , may be shown to be of the form

$$\beta = \frac{\beta_{01}}{R} + \frac{3.75}{\beta_{01} R \left[0.116 + \frac{\ell n R}{2.4 r_1} \right]} \quad , \quad (94)$$

where β_{01} is such that

$$J_0(\beta_{01}) = 0$$

If in the complete solution of the reactor equation for many partially inserted rods, we let $i = 1$ only, and form the limit

$$\lim_{\substack{h_1 \rightarrow H \\ d_1 \rightarrow 0}} \phi ,$$

we must get, because of the uniqueness of the solution, the same solution as in eq. (92). Carrying out the limiting process:

$$\lim_{\substack{h_1 \rightarrow H \\ d_1 \rightarrow 0}} \sum_{m,n,\ell} \left\{ \frac{a_{mn}}{\bar{a}_{mn}} \right\} b_\ell \frac{\prod_{s=1}^{\infty} \left(\Lambda^{(2s)} - \lambda_{mn\ell} \right)}{\prod_{s=0}^{\infty} \left(\Lambda^{(2s+1)} - \lambda_{mn\ell} \right)} J_m \left(\frac{\beta_{mn} r}{R} \right) \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} \sin \frac{\ell \pi z}{H}$$

we find after interchanging the sum and limit operations and applying this limiting process to eq. (83) that

$$\begin{aligned} & \frac{4r_1}{R} \sum_n \frac{\prod_{s=1}^{\infty} \left(\Lambda^{(2s)} - \lambda_{0n1} \right) J_1 \left(\frac{\beta_{0n} r_1}{R} \right)}{\prod_{s=0}^{\infty} \left(\Lambda^{(2s+1)} - \lambda_{0n1} \right) \beta_{01} J_1^2 \left(\frac{\beta_{0n}}{R} \right)} J_0 \left(\frac{\beta_{0n} r}{R} \right) \sin \frac{\pi z}{H} \\ &= A \left[J_0 \left(\frac{\beta r}{R} \right) - \frac{J_0(\beta)}{Y_0(\beta)} Y_0 \left(\frac{\beta r}{R} \right) \right] \sin \frac{\pi z}{H}, \end{aligned} \quad (95)$$

where

$$\lambda_{0n1} = (\beta_{0n}/R)^2 + (\pi/H)^2.$$

Since A is arbitrary, we may set $A = 4r_1/R$ and obtain from eq. (95) the identity

$$\begin{aligned} & \sum_n \frac{\prod_{s=1}^{\infty} \left(\Lambda^{(2s)} - \lambda_{0n1} \right) J_1 \left(\frac{\beta_{0n} r_1}{R} \right)}{\prod_{s=0}^{\infty} \left(\Lambda^{(2s+1)} - \lambda_{0n1} \right) \beta_{01} J_1^2 \left(\frac{\beta_{0n}}{R} \right)} J_0 \left(\frac{\beta_{0n} r}{R} \right) \\ &= J_0 \left(\frac{\beta r}{R} \right) - \frac{J_0(\beta)}{Y_0(\beta)} Y_0 \left(\frac{\beta r}{R} \right). \end{aligned} \quad (96)$$

IX. A METHOD FOR EVALUATING DEFINITE INTEGRALS

If we expand the right-hand side of eq. (96) in the Fourier-Bessel series:

$$\sum_n A_n J_0\left(\frac{\beta_{0n} r}{R}\right) = J_0\left(\frac{\beta r}{R}\right) - \frac{J_0(\beta)}{Y_0(\beta)} Y_0\left(\frac{\beta r}{R}\right) \quad , \quad (97)$$

we obtain by use of the formulas⁽¹⁵⁾

$$\int_{r_1}^R r J_\nu^2(kr) dr = \frac{r^2}{2} \left[\left\{ J'_\nu(kr) \right\}^2 + \left(\frac{1-\nu^2}{k^2 r^2} \right) \left\{ J_\nu(kr) \right\}^2 \right]_{r_1}^R$$

and

$$\begin{aligned} & \int_{r_1}^R Y_0(k_1 r) J_0(kr \sin \phi) r dr \\ &= \left[\frac{r}{k_1^2 - k^2 \sin^2 \phi} \left\{ k_1 J_0(kr \sin \phi) Y_1(k_1 r) - k \sin \phi Y_0(k_1 r) J_1(kr \sin \phi) \right\} \right]_{r_1}^R \end{aligned}$$

that

$$\begin{aligned} A_n &= \frac{R_2}{2} \left[J_1^2(\beta_{0n}) + J_0(\beta_{0n}) \right] - \frac{r_1^2}{2} \left[J_1^2\left(\frac{\beta_{0n} r_1}{R}\right) + J_0^2\left(\frac{\beta_{0n} r_1}{R}\right) \right] \\ &- \frac{J_0(\beta)}{Y_0(\beta)} \left[\frac{R^2}{\beta^2 - \beta_{0n}^2} \left\{ \beta J_0(\beta_{0n}) Y_1(\beta) - \beta_{0n} Y_0(\beta) J_1(\beta_{0n}) \right\} \right. \\ &\left. - \frac{r_1^2}{\beta^2 - \beta_{0n}^2} \left\{ \beta J_0\left(\frac{\beta_{0n} r_1}{R}\right) Y_1\left(\frac{\beta r_1}{R}\right) - \beta_{0n} Y_0\left(\frac{\beta r_1}{R}\right) J_1\left(\frac{\beta_{0n} r_1}{R}\right) \right\} \right] . \end{aligned}$$

Equating the left-hand side of eq. (96) to the left-hand side of eq. (97), we find that the Fourier-Bessel components A_n satisfy the equation

$$A_n = \frac{\prod_{s=1}^{\infty} \left(\Lambda^{(2s)}_{-\lambda_{0n1}} \right) J_1\left(\frac{\beta_{0n} r_1}{R}\right)}{\prod_{s=0}^{\infty} \left(\Lambda^{(2s+1)}_{-\lambda_{0n1}} \right) \beta_{01} J_1^2(\beta_{0n})} \quad . \quad (98)$$

In this way, we have derived the value of the sum of two integrals:

$$\int_{r_1}^R r J_0^2 \left(\frac{\beta_{0n}}{R} \right) dr + \frac{J_0(\beta)}{Y_0(\beta)} \int_{r_1}^R r Y_0(\beta r) J_0(\beta_{0n} r) dr$$

$$= \frac{\prod_{S=1}^{\infty} \left(\Lambda^{(2S)}_{-\lambda_{0n1}} \right) J_1 \left(\frac{\beta_{0n} r_1}{R} \right)}{\prod_{S=0}^{\infty} \left(\Lambda^{(2S+1)}_{-\lambda_{0n1}} \right) \beta_{01} J_1^2 \left(\frac{\beta_{0n}}{\beta_{01}} \right)}, \quad (100)$$

without carrying out the integrations themselves. This procedure may be helpful for evaluating definite integrals which do not admit of any analytic form. Note that the various orders of approximation, $\Lambda^{(k)}$, depend on R , since they were derived by integrating the eigenfunctions over Γ and $\overline{D-D_1}$.

X. DETERMINATION OF BUCKLING AND FLUX FOR A RECTANGULAR REACTOR WITH PARTIALLY INSERTED CROSS-SHAPED CONTROL RODS

We now will find the first eigenfunction and first eigenvalue for a finite, rectangular, bare, homogeneous thermal reactor with partially inserted cross-shaped control rods. The physical theory here is the same as presented in Sect. VII. Figure 3 shows how the x-, y-, and z-axes were chosen for this geometry. Only two rods are shown in Fig. 3, but, as before, the theory to be developed holds for k rods. Figure 4 shows how the parameters for the rod positions were chosen, as seen in the plane $z = 0$.

Fig. 3

Rectangular Parallelepiped
Reactor with Partially
Inserted Cross-shaped Rods.

Fig. 4

Control Rod Positions in the Plane
 $z = 0$.

We wish to determine the sink function, S , such that

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} + \Lambda \phi = S \text{ in } \bar{D}_1 \quad (101)$$

$$= 0 \text{ in } \bar{D} - \bar{D}_1, \quad (102)$$

where $\bar{D}$ denotes the extrapolated volume of the reactor and $\bar{D}_1$ the effective volume of the cross-shaped control rods and such that the eigenfunction ϕ is zero on Σ , the reactor surface, and on the effective control rod surfaces Σ_1 .

We assume as a first approximation to the sink function S_i ,

$$S_i^{(1)} = \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c} \text{ in } \bar{D}_i \quad (103)$$

$$= 0 \text{ in } \overline{D - D_i} \quad (i = 1, 2, \dots, k) \quad (104)$$

The unnormalized eigenfunctions f_{mnl} are

$$f_{mnl} = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{\ell\pi z}{c} \quad .$$

We form the eigenfunction expansion

$$S_i^{(1)} = \sum_{m,n,\ell}^{\infty} a_m b_n c_{\ell} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{\ell\pi z}{c} \quad . \quad (105)$$

Multiply eq. (105) by

$$\sin \frac{m'\pi x}{a} \sin \frac{n'\pi y}{b} \sin \frac{\ell'\pi z}{c}$$

and integrate over $\bar{D}$. By use of the orthogonality properties,

$$\begin{aligned} & \int_0^a \int_0^b \int_0^c \sin \frac{m\pi x}{a} \sin \frac{m'\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{n'\pi y}{b} \sin \frac{\ell\pi z}{c} \sin \frac{\ell'\pi z}{c} dx dy dz \\ &= \frac{abc}{8} \begin{cases} m = m' \\ n = n' \\ \ell = \ell' \end{cases} \\ &= 0 \quad \begin{cases} m \neq m' \\ n \neq n' \\ \ell \neq \ell' \end{cases} \end{aligned}$$

From eq. (103) and (104) we have

$$\begin{aligned} a_m^i b_n^i c_{\ell}^i &= \frac{8}{abc} \left[\left(\int_{x_1^i}^{x_2^i} \int_{y_2^i}^{y_3^i} \int_0^{z_i} + \int_{x_2^i}^{x_3^i} \int_{y_1^i}^{y_4^i} \int_0^{z_i} + \int_{x_3^i}^{x_4^i} \int_{y_2^i}^{y_3^i} \int_0^{z_i} \right) \right. \\ & \quad \left. x \sin \frac{\pi x}{a} \sin \frac{m\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{n\pi y}{b} \sin \frac{\pi z}{c} \sin \frac{\ell\pi z}{c} dx dy dz \right] . \end{aligned}$$

By repeated use of the equation

$$\int_{\alpha}^{\beta} \sin \frac{\pi \xi}{\gamma} \sin \frac{p\pi \xi}{\gamma} d\xi = \frac{\gamma}{2} A_p \Big|_{\alpha}^{\beta}$$

where

$$\begin{aligned} A_p \Big|_{\alpha}^{\beta} &= \frac{\gamma}{\pi} \left[\frac{\pi \xi}{2\gamma} - \frac{1}{4} \sin \frac{2\pi \xi}{\gamma} \right]_{\alpha}^{\beta}, \quad p = 1 \\ &= \frac{\gamma}{\pi} \left[\frac{1}{1-p^2} \right] \left[p \sin \frac{\pi \xi}{\gamma} \cos \frac{p\pi \xi}{\gamma} - \sin \frac{p\pi \xi}{\gamma} \cos \frac{\pi \xi}{\gamma} \right]_{\alpha}^{\beta}, \quad p > 1, \end{aligned}$$

we find that

$$a_m^i b_n^i c_{\ell}^i = A_m \begin{vmatrix} x_2^i & y_3^i & z_i \\ x_1^i & y_2^i & 0 \end{vmatrix} A_n \begin{vmatrix} y_3^i & A_{\ell} \\ y_2^i & 0 \end{vmatrix} + A_m \begin{vmatrix} x_3^i & y_4^i & z_i \\ x_2^i & y_1^i & 0 \end{vmatrix} A_n \begin{vmatrix} y_4^i & A_{\ell} \\ y_1^i & 0 \end{vmatrix} + A_m \begin{vmatrix} x_4^i & y_3^i & z_i \\ x_3^i & y_2^i & 0 \end{vmatrix} A_n \begin{vmatrix} y_3^i & A_{\ell} \\ y_2^i & 0 \end{vmatrix}.$$

The first approximation, $\Lambda^{(1)}$, of the first eigenvalue Λ is found by use of eq. (33b) and (33c). We find, therefore, that $\Lambda^{(1)}$ satisfies the equation

$$\sum_{i=1}^k \sum_{m,n,\ell}^{\infty} \frac{a_m^i b_n^i c_{\ell}^i}{\Lambda^{(1)} - \lambda_{mnl}} \sum_{j=1}^k \sum_{p=1}^{12} P_{mnl}^{jp} = 0, \quad (106)$$

where

$$\lambda_{mnl} = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{\ell^2}{c^2} \right),$$

$$P_{mnl}^{jp} = \int_{\sigma_j^P} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{\ell\pi z}{c} dx dy dz$$

and σ_j^P is the p^{th} part of the total surface σ_j of the j^{th} cross-shaped control rod. Knowing $\Lambda^{(1)}$, we have, from eq. (33c), the first approximation of the first eigenfunction:

$$\phi^{(1)} = \sum_{i=1}^k \sum_{m,n,\ell}^{\infty} \frac{a_m^i b_n^i c_{\ell}^i}{\Lambda^{(1)} - \lambda_{mnl}} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{\ell\pi z}{c}. \quad (107)$$

The second approximation, $\Lambda^{(2)}$, of Λ is obtained by use of Eq. (33d); $\Lambda^{(2)}$ satisfies the equation

$$\sum_{i=1}^k \sum_{m,n,\ell}^{\infty} \frac{a_m^i b_n^i c_\ell^i (\Lambda^{(2)} - \lambda_{mn\ell})}{\Lambda^{(2)} - \lambda_{mn\ell}} \left\{ Q_{mn\ell} - \sum_{j=1}^k \sum_{p=1}^4 R_{mn\ell}^{jP} \right\} \quad , \quad (108)$$

where

$$Q_{mn\ell} = \int_0^a \int_0^b \int_0^c \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{\ell\pi z}{c} dx dy dz \quad ,$$

$$R_{mn\ell}^{jP} = \int_{\bar{D}_j^P} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{\ell\pi z}{c} dx dy dz \quad ,$$

and $\bar{D}_j^P$ is the p^{th} of the total volume $\bar{D}_j$ of the j^{th} cross-shaped control rod. Processing in this manner, we may successively derive Eq. (33f) through (33k) for this geometry.

XI. NUMERICAL RESULTS FOR THE CYLINDRICAL REACTOR WITH TWO PARTIALLY INSERTED CONTROL RODS

Calculations were made with the formulas derived in Sect. VII and IX for the cases of one and two control rods inserted into the reactor in a variety of positions. The reactor and control rod parameters used were $R = 1$, $H = 2$, $r_1 = 0.02$, and r_2 equal 0.02, all in centimeters, where R is the extrapolated reactor radius, H is the extrapolated reactor height, and r_1 and r_2 are the effective radii of the control rods. The distances between the centers of the control rods and the reactor center in centimeters were designated by the parameters d_1 and d_2 . The lengths of insertion in centimeters were designated by the parameters h_1 and h_2 . The buckling or first eigenvalue calculated for these various configurations was in inverse square centimeters.

A complete description of how the equations for the cylindrical reactor with partially inserted control rods as derived in Sect. VII were coded for the IBM 704 is included in the appendix. This description includes convergence behavior of the equations in Sect. VII, the program listing the preparation of data, and operating procedures for the code.

The fluxes calculated by this code for a reactor with one and two partially inserted control rods in a variety of positions are plotted in Figs. 5 to 11. The calculated fluxes were multiplied by the arbitrary scale factor 10^4 . Before the flux calculation can be made, three different series must be evaluated a number of times. All of these series are, of course, partially summed and affect, in turn, the accuracy of the partially summed series for the calculation of the flux itself. The partial sums for the various approximations of the flux are designated as $\phi_{m \times n \times \ell}^{(k)}(r, z, \theta)$ and the location of where the flux was calculated may be determined by use of Fig. 1. Since the various orders of approximations of the flux and buckling require the repeated use of the series in eq. (85) through (91), all the series used in the calculations were summed over an equal number of terms, so the partial sums will be designated by the common symbol $T_{m \times n \times \ell}$. The point (r, z, θ) is given in terms of the reactor dimensions and is designated as $(r_0 R, z_1 H, \theta)$. The distances between the centers of the control rods and the center of the reactor are given in terms of the reactor radius R and are designated as $\delta_i R$, $i = 1, 2$.

Which order of approximation of the flux is calculated depends on the error ϵ_k stipulated. The resulting plots shown in Figs. 5 to 11 were made with $\epsilon_k = 0.01$.

In Fig. 6, calculations for the flux made by the author⁽¹⁶⁾ by a different but related approximate theory are also shown and are designated by $\Phi_{m \times n}(r, z)$. For an explanation of the meaning of $\Phi_{m \times n}(r, z)$, see reference 16.

Fig. 5 Radial Flux Plots for $Z = Z_1H$ for One Central Control Rod Inserted Effectively a Length 0.625

Fig. 6 Radial Flux Plots for $Z = Z_1H$ for One Central Control Rod Inserted Effectively a Length 1.00

Fig. 7 Radial Flux Plots for $Z = Z_1H$ for One Central Control Rod Inserted Effectively a Length 1.50

Fig. 8 Radial Flux Plots for $Z = Z_1H$ for One Central Control Rod Fully Inserted.

Fig. 9 Radial Flux for One Control Rod Inserted Effectively a Length 0.960 at $d_1 = 0.6$ for $Z = Z_1 H$ and $\alpha_1 = 0$.

Fig. 10 Radial Flux Plots for Two Control Rods with Axes at $d_1 = 0.4$, $\alpha_1 = 0$, and $d_2 = 0.4$, $\alpha_2 = \pi/2$ for $Z = Z_1 H$ Inserted Effective Lengths of 0.960

Fig. 11 Angular Flux Plots for Two Control Rods with Axes at $d_1 = 0.4$, $\alpha_1 = 0$ and $d_2 = 0.4$, $\alpha_2 = \pi/2$, for $\delta_1 R$ Inserted Effective Lengths of 0.960.

Table I shows how the various orders of approximation, $\Lambda^{(k)}$, for a half-way inserted control rod varied with the choice of m , n , and ℓ in the series used for these calculations

Table I

Orders of Approximation of the Buckling, $\Lambda^{(k)}$, for the
Central Control Rod Halfway Inserted vs
Partial Sums Taken[†]

Values of the Indices for the Partial Sum $T_{m \times n \times \ell}$			Orders of Approximation of $\Lambda^{(k)}$			
m	n	ℓ	$\Lambda^{(1)}$	$\Lambda^{(2)}$	$\Lambda^{(3)}$	$\Lambda^{(4)}$
1	50	5	9 8349791	9 6532998	9 6543206	9 6543207*

[†] The mathematically effective length was found to be 1.00 for the last partial sum taken. The author's previous value for the buckling of a halfway inserted control rod was 9.508 (see Ref. 16). Here the boundary condition at the tip was neglected. The mathematically effective length was 0.975, which was slightly smaller than in the above calculation.

*The actual length was 1.15

In Fig. 8, calculations made by use of eq. (92), which is the exact solution for this case for $A = \phi_{m \times n \times \ell}(0.3, 1, 2\pi)/\theta(0.3, 1)$ is also shown and is designated by $\theta(r, z)$, which was plotted using the value $\beta = 2.8842250$ found by use of eq. (92).*

Tables II-IV show how the various orders of approximation $\Lambda^{(k)}$ for control rods in a variety of positions varied with the choice of m , n , and ℓ .

Table II

Orders of Approximation of the Buckling, $\Lambda^{(k)}$, for the
Central Control Rod Fully Inserted vs Partial
Sums Taken

Values of the Indices for the Partial Sum $T_{m \times n \times \ell}$			Orders of Approximation of $\Lambda^{(k)}$			
m	n	ℓ	$\Lambda^{(1)}$	$\Lambda^{(2)}$	$\Lambda^{(3)}$	$\Lambda^{(4)}$
1	50	1	10 786831	10 555445	10 555555	10 555555
1	150	1	10 786236	10 694038	10 694055	10 694055
1	300	1	10 786165	10 736356	10 736366	10 736366*

*Buckling calculated using eq. (93) and (94) is 10.874274. Buckling calculated with the IBM 704 using eq. (92) and (93) is 10.786154 for a radius of $r_1 = 0.02$.

*Equation (92) was evaluated at $r = 0.02$, set equal to zero, and the value found by use of the IBM 704 to satisfy this equation was 2.8842250.

Table III

Orders of Approximation of the Buckling, $\Lambda^{(k)}$, for
One Halfway Inserted Control Rod at $d_1 = .4$ vs.
Partial Sums Taken†

Values of the Indices for the Partial Sum $T_{m \times n \times \ell}$			Orders of Approximation of $\Lambda^{(k)}$			
m	n	ℓ	$\Lambda^{(1)}$	$\Lambda^{(2)}$	$\Lambda^{(3)}$	$\Lambda^{(4)}$
15	40	2	9.3148143	8.7440069	8.7592590	8.7592585*
20	50	4	9.1456788	8.7813159	8.7814810	8.7814809**

† The mathematically effective length was found to be 0.960.

*The actual length was 1.25.

**The actual length was 1.13.

Table IV

Orders of Approximation of the Buckling, $\Lambda^{(k)}$, for
Two Halfway Inserted Control Rods at $d_1 = .4$,
 $d_2 = .4$, $\alpha_1 = 0$ and $\alpha_2 = 90^\circ$ vs. Partial
Sums Taken†

Values of the Indices for the Partial Sum $T_{m \times n \times \ell}$			Orders of Approximation of $\Lambda^{(k)}$			
m	n	ℓ	$\Lambda^{(1)}$	$\Lambda^{(2)}$	$\Lambda^{(3)}$	$\Lambda^{(4)}$
15	40	2	10.330864	9.6180513	9.6296294	9.6296357*

† The mathematically effective lengths were found to be 0.960.

Table III and Table IV illustrate the shadow effect or interaction effect of two control rods.

*The actual lengths were 1.25.

Figure 12 shows how the fractional change in buckling varied as a function of the mathematically effective (see discussion) control rod position for a central control rod. In addition, similar curves calculated at KAPL by use of CURE code,⁽¹⁷⁾ by a method developed by Schindler,⁽²¹⁾ and by a method developed by Murray⁽²²⁾ are also shown. These plots show the same S-shape, a property which is characteristic of such calculations. It should be noted that the KAPL calculation was made using slightly different reactor and control rod parameters. A calculation using exactly the KAPL parameter was also made and is shown in Fig. 12.

Figure 13 shows how the fractional change in buckling varies as a function of control rod position for a halfway inserted control rod moved away from the center a distance d_1 . The case of a fully inserted rod moved off center calculated by another method⁽²⁰⁾ is also shown. It should be noted that different parameters were used, yet both curves display the same characteristic shape.

Figure 14 shows the control rod worth for the central rod and off-center rod for a typical migration area of 300 cm^2 .

Discussion of Numerical Results

Since the method developed is an analytic one, the cutting off of the various series leads to certain inaccuracies. In order, for example, to represent the initial sink function, $S^{(1)}$, a great many terms would have to be taken since this function is a "step" function in three-space. Consequently, at the surfaces of discontinuity this function would have its poorest representation. In effect, this discrepancy in the partial-sum representation shrinks the subdomains which, in the case of control rods, shortens the length.

However, because of the iterative procedure developed for a fair amount of terms, the series solution will converge and be forced to satisfy the boundary conditions for somewhat smaller subdomains than initially stipulated in the given calculation. Thus, one is guaranteed in any case that the solution satisfy the partial differential equation and the boundary condition for a somewhat smaller domain, which will be designated as the mathematically effective subdomain.

In the case of control rods, one would have a mathematically effective insertion length corresponding to the length one wishes to consider. There is also some variation in diameter, of course, but this is very slight. One, then, in reactor calculations would perform a series of calculations with lengths in the vicinity of the mathematically effective lengths to be considered. This procedure would have to be adopted since to analyze mathematically this shrinkage or error introduced by partially summing is a formidable task.

$$\text{FRACTIONAL CHANGE IN BUCKLING} = \frac{B^2(r, z, \theta) - B_0^2}{B_1^2 - B_0^2}$$

Fig. 12 Fractional Change in Buckling vs. Effective Insertion Length of One Central Control Rod

Fig. 13 Fractional Change in Buckling vs. Distance of Control Rod from Center Inserted a Length 0.960

Fig. 14

Control Rod Worth vs. Effective Insertion Length of Central Control Rod (h/H) and Distance from Center of Control Rod (d/R) Inserted Effectively a Length 0.960

Certain methods exist for speeding up convergence of series; these may or may not work in the series developed by this method. These possibilities will be considered in the code developed.

The various tables and graphs illustrate the convergence of these solutions and, in addition, illustrate how, by using fewer terms and longer control rods, one gets effectively the same results as using many terms for slightly shorter control rods.

Fig. 12 indicates that after the control rod is inserted considerably past the halfway point, the theory is not quite as good. This behavior would become more marked for off-center control rods. However, for many more terms this situation would be alleviated. But since the theory is primarily for partially inserted control rods and not particularly for almost fully inserted control rods, this sort of calculation with many terms in the series would not have to be made, since there is a great deal of work in the literature for off-center fully inserted control rods.

XII. THEOREMS II AND III. EXTENSION OF THE THEORY TO THE WAVE EQUATION WITH VARIABLE COEFFICIENTS

We now extend the theory to solve the wave equation with variable coefficients which have one or more independent variables. We treat in this section the three-dimensional case again, but, as before, it is understood to apply to considerations involving n-dimensional problems as well.

First, we consider the somewhat special case in which the coefficients are step functions over the domains of interest. This special case of step-function variable coefficients is of interest in solving reactor physics problems involving multi-region calculations.

In this section we consider three-dimensional rectangular geometries only, but the treatment applies equally well to other geometries. We restrict our arguments to self-adjoint partial differential equations of the form

$$a(x,y,z) \frac{\partial^2 \phi}{\partial x^2} + b(x,y,z) \frac{\partial^2 \phi}{\partial y^2} + c(x,y,z) \frac{\partial^2 \phi}{\partial z^2} + d(x,y,z) \Delta \phi = 0 \quad ,$$

where the coefficients a , b , c , and d are to have certain restrictive properties as stated in the two theorems to follow. We consider the homogeneous Dirichlet problem, but, as before, we may solve other types as well.

Theorem II: Let the coefficients a , b , c , and d be constants in the open domain $D-D_1$. Then there exists a function S (sink) which has the following properties:

$$(i) \quad a \frac{\partial^2 \phi}{\partial x^2} + b \frac{\partial^2 \phi}{\partial y^2} + c \frac{\partial^2 \phi}{\partial z^2} + d \Delta \phi = S \text{ in } \overline{D}_1 \quad (110)$$

$$(ii) \quad = 0 \text{ in } \overline{D-D_1} \quad (111)$$

and is such that it makes ϕ satisfy the boundary condition IV (iii).

Here we are considering the simple case of a partial differential equation with constant coefficients. The coefficients are constant over the entire domain $D-D_1$; thus we do not expand these coefficients into their respective Fourier expansions.

The f_n in the following equation satisfy eq. (4) and the boundary condition in eq. (5). Further, we form the eigenfunction expansion:

$$\phi = \sum_n^{\infty} e_n f_n \quad (111a)$$

and the eigenfunction expansion of the first approximation, $S^{(1)}$, of the sink S :

$$S^{(1)} = \sum_n^{\infty} \alpha_n f_n, \quad (111b)$$

where, in all the above expansions, the Fourier coefficients are found by similar integrations as in eq. (19a). We now let the index n become the indices m, n, ℓ . Substituting these expansions into eq. (110) and interchanging summation and differentiation, we have by use of eq. (16) that

$$e_{mnl} = \alpha_{mnl} / \left(d_{mnl} \Lambda - a_{mnl} m^2 - b_{mnl} n^2 - c_{mnl} \ell^2 \right) \quad (112)$$

From eq. (112) one readily finds the first approximation to be

$$\phi^{(1)} = \sum_{m,n,\ell}^{\infty} \frac{\alpha_{mnl} f_{mnl}}{d_{mnl} \Lambda - a_{mnl} m^2 - b_{mnl} n^2 - c_{mnl} \ell^2} \quad (113)$$

The first approximation $\Lambda^{(1)}$ may also be found by use of eq. (23) again. As in eq. (29), $\Lambda^{(2)}$ may be found and so the same procedure is carried out as before by use of eq. (31) through (33k).

Since the denominators of the coefficients of the expansion in eq. (113) would make the expansion converge even more strongly than those derived for the proof of Theorem I, the proof of this theorem follows in exactly the same way.

A more general case involving variable coefficients is considered in the next theorem.

Theorem III. Let the coefficients a, b, c , and d be of the form $a(x,y,z)$, $b(x,y,z)$, $c(x,y,z)$, and $d(x,y,z)$. Further let these coefficients admit of the Fourier expansion(19)

$$\begin{aligned} a(x,y,z) &= \sum_{m,n,\ell}^{\infty} a_{mnl} f_{mnl} \\ b(x,y,z) &= \sum_{m,n,\ell}^{\infty} b_{mnl} f_{mnl} \\ c(x,y,z) &= \sum_{m,n,\ell}^{\infty} c_{mnl} f_{mnl} \\ d(x,y,z) &= \sum_{m,n,\ell}^{\infty} d_{mnl} f_{mnl}, \end{aligned}$$

in the domain $D-D_1$, where

$$f_{mn\ell} = f_m^{(1)}(x) f_n^{(2)}(y) f_\ell^{(3)}(z) \quad (114)$$

(The eigenfunction in eq.(114a) is, of course, the actual form of these eigenfunctions used throughout this section, but is displayed here to emphasize what is to follow.) Then there exists a function S (sink) which has the following properties:

$$(i) \quad a(x,y,z) \frac{\partial^2 \phi}{\partial x^2} + b(x,y,z) \frac{\partial^2 \phi}{\partial y^2} + c(x,y,z) \frac{\partial^2 \phi}{\partial z^2} + d(x,y,z) \Delta \phi = S \text{ in } \bar{D}_1 \quad (114a)$$

$$(ii) \quad = 0 \text{ in } \overline{D-D_1} \quad (115)$$

and is such that it makes ϕ satisfy the boundary condition IV (iii). We form the expansions in eq. (111a) and (111b) and substitute them and the expressions for the coefficients into eq. (114a). We now use for ϕ and $S^{(1)}$ the eigenfunctions

$$f_m^{(1)}(x) f_n^{(2)}(y) f_\ell^{(3)}(z)$$

and integrate over $\overline{D-D_1}$. We now directly integrate eq. (114a) after substituting into it the above expansions and after interchanging the various operations; we have by the orthogonality properties of

$$f_m^{(1)}(x) f_n^{(2)}(y) f_\ell^{(3)}(z)$$

that

$$e_{mn\ell} = \frac{\alpha_{mn\ell} \int_{\bar{D}_1} f_{mn\ell}}{d_{mn\ell} \Delta - a_{mn\ell} m^2 - b_{mn\ell} n^2 - c_{mn\ell} \ell^2} \quad (116)$$

as before. Eq. (116) illustrates that in this case as well the nature of the coefficients in the expansion of ϕ^1 and, consequently, succeeding derivations made will be essentially the same as those involved in Theorems I and II. Thus the proof of Theorem III is made exactly in the same manner as were the proofs of Theorems I and II.

XIII. AN APPLICATION OF THE METHOD TO A TIME-DEPENDENT PROBLEM AND THEOREM IV

We seek now the solution of the time-dependent wave equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} \quad (117)$$

subject to the following boundary and initial conditions.

$$\phi(x, y, t) \Big|_{\Gamma} = 0, \quad t > 0 \quad (118)$$

$$\phi(x, y, 0) = F(x, y), \quad (119)$$

where c is a constant and Γ is the boundary surface of an arbitrary domain $\overline{D-D_1}$. Assume the solution to be of the form

$$\phi(x, y, t) = \bar{\phi}(x, y) e^{-i\omega t} \quad (120)$$

Substituting eq. (120) into eq. (117), it is seen that $\bar{\phi}$ must satisfy the equation

$$\frac{\partial^2 \bar{\phi}}{\partial x^2} + \frac{\partial^2 \bar{\phi}}{\partial y^2} + \Lambda \bar{\phi} = 0, \quad (121)$$

where

$$\Lambda = \omega^2 / c^2.$$

Equation (118) now becomes equivalently the condition

$$\bar{\phi}(x, y) \Big|_{\Gamma} = 0 \quad (122)$$

Equation (122), then, is the same as boundary condition IV (iii), since Γ is the arbitrarily shaped surface for the arbitrarily shaped domain $\overline{D-D_1}$. We now find the sequence of eigenfunctions, $\{\phi_n\}$, and corresponding sequence of eigenvalues, $\{\Lambda_n\}$, by the methods already developed. The eigenfunctions $\bar{\phi}_n$ then have the representations

$$\bar{\phi}_n = \sum_m^{\infty} a_m^n f_m, \quad (123)$$

where the f_m satisfy eq. (4) and (5) and the a_m^n are the constants constructed by use of the theory developed so that each $\bar{\phi}_n$ satisfies the homogeneous boundary conditions on all other surfaces besides Γ . We now expand ϕ :

$$\phi = \sum_n \bar{\phi}_n e^{-i\Omega_n t} = \sum_{m,n} A_n a_m^n f_m e^{-i\Omega_n t}, \quad (124)$$

where

$$\Omega_n = c \Lambda_n^{\frac{1}{2}}.$$

Before finishing the solution to this problem by making ϕ satisfy eq. (119), we state and prove the following:

Theorem IV. The eigenfunctions of eq. (121) are orthogonal over $\overline{D-D_1}$.

Denote the operator in eq. (121) by L . Now let Λ_n and $\Lambda_{n'}$ be the two eigenvalues corresponding to the two eigenfunctions $\bar{\phi}_n$ and $\bar{\phi}_{n'}$, respectively. From eq. (121) we have, therefore, that

$$L\bar{\phi}_n = -\Lambda_n \bar{\phi}_n \quad (125)$$

and

$$L\bar{\phi}_{n'} = -\Lambda_{n'} \bar{\phi}_{n'} \quad (126)$$

We now multiply eq. (126) by $\bar{\phi}_n$, eq. (125) by $\bar{\phi}_{n'}$ and subtract eq. (126) from (125) to obtain

$$\bar{\phi}_{n'} L\bar{\phi}_n - \bar{\phi}_n L\bar{\phi}_{n'} = (\Lambda_{n'} - \Lambda_n) \bar{\phi}_{n'} \bar{\phi}_n \quad (127)$$

Since the operator L is self-adjoint, we have from Lagrange's formula:

$$\bar{\phi}_{n'} L\bar{\phi}_n - \bar{\phi}_n L\bar{\phi}_{n'} = \vec{\nabla} \cdot [\bar{\phi}_n \vec{\nabla} \bar{\phi}_{n'} - \bar{\phi}_{n'} \vec{\nabla} \bar{\phi}_n],$$

that

$$(\Lambda_{n'} - \Lambda_n) \bar{\phi}_{n'} \bar{\phi}_n = \vec{\nabla} \cdot [\bar{\phi}_n \vec{\nabla} \bar{\phi}_{n'} - \bar{\phi}_{n'} \vec{\nabla} \bar{\phi}_n] \quad (128)$$

We now integrate both sides of eq. (128) over $\overline{D-D_1}$ and obtain after an application of Green's theorem that

$$(\Lambda_{n'} - \Lambda_n) \int_{\overline{D-D_1}} \bar{\phi}_{n'} \bar{\phi}_n = \int_{\Gamma} (\bar{\phi}_n \vec{n} \cdot \vec{\nabla} \bar{\phi}_{n'} - \bar{\phi}_{n'} \vec{n} \cdot \vec{\nabla} \bar{\phi}_n) \quad (129)$$

But we have constructed the $\bar{\phi}_n$'s so that

$$\bar{\phi}_n \Big|_{\Gamma} = 0$$

and, therefore, we have that

$$(\Lambda_n - \Lambda_{n'}) \int_{\overline{D-D_1}} \bar{\phi}_{n'} \bar{\phi}_n = 0 \quad . \quad (130)$$

Since $n' \neq n$, we have

$$\int_{\overline{D-D_1}} \bar{\phi}_{n'} \bar{\phi}_n = 0 \quad . \quad (131)$$

Now, by the orthogonality property in eq. (131), we find from eq. (119) and (124) that, after interchanging summation and integration,

$$A_n = \sum_m^{\infty} a_m^n \int_{\overline{D-D_1}} F(x,y) f_m \quad ,$$

which thus completes the solution of this problem.

XIV. AN APPLICATION OF THE METHOD OF THE SOLUTION OF POISSON'S EQUATION

In a manner similar to the one in Sect. XIII we may employ the method developed to solve Poisson's equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \rho(x, y) \quad , \quad (132)$$

in an arbitrarily shaped domain $\overline{D-D_1}$, with ϕ subject to the homogeneous Dirichlet condition:

$$\phi \Big|_{\Gamma} = 0 \quad ,$$

where Γ is the surface of the domain $\overline{D-D_1}$.

We assume that $\rho(x, y)$ is a distributed source in the open domain $D-D_1$ and that it can be expanded into the series of eigenfunctions:

$$\rho(x, y) = \sum_n^{\infty} B_n \phi_n \quad , \quad (133)$$

where

$$\phi_n \Big|_{\Gamma} = 0$$

by an application of the method developed. Further, we make the expansion

$$\phi = \sum_n^{\infty} A_n \phi_n \equiv \sum_{m, n}^{\infty} A_{mn} \phi_{mn} \equiv \sum_{m, n, m', n'}^{\infty} A_{mn} a_{mn}^{m' n'} f_{m' n'} \quad (134)$$

for the solution. Substituting eq. (133) and (134) into eq. (132), we have by use of Theorem IV that

$$A_{mn} = -B_{mn}/(m^2 + n^2) \quad .$$

Since

$$\frac{\partial^2 \phi_n}{\partial x^2} = - \sum_{m, n, m', n'}^{\infty} m^2 A_{mn} a_{mn}^{m' n'} f_{m' n'} \quad ,$$

then

$$\frac{\partial^2 \phi_n}{\partial y^2} = - \sum_{m,n,m',n'}^{\infty} n^2 A_{mn} a_{mn}^{m'n'} f_{m'n'}$$

Therefore, the solution to Poisson's equation, subject to homogeneous Dirichlet boundary conditions, is

$$\phi = - \sum_{m,n,m',n'}^{\infty} \frac{B_{mn} a_{mn}^{m'n'}}{m^2 + n^2} f_{m'n'} \quad (135)$$

XV. REMARKS ON VERY IRREGULAR DOMAINS

If the domains are extremely irregular for the problems considered above, it is still possible to apply the method developed. In these cases, one may approximate these irregular domains by many small regular subdomains which allow one to perform the various integrations necessary in carrying out the solution.

REFERENCES

1. Titchmarsh, E. C., Some Theorems on Perturbation Theory, Proc. Roy. Soc. (London) A 200, 34 -36, (1949).
2. Kato, T., On the Convergence of the Perturbation Method II, 2 Progress Theoretical Physics 5, 207 - 212 (1950).
3. Feenberg, E., A Note on Perturbation Theory, Phys. Rev., 74, 206 (1948).
4. Brillouin, L., Perturbation by Boundary Deformation as a Problem of Proper Values, Compt. Rend., 204, 1863 (1937).
5. Morse and Feshbach, Methods of Theoretical Physics Vol. II, McGraw Hill Book Co. Inc., New York (1953), pp. 1052 - 1064.
6. Swanson, C. A., Differential Operators with Perturbed Domains Journal of Mathematics and Mechanics, U. of Indiana, 6, 832 (1957).
7. Weston, V. H., Solutions of the Helmholtz Equation for a Class of Non-separable Cylindrical and Rotational Coordinate Systems J. App. Math. 15, 420 (1957-58).
8. Greebler, Paul, TRIXY - A Computer Program for Multigroup Nuclear Reactor Calculations in Three-Space Dimensions, KAPL 1549 (August 1, 1956).
9. Churchill, R. V., Fourier Series and Boundary Value Problems, McGraw-Hill Book Co. Inc., New York and London (1941), pp. 80 - 81.
10. Courant and Hilbert, Methods of Mathematical Physics, Vol. I, Interscience, New York and London (1953), p. 409.
11. Glasstone, S. and Edlund, M. C., The Elements of Nuclear Reactor Theory, D. Van Nostrand Company, Inc., New York (1952) pp. 191 - 215.
12. Davisson, B. and Kushnerick, S., Linear Extrapolation Length for Black Sphere and Cylinder, MT-214 (March 3, 1946).
13. See reference 5, pp. 1259 - 1260.
14. Watson, G. N., Bessel Functions, The Macmillan Press Company, New York (1948), p. 361.

15. Maclaughlin, N. W., Bessel Functions for Engineers, Oxford University Press (1941), pp. 96 - 97.
16. Betinis, E. J., Flux Distribution for a Finite Cylindrical Bare Homogeneous Thermal Reactor with a Partially Inserted Control Rod, NAA-SR-1963 (July 1, 1957).
17. Jewett, J. P., Mullin, C. R., and Smith, J. H., Calculation of the Worth of a Partially Inserted Control Rod (Paper presented at the Meeting of the American Nuclear Society held in New York, October 29, 1957).
18. See reference 11, pp. 316 - 322.
19. Solkolnikoff, I. S., and E. S., Higher Mathematics for Engineers and Physicists, McGraw-Hill Book Company, Inc., New York and London (1941), p. 65.
20. Adler, F. T., MT 222, Efficiency of Control Rods as Function of Their Position in Cylindrical Pile in the One Group Picture, AECL - 253.
21. Schindler, G. M., On the Efficiency of a Concentric Cut-off Rod of A Thermal Reactor, as a Function of the Inserted Length of the Rod, Journal of Nuclear Energy, (1 - 3), 8 18 - 32 (November 1958).
22. Murray, R. L., Reactor Handbook (p. 604).

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APPENDIX

DESCRIPTION AND OPERATING PROCEDURE OF THE IBM 704 CODE
FOR THE BARE HOMOGENEOUS THERMAL REACTOR
WITH PARTIALLY INSERTED CONTROL RODS

I. Description of Code

The reactor control rod formulas derived in Section VII were coded by use of the Argonne National Laboratory version of FORTRAN II, an automatic coding system developed for use on the IBM 704. Since the nature of this calculation consisted primarily of summing series and evaluating formulas, it was felt that the FORTRAN coding technique would be rapid and effective for these calculations.

The complete code consists of ten decks. However, only three decks comprise the reactor calculation portion of the code, and the other six decks are auxiliary programs which would be used very infrequently at each installation; they do serve their purpose in preparing or correcting magnetic tapes, making preliminary calculations, interrupting the main reactor program, and restarting the main program. The main program is thus divided into three phases which will now be described along with the auxiliary programs.

In addition to this description and the FORTRAN listing of the code, a table defining the various FORTRAN counterparts of the mathematical symbols used is included just prior to the listing. The program listing included in the back of this Appendix is for calculations involving up to ten control rods inserted along the axis of the reactor at arbitrary angles, distances, and depths.

Phase 1. In phase 1, the Fourier-Bessel coefficients expressed by eq. (83), the expression in braces in eq. (88), i.e., the surface integral coefficients, and the expressions in brackets in eq. (90), i.e., the volume integral coefficients, are calculated and written on binary magnetic tape. The roots of the Bessel functions β_{mn} are calculated by use of the formula*

$$\beta_{mn} = \beta - \frac{q-1}{8\beta} \left(1 + \frac{Q_1}{3(4\beta)^2} + \frac{2Q_2}{15(4\beta)^4} + \frac{Q_3}{105(4\beta)^6} + \dots \right)$$

where

$$q = 4 m^2$$

$$\beta = (m - 0.5 + 2n) \pi / 2$$

$$Q_1 = 7q - 31$$

$$Q_2 = 83q^2 - 982q + 3779$$

$$Q_3 = 6949q^3 - 153855q^2 + 1585743q - 6277237$$

*See Jahnke and Emde, Theory of Functions, Dover Publications, Inc. pp. 135-141.

and are used as part of the arguments of the Bessel function subroutines throughout phase 1. The eigenvalues $\lambda_{mn\ell}$ are also written on binary magnetic tape.

In addition, the following operations are performed:

- (1) The dimension statement sets aside storage for the subscripted variables appearing in the calculation of β_{mn} , floating point integers q_n , and for the subscripted variables necessary to calculate the Bessel functions using the N.Y.U. Bessel function subroutine. These latter subscripted variables appear as the various TABLE variables.
- (2) FORTRAN statements are used to define additional Bessel function subroutines for large argument and index of approximately equal magnitude.
- (3) The data describing the rod positions, initial flux points, increments in flux plots, number of rods, etc., are read in (see operating procedure for phase 1). All input concerning rod positions, number of terms, number of rods, date, and heading of table for Fourier-Bessel coefficients, surface integral coefficients, eigenvalues, volume integral coefficients, current values of indices and the current rod number are printed on line.
- (4) The coefficients $c_{mn\ell}$ are set equal to one and written on binary tape. Since the first root of the first Bessel function is in error by a considerable amount, this root is set equal to the correct value in the program.
- (5) When phase 1 is finished, all input data necessary to start phase 2 is punched out.

Phase 2. The data from phase 1 is read in. The various coefficients calculated in phase 1 are used to sum the series in eq. (88) and (90) for the various orders of approximations of the first eigenvalue and eigenfunction. Once again pertinent input data regarding rod positions, terms, etc., is printed on line.

The following operations are performed:

- (1) For a some initial guess on Λ , namely, $\Lambda_0^{(1)}$, the series in eq. (88) is summed. Since the theory shows that this series is either positive or negative (and one can always start positively by making $\Lambda_0^{(n)}$ close to Λ_{011}), the program tests for a negative value. If this series is negative, the increment

$\Delta\Lambda$ is subtracted from $\Lambda^{(1)}$. This increment is then divided by an arbitrary number S and the series in eq. (88) is resummed using $(\Delta\Lambda/S) + \Lambda^{(1)} = \Lambda^{(2)}$ until the sign of the series changes again. Thus one may increment to the accuracy $\Delta\Lambda/S^{J_4+1}$, where J_4 is the number of sign changes requested.

- (2) Once $\Lambda^{(1)}$ is known, then $\Lambda^{(2)}$ is found from eq. (90) as the ratio of two series, as is shown more clearly in eq. (64).
- (3) $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are compared to see if $|\Lambda^{(1)} - \Lambda^{(2)}| \leq \epsilon$, where ϵ is previously specified as input data.
- (4) If $|\Lambda^{(1)} - \Lambda^{(2)}| > \epsilon$, then the iterative procedure is repeated.
- (5) When phase 2 is completed, the necessary data to start phase 3 is punched out.

Phase 3. Phase 3 consists of primarily making flux plots in various regions of the reactor. Once phases 1 and 2 are completed, the necessary binary tapes for calculations in phase 3 for the case of interest are prepared and may be used repeatedly to make flux calculations. Thus it is possible to make flux traverses in any desired region of the reactor by simply changing some of the input data cards (such as initial values for starting flux plots or the increments used in these flux plots).

The following operations are performed:

- (1) Pertinent data from phase 2 is read in.
- (2) Pertinent data regarding rod position parameters, etc., is printed on line.
- (3) The series of the form expressed in eq. (91) on a higher-order approximation is summed for any choice of parameters stipulated by the user.
- (4) All final information is written on B.C.D. tape and is printed off-line. Intermediate print outs may also be had by proper sense switch setting.
- (5) Phase 3 may be interrupted, in which case current data is then punched out to either be used for continuing the calculation or it may be altered to start a new flux plot.

Auxiliary Program No. 1. This program is used to calculate the buckling for fully inserted central rod by the Nordheim-Scalletar theory, i.e., by use of eq. (93) and (94) and by use of the exact equation as expressed in eq. (92). The flux is also calculated using eq. (92) and normalized according to a given constant.

Auxiliary Program No. 2. This program is used to initialize magnetic tape number four if for some reason phase 2 is to be restarted from its beginning any time after iteration number one for the buckling has been completed. This sort of situation may occur in the event that because iteration number one for the buckling was not calculated accurately enough, leading consequently to a larger inaccuracy in the determination of iteration number two, one may wish to restart phase 2 for a more accurate determination of the buckling.

Auxiliary Program No. 3. In the event that calculations involving more than one control rod are to be made, one uses this program after using phase 1, if phase 1 was interrupted after the various coefficients were calculated for k control rods, where $1 < k \leq 10$. For a large number of control rods one would use the interrupt features available in phase 1 and then use this auxiliary program to reset the various tapes used in phase 1 in order to continue the phase 1 calculation for the coefficients of the remaining control rods. In addition to resetting the proper tapes, this program also prints on line the last records which contain the pertinent coefficients calculated by phase 1, so that one has a way of checking to be sure the tapes have been reset properly.

Auxiliary Program No. 4. (In three parts) Since the formula for calculating the roots β_{mn} of the Bessel Functions $J_m(\beta_{mn})$ was inaccurate for indices of the order three and up, it was necessary to devise this program to find out precisely which of these roots were in error. These discrepancies were determined by calculating the roots from the formula and then substituting the appropriate root or zero into the appropriate Bessel function. Once this was done, one could readily see which roots went astray.

Once this was known, these roots were corrected by use of the Harvard tables for Bessel functions and these corrections were punched on cards. Using these cards and the rest of auxiliary program No. 4, one makes up a master binary tape having $m \times n = 50 \times 100$ roots. As checks on these roots, the root and the value of the corresponding Bessel function, i.e., the Bessel function which has this root as a zero, are written on B.C.D. tape for off-line printing.

Auxiliary Program No. 5. Once the master tape containing 5000 correct roots β_{mn} has been made, then this program allows one to make a B.C.D. and binary tape of $m \times n$ roots where $m \leq 50$ and $n \leq 100$. The binary tape is then used as tape number seven for calculations in phases 1 and 2.

II. Operating Procedures

Phase 1.

- (1) Preparation of Data. The data $R, H, z_{in}, \theta_{in}, \Delta\Lambda, \Delta r, \Delta z, \Delta\theta, \epsilon, S, \Lambda_0, \Lambda_{in}, \Delta'r, \Delta'z, \Delta'\theta, (h_i, i = 1, 10), (r_i, i = 1, 10), (d_i, i = 1, 10), (\alpha_i, i = 1, 10), (M/R_1)^2$ and TEST are punched on the first twelve cards according to FORMAT (1H 1P5E14.7).

The data $J_1, J_2, J_3, J_4, n_r, i_1, i_2$ and J_k are punched on the thirteenth card according to FORMAT (8I4).

The total number of terms to be taken in summing the various series, namely, $J_{15} = n_r \times m \times n \times \ell$, is to be punched on the fourteenth card according to FORMAT (1I6).

On the fifteenth card the information for the date, namely, J_{12}, J_{13} , and J_{14} , will be punched according to FORMAT (3I3).

The sixteenth card allows up to seventy-one Hollerith characters to be read in so as to be used for problem identification. Such information as the originator and the originator's affiliation may be punched in according to FORMAT (72HO 71).

- (2) Magnetic Tapes. The magnetic tapes needed in phase 1 are tapes set to logical 2, 4, 8 and 7. Tape number 7 is prepared by use of auxiliary programs 4 and 5.
- (3) On-Line Punch. Put in sixteen cards and ready punch.
- (4) Sense Switch Settings

Sense Switch #1	Up - No print out of table entries - Fourier-Bessel coefficients, rod number , etc.
	Down - Print out table entries and monitor progress of calculation.
Sense Switch #2	Not used
Sense Switch #3	Not used
Sense Switch #4	Up - No print out of Γ_0 and Γ_k .
	Down - Print out Γ_0 and Γ_k .

Sense Switch #5 Up - No print out of B_0^2 , $RT(1,1)$, $RT(1,2)$,
 $\dots$, $RT(1,5)$.
 Down - Print out B_0^2 , $RT(1,1)$, $RT(1,2)$,
 $\dots$, $RT(1,5)$.

Sense Switch #6 Up - No print out of Bessel functions
 and argument.
 Down - Print out the following information:
 $\beta_{mn} d_i / R$, $J_m \left(\frac{\beta_{mn} d_i}{R} \right)$,
 $J_{m+1}(\beta_{mn})$, $J_{m+k} \left(\frac{\beta_{mn} d_i}{R} \right)$,
 $J_{m-k} \left(\frac{\beta_{mn} d_i}{R} \right)$, $J_k \left(\frac{\beta_{mn} r_i}{R} \right)$,
 m , n , k , and i .

(5) Stops

- (a) HPR 10₈ - Unacceptable argument for the G.E. Bessel function subroutine. Probably one of the control rod position parameters is identically zero.
- (b) N.Y.U. Bessel Function Error - An error may occur in the execution of the N.Y.U. Bessel function subroutine due to a very small argument also. The program will then print out this information. However, there is a possibility that this is a machine error. This machine error may be due to the core malfunctioning or because of an error on one of the tapes. The tape error probably occurred in preparation of tape #7. An HPR 66666₈ will then occur.
- (c) HPR 55555₈ - End of phase 1 calculation.
- (d) For other object program stops see FORTRAN II operator's manual.

- (6) Interrupt and Restart Procedures. If in phase 1 coefficients are to be calculated for n_r control rods starting at the i_1^{th} and ending at the i_2^{th} , one may accomplish this by punching data card number 15 with this information. The program will then interrupt itself and print out the pertinent information as to how far the calculation has proceeded and as to how many control rod coefficients are ultimately to be calculated. By use of auxiliary program number 3,

one resets the tapes used in phase 1 so that the coefficients for the remaining rods may be calculated. At this point one replaces i_1 by $i_1 + 1$ and i_2 by $i_2 + 1$ in data card number 15, reads in the phase 1 deck and continues the calculation. This procedure is continued until the coefficients of the n_r^{th} rod are calculated.

If one wishes not to initialize tape #4 in phase 1, one makes the data TEST on data card number 12 non-zero. If it is zero then tape #4 will be initialized, i.e., all coefficients $c_{mn\ell}$ will be set equal to one. This feature probably will not be used very much.

When phase 1 is finished, all pertinent data to start phase 2 is punched on line.

Note: Since this program is a FORTRAN II written program, the data cards are put in their proper order behind the object deck. These cards are put into the reader, readied, and loaded into the 704 in the ordinary manner.

Phase 2.

- (1) Preparation of Data. The necessary data cards to start phase 2 are obtained from phase 1. However, the third and fourth words on card number 12 must be punched according as to how phase 2 is to be restarted or interrupted. In this regard, see the section on "Interrupt and Restart Procedures" for this phase. If phase 2 is to be started, a blank card is put behind the sixteenth data card.
- (2) Magnetic Tapes. Mount tapes 2, 4, and 8 from phase 1. In addition, set two tapes to logical 5 and 6.
- (3) On-Line Punch. Put in seventeen cards and ready punch.
- (4) Sense Switch Settings

Sense Switch #1 Up - does not interrupt phase 2.
 Down - interrupts phase 2 in the phase 2(a) portion, i.e., after the series is summed using the value $\Lambda^{(2k+1)}$.
 The purpose here is to avoid unnecessary machine time if a poor initial guess for the odd-order approximation of the eigenvalue was made. Pertinent data is then punched out to restart phase 2.

- Sense Switch #2 Not used
- Sense Switch #3 Up - proceed from phase 2(a) to phase 2(b).
Down - interrupt phase 2 after phase 2(a) is completed and punch out pertinent data.
- Sense Switch #4 Up - does not print
Down - print iteration number for $\Lambda^{(2k+1)}$, i.e., print $2k+1$ and print the value of $\Lambda^{(2k+1)}$. The odd-order approximation is printed out.
- Sense Switch #5 Up - does not print.
Down - prints out $\Lambda^{(2k+1)}$ and value of corresponding series utilizing $\Lambda^{(2k+1)}$ (odd-order approximation) when program is in phase 2(a). Prints out $\Lambda^{(2k+2)}$ (even-order approximation) and the values of the two series whose ratio is $\Lambda^{(2k+2)}$. The iteration number for the odd-order approximation is also printed out.
- Sense Switch #6 Up - does not write B.C.D. on tape #5.
Down - writes in B.C.D. on tape #5, the partial sums of the various series summed and the current values of the indices.

- (5) Stops. HPR 11111₈ designates end of phase 2(a), i.e., completion of calculation of $\Lambda^{(2k+1)}$ and $\Lambda^{(2k+2)}$.

If $|\Lambda^{(2k+1)} - \Lambda^{(2k+2)}| \not< \epsilon$, then program will stop after

$$c_{mn\ell} \rightarrow c_{mn\ell} (\Lambda^{(2k+2)} - \lambda_{mn\ell}) / (\Lambda^{(2k+1)} - \lambda_{mn\ell}).$$

This calculation is performed in phase 2(b). If sense switch #3 is down, then phase 2 will be interrupted after START is hit and pertinent data will be punched out to restart phase 2. If sense switch #3 is up, then phase 2 will continue until $|\Lambda^{(2k+1)} - \Lambda^{(2k+2)}| < \epsilon$. HPR 77777₈ designates end of phase 2 calculation.

- (6) Interrupt and Restart Procedures. Phase 2 as previously mentioned requires a blank card behind the sixteenth data card. The third and fourth words on data card number 12 are used to tell the program where in phase 2 the calculation should be started. These words are designated as TEST 1 and TEST 2.

If TEST 1 ≤ 0 and TEST 2 > 0 then phase 2 will start with the calculation of $\Lambda^{(2k+1)}$ and proceed to the calculation of $\Lambda^{(2k+2)}$, where $k = 1$. An HPR 11111₈ will then occur and phase 2 may be interrupted at this point, if the error stipulated is still too great, by putting sense switch #3 down and hitting START.

Now phase 2 may be restarted, i.e., the calculation will proceed from the calculation of $\Lambda^{(2k+1)}$ and go on to the calculation of $\Lambda^{(2k+2)}$, where $k = 2$, until the difference in these approximations is within the error stipulated if TEST 1 > 0 and TEST 2 may be arbitrary. At the completion of phase 2, seventeen data cards are punched out as input for phase 3.

Note: Once again this program is loaded into the 704 as is done in phase 2.

Phase 3.

- (1) Preparation of Data. All necessary data to start phase 3 is punched out by phase 2 except for the third word on card number 12. This word designates the actual radius of the reactor in centimeters and is left available to the problem requestor's liking. Initial values for the flux and increments on the flux plots may be changed before starting phase 3 or after phase 3 is interrupted depending once again on the originator's inclinations.
- (2) Magnetic Tapes. Tapes 2 and 4 prepared by phase 2 are mounted and used in the phase 3 calculations. A blank tape set to logical 5 is also mounted and all input and output is written on tape number 5 for off-line printing.
- (3) On-Line Punch. Put in seventeen cards and ready punch.
- (4) Sense Switch Settings

Sense Switch #1	Not used
Sense Switch #2	Up - no print out
	Down - print out r , z , θ , and flux
Sense Switch #3	Up - do not interrupt phase 3
	Down - interrupt phase 3 punch out current data, put end of file on tape number 5, and rewind all tapes. Interrupts after series for flux is summed.

Sense Switch #4 Same as sense switch #3 except interrupts before series for flux is summed
 Sense Switch #5 Not used
 Sense Switch #6 Up - no print out
 Down - print out (h_i , $i = 1, 10$) and $\Lambda^{(2k+2)}$.

- (5) Stops. HPR 22222₈ occurs after N.Y.U. Bessel function subroutine fails (see phase 1 stops). HPR 10₈ occurs if G.E. Bessel function subroutine fails.

Object program stops (see FORTRAN II operator's manual).

HPR 11111₈ - end of phase 3 calculation.

- (6) Interrupt and Restart Procedures. If program is interrupted by sense switch number 3 or 4, the punch out may simply be used as data once again as before if the program is to be resumed. Once again this data may be altered to start the flux plot in a different region. One must, of course, keep tapes 2 and 4 for phase 3 calculations. These tapes may be used over again depending on what flux calculations are to be made.

Load and ready program and data cards as in the above programs.

Auxiliary Program No. 1

- (1) Preparation of Data. The following information r_0 , z_{in} , Δr , Δz , $\Delta \alpha$, α_{in} , R , H , $\Delta' r$ and C_2 is punched on the first two data cards according to FORMAT (1P5E14.7). On the third card J_4 is punched according to FORMAT (I4).

- (2) Sense Switch Settings

Sense Switch #1 Up - no print
 Down - print α_{in} and value of the transcendental function

$$J_0\left(\frac{\alpha r_0}{R}\right) - \frac{J_0(\alpha)}{Y_0(\alpha)} Y_0\left(\frac{\alpha r_0}{R}\right).$$

- (3) Stops. HPR 77777₈ - end of program.

Auxiliary Program No. 2

- (1) Preparation of Data. The following information: J_1, J_2, J_3, n_r , and i_3 , is punched on one card according to FORMAT (5I4). This program is used to initialize tape number 4 if, after $\Lambda_{(2k+2)}$ is calculated, the accuracy is not satisfactory, which means starting phase 2 from the beginning without preparing tapes 2 and 8 by use of phase 1. The total number of coefficients initialized is equal to $n_r \times J_1 \times J_2 \times J_3$.
- (2) Magnetic Tapes. Mount blank tape and set to logical 4.
- (3) Stops. HPR 11111₈ - end of program.

Auxiliary Program No. 3

- (1) Preparation of Data. The following information: J_1, J_2, J_3, n_r , and i_3 , is punched on the first card according to FORMAT (5I4). On the second card is punched the information, J_{12}, J_{13} , and J_{14} according to FORMAT (3I3). This program positions tapes 2, 4, and 8 in phase 1 when coefficients ultimately are to be calculated for n_r rods, but the calculation has been interrupted after the coefficients for i_2 rods have been made. Then i_3 is equal to $i_2 + 1$, and $\leq n_r$.
- (2) Stops. HPR 44444₈ - end of program.

Auxiliary Program No. 4

Part 1.

- (1) Preparation of Data. Punch on one card the information $J_1, J_2, J_3, J_4, n_r, i_1, i_2$ and J_k according to FORMAT (8I4).
- (2) Magnetic Tapes. Mount a blank tape and set to logical 5. On this tape will be written the values of $J_m (\beta_{mn}), \beta_{mn}, m$, and n , where β_{mn} is calculated by the formulas mentioned in the description of phase 1. This information is written in B.C.D. and is printed off-line. From this output one can find which roots are unsatisfactory. An end of file is written by the program on tape 5. This tape is saved for use in part 2.
- (3) Stops. HPR 33333₈ - end of program.

Part 2.

- (1) Preparation of Data. From the results on tape 5 in part 1, one finds, by use of the Harvard Tables on Bessel functions, which roots cannot be accurately calculated by the usual formulas. These corrected roots designated as CORTMN, the indices designated as MM and NN, and a tag IND used to designate end of corrections are all punched on cards (one at a time) according to FORMAT (1H1PE14.7, 3I4). The tag IND if not equal to zero means that no more corrections are to be made after that card and if equal to zero means that corrections are to be continued.
- (2) Magnetic Tapes. From part 1, tape number 5 is used as an input tape for examination of which roots are to be corrected. The corrected roots and appropriate indices are written on tape number 3, which must be mounted. A manual end of file must be put on tape number 3. Tape number 3 is in B.C.D. and may be printed off line as a check to see if the proper roots have been corrected when compared with the output of tape 5.
- (3) Stops. No machine stops have been designated.

Part 3.

- (1) Preparation of Data. See (2) below.
- (2) Magnetic Tapes. Tape Number 3 from Part 2, containing the corrected roots β_{mn} , is mounted and set to logical 3. This tape is now read in as an input tape. Mount two blank tapes and set to logical 1 and 5. The program will now write in binary on tape number 1 the roots β_{mn} and in B.C.D. the information $J_m(\beta_{mn})$, β_{mn} , m and n, which can be printed off-line for final inspection as to how good the roots β_{mn} are. Manual end of files are put on tapes 1 and 5.
- (3) Stops. No stops are designated for this program.

Auxiliary Program No. 5

- (1) Preparation of Data. The following information: J_{11} , J_{22} , J_1 and J_2 , is punched on one card according to FORMAT (1H4I4).

- (2) Magnetic Tapes. Mount tape number 1 from Auxiliary Program No. 4, Part 2, and set to logical 1. Mount a blank tape and set to logical 7. Mount a blank tape and set to logical 6. The program will read the 5000 roots β_{mn} from tape 1 into the memory. It will then make up a tape 7 with $m \times n$ roots, where $m \leq 50$ and $n \leq 100$ according to the requestor's demands for the partial sums to be calculated in the various series. As a check to see if tape 7 is properly written, the information β_{mn} , m , and n is written on tape 6 in B.C.D. and may be printed off-line.

- (3) Stops. HPR 77777₈ - end of program.

Procedure for the Determination of the Number of Terms to be Taken in Summing the Various Series. Since, as previously mentioned, the number of terms to be taken is a function of the geometry being considered, it is felt that it would be more efficient to run a few test cases. The procedure to be outlined to run test cases for one or two control rods will save a great deal of machine time when calculations are made for many control rods.

A particular calculation may be considered reliable if, with increasing numbers of terms taken in evaluating the various series, the buckling or first eigenvalue shows no substantial change and the flux along the control surfaces satisfies the homogeneous Dirichlet boundary conditions. Of particular importance is the satisfaction of the boundary condition across the control rod tips.

There are two reasons which make the satisfaction of the boundary condition across the control tip extremely important. The first is that the actual mathematical control rod length may be determined by this procedure. In order to determine this length, one may form flux plots along the effective radii of the control; the position where the flux begins to increase away from the zero boundary condition is the point where the actual mathematical control rod ends. Of course, because a Fourier-type series is involved here this point may fluctuate a little for various numbers of terms taken. However, after a certain number of terms, this point essentially does not change. As a result of this procedure, one may determine the least amount of terms to get high accuracy.

The second reason that makes this approach very important is that the buckling will automatically be determined for the exactly or very close to the exactly correct control rod length considered.

The various calculations made in this paper will serve as useful guides in these preliminary studies. Three cases that certainly should be tried first are the fully inserted central control rod using the Auxiliary

Program No. 1 as a checking feature, the central halfway inserted control rod, and the halfway off-centered control rod. In this way, a typical picture may be obtained for the determination of the proper choice of the indices involved. In particular, a helpful check point is that the buckling for a small ($r_0 \ll R$) control rod inserted halfway is equal to ~ 0.55 of the total change in buckling due to a fully inserted central control rod.

There is no provision to take care of underflow except by a built-in device. The limits that the indices may take are $m \leq 50$, $n \leq 100$ and $l \leq 100$.

To further aid the user of this code some general information will be given regarding limitations on the rod position parameters, general behavior of the mathematically effective length as a function of the number of terms used, and on estimates of the running times for phases 1, 2, and 3.

In regard to the range of validity of the position parameters on what in essence is the same thing the ranges of the arguments of the subroutines, it was found that the following bounds on these parameters yielded acceptable calculations, namely,

$$\begin{aligned} 10^{-4} &\leq d_i \leq 0.65R \\ 0.25 H &\leq h_i \leq 0.75H \\ 0.01 R &\leq r_i \leq 0.10R \\ 0 &\leq \alpha_i \leq 2\pi \end{aligned}$$

It was also found that one had to specify a mathematically effective control rod length of 10 to 25% longer than the actual length one wished to consider. Some approximate, but indicative, figures are given as follows:

Percent Increase of Mathematically Effective Length over Actual Length	Total Number of Terms Number of Dimensions		
	One	Two	Three
0	1 250 - 500 250 - 500	$n \times 1$	$m \times n \times 1$
15 - 10		250 - 500 $50 \times 5 - 100 \times 5$	
30 - 20			800 - 4000 $10 \times 40 \times 2 - 20 \times 50 \times 4$

There is some irregularity or change of rod radius also, but this is really a much smaller effect than the length of the rod being shortened because of the partial sum representation of the sink absorption distribution

assumed initially in the theory. It was also found that the number of terms for the m and n indices was always much greater than the number of terms required for the l index.

Since tape 8 in phase 1 will have $n_r \times m \times n \times l$ records on it, the maximum number of terms for $n_r = 10$ would then have to be about $m \times n \times l = 320$. Thus, for a ten-rod calculation one would pick something like $10 \times 16 \times 2$ terms, but chances are that the accuracy of the calculation would suffer somewhat. However, the total change in buckling would be great and might result in at least a fair order of approximation of the buckling and flux. The restriction is due to the smallness of the particular IBM 704 for which this code was written and not due to the theory. Thus, on larger machines more rods and more terms may be easily added by altering the code slightly. As it stands, however, it is not strongly recommended that ten-rod calculations be made. Four-six rod calculations would be more reasonable problems to consider.

Some idea of the running times for phases 1, 2, and 3 will now be given. For phase 1 the running time is about

$$n_r \times m \times n \times l \times (2 \times 10^{-4}) \text{ hr.}$$

For phase 2 the running time is about

$$n_r \times m \times n \times l \times J_4 \times J_{15} \times (4 \times 10^{-5}) \text{ hr.}$$

For phase 3 the running time per flux point is about

$$n_r \times m \times n \times l \times (4 \times 10^{-5}) \text{ hr.}$$

These running times are estimates, it must be remembered, and would tend to be slightly less if one- or two-rod calculations were made. They would tend to be a little greater if three- to six-rod calculations were made.

It was also found that J_k gave good results when set equal to one, i.e., one term in the series arising from use of the addition formulas was usually enough for the geometries considered.

Certain techniques for improving the convergence of series* have been tried. These techniques, however, did not seem to improve the convergence of the series developed in the above theory and were thus not incorporated in the code.

*Shanks, Daniel, Journal of Mathematics and Physics, 34 (1955-56)

Table V

FORTRAN Symbols and Meaning in Code		
FORTRAN Symbol	Meaning or Mathematical Symbol Represented	Remarks
U(M)	Numbers 0, 1, 2, ..., M-1	Floating point integers generated.
$Q\phi(M)$	q	Used to calculate roots of Bessel functions.
Q1(M)	$7q - 31$	" " "
Q2(M)	$83q^2 - 982q + 3779$	" " "
Q3(M)	$6949q^3 - 153855q^2$ $+ 1585743q - 6277237$	" " "
Q(J)	Numbers 1, 2, ..., J	Floating point integers generated.
Q(N)	" 1, 2, ..., N	
Q(L)	" 1, 2, ..., L	
RT(M, N)	β_{mn}	Roots of Bessel Functions.
$C\phi(I)$	$C_i' = \frac{4r_i}{R}; C_i = \frac{8\pi r_i}{R}$	
RR(I)	r_i	Control rod radii.
D(I)	d_i	Distances between control rod centers and reactor center.
HR(I)	h_i	Insertion length of control rods.
ALPH(I)	α_i	Control rod angles made with reference rod.
R	R	Reactor radius.

Table V (Cont'd.)

<u>FORTTRAN Symbol</u>	<u>Meaning or Mathematical Symbol Represented</u>	<u>Remarks</u>
H	H	Reactor height.
RIN	r_{in}	Initial point for radial plot.
ZIN	z_{in}	Initial point for vertical plot.
THETIN	θ_{in}	Initial point for angular plot.
DELTA E	$\Delta\Lambda$	Used to increment Λ when finding root.
DELTA R	Δr	Increment for radial plot.
DELTA Z	Δz	Increment for vertical plot.
DELTA O	$\Delta\theta$	Increment for angular plot.
ERROR	ϵ	Error stipulated between $\Lambda^{(2k+1)}$ and $\Lambda^{(2k+2)}$.
S	S	Variable used to divide $\Delta\Lambda$.
EO	Λ_0	First guess of eigenvalue $\Lambda^{(1)} \cdot \Lambda_0 \gtrsim \lambda_{mnl}$.
EIN	Λ_{in}	First guess of eigenvalue $\Lambda^{(2)} \cdot \Lambda^{(2)} \gtrsim \lambda_{mnl}$.
DELPR	$\Delta' r$	Used to initialize radial plot.
DELPZ	$\Delta' z$	Used to initialize vertical plot.
DELPO	$\Delta' \theta$	Used to initialize angular plot.
AREA	M	Migration area.
J1	m	Index m in series.

Table V (Cont'd.)

<u>FORTTRAN Symbol</u>	<u>Meaning or Mathematical Symbol Represented</u>	<u>Remarks</u>
J2	n	Index n in series.
J3	ℓ	Index l in series.
J4	$\Delta\Lambda/S^J$ and no. of sign changes.	Number of sign changes near roots.
NR	n_r	Total number of rods.
J15	$J_{15} = m \times n \times \ell$	Total number of coefficients $c_{mn\ell}$ to be calculated using tape 4.
J12	Mo.	Integers for date.
J13	Day	" " "
J14	Yr.	" " "
PI	π	π set equal to 3.1415926 in program.
DELTIN	$\Delta\Lambda_{in}$	Used to initialize $\Delta\Lambda$ after iteration.
BE	β	Dummy variable to conserve storage.
BB	λ_{011}	$\lambda_{011} = \left(\frac{\beta_{01}}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2$.
JJ5	J_{J5}	Counter for number of iterations made.
ROOTMN	β_{mn}	Dummy variable for roots of Bessel functions used to conserve storage.
JBESSF(X, N)	$J_n(x)$	Bessel function of index n.G.E.

Table V (Cont'd.)

<u>FORTTRAN Symbol</u>	<u>Meaning or Mathematical Symbol Represented</u>	<u>Remarks</u>
SINF(X)	$\sin(x)$	Sine function.
A	a_{mn}	Dummy variable used to conserve storage.
B	b_{ℓ}	" " "
QU	$Q_{mn\ell}^j = J_m \left(\frac{\beta_{mn} d_j}{R} \right) R J_1 \left(\frac{\beta_{mn} r_j}{R} \right) *$ $\sin \frac{\ell \pi h_1}{H} / \beta_{mn} + \frac{H}{\ell \pi} J_0 \left(\frac{\beta_{mn} r_j}{R} \right) *$ $\left(1 - \cos \frac{\ell \pi h_j}{H} \right) (\cos m \alpha_i \cos m \alpha_j + \sin m \alpha_i \sin m \alpha_j)$	Dummy variable used to conserve storage--arises from surface integration.
P	$P_{mn\ell} = R^2 H J_1 (\beta_{on}) / \beta_{on\ell} *$ $(1 - \cos \ell \pi)$	Dummy variable used to conserve storage--arises from volume integrations.
E	$\lambda_{mn\ell}$	Dummy variable used to conserve storage.
C	$C_{mn\ell} = \left \frac{\Lambda(2k+2) - \lambda_{mn\ell}}{\Lambda(2k+1) - \lambda_{mn\ell}} \right $	" " "
SUMA	$\sum_{i=1}^{n_r} c_i' \sum_{m=0, n, \ell}^{0, \infty, \infty} \frac{a_{mn} b_{\ell} c_{mn\ell}}{\Lambda^{(1)} - \lambda_{mn\ell}} \sum_j^{n_r} r_j Q_{mn\ell}^j$	Partial sum for odd order approximation of eigenvalue.
El	$\Lambda(2k+1)$	Dummy variable for odd order approximations of the eigenvalue.
SUMC	$\sum_{i=1}^{n_r} c_i' \sum_{m=0, n, \ell}^{0, \infty, \infty} \frac{a_{mn} b_{\ell} c_{mn\ell}}{\Lambda^{(1)} - \lambda_{mn\ell}} \left(P_{mn\ell} + \sum_{j=1}^{n_r} z_{jmn\ell}^1 \right)$	Partial sum for odd order approximation for eigenvalue.

Table V (Cont'd.)

<u>FORTRAN Symbol</u>	<u>Meaning or Mathematical Symbol Represented</u>	<u>Remarks</u>
SUME	$\sum_{i=1}^{n_r} c_i' \sum_{m,n,\ell}^{\infty} \frac{a_{mn} b_{\ell} c_{mn\ell}}{\Lambda^{(i)} - \lambda_{mn\ell}} \left(P_{mn\ell} + \sum_{j=1}^{n_r} z_{mn\ell}^j \right)$	Partial sum for even order approximations for eigenvalue.
E2	$\Lambda^{(2k+2)}$	Dummy variable for even order approximations of the eigenvalue.
DELKOK	$\Delta k/k$	Reactivity worth.
SUMH	$\sum_{i=1}^{n_r} c_i \sum_{m,n,\ell}^{\infty} \frac{a_{mn} b_{\ell} c_{mn\ell}}{\Lambda^{(i)} - \lambda_{mn\ell}} J_m \left(\frac{\beta_{mn} r_{in}}{R} \right) \left(\cos m\alpha_1 \cos m\theta_{in} + \sin m\alpha_1 \sin \theta_{in} \right) \frac{\ell \pi z_{in}}{\sin \frac{\ell \pi z_{in}}{H}}$	Partial sum for flux.
NOFLX	n_{flux}	Number of approximations made for eigenfunction.
TEST	Constant for altering path of calculation.	Variable used in phase 1. to set all coefficients on tape 4. equal to one if TEST = 0. If TEST $\neq$ 0 initializing tape 4. will be omitted and calculation will proceed.
TABLE(M)	Reserves storage.	Used in conjunction with N.Y.U. Bessel function subroutine.
TABL2(M)	" "	" " "
TABL3(M)	" "	" " "
TABL4(M)	" "	" " "
TABL5(M)	" "	" " "
ALPF(X, Y)	Function statement.	Used to define a subroutine which in turn is used in a Bessel function subroutine.

Table V (Cont'd.)

<u>FORTTRAN Symbol</u>	<u>Meaning or Mathematical Symbol Represented</u>	<u>Remarks</u>
BETF(X, Y)	Function statement.	Used to define a subroutine which in turn is used in a Bessel function subroutine.
JBES1(X, Y)	" "	Used to define a subroutine for the asymptotic form of $J_m(X)$ for large index m , where $m < x$. See Morse and Feshbach, Vol. II.
JBES2(X, Y)	" "	Used to define a subroutine for the asymptotic form of $J_m(X)$ for large index m , where $m > x$. See Morse and Feshbach, Vol. II.
GNU	Part of N.Y.U. Bessel Function subroutine.	Fractional part ν in the function $J_n \pm \nu(X)$ where n is an integer.
LE	" "	
LEE	" "	
LEEE	" "	
LFEE	" "	
LFFE	" "	
JK	Index in series.	Index k in the series arising from using the additional formulas for Bessel function.
I1	i_1	Index used to calculate coefficients in phase 1. from i_1 to n_r rods.
I2	i_2	Index used to interrupt the calculation of coefficients in phase 1. when calculating from i_1 to i_2 rods where $i_2 \leq n_r$.

Table V (Cont'd.)

<u>FORTTRAN Symbol</u>	<u>Meaning or Mathematical Symbol Represented</u>	<u>Remarks</u>
S8	$S_8 = J_1 \left(\frac{\beta_{01} r_i}{R} \right)$	Dummy variable for the value of G.E. Bessel function subroutine.
S9	$S_9 = J_0 \left(\frac{\beta_{01} r_i}{R} \right)$	" " "
S16	$S_{16} = J_0 \left(\frac{\beta_{01} d_i}{R} \right)$	" " "
ARG	$\beta_{mn} d_i$	Argument of Bessel function subroutines.
AS		Exit location for N.Y.U. Bessel function subroutine. If AS = 0 subroutine has been properly executed. If AS $\neq$ 0, subroutine incorrectly executed.
S6	$S_6 = J_{m+1} (\beta_{mn})'$	Dummy variable for N.Y.U. Bessel function subroutine of argument β_{mn} .
S7	$S_7 = J_m (\beta_{mn})$	Dummy variable for G.E. Bessel function subroutine of argument β_{mn} .
GAMK	$\Gamma_k = \sum_{k=1}^{\infty} \left(\frac{1}{\beta_{01}^2 - \beta_{mn}^2} \right) (\beta_{01} s_{10} s_{13} - \beta_{mn} s_{12} s_{11}) s_{17} s_{14} (-1)^{2k-1} \beta_{01} s_{10} s s_{12} - \beta_{mn} s_{12} s s_{10} (-1)^k s_{17} s s_{14})$	Partial sum of series.
GAMO	$\Gamma_0 = \left(\frac{1}{\beta_{01}^2 - \beta_{mn}^2} \right) (\beta_{01} s_7 s_8 - \beta_{mn} s_9 s_2)$	First term in series.

Table V (Cont'd.)

<u>FORTTRAN Symbol</u>	<u>Meaning or Mathematical Symbol Represented</u>	<u>Remarks</u>
BS		Similar to AS.
SS10	$S_{S10} = J_{k-1} \left(\frac{\beta_{mn} r_i}{R} \right)$	Dummy variable for N.Y.U. Bessel function subroutine of argument $\beta_{mn} r_i / R$
S10	$S_{10} = J_k \left(\frac{\beta_{mn} r_i}{R} \right)$	" " "
S11	$S_{11} = J_{k+1} \left(\frac{\beta_{mn} r_i}{R} \right)$	" " "
S17	$S_{17} = J_k \left(\frac{\beta_{01} d_i}{R} \right)$	Dummy variable for G.E. Bessel function subroutine of argument $\beta_{01} d_i / R$
DS		Same as AS.
NEWMK	$new_{mk} = m + k$	Change of variable to allow index $m + k$.
S14	$S_{14} = J_{m+k} \left(\frac{\beta_{mn} d_i}{R} \right)$	Dummy variable for N.Y.U. Bessel function subroutine.
T	$t = m + k$	Variable used to float fixed point variable.
TT	$tt = m - k$	" " "
DS		Same function as AS.
SS14	$S_{S14} = (-1)^{m-k} J_{m-k} \left(\frac{\beta_{mnd_i}}{R} \right)$	Dummy variable for Bessel function subroutines depend- ing on argument and index.

Table V (Cont'd.)

<u>FORTTRAN Symbol</u>	<u>Meaning or Mathematical Symbol Represented</u>	<u>Remarks</u>
L1	$\ell_1 = k - m$	Change of variable to allow index $k - m$.
L2	$\ell_2 = m - k$	Change of variable to allow index $m - k$.
XLOCF	Part of N.Y.U. Bessel function.	Location function subroutine.
FK	$f_k = k$	Float fixed point variable.
S5	$S_5 = J_m \left(\frac{\beta_{mn} d_i}{R} \right)$	Dummy variable for G.E. Bessel function subroutine.
ES		Same function as AS.
SS5	$S_{S5} = J_m \left(\frac{\beta_{mn} d_j}{R} \right)$	Dummy variable for Bessel function subroutines for j^{th} control rod.
Z	$Z_{mn\ell}^j = \frac{-Rr_j H}{\ell \beta_{mn}} s_{S5} J_1 \left(\frac{\beta_{mn} r_j}{R} \right)^*$ $\left(1 - \cos \frac{\ell \pi h_j}{H} \right) (\cos m\alpha_i \cos m\alpha_j + \sin m\alpha_i \sin m\alpha_j)$	Term arising when interaction of i^{th} and j^{th} control rod is considered.
TEST1	Test constant.	If $\text{TEST1} > 0$ then the phase 2. calculation will begin iterations at $\Lambda^{(2k+1)}$ where $k > 1$.
TEST2	" "	If $\text{TEST1} \leq 0$ and $\text{TEST 2} > 0$ then the phase 2. calculation will begin at $\Lambda^{(2k+1)}$, where $k = 0$. If $\text{TEST2} \leq 0$, then the phase 2. calculation will begin with making up the coefficient $C_{mn\ell}$.

Table V (Cont'd.)

<u>FORTTRAN Symbol</u>	<u>Meaning or Mathematical Symbol Represented</u>	<u>Remarks</u>
R1	R_1	Actual radius of reactor considered in centimeters.
THETAF(X,Y)	$\theta(x,y) = J_0(x) - \frac{J_0(y)}{Y_0(y)} Y_0(x)$	Subroutine defining the exact solution for one central rod fully inserted.
RO	r_o	Effective radius for fully inserted central rod.
DETAL	$\Delta\alpha$	Increment for root of transcendental equation for fully inserted central rod.
ALPHIN	α_{in}	Initial guess for root of transcendental equation for fully inserted central rod.
C2	$C_2 = C_1 \theta(\alpha_{in} * 15 r_o, \alpha_{in})$	Normalization constant used to normalize solution obtained by use of the transcendental equation for the fully inserted central rod so as to compare flux plots obtained by use of phases 1., 2., and 3.
FUNCT	$F = \theta(x,y)$	Value of transcendental function for the flux for the fully inserted central rod at various points r, and Z.
EIGVL1	$B^2 = \left(\frac{\beta}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2$	Buckling obtained by the use of the Nordheim-Scalletar theory for a fully inserted central rod.
EIGVL2	$B^2 = \left(\frac{\alpha}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2$	Buckling obtained by use of the transcendental equation for a fully inserted central rod.

Table V (Cont'd.)

<u>FORTTRAN Symbol</u>	<u>Meaning or Mathematical Symbol Represented</u>	<u>Remarks</u>
DELBE	$\Delta\beta$	Change in radial component of buckling using Nordheim-Scalletar Theory.
BETA	$\beta = \left(\frac{2.4048}{R} + \Delta\beta \right)$	Radial component of buckling using Nordheim-Scalletar Theory.
I3	i_3	Used in conjunction with J1, J2, and J3 to initialize tape 4. in phase 1. It is also used to reset tapes 2., 4., and 8. in phase 1. if only part of the control rod coefficients have been calculated.
CORTMN	β'_{mn}	Corrected roots of J_m found by use of Harvard Tables for Bessel functions.
MM	m'	Index used to tell program which roots are to be corrected.
NN	n'	" " "
IND	Index	Tag to signal program when to stop correcting roots.
S6	$S_6 = J_m (\beta_{mn})$	S_6 has this slightly different meaning in Auxiliary Program No. 4. than in phase 1.
J11	J_{11}	Used in Auxiliary Program No. 5. to read in m roots.
J22	J_{12}	Used in Auxiliary Program No. 5 to read in n roots.
J1	J_1	Used in Auxiliary Program No. 5. to write m roots.

Table V (Cont'd.)

<u>FORTTRAN Symbol</u>	<u>Meaning or Mathematical Symbol Represented</u>	<u>Remarks</u>
J2	J_2	Used in Auxiliary Program No. 5. to write n roots.
S13	$S_{13} = J_{k+1} \left(\frac{\beta_{01} r_i}{R} \right)$	Dummy Variable
S12	$S_{12} = J_k \left(\frac{\beta_{01} r_i}{R} \right)$	" "
SS12	$S_{s12} = J_{k-1} \left(\frac{\beta_{01} r_i}{R} \right)$	" "
S2	$S_2 = J_1 \left(\frac{\beta_{mn} r_i}{R} \right)$	" "

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C      PROGRAM FOR THE CALCULATION OF THE BUCKLING AND FLUX FOR A BARE PHS1 001
C      CYLINDRICAL HOMOGENEOUS REACTOR WITH PARTIALLY INSERTED CONTROL PHS1 002
C      RODS PHS1 003
C      PHS1 004
C      PHASE 1-CALCULATION OF FOURIER-BESSEL COEFFICIENTS PHS1 005
C      SURFACE INTEGRAL COEFFICIENTS,AND VOLUME INTEGRAL PHS1 006
C      COEFFICIENTS. PHS1 007
C      4/30/59 PHS1 008
C      PHS1 009
C      E.J.BETINIS PHS1 010
C      PHS1 011
C      DIMENSION U(51),Q0(50),Q1(50),Q2(50),Q3(50),Q(100),RT(1,100), PHS1 012
C      1CO(10),RR(10),D(10),HR(10),ALPH(10),TABLE(120),TABL2(120),TABL3(120), PHS1 013
C      20),TABL4(120),TABL5(120) PHS1 014
C      ALPF(X,Y)=.5*LOGF((1.+SQRTF(1.-(X/Y)**2))/(1.-SQRTF(1.-(X/Y)**2))) PHS1 015
C      BETF(X,Y)=ATANF(SQRTF((X/Y)**2-1.)) PHS1 016
C      JBES1F(X,Y)=EXPF(Y*(TANH(F(ALPF(X,Y))-ALPF(X,Y)))/SQRTF(2.*PI PHS1 017
C      1*Y*TANH(F(ALPF(X,Y)))) PHS1 018
C      JBES2F(X,Y)=SQRTF(2./(PI*Y*SINF(BETF(X,Y))/COSF(BETF(X,Y))))*COSF PHS1 019
C      1Y*SINF(BETF(X,Y))/COSF(BETF(X,Y))-Y*BETF(X,Y)-PI/4.) PHS1 020
C      READ 5,R,H,RIN,ZIN,THETIN,DELTA E,DELTA R,DELTA Z,DELTA O, PHS1 021
C      1ERROR,S,E0,EIN,DELPR,DELPZ,DELPO,(HR(I),I=1,10),(RR(I),I=1,10), PHS1 022
C      2(D(I),I=1,10),(ALPH(I),I=1,10),AREA,TEST PHS1 023
C      5 FORMAT(1H 1P5E14.7) PHS1 024
C      READ 3,J1,J2,J3,J4,NR,I1,I2,JK PHS1 025
C      3 FORMAT(8I4) PHS1 026
C      READ 304,J15 PHS1 027
C      304 FORMAT(1I6) PHS1 028
C      READ 303,J12,J13,J14 PHS1 029
C      303 FORMAT(3I3) PHS1 030
C      READ 370 PHS1 031
C      370 FORMAT(72H0 PHS1 032
C      1 ) PHS1 033
C      PRINT 301 PHS1 034
C      301 FORMAT(1H1) PHS1 035
C      PRINT 302,J12,J13,J14 PHS1 036
C      302 FORMAT(6H DATE 12,1H/I2,1H/I2) PHS1 037
C      PRINT 370 PHS1 038
C      PRINT 307 PHS1 039
C      307 FORMAT(26H0BEGIN MONITORING PHASE 1.) PHS1 040
C      PRINT 306 PHS1 041
C      306 FORMAT(24H0MATHEMATICAL PARAMETERS) PHS1 042
C      PRINT 3,J1,J2,J3,J4,NR,I1,I2,JK PHS1 043
C      PRINT 314,J1,J2,J3 PHS1 044
C      314 FORMAT(31H TOTAL NUMBER OF TERMS PER ROD=I3,1HXI3,1HXI3) PHS1 045
C      PRINT 345,NR PHS1 046
C      345 FORMAT(22H TOTAL NUMBER OF RODS=I3) PHS1 047
C      PRINT 312 PHS1 048
C      312 FORMAT(24H0ROD POSITION PARAMETERS) PHS1 049
C      PRINT 313,(HR(I),I=1,10),(RR(I),I=1,10),(D(I),I=1,10),(ALPH(I),I=1 PHS1 050
C      1,10) PHS1 051
C      313 FORMAT(1H 10F10.3) PHS1 052
C      GNU=0.0 PHS1 053
C      LE=120 PHS1 054
C      LEE=120 PHS1 055
C      LEEE=120 PHS1 056
C      LFEE=120 PHS1 057
C      LFFE=120 PHS1 058
C      LM=XLOC(F(TABLE(1)) PHS1 059
C      LMM=XLOC(F(TABL2(1)) PHS1 060
C      LMMM=XLOC(F(TABL3(1)) PHS1 061
C      LNMM=XLOC(F(TABL4(1)) PHS1 062
C      LNNM=XLOC(F(TABL5(1)) PHS1 063
C      U(1)=0.0 PHS1 064
C      DO 4 K=1,J1 PHS1 065
C      4 U(K+1)=U(K)+1.0 PHS1 066
C      PI=3.1415926 PHS1 067
C      DO 6 M=1,J1 PHS1 068
C      Q0(M)=4.*U(M)**2 PHS1 069
C      Q1(M)=7.*Q0(M)-31. PHS1 070
C      Q2(M)=83.*Q0(M)**2-982.*Q0(M)+3779. PHS1 071
C      6 Q3(M)=6949.*Q0(M)**3-153855.*Q0(M)**2+1585743.*Q0(M)-6277237. PHS1 072
C      Q(1)=1.0 PHS1 073
C      DO 7 J=2,100 PHS1 074
C      7 Q(J)=Q(J-1)+1.0 PHS1 075
C      DO 8 M=1,1 PHS1 076
C      DO 8 N=1,100 PHS1 077
C      BE=(U(M)-0.5+2.*Q(N))*PI/2. PHS1 078
C      8 RT(M,N)=BE-(Q0(M)-1.)/(8.*BE)*(1.+Q1(M)/(3.*(4.*BE)**2)+ PHS1 079
C      12.*Q2(M)/(15.*(4.*BE)**4)+Q3(M)/(105.*(4.*BE)**6)) PHS1 080

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      RT(1,1)=2.4048256000
      IF(SENSE SWITCH 5) 9,10
9 PRINT 308,RT(1,1),RT(1,2),RT(1,3),RT(1,4),RT(1,5)
308 FORMAT(26H0THE FIRST FIVE ROOTS ARE 1P5E14.7)
10 BB=(RT(1,1)/R)**2+(Q(1)*PI/H)**2
      IF(SENSE SWITCH 5) 12,13
12 PRINT 309,BB
309 FORMAT(56H0THE GEOMETRIC BUCKLING FOR THE REACTOR WITHOUT RODS IS
11PE14.7)
13 IF(TEST) 3006,2,3006
2 J5=1
  J16=1
  C=1.0
305 WRITE TAPE 4,C
  IF(J16-J15) 100,101,101
100 J16=J16+1
  GO TO 305
101 END FILE 4
  REWIND 4
  PRINT 310
310 FORMAT(102H0      A(M,N)      B(L)      Q(M,N,L)      P(M,N,L)
1      E(M,N,L)      Z(M,N,L)      M+1      N      L RDNO)
3006 DO 1119 I=1,NR
  REWIND 7
  IF(I-I2) 1200,1200,555
1200 S8=JBESSF(RT(1,1)*RR(I)/R,1)
  S9=JBESSF(RT(1,1)*RR(I)/R,0)
  S16=JBESSF(RT(1,1)*D(I)/R,0)
  DO 1119 M=1,J1
  Y=M
  DO 1119 N=1,J2
  READ TAPE 7,ROOTMN
  IF(M-1) 1111,1111,1113
1111 IF(N-1) 1112,1112,1113
1112 ROOTMN=2.4048256000
1113 S2=JBESSF(ROOTMN*RR(I)/R,1)
  ARG=D(I)*ROOTMN
  IF(M-15) 3677,3667,3667
3677 IF(ROOTMN-50.) 3678,3678,3698
3678 AS=BESJF(ROOTMN,GNU,M,LE,LM)
  S6=TABLE(M+1)
  GO TO 3699
3667 IF(ROOTMN-50.) 8801,8801,3888
8801 AS=BESJF(ROOTMN,GNU,M,LE,LM)
  S6=TABLE(M+1)
  GO TO 3699
3888 IF(ROOTMN-4.*Y) 8882,8882,3698
8882 S6=JBESSF(ROOTMN,Y)
  GO TO 3699
3698 S6=JBESSF(ROOTMN,M)
3699 S7=JBESSF(ROOTMN*RR(I)/R,0)
  GAMK=0.0
  IF(RT(1,1)-ROOTMN) 2222,3333,2222
2222 IF(M-1) 2223,2223,2224
2223 GAM0=1./(RT(1,1)**2-ROOTMN**2)*(RT(1,1)*S7*S8-ROOTMN*S9*S2)
  GO TO 9100
2224 GAM0=2./(RT(1,1)**2-ROOTMN**2)*(RT(1,1)*S7*S8-ROOTMN*S9*S2)
  GO TO 9100
3333 GAM0=RR(I)/(2.*R)*(S2**2+S7**2)
9100 IF(D(I)-.001) 3130,3130,7777
7777 DO 4444 K=1,JK
  BS=BESJF(ROOTMN*RR(I)/R,GNU,K+1,LEE,LMM)
  SS10=TABL2(K)
  S10=TABL2(K+1)
  S11=TABL2(K+2)
  T=M-1+K
  CS=BESJF(RT(1,1)*RR(I)/R,GNU,K+1,LEEE,LMMM)
  SS12=TABL3(K)
  S12=TABL3(K+1)
  S13=TABL3(K+2)
  S17=JBESSF(RT(1,1)*D(I)/R,K)
  IF(M+K-16) 1234,5678,5678
5678 IF(ROOTMN*D(I)/R-50.) 7789,7789,8701
7789 DS=BESJF(ROOTMN*D(I)/R,GNU,M-1+K,LFEE,LNMM)
  NEWMK=M+K
  S14=TABL4(NEWMK)
  GO TO 9104
8701 IF(ROOTMN*D(I)/R-.5*T) 9102,9102,8802
8802 IF(ROOTMN*D(I)/R-4.*T) 9103,9101,9101
1234 IF(ROOTMN*D(I)/R-50.) 789,789,9101

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PHS1 081
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 PHS1 083
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789 DS=BESJF(ROOTMN*D(I)/R,GNU,M-1+K,LFEE,LNMM) PHS1 161
    NEWMK=M+K PHS1 162
    S14=TABL4(NEWMK) PHS1 163
    GO TO 9104 PHS1 164
9101 S14=JBESSF(ROOTMN*D(I)/R,M-1+K) PHS1 165
    GO TO 9104 PHS1 166
9102 S14=JBES1F(ROOTMN*D(I)/R,T) PHS1 167
    GO TO 9104 PHS1 168
9103 S14=JBES2F(ROOTMN*D(I)/R,T) PHS1 169
9104 TT=K-M+1 PHS1 170
    IF(TT) 9905,9904,9904 PHS1 171
9904 IF(TT-16.) 9234,9678,9678 PHS1 172
9678 IF(ROOTMN*D(I)/R-50.) 9989,9989,8702 PHS1 173
9989 L1=K-M+2 PHS1 174
    DS=BESJF(ROOTMN*D(I)/R,GNU,L1-1,LFEE,LNMM) PHS1 175
    SS14=-1.**TT*TABL4(L1) PHS1 176
    GO TO 9944 PHS1 177
8702 IF(ROOTMN*D(I)/R-.5*TT) 9972,9972,8803 PHS1 178
8803 IF(ROOTMN*D(I)/R-4.*TT) 9993,9971,9971 PHS1 179
9234 IF(ROOTMN*D(I)/R-50.) 989, 989,9971 PHS1 180
989 L1=K-M+2 PHS1 181
    DS=BESJF(ROOTMN*D(I)/R,GNU,L1-1,LFEE,LNMM) PHS1 182
    SS14 = -1.**TT*TABL4(L1) PHS1 183
    GO TO 9944 PHS1 184
9971 SS14=JBESSF(ROOTMN*D(I)/R ,L1-1)*(-1.**TT) PHS1 185
    GO TO 9944 PHS1 186
9972 SS14=JBES1F(ROOTMN*D(I)/R,TT)*(-1.**TT) PHS1 187
    GO TO 9944 PHS1 188
9993 SS14=JBES2F(ROOTMN*D(I)/R,TT)*(-1.**TT) PHS1 189
    GO TO 9944 PHS1 190
9905 TT=M-1-K PHS1 191
    IF(TT-16.) 9334,9378,9378 PHS1 192
9378 IF(ROOTMN*D(I)/R-50.) 9939,9939,8703 PHS1 193
9939 L2=M-K PHS1 194
    DS=BESJF(ROOTMN*D(I)/R,GNU,L2-1,LFEE,LNMM) PHS1 195
    SS14=-1.**TT*TABL4(L2) PHS1 196
    GO TO 9944 PHS1 197
8703 IF(ROOTMN*D(I)/R-.5*TT) 9392,9392,8804 PHS1 198
8804 IF(ROOTMN*D(I)/R-4.*TT) 9393,9391,9391 PHS1 199
9334 IF(ROOTMN*D(I)/R-50.) 939, 939,9391 PHS1 200
939 L2=M-K PHS1 201
    DS=BESJF(ROOTMN*D(I)/R,GNU,L2-1,LFEE,LNMM) PHS1 202
    SS14=-1.**TT*TABL4(L2) PHS1 203
    GO TO 9944 PHS1 204
9391 SS14=JBESSF(ROOTMN*D(I)/R ,L2-1)*(-1.**TT) PHS1 205
    GO TO 9944 PHS1 206
9392 SS14=JBES1F(ROOTMN*D(I)/R,TT)*(-1.**TT) PHS1 207
    GO TO 9944 PHS1 208
9393 SS14=JBES2F(ROOTMN*D(I)/R,TT)*(-1.**TT) PHS1 209
9944 IF(RT(1,1)-ROOTMN) 9105,9106,9105 PHS1 210
9105 GAMK=GAMK+1./(RT(1,1)**2-ROOTMN**2)*((RT(1,1)*S10*S13-ROOTMN*S12 PHS1 211
    1*S11)*S17*S14+(-1.***(2*K-1))*((RT(1,1)*S10*SS12-ROOTMN*S12*SS10)*(- PHS1 212
    21.**K)*S17*SS14) PHS1 213
    GO TO 9188 PHS1 214
9106 FK=K PHS1 215
    GAMK=GAMK+RR(I)/(2.*R)*((S10**2-2.*FK*R/(ROOTMN*RR(I))*S10 PHS1 216
    1*S11+S11**2)*S17*S14+(S10**2+(-1.***(2*K-1))*2.*FK*R/(ROOTMN* PHS1 217
    2RR(I))*S10*SS10+SS10**2)*(-1.**K)*S17*SS14) PHS1 218
9188 IF(SENSE SWITCH 4)9108,4444 PHS1 219
9108 PRINT 311,GAM0,GAMK PHS1 220
311 FORMAT(14H GAMMASUBZERO=1PE14.7,11H GAMMASUBK=1PE14.7) PHS1 221
4444 CONTINUE PHS1 222
3130 IF(M-16) 3116,3113,3113 PHS1 223
3113 IF(ROOTMN*D(I)/R-50.)3998,3998,8704 PHS1 224
3998 AS=BESJF(ROOTMN*D(I)/R,GNU,M,LE,LM) PHS1 225
    S5=TABLE(M) PHS1 226
    GO TO 3133 PHS1 227
8704 IF(ROOTMN*D(I)/R-.5*Y ) 3213,3213,8805 PHS1 228
8805 IF(ROOTMN*D(I)/R-4.*Y) 3223,3129,3129 PHS1 229
3213 S5=JBES1F(ROOTMN*D(I)/R,Y-1.) PHS1 230
    GO TO 3133 PHS1 231
3223 S5=JBES2F(ROOTMN*D(I)/R,Y-1.) PHS1 232
    GO TO 3133 PHS1 233
3116 IF(ROOTMN*D(I)/R-50.) 3118,3118,3129 PHS1 234
3129 S5=JBESSF(ROOTMN*D(I)/R,M-1) PHS1 235
    GO TO 3133 PHS1 236
3118 AS=BESJF(ROOTMN*D(I)/R,GNU,M,LE,LM) PHS1 237
    S5=TABLE(M) PHS1 238
3133 A=(S16*S5*GAM0+GAMK)/(S6**2) PHS1 239
9990 DO 1119 L=1,J3 PHS1 240

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      E=(ROOTMN /R)**2+(Q(L)*PI/H)**2
      IF(L-1) 800,800,801
800 B      =1./PI*(PI*HR(I)/(2.*H)-SINF(2.*PI*HR(I)/H)/4.)
      GO TO 199
801 B      =1./PI*1./((1.-Q(L)**2)*(Q(L)*SINF(PI*HR(I)/H)*COSF(Q(L)*PI*HR(
11)/H)-SINF(Q(L)*PI*HR(I)/H)*COSF(PI*HR(I)/H))
199 P=R**2*H*JBESSF(RT(1,N),1)/(RT(1,N)*Q(L))*(1.-COSF(Q(L)*PI))
      WRITE TAPE 2,A,B,P,E
      DO 1119 J=1,NR
      IF(M-16) 4114,4113,4113
4113 IF(ROOTMN*D(J)/R-50.) 4115,4115,8705
4115 ES=BESJF(ROOTMN*D(J)/R,GNU,M,LFFE,LNNM)
      SS5=TABL5(M)
      GO TO 9991
8705 IF(ROOTMN*D(J)/R-.5*Y ) 4213,4213,8806
8806 IF(ROOTMN*D(J)/R-4.*Y) 4223,5416,5416
4213 SS5=JBES1F(ROOTMN*D(J)/R,Y-1.)
      GO TO 9991
4223 SS5=JBES2F(ROOTMN*D(J)/R,Y-1.)
      GO TO 9991
4114 IF(ROOTMN*D(J)/R-50.) 5415,5415,5416
5415 ES=BESJF(ROOTMN*D(J)/R,GNU,M,LFFE,LNNM)
      SS5=TABL5(M)
      GO TO 9991
5416 SS5=JBESSF(ROOTMN*D(J)/R,M-1)
9991 QU=SS5*(R*JBESSF(ROOTMN*RR(J)/R,1)*SINF(Q(L
1)*PI*HR(J)/H)/ROOTMN+H/(Q(L)*PI)*JBESSF(ROOTMN*RR(J)/R,0)*(1.-COSF
2(Q(L)*PI*HR(J)/H)))*(COSF(U(M)*ALPH(I))*COSF(U(M)*ALPH(J))
3+SINF(U(M)*ALPH(I))*SINF(U(M)*ALPH(J)))
1992 Z=-R*RR(J)*H/(Q(L)*ROOTMN)*SS5*JBESSF(ROOTMN*RR(J)/R,1)*
1(1.-COSF(Q(L)*PI*HR(J)/H))*(COSF(U(M)*ALPH(I))*COSF(U(M)*ALPH
2(J))+SINF(U(M)*ALPH(I))*SINF(U(M)*ALPH(J)))
      IF(AS+BS+CS+DS+ES) 666,9992,666
9992 IF(SENSE SWITCH 1) 1129,1119
1129 PRINT 55,A,B,QU,P,E,Z,M,N,L,I
      55 FORMAT(1H 1P6E14.7,4I4)
      IF( SENSE SWITCH 6) 3019,1119
3019 PRINT 55,ARG,SS,S6,S14,SS14,S10,M,N,K,I
1119 WRITE TAPE 8,QU,Z
      END FILE 2
      REWIND 2
      END FILE 8
      REWIND 8
      REWIND 7
1000 PRINT 320
320 FORMAT(24H END MONITORING PHASE 1.)
      PRINT 301
      PUNCH 5,R,H,RIN,ZIN,THETIN,DELTA E,DELTA R,DELTA Z,DELTA O,
1ERROR,S,E,O,EIN,DELPR,DELPZ,DELPO,(HR(I),I=1,10),(RR(I),I=1,10),
2(D(I),I=1,10),(ALPH(I),I=1,10),AREA
      PUNCH 3,J1,J2,J3,J4,NR,I1,I2,JK
      PUNCH 304,J15
      PUNCH 303,J12,J13,J14
      STOP 7777
666 PRINT 6666
6666 FORMAT(117H0AN ERROR OCCURRED IN THE EXECUTION OF NYU BESSEL FUNCT
1ION SUBROUTINE.THIS IS EITHER A MACHINE ERROR OR A DATA ERROR.)
      PRINT 330,M,N,L,I,J
330 FORMAT(39H0THE CURRENT VALUES OF THE INDICES ARE 513)
      PRINT 301
      END FILE 2
      REWIND 2
      END FILE 8
      REWIND 8
      STOP 66666
555 I=I-1
      PRINT 5555,I
5555 FORMAT(67H0PHASE 1. HAS BEEN INTERRUPTED.THE COEFFICIENTS WERE CAL
1CULATED FOR13,14H CONTROL RODS.)
      PRINT 301
      END FILE 2
      REWIND 2
      END FILE 8
      REWIND 8
      STOP 55555
      END(0,1,0,0,0)

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C
C      PROGRAM FOR THE CALCULATION OF THE BUCKLING AND FLUX FOR A BARE
C      CYLINDRICAL HOMOGENEOUS REACTOR WITH PARTIALLY INSERTED CONTROL
C      RODS
C      PHASE 2 (PHASE 2(A) AND PHASE 2(B) )-CALCULATION OF THE VARIOUS
C      ORDERS OF APPROXIMATION OF THE GEOMETRIC BUCKLING.
C      4/30/59
C
C      E.J. BETINIS
C
C      DIMENSION CO(10),RR(10),D(10),HR(10),ALPH(10),RT(1,1)
C      READ 5,R,H,RIN,ZIN,THETIN,DELTAET,DELTAR,DELTAZ,DELTAO,
C      1ERROR,S,EO,EIN,DELPR,DELPZ,DELPO,(HR(I),I=1,10),(RR(I),I=1,10),
C      2(D(I),I=1,10),(ALPH(I),I=1,10),AREA,TEST1,TEST2
C      5 FORMAT(1H 1P5E14.7)
C      READ 3,J1,J2,J3,J4,NR,I1,I2,JK
C      3 FORMAT(8I4)
C      READ 304,J15
C      304 FORMAT(1I6)
C      READ 303,J12,J13,J14
C      303 FORMAT(3I3)
C      READ 370
C      370 FORMAT(72H0
C      1
C      )
C      READ 556,E2,BB,JJ5
C      PI=3.1415926
C      DELTIN=DELTAET
C      RT(1,1)=2.4048256
C      13 DO 14 K=1,NR
C      14 CO(K)=8.*RR(K)/R*PI
C      PRINT 306
C      306 FORMAT(1H1)
C      PRINT 307,J12,J13,J14
C      WRITEOUTPUTTAPE5,307,J12,J13,J14
C      307 FORMAT(6H DATE 12,1H/I2,1H/I2)
C      PRINT 370
C      PRINT 308
C      308 FORMAT(26H0BEGIN MONITORING PHASE 2.)
C      PRINT 360
C      360 FORMAT(24H0MATHEMATICAL PARAMETERS)
C      PRINT 344,J1,J2,J3
C      344 FORMAT(31H TOTAL NUMBER OF TERMS PER ROD=I3,1HXI3,1HXI3)
C      PRINT 3,J1,J2,J3,J4,NR,I1,I2,JK
C      PRINT 345,NR
C      345 FORMAT(22H TOTAL NUMBER OF RODS=I3)
C      PRINT 312
C      312 FORMAT(24H0ROD POSITION PARAMETERS)
C      PRINT 313,(HR(I),I=1,10),(RR(I),I=1,10),(D(I),I=1,10),(ALPH(I),I=1,10)
C      313 FORMAT(1H 10F10.3)
C      PRINT 6000
C      6000 FORMAT(24H0BEGINNING OF ITERATIONS)
C      IF(TEST1) 1146,1146,6002
C      1146 IF(TEST2) 1046,1046,115
C
C      PHASE 2(A)
C
C      115 SUMA=0.0
C      DO 19 I=1,NR
C      DO 19 M=1,J1
C      DO 19 N=1,J2
C      DO 19 L=1,J3
C      READ TAPE 2,A,B,P,E
C      READ TAPE 4,C
C      DO 19 J=1,NR
C      READ TAPE 8,QU,Z
C      IF(SENSE SWITCH 6) 99,19
C      99 WRITEOUTPUTTAPE5,9911
C      9911 FORMAT(5H0SUMA)
C      WRITEOUTPUTTAPE5,9999,SUMA,M,N,L,J,I
C      19 SUMA=SUMA+CO(I)*A*B*C/(EO-E)*(RR(J)*QU)
C      REWIND 8
C      REWIND 2
C      REWIND 4
C      IF(SENSE SWITCH 5) 221,21
C      221 PRINT 5,EO,SUMA
C      21 IF(SENSE SWITCH 1) 5022,5021
C      5021 IF(SUMA) 23,22,22
C      22 EO=EO+DELTAET
C      GO TO 115

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PHS2 000
PHS2 001
PHS2 002
PHS2 003
PHS2 004
PHS2 005
PHS2 006
PHS2 007
PHS2 008
PHS2 009
PHS2 010
PHS2 011
PHS2 012
PHS2 013
PHS2 014
PHS2 015
PHS2 016
PHS2 017
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PHS2 077
PHS2 078
PHS2 079

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771 END FILE 4
REWIND 4
REWIND 6
GO TO 53
48 PRINT 6001
6001 FORMAT(18H0END OF ITERATIONS)
BB=(RT(1,1)/R)**2+(PI/H)**2
GO TO 1010
53 PAUSE 11111
IF(SENSE SWITCH 3) 1000,6002
6002 EO=EIN
J5=1
DELTA E=DELTA IN
GO TO 115
1000 PRINT 321
321 FORMAT(31H PHASE 2. HAS BEEN INTERRUPTED.)
1010 PRINT 322
322 FORMAT(24H END MONITORING PHASE 2.)
5024 PRINT 306
PUNCH 5,R,H,RIN,ZIN,THETIN,DELTA IN,DELTA R,DELTA Z,DELTA O,
1ERROR,S,EO,EIN,DELPR,DELPZ,DELPO,(HR(I),I=1,10),(RR(I),I=1,10),
2(D(I),I=1,10),(ALPH(I),I=1,10),AREA
PUNCH 3,J1,J2,J3,J4,NR,I1,I2,JK
PUNCH 304,J15
PUNCH 303,J12,J13,J14
PUNCH 556,E2,BB,JJ5
556 FORMAT(1H 1P2E14.7,I3)
STOP 77777
5022 PRINT 5023,EO
5023 FORMAT(87H0PHASE 2. HAS BEEN INTERRUPTED. THE LAST APPROXIMATION
1FOR THE GEOMETRIC BUCKLING WAS 1PE14.7)
GO TO 5024
END(0,1,0,0,1)
C PROGRAM FOR THE CALCULATION OF THE BUCKLING AND FLUX FOR A BARE
C CYLINDRICAL HOMOGENEOUS REACTOR WITH PARTIALLY INSERTED CONTROL
C RODS
C
C PHASE 3-CALCULATION OF THE FLUX IN VARIOUS
C REGIONS OF THE REACTOR,THE REACTIVITY WORTH AND THE
C BUCKLING FOR THE REACTOR CONSIDERED.
C 4/30/59
C
C E.J. BETINIS
C
C DIMENSION U(51),Q(100),CO(10),RR(10),D(10),HR(10),ALPH(10),TABLE
X(120),RT(1,1)
ALPF(X,Y)=.5*LOGF((1.+SQRTF(1.-(X/Y)**2))/(1.-SQRTF(1.-(X/Y)**2)))
BETF(X,Y)=ATANF(SQRTF((X/Y)**2-1.))
JBES1F(X,Y)=EXP F(Y*(TANH F(ALPF(X,Y))-ALPF(X,Y)))/SQRTF(2.*PI
1*Y*TANH F(ALPF(X,Y)))
JBES2F(X,Y)=SQRTF(2./(PI*Y*SINF(BETF(X,Y))/COSF(BETF(X,Y))))*COSF(
1Y*SINF(BETF(X,Y))/COSF(BETF(X,Y))-Y*BETF(X,Y)-PI/4.)
READ 5,R,H,RIN,ZIN,THETIN,DELTA E,DELTA R,DELTA Z,DELTA O,
1ERROR,S,EO,EIN,DELPR,DELPZ,DELPO,(HR(I),I=1,10),(RR(I),I=1,10),
2(D(I),I=1,10),(ALPH(I),I=1,10),AREA,R1
5 FORMAT(1H 1P5E14.7)
READ 3,J1,J2,J3,J4,NR,I1,I2,JK
3 FORMAT(8I4)
READ 304,J15
304 FORMAT(1I6)
READ 303,J12,J13,J14
303 FORMAT(3I3)
READ 370
370 FORMAT(72H0
1 )
READ 556,E2,BB,JJ5
PRINT 306
306 FORMAT(1H1)
PRINT 307,J12,J13,J14
307 FORMAT(6H DATE I2,1H/I2,1H/I2)
PRINT 370
PRINT 308
308 FORMAT(26H0BEGIN MONITORING PHASE 3.)

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PHS2 160
PHS2 161
PHS2 162
PHS2 163
PHS2 164
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PHS2 166
PHS2 167
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PHS3 000
PHS3 001
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PHS3 038
PHS3 039

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PRINT 360
360 FORMAT(24H0MATHEMATICAL PARAMETERS)
PRINT3,J1,J2,J3,J4,NR,I1,I2,JK
PRINT 344,J1,J2,J3
344 FORMAT(31H TOTAL NUMBER OF TERMS PER ROD=I3,1HXI3,1HXI3)
PRINT 312
312 FORMAT(24H0ROD POSITION PARAMETERS)
PRINT 313,(HR(I),I=1,10),(RR(I),I=1,10),(D(I),I=1,10),(ALPH(I),I=1,10)
313 FORMAT(1H 10F10.3)
GNU=0.0
LE=120
LM=XLOC(F(TABLE(1)))
DO 4 K=1,J1
4 U(K+1)=U(K)+1.0
PI=3.1415926
Q(1)=1.0
DO 7 J=2,100
7 Q(J)=Q(J-1)+1.0
AB=E2/(R1**2)
RT(1,1)=2.4048256
13 DO 14 K=1,NR
14 CO(K)=4.*RR(K)/R
2 DELKOK=AREA*(E2-BB)/(1.+AREA*E2)
WRITE OUTPUT TAPE 5,306
WRITE OUTPUT TAPE 5,949,J12,J13,J14
949 FORMAT(6H DATE 12,1H/12,1H/12)
WRITE OUTPUT TAPE 5,370
WRITE OUTPUT TAPE 5,661
WRITE OUTPUT TAPE 5,950
950 FORMAT(24H MATHEMATICAL PARAMETERS)
WRITE OUTPUT TAPE 5,49,EIN,DELTA,E,ERROR,S,J1,J2,J3,J4,
1NR,I1,I2,JK
49 FORMAT(16H INT. GEO. BUCK=F6.3,7H DELTA=F6.4,7H ERROR=F6.4,3H S=F5
1.2,11H INDICES M=I3,3H N=I3,3H L=I3,4H J4=I2,11H NO.OF RDS=I3,4H I
21=I2,4H I2=I2,3H K=I3)
WRITE OUTPUT TAPE 5,951,R,H
951 FORMAT(31H REACTOR DIMENSIONS (IN CM.) R=F6.3,3H H=F6.3)
WRITE OUTPUT TAPE 5,309,BB
309 FORMAT(56H0THE GEOMETRIC BUCKLING FOR THE REACTOR WITHOUT RODS IS
11PE14.7)
WRITE OUTPUT TAPE 5,661
WRITE OUTPUT TAPE 5,3003,JJ5
3003 FORMAT(50H NO. OF ITERATIONS MADE FOR GEOMETRIC BUCKLING IS I3)
NOFLX=JJ5/2
WRITE OUTPUT TAPE 5,3004,NOFLX
3004 FORMAT(36H NO. OF ITERATIONS MADE FOR FLUX IS I3)
WRITE OUTPUT TAPE 5,661
548 WRITE OUTPUT TAPE5,549
549 FORMAT(29H FOR THE ROD LENGTHS (IN CM.))
WRITE OUTPUT TAPE 5,51,(HR(I),I=1,10),E2
51 FORMAT(1H010F6.3,24H THE GEOMETRIC BUCKLING=F10.4)
WRITE OUTPUT TAPE 5,850
850 FORMAT(27H FOR THE ROD RADII (IN CM.))
WRITE OUTPUT TAPE 5,551,(RR(I),I=1,10)
WRITE OUTPUT TAPE 5,849
849 FORMAT(31H FOR THE ROD DISTANCES (IN CM.))
WRITE OUTPUT TAPE 5,551,(D(I),I=1,10)
551 FORMAT(1H 10F6.3)
WRITE OUTPUT TAPE 5,649
649 FORMAT(32H FOR THE ROD ANGLES (IN RADIAN))
WRITE OUTPUT TAPE 5,551,(ALPH(I),I=1,10)
WRITE OUTPUT TAPE 5,662,DELKOK,AREA
662 FORMAT(18H REACTIVITY WORTH=1PE14.7,16H MIGRATION AREA=1PE14.7)
WRITE OUTPUT TAPE 5,501,AB
501 FORMAT(48H0THE ACTUAL GEOMETRIC BUCKLING FOR THIS REACTOR=1PE14.7)
WRITE OUTPUT TAPE 5,661
661 FORMAT(1H0)
WRITE OUTPUT TAPE 5,751
751 FORMAT(50H FLUX PLOTS IN VARIOUS REGIONS OF THE REACTOR)
IF(SENSE SWITCH 6) 449,54
449 PRINT 51,(HR(I),I=1,10),E2
PRINT 501,AB
54 IF(SENSE SWITCH 3) 1000,55
55 SUMH=0.0
DO 956 I=1,NR
REWIND 7
DO 956 M=1,J1
Y=M
DO 956 N=1,J2

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      READ TAPE 7,ROOTMN
      DO 956 L=1,J3
      READ TAPE 2,A,B,P,E
      READ TAPE 4,C
9566 IF(M-16) 9114,9113,9113
9113 IF(ROOTMN*RIN/R-50.) 9115,9115,8905
9115 AS=BESJF(ROOTMN*RIN/R,GNU,M,LE,LM)
      IF(AS) 1111,9006,1111
9006 S5=TABLE(M)
      GO TO 9125
8905 IF(ROOTMN*RIN/R-.5*Y) 9213,9213,8906
8906 IF(ROOTMN*RIN/R-4.*Y) 9223,9134,9134
9213 SUMH=SUMH+CO(I)*C*A*B*JBES1F(ROOTMN*RIN/R,Y-1.)*
      1(COSF(U(M)*ALPH(I))*COSF(U(M)*THETIN)+SINF(U(M)*ALPH(I))*SINF(
      2U(M)*THETIN))*SINF(Q(L)*PI*ZIN/H)/(EO-E)
      GO TO 956
9223 SUMH=SUMH+CO(I)*C*A*B*JBES2F(ROOTMN*RIN/R,Y-1.)*
      1(COSF(U(M)*ALPH(I))*COSF(U(M)*THETIN)+SINF(U(M)*ALPH(I))*SINF(
      2U(M)*THETIN))*SINF(Q(L)*PI*ZIN/H)/(EO-E)
      GO TO 956
9114 IF(ROOTMN*RIN/R-50.) 9124,9124,9134
9124 AS=BESJF(ROOTMN*RIN/R,GNU,M,LE,LM)
      IF(AS) 1111,9125,1111
9125 S5=TABLE(M)
      SUMH=SUMH+CO(I)*A*B*C*S5*(COSF(U(M)*ALPH(I))*COSF(U(M)*THETIN)
      1+ SINF(U(M)*ALPH(I))*SINF(U(M)*THETIN))*SINF(Q(L)*PI*ZIN/H)/(EO-E)
      GO TO 956
9134 SUMH=SUMH+CO(I)*C*A*B*JBESSF(ROOTMN*RIN/R,M-1)*
      1(COSF(U(M)*ALPH(I))*COSF(U(M)*THETIN)+SINF(U(M)*ALPH(I))*SINF(
      2U(M)*THETIN))*SINF(Q(L)*PI*ZIN/H)/(EO-E)
956 CONTINUE
      REWIND 4
      REWIND 2
      REWIND 7
4501 IF(SENSE SWITCH 2) 60,62
      60 PRINT 61,RIN,ZIN,THETIN,SUMH
      61 FORMAT(3H R=F6.3,3H Z=F6.3,7H THETA=F6.3,6H FLUX=1PE14.6)
      62 WRITE OUTPUT TAPE 5,61,RIN,ZIN,THETIN,SUMH
      IF(SENSE SWITCH 4) 1000,4500
4500 RIN=RIN+DELTAR
      IF(RIN-R) 55,55,64
      64 RIN=RIN-DELPR
      ZIN=ZIN+DELTAZ
      IF(ZIN-H) 55,66,66
      66 ZIN=ZIN-DELPZ
      THETIN=THETIN+DELTAO
      IF(THETIN-6.2831852) 55,68,68
      68 THETIN=THETIN-DELPO
      END FILE 5
      REWIND 5
      PRINT 320
      320 FORMAT(25H END MONITORING PHASE 3.)
      PRINT 306
      STOP 77777
1000 PRINT 321
      321 FORMAT(31HOPHASE 3. HAS BEEN INTERRUPTED.)
      END FILE 5
      REWIND 5
      PRINT 306
      PUNCH 5,R,H,RIN,ZIN,THETIN,DELTAE,DELTAR,DELTAZ,DELTAO,
      1ERROR,S,EO,EIN,DELPR,DELPZ,DELPO,(HR(I),I=1,10),(RR(I),I=1,10),
      2(D(I),I=1,10),(ALPH(I),I=1,10),AREA,R1
      PUNCH 3,J1,J2,J3,J4,NR,I1,I2,JK
      PUNCH 304,J15
      PUNCH303,J12,J13,J14
      PUNCH 556,E2,BB,JJ5
      556 FORMAT(1H 1P2E14.7,I3)
      STOP 22222
1111 PRINT 6666
6666 FORMAT(117H0AN ERROR OCCURRED IN THE EXECUTION OF NYU BESSEL FUNCTPHS3
      1ION SUBROUTINE.THIS IS EITHER A MACHINE ERROR OR A DATA ERROR.)
      PRINT 306
      STOP 11111
      END (0,1,0,0,0)

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PHS3 120
 PHS3 121
 PHS3 122
 PHS3 123
 PHS3 124
 PHS3 125
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 PHS3 127
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 PHS3 189
 PHS3 190
 PHS3 191
 PHS3 192
 PHS3 193

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C      AUXILIARY PROGRAM NO. 1.-FOR CALCULATING THE          AUX1 000
C      BUCKLING OF A FULLY INSERTED CENTRAL ROD BY THE      AUX1 001
C      NORDHEIM-SCALLETAR THEORY AND FROM THE EXACT         AUX1 002
C      SOLUTION,THE FLUX FROM THE EXACT SOLUTION IS ALSO    AUX1 003
C      CALCULATED.                                           AUX1 004
C      4/30/59                                               AUX1 005
C                                                           AUX1 006
C      E.J.BETINIS                                           AUX1 007
C                                                           AUX1 008
C      THETAF(X,Y)=JBESSF(X,0)-JBESSF(Y,0)/YBESSF(Y,0)*YBESSF(X,0)
C      READ3,RO,ZIN,DELTAR,DELTAR,DELTAL,ALPHIN,R,H,DELPR,C2
C      3 FORMAT(1P5E14.7)                                     AUX1 009
C      READ 4,J4                                              AUX1 010
C      4 FORMAT(I4)                                           AUX1 011
C      ROOT=2.4048256                                         AUX1 012
C      PI=3.1415926                                           AUX1 013
C      PRINT 76,RO,R,H                                         AUX1 014
C      76 FORMAT(26H0FOR THE EFFECTIVE RADIUS F7.4,7H AND R=F6.3,3H H=F6.3)
C      DELBE=3.75/(ROOT*(.116+LOGF(R/(2.4*RO))))           AUX1 015
C      BETA=ROOT+DELBE                                         AUX1 016
C      EIGVL1=(BETA/R)**2+(PI/H)**2                           AUX1 017
C      J5=1                                                    AUX1 018
C      115 X=ALPHIN*RO                                         AUX1 019
C      Y=ALPHIN                                                AUX1 020
C      FUNCT=THETAF(X,Y)                                       AUX1 021
C      IF(SENSE SWITCH 1) 555,556                             AUX1 022
C      555 PRINT 5,ALPHIN,FUNCT                                AUX1 023
C      5 FORMAT(7H ALPHA=1PE15.7,25H TRANSCENDENTAL FUNCTION=1PE15.7)
C      556 IF(FUNCT) 23,22,22                                  AUX1 024
C      22 ALPHIN=ALPHIN+DELTAL                                  AUX1 025
C      GO TO 115                                                AUX1 026
C      23 IF(J5-J4) 27,27,26                                   AUX1 027
C      27 ALPHIN=ALPHIN-DELTAL                                  AUX1 028
C      DELTAL=DELTAL/2.                                         AUX1 029
C      J5=J5+1                                                  AUX1 030
C      GO TO 22                                                 AUX1 031
C      26 EIGVL2=(ALPHIN/R)**2+(PI/H)**2                       AUX1 032
C      PRINT 66,BETA,EIGVL1,ALPHIN,EIGVL2                     AUX1 033
C      66 FORMAT(22H0ROOT NORD-SCALL THEO=1PE14.7,12H AND GEO BK=1PE14.7,12H
C      X EXACT ROOT=1PE14.7,18H AND EXACT GEO BK=1PE14.7)     AUX1 034
C      X=ALPHIN*15.*RO                                         AUX1 035
C      Y=ALPHIN                                                AUX1 036
C      F=THETAF(X,Y)                                           AUX1 037
C      C1=C2/F                                                  AUX1 038
C      700 X=ALPHIN*RO                                         AUX1 039
C      F=THETAF(X,Y)*SINF(PI*ZIN/H)                            AUX1 040
C      F=C1*F                                                   AUX1 041
C      PRINT 666,RO,ZIN,F                                       AUX1 042
C      666 FORMAT(3H R=F6.3,3H Z=F6.2,10H FUNCTION=1PE14.6)   AUX1 043
C      RO=RO+DELTAR                                             AUX1 044
C      IF(RO-R) 700,700,64                                       AUX1 045
C      64 RO=RO-DELPR                                           AUX1 046
C      ZIN=ZIN+DELTAR                                           AUX1 047
C      IF(ZIN-H) 700,667,667                                     AUX1 048
C      667 STOP 77777                                           AUX1 049
C      END(0,1,0,0,1)                                           AUX1 050

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C	AUXILIARY PROGRAM NO. 2.-FOR INITIALIZING TAPE NO. 4	AUX2 000
C	IF PHASE 2(A) IS TO BE REPEATED.	AUX2 001
C		AUX2 002
C	4/30/59	AUX2 003
C		AUX2 004
C	E.J.BETINIS	AUX2 005
C		AUX2 006
	READ 1,J1,J2,J3,NR,I3	AUX2 007
	1 FORMAT(5I4)	AUX2 008
	C=1.0	AUX2 009
	DO 2 I=1,I3	AUX2 010
	DO 2 M=1,J1	AUX2 011
	DO 2 N=1,J2	AUX2 012
	DO 2 L=1,J3	AUX2 013
	2 WRITE TAPE 4,C	AUX2 014
	END FILE 4	AUX2 015
	REWIND 4	AUX2 016
	STOP11111	AUX2 017
	END(0,1,0,0,1)	AUX2 018
C	AUXILIARY PROGRAM NO. 3.-FOR RESETTING TAPES 2,4,AND 8	AUX3 000
C	WHEN NR RODS ARE CONSIDERED AND PHASE 1. IS CALCULATED	AUX3 001
C	FOR I1 TO NR RODS TAKEN I2 AT A TIME.	AUX3 002
C		AUX3 003
C	4/30/59	AUX3 004
C		AUX3 005
C	E.J.BETINIS	AUX3 006
C		AUX3 007
	READ 1,J1,J2,J3,NR,I3	AUX3 008
	1 FORMAT(5I4)	AUX3 009
	READ 303,J12,J13,J14	AUX3 010
	303 FORMAT(3I3)	AUX3 011
	PRINT 302,J12,J13,J14	AUX3 012
	302 FORMAT(6H DATE I2,1H/I2,1H/I2)	AUX3 013
	DO 2 I=1,I3	AUX3 014
	DO 2 M=1,J1	AUX3 015
	DO 2 N=1,J2	AUX3 016
	DO 2 L=1,J3	AUX3 017
	II=I	AUX3 018
	MM=M	AUX3 019
	NN=N	AUX3 020
	LL=L	AUX3 021
	READ TAPE2,A,B,P,E	AUX3 022
	READ TAPE 4 ,C	AUX3 023
	DO 2 J=1,NR	AUX3 024
	2 READ TAPE 8,QU,Z	AUX3 025
	PRINT 3	AUX3 026
	3 FORMAT(101H0THE LAST RECORDS NOW ON TAPES 2 AND 8 ALONG WITH RESPE	AUX3 027
	1CTIVE INDICES ARE GIVEN IN THE FOLLOWING TABLE.)	AUX3 028
	PRINT 310	AUX3 029
	310 FORMAT(102H0 A(M,N) B(L) Q(M,N,L) P(M,N,L)	AUX3 030
	1 E(M,N,L) Z(M,N,L) M+1 N L RDNO)	AUX3 031
	PRINT 55,A,B,QU,P,E,Z,MM,NN,LL,II	AUX3 032
	55 FORMAT(1H 1P6E14.7,4I4)	AUX3 033
	PRINT 4,C	AUX3 034
	4 FORMAT(31H0THE LAST RECORD ON TAPE 4 WAS 1PE14.7)	AUX3 035
	STOP 44444	AUX3 036
	END(0,1,0,0,1)	AUX3 037

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C      AUXILIARY PROGRAM NO. 4.-FOR GENERATING MXN          AUX4 000
C      ROOTS RT(M,N) SUCH THAT J(RT(M,N),M)=0,WHERE          AUX4 001
C      M=50,N=100,AND J IS THE BESSEL FUNCTION OF            AUX4 002
C      THE FIRST KIND.IN THREE PARTS.                         AUX4 003
C      4/30/59                                                AUX4 004
C                                                            AUX4 005
C      E.J.BETINIS                                           AUX4 006
C                                                            AUX4 007
C      PART 1.TAPE 5.                                         AUX4 008
C                                                            AUX4 009
C      DIMENSION U(51),QO(50),Q1(50),Q2(50),Q3(50),Q(100),Q4(50),TABLE(12)AUX4 010
C      10)                                                    AUX4 011
C      BETF(X,Y)=ATANF(SQRTF((X/Y)**2-1.))                    AUX4 012
C      JBES2F(X,Y)=SQRTF(2./(PI*Y*SINF(BETF(X,Y))/COSF(BETF(X,Y))))*COSF(AUX4 013
C      1Y*SINF(BETF(X,Y))/COSF(BETF(X,Y))-Y*BETF(X,Y)-PI/4.)  AUX4 014
C      READ 3,J1,J2,J3,J4,NR,I1,I2,JK                        AUX4 015
C      3 FORMAT(8I4)                                           AUX4 016
C      GNU=0.0                                                 AUX4 017
C      LE=120                                                  AUX4 018
C      LM=XLOC(F(TABLE(1)))                                    AUX4 019
C      U(1)=0.0                                                AUX4 020
C      DO 4 K=1,J1                                             AUX4 021
C      4 U(K+1)=U(K)+1.0                                        AUX4 022
C      PI=3.1415926                                           AUX4 023
C      DO 6 M=1,J1                                             AUX4 024
C      QO(M)=4.*U(M)**2                                         AUX4 025
C      Q1(M)=7.*QO(M)-31.                                       AUX4 026
C      Q2(M)=83.*QO(M)**2-982.*QO(M)+3779.                   AUX4 027
C      Q4(M)=70197.*QO(M)**4-2479316.*QO(M)**3+48010494.*QO(M)**2-5120625AUX4 028
C      148.*QO(M)+2092163573.                                  AUX4 029
C      6 Q3(M)=6949.*QO(M)**3-153855.*QO(M)**2+1585743.*QO(M)-6277237.  AUX4 030
C      Q(1)=1.0                                               AUX4 031
C      DO 7 J=2,100                                           AUX4 032
C      7 Q(J)=Q(J-1)+1.0                                       AUX4 033
C      DO 8 M=1,J1                                             AUX4 034
C      DO 8 N=1,J2                                             AUX4 035
C      BE=(U(M)-0.5+2.*Q(N))*PI/2.                             AUX4 036
C      EM=FLOATF(M)                                           AUX4 037
C      Y=M                                                     AUX4 038
C      ROOTMN=BE-(QO(M)-1.)/(8.*BE)*(1.+Q1(M)/(3.*(4.*BE)**2)+2.*Q2(M)/(1AUX4 039
C      15.*(4.*BE)**4)+Q3(M)/(105.*(4.*BE)**6)+Q4(M)/(2520.*8.**4*BE**8))  AUX4 040
C      IF(M-15) 3677,3667,3667,3667                          AUX4 041
C      3677 IF(ROOTMN-50.) 3678,3678,3698                      AUX4 042
C      3678 AS=BESJF(ROOTMN,GNU,M,LE,LM)                       AUX4 043
C      S6=TABLE(M)                                             AUX4 044
C      GO TO 8                                                 AUX4 045
C      3667 IF(ROOTMN-50.) 8801,8801,3888                      AUX4 046
C      8801 AS=BESJF(ROOTMN,GNU,M,LE,LM)                       AUX4 047
C      S6=TABLE(M)                                             AUX4 048
C      GO TO 8                                                 AUX4 049
C      3888 IF(ROOTMN-4.*EM) 8802,8802,3698                   AUX4 050
C      8802 S6=JBES2F(ROOTMN,Y-1.)                            AUX4 051
C      GO TO 8                                                 AUX4 052
C      3698 S6=JBESSF(ROOTMN,M-1)                              AUX4 053
C      8 WRITE OUTPUT TAPE 5,3030,S6,ROOTMN,M,N              AUX4 054
C      3030 FORMAT(1H 1P2E14.7,2I4)                           AUX4 055
C      ENDFILE5                                               AUX4 056
C      REWIND 5                                               AUX4 057
C      STOP33333                                              AUX4 058
C      END(0,1,0,0,1)                                         AUX4 059
C      PART 2.TAPES 3,5                                       AUX4 060
C                                                            AUX4 061
C      11 READ 1000,CORTMN,MM,NN,IND                          AUX4 062
C      1000 FORMAT(1H 1PE14.7,3I4)                             AUX4 063
C      1001 READ INPUT TAPE 5,3030,S6,ROOTMN,M,N              AUX4 064
C      3030 FORMAT(1H 1P2E14.7,2I4)                             AUX4 065
C      IF(MN-M) 4008,4003,4008                                AUX4 066
C      4003 IF(NN-N) 4008,4000,4008                             AUX4 067
C      4008 WRITE OUTPUT TAPE 3,1000,ROOTMN,M,N               AUX4 068
C      GO TO 1001                                              AUX4 069
C      4000 WRITE OUTPUT TAPE 3,1000,CORTMN,M,N               AUX4 070
C      IF(IND) 11,1001,11                                      AUX4 071
C      MANUAL END OF FILE TAPE 3                              AUX4 072
C      END(0,1,0,0,1)                                         AUX4 073
C      PART 3. TAPES 1,3,5                                    AUX4 074
C                                                            AUX4 075
C      DIMENSION TABLE(120)                                  AUX4 076
C      BETF(X,Y)=ATANF(SQRTF((X/Y)**2-1.))                    AUX4 077
C      JBES2F(X,Y)=SQRTF(2./(PI*Y*SINF(BETF(X,Y))/COSF(BETF(X,Y))))*COSF(AUX4 078
C      1Y*SINF(BETF(X,Y))/COSF(BETF(X,Y))-Y*BETF(X,Y)-PI/4.)  AUX4 079

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PI=3.1415926	AUX4 080
GNU=0.0	AUX4 081
LE=120	AUX4 082
LM=XLOC(F(TABLE(1)))	AUX4 083
2000 READ INPUT TAPE 3,3031,ROOTMN,M,N	AUX4 084
Y=M	AUX4 085
4008 WRITE TAPE 1,ROOTMN	AUX4 086
IF(M-15) 3677,3667,3667	AUX4 087
3677 IF(ROOTMN-50.) 3678,3678,3698	AUX4 088
3678 AS=BESJF(ROOTMN,GNU,M,LE,LM)	AUX4 089
S6=TABLE(M)	AUX4 090
GO TO 8	AUX4 091
3667 IF(ROOTMN-50.) 8801,8801,3888	AUX4 092
8801 AS=BESJF(ROOTMN,GNU,M,LE,LM)	AUX4 093
S6=TABLE(M)	AUX4 094
GO TO 8	AUX4 095
3888 IF(ROOTMN-4.* Y) 8802,8802,3698	AUX4 096
8802 S6=JBES2F(ROOTMN,Y-1.)	AUX4 097
GO TO 8	AUX4 098
3698 S6=JBESSF(ROOTMN,M-1)	AUX4 099
8 WRITE OUTPUT TAPE 5,3030,S6,ROOTMN,M,N	AUX4 100
GO TO 2000	AUX4 101
3030 FORMAT(1H 1P2E14.7,2I4)	AUX4 102
3031 FORMAT(1H 1PE14.7,2I4)	AUX4 103
C MANUAL END OF FILE TAPES 1,5	AUX4 104
END(0,1,0,0,1)	AUX4 105
C AUXILIARY PROGRAM NO. 5.-FOR PUTTING MXN	AUX5 000
C ROOTS RT(M,N) ON TAPE 7,WHERE M AND N ARE VARIABLE.	AUX5 001
C 4/30/59	AUX5 002
C	AUX5 003
C E.J.BFTINIS	AUX5 004
C	AUX5 005
DIMENSIONRT(50,100)	AUX5 006
READ 1,J11,J22,J1,J2	AUX5 007
1 FORMAT(1H 4I4)	AUX5 008
DO 2 M=1,J11	AUX5 009
DO 2 N=1,J22	AUX5 010
2 READ TAPE 1,RT(M,N)	AUX5 011
REWIND 1	AUX5 012
DO 3 M=1,J1	AUX5 013
DO 3 N=1,J2	AUX5 014
WRITE OUTPUT TAPE 6,4,RT(M,N),M,N	AUX5 015
3 WRITE TAPE 7,RT(M,N)	AUX5 016
END FILE 6	AUX5 017
REWIND 6	AUX5 018
END FILE 7	AUX5 019
REWIND 7	AUX5 020
4 FORMAT(1H 1PE14.7,2I4)	AUX5 021
STOP 77777	AUX5 022
END(0,1,0,0,1)	AUX5 023