ANALYTIC SOLUTION OF THE STEADY-STATE WAVE EQUATION IN THREE DIMENSIONS AND IN IRREGULARLY SHAPED DOMAINS WITH AN APPLICATION TO A REACTOR WITH PARTIALLY INSERTED CONTROL RODS

by

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I. Introduction ......................................................... 5
II. Statement of the Problem ......................................... 7
III. Restatement of the Problem ....................................... 8
IV. Consideration of the Steady-state Wave
    Equation and Statement of Theorem I ......................... 9
V. Proof of Theorem I by a Method of Successive
    Approximations .................................................. 10
VI. Perturbation Formulas for the Eigenvalues
    and Eigenfunctions .............................................. 22
VII. Determination of the Flux and Buckling
    for a Homogeneous Reactor with Partially
    Inserted Control Rods .......................................... 24
VIII. The Special Case of a Fully Inserted
    Control Rod and a Resulting Identity ......................... 34
IX. A Method for Evaluating Definite Integrals .................. 36
X. Determination of Buckling and Flux for a
    Rectangular Reactor with Partially Inserted
    Cross-shaped Control Rods .................................... 38
XI. Numerical Results for the Cylindrical Reactor
    with Two Partially Inserted Control Rods.
    Discussion of Numerical Results ............................. 42
XII. Theorems II and III. Extension of the Theory to
    the Wave Equation with Variable Coefficients .............. 50
XIII. An Application of the Method to a Time-dependent
    Problem and Theorem IV ....................................... 53
<table>
<thead>
<tr>
<th>Table of Contents</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>XIV. An Application of the Method to the Solution</td>
<td>56</td>
</tr>
<tr>
<td>of Poisson's Equation</td>
<td></td>
</tr>
<tr>
<td>XV. Remarks on Very Irregular Domains</td>
<td>57</td>
</tr>
<tr>
<td>References</td>
<td>58</td>
</tr>
<tr>
<td>Acknowledgement</td>
<td>60</td>
</tr>
<tr>
<td>Appendix - Description of Reactor Code</td>
<td>88</td>
</tr>
</tbody>
</table>
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E. J. Betinis

I. INTRODUCTION

Many perturbation theories for boundary-value problems have been developed in the past, but the vast majority have been concerned with the effect on eigenvalues and eigenfunctions when operators of the form $\nabla^2 + \Delta$ have been perturbed by the addition of a small perturbing operator of the form $\varepsilon$ while the domain of the operator remained unchanged. In this regard, the literature is very extensive (for example, see references 1-3).

A paper by Brillouin\(^{(4)}\) is concerned with the problem of finding the corresponding distorted domain for a perturbed operator.

Other works regarding the perturbations on the eigenvalues and eigenfunctions due to perturbations in the boundary shape may be found in reference 5. In this same vein, a recent paper\(^{(6)}\) concerns itself explicitly with this type of perturbation for second-order operators in one-dimensional geometry. Since boundary-value problems of the eigenvalue type for distorted regions are actually concerned with geometries in which the differential equation is not separable, it is pertinent to cite reference 7 in this regard.

In addition, many numerical techniques have been developed for two-dimensional problems with good success. But for three-dimensional problems involving irregularly shaped boundaries there has been very little successful work done. There is an IBM 704 nuclear code, called Trixy,\(^{(8)}\) written by a group at The General Electric Company, which employs numerical techniques for solving three-dimensional problems and is successful for certain geometries.

A method of successive approximations has been developed for solving the steady-state wave equation

$$\left(\nabla^2 + \Delta\right) \phi = 0 \tag{1}$$

in an arbitrarily shaped $n$-dimensional domain with $\phi$ subject to homogeneous boundary conditions. For the most part, the Helmholtz operator $\nabla^2 + \Delta$ is considered; however, extensions to more general operators are also treated.
In particular, the technique developed has been extensively applied in solving the one-group diffusion equation, which is the same as eq. (1), for a bare, homogeneous cylindrical thermal reactor with many partially or fully inserted control rods. These calculations were made on the IBM 704 and have been included in this paper.

In addition, formulas were derived for the solution of eq. (1) for irregularly shaped rectangular domains.

The solutions of eq. (1) and related equations are in the form of a Fourier series. In the method of successive approximations developed in this paper, the Fourier coefficients necessary to solve a given boundary-value problem of the eigenvalue type mentioned above have been derived. As a further consequence of the technique developed, perturbation formulas for the eigenvalues of eq. (1) and related equations for the above boundary-value problem have also been derived.
II. STATEMENT OF THE PROBLEM

The eigenvalue problem to be considered may be solved subject to a variety of homogeneous boundary conditions. In the treatment that follows the homogeneous Dirichlet boundary conditions will be of concern in the main, although utilization of the method to be developed for the solution of the problem subject to other boundary conditions follows quite trivially.

We wish to determine the eigenfunctions and eigenvalues of the self-adjoint partial differential equation

$$\nabla^2 \phi + \Lambda \phi = 0 \text{ in } G$$

subject to one of the following homogeneous boundary conditions:

$$\phi \bigg|_{\Gamma} = 0 \quad \text{(Dirichlet)} \quad (3a)$$

$$\mathbf{n} \cdot \nabla \phi \bigg|_{\Gamma} = 0 \quad \text{(Neumann)} \quad (3b)$$

or

$$\left( \phi + K \mathbf{n} \cdot \nabla \phi \right) \bigg|_{\Gamma} = 0 \quad \text{(Mixed)} \quad (3c)$$

where $G$ is an arbitrarily shaped, $n$-dimensional domain with continuous boundary surface $\Gamma$, $\nabla^2$ the Laplacian for this domain, $\mathbf{n} \cdot \nabla \phi$ the normal derivative of $\phi$, and $K$ is an arbitrary constant. In the sequel, the discussion and applications will be confined to three-dimensional considerations for the sake of clarity because in all that follows the $n$-dimensional considerations are equally valid. Actually in the treatment to follow the above boundary conditions are replaced by boundary conditions which are very similar if not in many cases equivalent. In this regard see page 11, eq. (23).
III. RESTATEMENT OF THE PROBLEM

Let \( \overline{D} \) be a closed non-null subset of a three-dimensional Euclidean space, \( \mathbb{E}_3 \), with boundary surface \( \Sigma \) such that the eigenfunctions and eigenvalues of the equation

\[
\nabla^2 f_n + \lambda_n f_n = 0 \text{ in } \overline{D},
\]

subject to the homogeneous Dirichlet boundary condition

\[
f_n|_{\Sigma} = 0,
\]

may be determined. The index \( n \) has been written in place of the indices \( m, n, \) and \( l \) for the sake of clarity and brevity here and in what is to follow. Further let \( \overline{D}_i \) with \( i = 1, 2, \ldots, k \) be \( k \) closed proper non-null subsets with continuous boundary surfaces \( \sigma_i \), and define the union of these subsets as

\[
\overline{D}_1 = \bigcup_{i=1}^{k} \overline{D}_i.
\]

Define the union of the \( k \) surfaces of these proper subsets as

\[
\Sigma_1 = \bigcup_{i=1}^{k} \sigma_i.
\]

If we now let

\[
G = \overline{D} - \overline{D}_1 = \overline{D} - \overline{D}_1
\]

and

\[
\Gamma = \Sigma \cup \Sigma_1,
\]

the problem as stated in Sect. II may be restated as follows: we wish to determine the eigenvalues and eigenfunctions of the equation

\[
\nabla^2 \phi_n + \Lambda_n \phi_n = 0 \text{ in } \overline{D} - \overline{D}_1
\]

subject to the boundary condition

\[
\phi_n|_{\sigma_1} = \phi_n|_{\sigma_2} = \ldots \phi_n|_{\sigma_k} = \phi_n|_{\Sigma} = 0.
\]

The boundary condition in eq. (11) could have been replaced equally well by either of the boundary conditions in eq. (3b) or eq. (3c). Furthermore, as will later be pointed out, the treatment to follow could apply equally well
to problems in which the boundary condition in eq. (11) would take on the form

\[
\phi_n\left|_{\sigma_1} = \phi_n\right|_{\sigma_2} = \cdots = \phi_n\left|_{\sigma_k} = \nabla \cdot \nabla \phi_n\right|_{\Sigma} = 0, \tag{11a}
\]

or, possibly, of the form

\[
\phi_n + K \nabla \cdot \nabla \phi_n\left|_{\sigma_1} = \phi_n + K \nabla \cdot \nabla \phi_n\right|_{\sigma_2} = \cdots \phi_n + K \nabla \cdot \nabla \phi_n\left|_{\sigma_k} = \phi_n\right|_{\Sigma} = 0. \tag{11b}
\]

IV. CONSIDERATION OF THE STEADY-STATE WAVE EQUATION
AND STATEMENT OF THEOREM I

There exists a function \( S(\text{sink}) \) which has the following properties:

(i) \( \nabla^2 \phi + \Lambda \phi = S \) in \( \overline{D_1} \)

(ii) \( \nabla^2 \phi + \Lambda \phi = 0 \) in \( \overline{D-D_1} \)

such that it makes \( \phi \) satisfy the boundary condition

(iii) \( \phi\left|_{\sigma_1} = \phi\right|_{\sigma_2} = \cdots \phi\left|_{\sigma_k} = \phi\right|_{\Sigma} = 0 \)

and such that

(iv) \( 0 < |S| < K \) \( \tag{14} \)

where \( K \) is a finite constant, and \( \phi \) is analytic in \( \overline{D-D_1} \).

The satisfaction of eq. (13) along with the boundary condition (iii) will then yield the eigenfunction solution \( \phi_n \) and corresponding eigenvalue \( \Lambda_n \).
V. PROOF OF THEOREM I BY A METHOD OF SUCCESSIVE APPROXIMATIONS

We assert, at this point, that the solution of eq. (13), subject to the satisfaction of boundary condition IV. (iii), may be expressed as the orthonormal expansion

\[ \phi = \sum_{n}^{\infty} a_n f_n \]  

(15)

The \( f_n \)'s satisfy the orthonormality condition

\[ \int_{D} f_m f_n = \delta_{mn} \]  

(16)

and, in addition, satisfy eq. (4) and (5). The integration in eq. (16) and in the following equations will be understood to be Riemannian and over the domains indicated by the integral sign. Thus, all integrations pertain to bounded functions over the domains considered. The main argument of the theory to be presented here is that the \( a_n \)'s in eq. (16) may be determined by a series of successive approximations in such a way that \( \phi \) will satisfy the boundary condition IV. (iii). The choice of the \( f_n \)'s by virtue of eq. (5) make \( \phi \) satisfy the same boundary condition as eq. (5) with the properly constructed \( a_n \)'s, making \( \phi \) satisfy the remaining boundary conditions in IV. (iii).

Denote by \( S^{(k)} \) the \( k \)th approximation of the sink function \( S \). Define the first approximation of \( S \) so as to have the following properties:

\[ S^{(l)} = f_n \text{ in } D_1 \]  

\[ S^{(l)} = 0 \text{ in } \overline{D-D_1} \]  

(17)  

(18)

where \( f_n \) satisfies eq. (4) and boundary condition (5). Furthermore, assume that \( S^{(l)} \) is continuous in the interior of \( \overline{D_1} \), which we designate as \( D_1 \), and so admits of the Fourier expansion

\[ S^{(l)} = \sum_{n}^{\infty} b_n f_n \]  

(19)

where, by virtue of eq. (16), (17), and (18),

\[ b_n = \int_{D_1} S^{(l)} f_n \]  

(19a)

This choice of \( S \) thus satisfies eq. (12) and (13) in the respective interiors of the domains \( \overline{D_1} \) and \( \overline{D-D_1} \).
In order to make valid the interchange of summation and differentiation which is to follow, we assume that the solution \( \phi \), the first partial derivative of \( \phi \), and the second partial derivative of \( \phi \) are continuous in the interior of \( D - D_1 \), which we designate as \( D - D_1 \). Later, we will show that this restriction may be removed, for the series solution derived by this method is actually uniformly convergent and uniformly continuous. In addition, it will be shown that the series representations of the first and second-order partial derivatives of \( \phi \) are uniformly convergent and uniformly continuous.

Denote the \( k^{th} \) approximation of the eigenvalue \( \Lambda \) by \( \Lambda^{(k)} \) and the \( k^{th} \) approximation of the corresponding eigenfunction by \( \phi^{(k)} \). We now find a first approximation \( \phi^{(1)} \) of \( \phi \) as follows: substitute eq. (15) and (19) into eq. (12) to get, after interchanging summation and differentiation,

\[
\sum_{n} \alpha_n (\Lambda - \lambda_n) f_n = \sum_{n} b_n f_n \quad (20)
\]

Here \( \phi \) is defined over \( \overline{D} \) so that by application of the orthonormality property of the \( f_n \)'s

\[
\alpha_n = \frac{b_n f_n}{\Lambda - \lambda_n} \quad (21)
\]

and, therefore, that

\[
\phi^{(1)} = \sum_{n} \frac{b_n f_n}{\Lambda - \lambda_n} \quad (22)
\]

However, we seek a \( \phi^{(1)} \) such that boundary condition IV. (iii) holds as well. In place of the boundary condition IV. (iii), we impose the boundary condition

\[
\int_{\Sigma_1} \phi^{(1)} = \left( \int_{\sigma_1} + \int_{\sigma_2} + \cdots + \int_{\sigma_k} \right) \phi^{(1)} = 0 \quad (23)
\]

Because of the choice of the \( f_n \)'s, we have that

\[
\int_{\Sigma} \phi = 0 \quad .
\]

If the series expression for \( \phi^{(1)} \) in eq. (22) were a "pure" Fourier series, one could interchange integration and summation because the integration of the eigenfunction would introduce the additional factor \( 1/n \) in \( b_n \). Since we have introduced another additional factor, \( \Lambda - \lambda_n \), we are allowed to interchange integrations and summations a fortiori.
We now obtain the first approximation $\Lambda^{(1)}$ of the eigenvalue $\Lambda$ by integrating $\phi^{(1)}$ in eq. (22) over the surface $\Gamma$ and, after interchanging integration and summation, we have that

$$\sum_{n}^{\infty} \frac{b_n C_n}{\Lambda^{(1)} - \lambda_n} = 0$$

where

$$C_n = \int_{\Gamma} f_n$$

Note that $\Lambda^{(1)}$ could never equal $\lambda_n$. If $\Lambda^{(1)}$ equaled $\lambda_n$, then this would imply that one of the eigenfunctions $\phi_n$ equals one of the eigenfunctions $f_n$. This would only be possible if the subdomains $\overline{D}_i$ vanished, but these have been assumed to be non-null sets.

Previously, we have mentioned that the $f_n$'s are bounded so that, in particular,

$$|C_n| = \left| \int_{\Gamma} f_n \right| \leq \frac{M}{g(n)}$$

where $g(n)$ is a linear function of $n$ which varies with the eigenfunction integrated. Therefore, the series in eq. (25) would converge absolutely, since

$$|b_n| = \left| \int_{\overline{D}_i} S^{(1)} f_n \right| \leq \frac{KM}{g(n)}$$

and since

$$\left| \sum_{n}^{\infty} \frac{b_n C_n}{\Lambda^{(1)} - \lambda_n} \right| \leq \sum_{n}^{\infty} \left| \frac{b_n C_n}{\Lambda^{(1)} - \lambda_n} \right| \leq KM^2 \sum_{n}^{\infty} \frac{1}{|\Lambda^{(1)} - \lambda_n| g^2(n)}$$

$$= KM^2 \sum_{n}^{\infty} \frac{1}{|\lambda_k + \epsilon - \lambda_n| g^2(n)} = KM^2 \sum_{n}^{\infty} \frac{1}{|k^2 - n^2 + \epsilon| n^2}$$

$$= \frac{KM^2}{\epsilon k^2} + KM^2 \sum_{n \neq k}^{\infty} \frac{1}{|k^2 - n^2 + \epsilon| n^2} < \frac{KM^2}{\epsilon k^2} + KM^2 \sum_{n \neq k}^{\infty} \frac{1}{n^p}$$
where \( \lambda_k + \epsilon \) represents the first approximation of the \( k \)th perturbed eigenvalue \( \Lambda_\alpha \), \( \lambda_n \) is known to be proportional to \( n^2 \), and \( 1 < p < 2 + \frac{\log [(k^2 + \epsilon) + n^2]}{\log n} \) with \( k \neq n \neq 1 \). Further, the series

\[
\sum_{n} \frac{b_n C_n}{\Lambda^{(1)}_\alpha - \lambda_n}
\]

would have one term, namely, the term in which \( k = n \) dominates by far the entire series; that is, a term of the form

\[
\frac{b_k C_k}{\epsilon k^2} = \frac{KM^2}{\epsilon k^2}
\]

would always in absolute value be the largest term in the series. Consequently, the series in eq. (25) would have to converge to a positive number, a negative number, or zero depending on how much this term is dominating the series (in other words, depending on the choice of \( \epsilon \) or, what is the same thing, the choice of \( \Lambda^{(1)}_\alpha \)).

The first approximation of the eigenfunction is thus given by

\[
\phi^{(1)} = \sum_{n} \frac{b_n f_n}{\Lambda^{(1)}_\alpha - \lambda_n}
\]

(27)

The series in eq. (27) also would have one dominating term regardless of the value of \( \overline{X} \) (\( \overline{X} \) denotes the \( n \)-truple \( (x_1, x_2, \ldots, x_n) \)) \( \in \Gamma \), for the \( f_n \) are bounded. Therefore, \( \phi^{(1)} \) would not oscillate along the surface \( \Gamma \), so that eq. (25) would not hold because of negative and positive terms cancelling each other. The integral boundary condition therefore assures that \( \phi^{(1)} \) be zero along \( \Gamma \) as well as that the integral of \( \phi^{(1)} \) be zero.

The choice of \( S^{(1)} \) expressed in eq. (17) and (18) does not necessarily guarantee that \( \phi^{(1)} \), which now satisfies the boundary conditions imposed, still satisfies eq. (13). The argument here is that \( S \) may be a more complicated function than a simple "step" type of inhomogeneity. However, the first approximation of \( S \), namely, \( S^{(1)} \), admits of a Fourier expansion, whereas \( S \) itself could not be obtained by a simple Fourier expansion if it were a function of the form

\[
S = S(\overline{x})
\]

instead of the form

\[
S = F(x_1) F(x_2) \cdots F(x_n)
\]
The ultimate form of $S$ is not known initially and no assumption of its form is made initially. Assuming, instead, the form $S^{(1)}$ and getting $\phi^{(1)}$, we check to see if $S^{(1)}$ still allows $\phi^{(1)}$ to satisfy eq. (13). This step is accomplished by substituting $\phi^{(1)}$ into eq. (13) to obtain, after interchanging summation and differentiation, the equation

$$S^{(2)} = \sum_{n}^{\infty} \frac{b_n (\Lambda - \lambda_n) f_n}{\Lambda^{(1)} - \lambda_n} = 0 \text{ in } D - D_1 \quad (28)$$

From eq. (28), we may, after interchanging summation and integration, obtain for the determination of $\Lambda^{(2)}$, the condition

$$\sum_{n}^{\infty} \frac{b_n (\Lambda^{(2)} - \lambda_n) d_n}{\Lambda^{(1)} - \lambda_n} = 0 \quad (29)$$

where

$$d_n = \int_{D - D_1} f_n \quad (30)$$

Knowing $\Lambda^{(2)}$, we have $S^{(2)}$ from eq. (28). Repeating the above procedure with $S^{(2)}$, we find that $\Lambda^{(3)}$ must satisfy

$$\sum_{n}^{\infty} \frac{b_n (\Lambda^{(2)} - \lambda_n) C_n}{(\Lambda^{(3)} - \lambda_n)(\Lambda^{(1)} - \lambda_n)} = 0 \quad (31)$$

and immediately we have that

$$\phi^{(2)} = \sum_{n}^{\infty} \frac{b_n (\Lambda^{(2)} - \lambda_n) f_n}{(\Lambda^{(3)} - \lambda_n)(\Lambda^{(1)} - \lambda_n)} \quad (32)$$

Proceeding in this manner, we obtain the following successive approximations for $S$, $\Lambda$, and $\phi$:

$$\sum_{n}^{\infty} b_n f_n = S^{(1)} \quad (33a)$$

$$\sum_{n}^{\infty} \frac{b_n C_n}{\Lambda^{(1)} - \lambda_n} = 0 \quad (33b)$$
We have demonstrated the existence of the k approximations of the sink functions $S$, and by construction each approximation yielded the $k^{th}$ approximation of the eigenfunction and eigenvalue.

By this method of successive approximations, we have also constructed the eigenfunction so as to be zero on the boundary surfaces, provided the resulting series actually converges uniformly in $D-D_1$ and, in particular, uniformly to zero on the boundary surfaces. In proving that

$$\lim_{k \to \infty} \phi^{(k)} = \phi$$

(34a)
\[ \lim_{k \to \infty} \Lambda^{(k)} = \Lambda \]  
\[ \lim_{k \to \infty} S^{(k)} = S \]

and further that

\[ \lim_{k \to \infty} \left[ \nabla^2 \phi^{(k)} + \Lambda^{(k)} \phi^{(k)} \right] = \nabla^2 \phi + \Lambda \phi = \lim_{k \to \infty} S^{(k)} = S \]

we also prove uniform convergence of the various series as is needed.

First, we quote the following theorem concerning the eigenvalue problem being treated here.

**Theorem 1.** (10) Under the boundary condition \( \phi |_{\Gamma} = 0 \), the \( n \text{th} \) eigenvalue for a domain \( G \) never exceeds the \( n \text{th} \) eigenvalue for a subdomain \( G' \).*

In order to prove eq. (34a), the consider \( \phi \) to be the first eigenfunction designated as \( \phi_1 = \phi \) with the corresponding first eigenvalue \( \Lambda_1 = \Lambda \). The argument to follow applies to all other eigenfunctions as well.

From Theorem 1, we have the inequality

\[ \lambda_1 < \Lambda < \lambda_2 \]

with \( G = D \) and \( G' = D - D_1 \). Let \( \varepsilon_k \) be the \( k \text{th} \) error in the \( k \text{th} \) approximation of \( \Lambda \). Further, write the \( k \text{th} \) approximation of \( \Lambda \) in the form

\[ \Lambda^{(k)} = \lambda_1 + K + \varepsilon_k \]

First, we will show that eq. (34a) holds. From eq. (33k) we must show that

\[ \lim_{k \to \infty} \sum_{n} \frac{b_n}{\Lambda^{(n)} - \lambda_n} \prod_{s=1}^{k} \left\{ \frac{\Lambda^{(s)} - \lambda_n}{\Lambda^{(s+1)} - \lambda_n} \right\} f_n = \Phi \]

and that \( \Phi = \phi \). We must, therefore, show that the product

\[ \lim_{k \to \infty} \left\{ \frac{1}{\Lambda^{(1)} - \lambda_n} \prod_{s=1}^{k} \frac{\Lambda^{(s)} - \lambda_n}{\Lambda^{(s+1)} - \lambda_n} \right\} = \lim_{k \to \infty} P_k = P \]

*In fact, it is always smaller when we are dealing with a proper subdomain.*
yields a result which makes the series in eq. (37) convergent. Substituting eq. (36) into eq. (38), we have to show that

\[
\frac{1}{\lambda_1 - \lambda_n + K + \epsilon_1} \prod_{k=1}^{\infty} \left( \frac{\lambda_1 - \lambda_n + K + \epsilon_{2k}}{\lambda_1 - \lambda_n + K + \epsilon_{2k+1}} \right) = P
\]  

(39)

yields a convergent infinite product which is a function of the behavior of the errors \( \epsilon_k \).

Assume, first, that each error term becomes progressively larger, so that

\[
\lim_{k \to \infty} \epsilon_k = \infty .
\]  

(40)

The product in eq. (39) would then take on the indeterminate form

\[
\frac{1}{\lambda_1 - \lambda_n + K + \epsilon_1} \prod_{k=1}^{\infty} \left( \frac{\lambda_1 - \lambda_n + K + \epsilon_{2k}}{\lambda_1 - \lambda_n + K + \epsilon_{2k+1}} \right) = \infty
\]  

(41)

The values of the indeterminate form in eq. (41) could take on the following values depending on how the errors behave, namely,

Case (i) \( P = A \),

Case (ii) \( P = 0 \),

and

Case (iii) \( P = \infty \),

where \( A \) is a constant, independent of \( n \) and greater than 0. Case (i) yields, then, that

\[
\lim_{k \to \infty} \phi^{(k)} = A \sum_{n} b_n f_n = AS^{(1)}
\]  

(42)

where \( S^{(1)} \) is again the sink function in eq. (19). Equation (42) means that, given \( \epsilon > 0 \),

\[
| \phi^{(k)} - AS^{[k]} | < \epsilon
\]  

(43)

for all \( \mathbf{X} \) \( [\mathbf{X} \text{ denotes the n-tuple } (x_1, x_2, \cdots, x_n) \] and for \( k > N \). But for any \( \mathbf{X} \in \overline{D-D_1} \), \( S^{(1)} = 0 \); therefore, eq. (43) yields that

\[
| \phi^{(k)} | < \epsilon
\]  

(44)
Equation (44) in turn implies that for large $k$

$$\nabla^2 \phi^{(k)} + \Delta^{(k)} \phi^{(k)} = S^{(k)} \to 0$$

(45)

and, in particular, for $x \in \Sigma_1$, which contradicts eq. (14).

Because $P = 0$ in case (ii), we have, immediately from eq. (43), the inequality

$$|\phi^{(k)}| < \varepsilon$$

(46)

By the same argument as presented for case (i), we have the same contradiction of eq. (14) by case (ii).

For case (iii), we have, from eq. (42), the inequality

$$|\phi^{(k)}| > M,$$

(47)

where $M$ is a constant which, for increasing $k$, becomes arbitrarily large. Thus, for large $k$ we would have that

$$\nabla^2 \phi^{(k)} + \Delta^{(k)} \phi^{(k)} = S^{(k)} \to \infty$$

(48)

which again contradicts eq. (14).

If the $\varepsilon_k$'s did not change with all $k$'s, we would have the situations corresponding to case (i) or case (ii). If the $\varepsilon_k$'s increased without bound, we would have case (iii).

Therefore, the $\varepsilon_k$'s must decrease with increasing $k$. So we must have that

$$\lim_{k \to \infty} \varepsilon_k = 0$$

(49)

and eq. (49) implies that all the ratios in eq. (41) tend to one, so that

$$\prod_{k=1}^{\infty} \left( \frac{\lambda_1 - \lambda_n + K + \varepsilon_{2k}^{(k)}}{\lambda_1 - \lambda_n + K + \varepsilon_{2k+1}^{(k)}} \right) = B_n$$

(50)

Therefore, we have that

$$\lim_{k \to \infty} \phi^{(k)} = \sum_{n} \frac{b_n B_n f_n}{\Lambda^{(n)} - \lambda_n} = \Phi$$

(51)
Since the $\epsilon_k$'s tend to zero with increasing $k$, we see from eq. (50) that

$$\lim_{n \to \infty} B_n \leq 1.$$  

The series expression for $\phi$ in eq. (51) will now be shown to converge uniformly. Since $|S|$ is bounded,

$$|b_n| = \left| \int_{\overline{D}_1} S^{(1)} f_n \right| \leq \int_{\overline{D}_1} \left| S^{(1)} f_n \right| < K |\overline{D}_1|,$$  

(51a)

where $K$ is a fixed number $\leq 1$ independent of $n$ and $|\overline{D}_1|$ is the magnitude of $\overline{D}_1$. Consequently,

$$\left| \frac{b_n B_n f_n}{\Lambda^{(1)} - \lambda_n} \right| < \frac{MB_n}{\Lambda^{(1)} - \lambda_n},$$

where $M = K |\overline{D}_1|$. The series of constant terms $B_n / \Lambda^{(1)} - \lambda_n$ converges since $\lambda_n$ is of the form $n^2$; hence, according to the Weierstrass M-test, the series in eq. (51) converges uniformly with respect to $\mathfrak{x}$. Also, the terms of this series are continuous with respect to $\mathfrak{x}$, so that $\phi$ is uniformly continuous.

The terms of the series obtained by integrating eq. (51) over $\Gamma$ satisfy the inequality

$$\left| \frac{b_n B_n f_n}{n(\Lambda^{(1)} - \lambda_n)} \right| \leq \frac{KM'B_n}{n |\Lambda^{(1)} - \lambda_n|},$$

where $K$ is defined in eq. (36) and $M'$ is defined in the equation

$$\int_{\Gamma'} f_n \leq \frac{1}{n} \int_{\Gamma} \left| F_n \right| = \frac{N |\Gamma'|}{n} = \frac{M'}{n}.$$  

Here $|\Gamma'|$ is the magnitude of the surface $\Gamma'$ over which the integration is carried out after a change of variables $\mathbf{x} \to \mathbf{x}' = \mathbf{x}/n$. As in the above argument, this series is uniformly convergent. Since $\phi$ is of the same form as $\phi^{(1)}$, then by the same argument that $\phi^{(1)}$ is zero on $\Gamma$, also $\phi$ is zero on $\Gamma$.

If the series in eq. (51) is differentiated once and then again, the following inequalities are derived:

$$\left| \frac{nb_n B_n C_n}{\Lambda^{(1)} - \lambda_n} \right| \leq \frac{A'n B_n}{\Lambda^{(1)} - \lambda_n}.  \quad (51b)$$
and
\[
\frac{n^2 b_n B_n f_n}{\Lambda^{(1)} - \lambda_n} \leq \frac{M n^2 B_n}{\Lambda^{(1)} - \lambda_n} ,
\]
(51c)
where
\[
nA' = |\nabla f_n| \leq n |G_n b_n|
\]
and M is as previously defined. Since A' and M are constants, we test the terms involving n in eq. (51b) and (51c) for uniform convergence by the Cauchy Ratio Test and find that
\[
A' \lim_{n \to \infty} \left| \frac{(n + 1) B_{n+1} (\Lambda^{(1)} - \lambda_n)}{n B_n (\Lambda^{(1)} - \lambda_{n+1})} \right| = A' < 1
\]
(52)
and that
\[
M \lim_{n \to \infty} \left| \frac{(n+1)^2 B_{n+1} (\Lambda^{(1)} - \lambda_n)}{n^2 B_n (\Lambda^{(1)} - \lambda_{n+1})} \right| = M = K |D_1| ,
\]
(53)
respectively. Since |D_1| is less than one, then for eigenfunctions having a bound K < 1, the right-hand side of eq. (53) is less than one, so that once and twice-differentiated series are uniformly convergent and may be differentiated term by term.

Next, we prove that \(\Lambda^{(k)}\) converges, i.e., that
\[
\lim_{k \to \infty} \Lambda^{(k)} = \bar{\Lambda}
\]
(54)
Now, we know that
\[
\Lambda^{(k)} = \lambda_1 + K + \varepsilon_k
\]
(55)
and, therefore, that
\[
\Lambda^{(k+1)} - \Lambda^{(k)} = \varepsilon_{k+1} - \varepsilon_k
\]
(56)
From eq. (53), we have
\[
|\Lambda^{(k+1)} - \Lambda^{(k)}| = |\varepsilon_{k+1} - \varepsilon_k|
\]
(57)
which, for \(\varepsilon > 0\) and large \(k\), gives
\[
|\Lambda^{(k+1)} - \Lambda^{(k)}| < \varepsilon
\]
(58)
Since the $\epsilon_k$'s $\to 0$ with increasing $k$, we have as in the above arguments that, in addition,

$$\lim_{k \to \infty} S^{(k)} = S.$$ \hfill (59)

Proceeding from the above arguments, we have then that

$$\nabla^2 \phi + \overline{\Lambda} \phi = \overline{S} \text{ in } \overline{D_1}$$ \hfill (60)

$$= 0 \text{ in } \overline{D-D_1}$$ \hfill (61)

and that

$$\phi \big|_{T} = 0.$$ \hfill (62)

Therefore, by the Cauchy-Kowalewski theorem, we have

$$\phi = \Phi$$

$$\Lambda = \overline{\Lambda}$$ \hfill (63)

Q.E.D.
VI. PERTURBATION FORMULAS FOR THE EIGENVALUES AND EIGENFUNCTIONS

From eq. (33f) and (33k), one may easily obtain a perturbation-like formula for the eigenvalues. In particular, we may rearrange eq. (33d) to obtain an expression for the second-order approximation:

\[ \lambda^{(2)} = \sum_n^\infty \frac{b_n d_n \lambda_n}{\Lambda^{(1)} - \lambda_n} \left/ \sum_n^\infty \frac{b_n d_n}{\Lambda^{(1)} - \lambda_n} \right. \]  

(64)

In practice, if one, for example, were looking for the \( n \)-th eigenvalue of the sequence \( \{\Lambda_n\} \), he would start at \( \Lambda_n^{(1)} \), given by

\[ \lambda_n < \Lambda_n^{(1)} < \lambda_n + 1 \]  

(65)

and increment \( \Lambda_n^{(1)} \) for \( n=1 \) until eq. (64) was satisfied within some desired error.

In using eq. (33f) to find the third approximation \( \Lambda^{(3)} \), and knowing the values of \( \Lambda^{(2)} \) and \( \Lambda^{(1)} \), we replaced \( d_n \) by

\[ \frac{d_n (\Lambda^{(2)} - \lambda_n)}{(\Lambda^{(1)} - \lambda_n)(\Lambda^{(3)} - \lambda_n)} \]  

(66)

and proceed as in the above manner to obtain

\[ \Lambda^{(4)} = \sum_n^\infty \frac{b_n d_n (\Lambda^{(2)} - \lambda_n) \lambda_n}{(\Lambda^{(1)} - \lambda_n)(\Lambda^{(3)} - \lambda_n)} \left/ \sum_n^\infty \frac{b_n d_n (\Lambda^{(2)} - \lambda_n)}{(\Lambda^{(1)} - \lambda_n)(\Lambda^{(3)} - \lambda_n)} \right. \]  

(67)

that is, we increment

\[ \Lambda^{(3)} = \lambda_1 + K + \epsilon_3 \]  

(68)

until \( \Lambda^{(4)} \) times the denominator of eq. (67) equals the numerator.

Continuing in this manner, we find that the even-order approximation of the eigenvalue \( \Lambda \) may be found from the equation

\[ \Lambda^{(2k+2)} = \sum_n^\infty \frac{b_n d_n}{\prod_{s=0}^{k} (\Lambda^{(2s+1)} - \lambda_n)} \left/ \sum_n^\infty \frac{b_n d_n}{\prod_{s=0}^{k} (\Lambda^{(2s+1)} - \lambda_n)} \right. \]  

(69)
with the values of all the even-order approximations being found by use of the equations for odd-order approximations.

If one expands eq. (67), one gets a perturbation-like formula of the form

$$
\Lambda^{(2k+2)} = \frac{b_n D_n^k}{\sigma^k} \left[ \lambda_n + \sum_{j \neq n}^{\infty} \frac{b_j D_j^k \lambda_j}{b_n D_n^k} \right],
$$

(70)

where

$$
D_n^k = \frac{d_n \prod_{s=1}^{k} (\Lambda^{(2s)} - \lambda_n) \lambda_n}{\prod_{s=0}^{k} (\Lambda^{(2s+1)} - \lambda_n)}.
$$

and

$$
\sigma^k = \frac{\sum_{n}^{\infty} b_n d_n \prod_{s=1}^{k} (\Lambda^{(2s)} - \lambda_n)}{\prod_{s=0}^{k} (\Lambda^{(2s+1)} - \lambda_n)}.
$$

In a similar manner, a perturbation-like formula

$$
\phi_n^k = a_n E_n^k \left[ f_n + \sum_{j \neq n}^{\infty} \frac{a_j E_j^k f_j}{a_n E_n^k} \right],
$$

(71)

where

$$
E_n^k = \prod_{s=1}^{k} (\Lambda^{(2s)} - \lambda_n) \prod_{s=0}^{k} (\Lambda^{(2s+1)} - \lambda_n)
$$

may be derived for eigenfunctions.
VII. DETERMINATION OF THE FLUX AND BUCKLING FOR A HOMOGENEOUS REACTOR WITH PARTIALLY INSERTED CONTROL RODS

The method developed will now be applied to determine the flux and buckling for a finite, cylindrical, bare, homogeneous thermal reactor with partially inserted control rods. In the approximation given by diffusion theory,\(^{(11)}\) we wish to solve eq. (1) for the first eigenfunction subject to homogeneous Dirichlet boundary conditions on the extrapolated boundaries of the reactor and on the effective boundaries of the control rods.\(^{(12)}\) We also wish to determine the first eigenvalue. The first eigenfunction and corresponding eigenvalue are known as the flux and buckling, respectively, in reactor parlance.

Figure 1 shows how the r- and z- axes were chosen, where \(H\) is the extrapolated height, \(R\) the extrapolated radius of the reactor, \(h_i\) is the effective length of insertion of the \(i^{th}\) control rod, and \(r_i\) its effective radius. Only two control rods are shown in Fig. 1 for the sake of clarity since the theory applies equally well for many rods. We denote by \(\Omega\) the extrapolated volume of the reactor, and by \(S\) its surfaces. We denote by \(\Omega_i\) the effective volume of the \(i^{th}\) control rod, and by \(\sigma_i\) its surface. With these definitions we wish to solve eq. (12) and (13) subject to boundary conditions IV (iii).

Equations (12) and (13) become, for the geometry under consideration,

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} + \Lambda \phi = S \text{ in } \Omega_i \tag{72}
\]

\[
= 0 \text{ in } \Omega - \Omega_i \tag{73}
\]
The eigenfunctions $f_n$ are taken to be the unnormalized eigenfunctions (13)

$$f_{mn\ell} = J_m\left(\frac{\beta_{mn}}{R}\right) \left( a_{mn} \cos m\theta + i^{-\bar{a}_{mn}} \sin m\theta \right) \sin \frac{l\pi z}{H}, \quad (73a)$$

and the eigenvalues $\lambda_n$ become the eigenvalues

$$\lambda_{mn\ell} = \left(\frac{\beta_{mn}}{R}\right)^2 + \left(\frac{l\pi}{H}\right)^2,$$

where the $\beta_{mn}$ are the $n$ roots of the Bessel function of the first kind of order $m$.

In the reactor without any rods, the first eigenfunction (which is $\geq 0$ in $D$) is then of the form

$$f_{011} = J_0\left(\frac{\beta_{01}}{R}\right) \sin \frac{\pi z}{H}.$$

Thus, we define the first approximation of the sink $S_1$ to be

$$S_1(r,z,\theta) = J_0\left(\frac{\beta_{01}}{R}\right) \sin \frac{\pi z}{H} \text{ in } D_i$$

$$= 0 \text{ in } D-D_i \quad (i = 1, 2, \ldots, k) \quad (74)$$

$$= 0 \text{ in } D-D_i \quad (i = 1, 2, \ldots, k) \quad (75)$$

Figure 2 is a view of the reactor in the plane $z = 0$ and shows how
the various parameters were chosen to describe the positions of the $i^{th}$ and $i+1$ control rods. The effective control rod heights are understood to be $h_i$ and $h_i + 1$, respectively. Figure 2 also shows how the variables $r$ and $\theta$ are changed to $\rho$ and $\psi$ when the origin is moved from $O$ to $O'$, the center of the $i^{th}$ rod. This change of variables and the other angles shown are to be used in conjunction with the addition theorems for Bessel functions, which in turn are needed to carry out the various integrations. The general form for the addition theorem is:

$$\sum_{k=0}^{\infty} J_{m+k}(\lambda r) J_{m+k}(\lambda \rho) e^{ik\psi} = \frac{1}{\sqrt{2\pi}} J_{m}(\lambda r) J_{m}(\lambda \rho) e^{im\theta}, \quad (75a)$$

where $\lambda$ is a constant.

Since more than one control rod is present, the interaction effects of these control rods must be considered. An arbitrary reference radius may be taken as one side of the various angles $\alpha_1, \alpha_2, \ldots, \alpha_i, \alpha_{i+1}$, but once chosen must remain fixed. Thus eq. (75a) now takes on the form

$$J_{m}(\lambda r) e^{im(\theta - \alpha_i)} = \sum_{k=-\infty}^{\infty} J_{m+k}(\lambda d_i) J_k(\lambda \rho) e^{ik\psi} \quad (76)$$

Multiplying both sides of eq. (76) by $e^{im\alpha_i}$ and taking the real and imaginary parts of both sides of eq. (76), we have

$$J_{m}(\lambda r) \left\{ \cos m\theta \right\} = \sum_{k=-\infty}^{\infty} J_{m+k}(\lambda d_i) J_k(\lambda \rho) \left\{ \cos(k\psi + m\alpha_i) \right\} \quad (77)$$

If $r$ varies very little over $D_i$, we could replace $r$ by $d_i$ in eq. (74) to get for the $i^{th}$ sink function:

$$S_i(z, \theta) = \frac{\beta_0 d_i}{R} \sin \frac{\pi z}{H} \text{ in } \overline{D_i}$$

$$= 0 \text{ in } D_i \setminus \overline{D_i} \quad (i = 1, 2, \ldots, k) \quad (78)$$

However, in what follows, we will not make the assumption that $\overline{D_i}$ is very small relative to $\overline{D}$.

The sink function $S$ in eq. (68) is actually the sum of all the sink functions $S_i$, since eq. (68) is to be solved in the domain $\overline{D - \bigcup_{i=1}^{k} D_i}$. 


We will thus consider the sink $S_i$ first and sum the effects of the sinks from $i = 1$ to $k$ to get the total sink function, $S$. In so doing, the interactions of the rods will also appear implicitly because of the angles $\alpha_i$ appearing in the function $S$.

Using the functions $f_{mn}^l$, we form the eigenfunction expansion

$$S_i(t) = \sum_{m,n,l} \left\{ \frac{a_{mn}}{\hat{a}_{mn}} \right\} b_l J_m \left( \frac{\beta_{mn}^l R}{R} \right) \sin \frac{\ell \pi z}{H} \left\{ \cos m \theta \right\} , \quad (80)$$

where $\left\{ \right\}$ denote the linear combination expressed in eq. (73a). We now multiply eq. (80) by the linear combination

$$\left\{ \frac{a_{m'n'}}{\hat{a}_{m'n'}} \right\} J_m \left( \frac{\beta_{m'n'}^l R}{R} \right) \sin \frac{\ell' \pi z}{H} \left\{ \cos m' \theta \right\}$$

and integrate over $D$. By use of the orthogonality properties,

$$\int_0^R r J_m \left( \frac{\beta_{mn}^l R}{R} \right) J_m \left( \frac{\beta_{mn'}^l R}{R} \right) dr = \frac{R^2}{2} J_{m+1}^2 \left( \frac{\beta_{mn}^l R}{R} \right) ; \quad n = n', \quad 0 = 0 ; \quad n \neq n'$$

$$\int_0^H \sin \frac{\ell \pi z}{H} \frac{\ell' \pi z}{H} dz = H/2 \quad ; \quad \ell = \ell'$$

$$= 0 \quad ; \quad \ell \neq \ell'$$

$$\int_0^{2\pi} \cos m \theta \cos m' \theta d\theta = 2\pi \quad ; \quad m = m' = 0$$

$$= \pi \quad ; \quad m = m' \neq 0$$

$$= 0 \quad m \neq m'$$

and

$$\int_0^{2\pi} \sin m \theta \sin m' \theta d\theta = \pi \quad ; \quad m = m'$$

$$= 0 \quad ; \quad \text{otherwise.}$$
We have that
\[
\begin{align*}
\left\langle a_{mn} \right| b_{kl} &= \frac{4}{\pi R^2 H J^2} \left( \frac{\beta}{\beta_{mn}} \right) \int_0^R \int_0^H \int_0^{2\pi} S_i(r, z, \theta) J_m \left( \frac{\beta_{mn} r}{R} \right) \\
&\times \begin{bmatrix}
\cos m\theta \\
\sin m\theta
\end{bmatrix} \sin \frac{l\pi z}{H} r \, dr \, dz \, d\theta.
\end{align*}
\]
\[ (81) \]

From eq. (74) and (75), \( S_i \neq 0 \) in \( D_i \) and \( S_i = 0 \) in \( \overline{D-D_i} \) and by use of eq. (77), we have eq. (81) in the form
\[
\begin{align*}
\left\langle a_{mn} \right| b_{kl} &= \frac{4}{\pi R^2 H J^2} \left( \frac{\beta}{\beta_{mn}} \right) \int_0^{r_1} \int_0^{h_1} \int_0^{2\pi} \sum_{k, k' = -\infty}^{\infty} J_{m+k} \left( \frac{\beta_{mn} d_i}{R} \right) \\
&\times J_k \left( \frac{\beta_{mn}}{R} \right) J_{k'} \left( \frac{\beta_{mn}}{R} \right) \begin{bmatrix}
\cos (k\psi + m \alpha_i) \\
\sin (k\psi + m \alpha_i)
\end{bmatrix} \\
&\times \begin{bmatrix}
E_{k'} \cos k' \psi + \overline{E}_{k'} \sin k' \psi \\
E_{k'} \cos k' \psi + \overline{E}_{k'} \sin k' \psi
\end{bmatrix} \sin \frac{\pi z}{H} \sin \frac{l\pi z}{H} \rho \, d\rho \, dz \, d\psi.
\end{align*}
\]
\[ (82) \]

If we first carry out the integrations with respect to \( \psi \), we have integrals of the form
\[
\int_0^{2\pi} \left( E_{k'} \cos k' \psi + \overline{E}_{k'} \sin k' \psi \right) \begin{bmatrix}
\cos (k\psi + m \alpha_i) \\
\sin (k\psi + m \alpha_i)
\end{bmatrix}
\]
\[
\begin{cases}
2\pi \left\{ \begin{array}{c}
\cos m \alpha_i \\
\sin m \alpha_i
\end{array} \right\} & k' = k = 0 \\
2\pi \left\{ \begin{array}{c}
\cos m \alpha_i \\
\sin m \alpha_i
\end{array} \right\} & k' = k \\
\pi \left\{ \begin{array}{c}
\cos m \alpha_i \\
\sin m \alpha_i
\end{array} \right\} & k' \neq k
\end{cases}
\]

where \( E_{k'} \), being an arbitrary constant, is taken equal to one.

The integration with respect to \( \rho \) yields integrals of the form (15)
\[
\int_{0}^{r_i} J_k\left(\frac{\beta_{01} r}{R}\right) J_k\left(\frac{\beta_{mn} r}{R}\right) r \, dr = \frac{r_i R}{\beta_{01}^2 - \beta_{mn}^2} \left\{ \beta_{01} J_k\left(\frac{\beta_{mn} r}{R}\right) J_{k+1}\left(\frac{\beta_{01} r}{R}\right) \right. \\
- \beta_{mn} J_k\left(\frac{\beta_{01} r}{R}\right) J_{k+1}\left(\frac{\beta_{mn} r}{R}\right) \right\} r_{01} \neq \beta_{mn}
\]
\[
= \frac{r_i^2}{2} \left[ J_k^2\left(\frac{\beta_{01} r}{R}\right) \right] + \left[ 1 - \frac{k^2}{\left(\frac{\beta_{01} r}{R}\right)^2} \right] J_k^2\left(\frac{\beta_{01} r}{R}\right) \beta_{01} = \beta_{mn}
\]
by use of the relation (15)
\[
\left[ J_{\nu}(kz) \right]^2 = \left[ \frac{\nu J_{\nu}(kz)}{kz} - J_{\nu+1}(kz) \right]^2,
\]
the latter integration is equal to
\[
\frac{r_i^2}{2} \left[ J_k^2\left(\frac{\beta_{01} r}{R}\right) - \frac{2kR}{\beta_{mn} r_i} J_m\left(\frac{\beta_{01} r}{R}\right) J_{k+1}\left(\frac{\beta_{mn} r}{R}\right) + J_k^2\left(\frac{\beta_{01} r}{R}\right) \right].
\]
Substituting these equations and the equation
\[
\frac{2}{H} \int_{0}^{h_i} \sin \frac{\pi z}{H} \sin \frac{\pi l z}{H} \, dz = b_\ell,
\]
where
\[
b_\ell = \frac{2}{\pi} \left( \frac{\pi h_i}{H} - \frac{1}{2} \sin \frac{2\pi h_i}{H} \right), \quad \ell = 1
\]
\[
= \frac{2}{\pi} \left( \frac{1}{1 - \ell^2} \right) \left( \ell \sin \frac{\pi h_i}{H} \cos \frac{\pi h_i}{H} - \sin \frac{\ell \pi h_i}{H} \cos \frac{\ell \pi h_i}{H} \right), \quad \ell > 1,
\]
into eq. (82), we have that
\[
\left\{ \begin{array}{c}
\frac{a_{mn}}{\pi mn} \\
\frac{2}{\pi mn}
\end{array} \right\} b_\ell = \frac{4r_i}{R J^{2m+1}(\beta_{mn})} \left[ J_0\left(\frac{\beta_{01} d_i}{R}\right) J_m\left(\frac{\beta_{mn} d_i}{R}\right) \Gamma_0 \\
+ \sum_{k=1}^{\infty} J_k\left(\frac{\beta_{01} d_i}{R}\right) X J_{m+k}\left(\frac{\beta_{mn} d_i}{R}\right) \Gamma_k + (-1)^{2k-m} \right]
\]
\[
J_{k-m}\left(\frac{\beta_{mn} d_i}{R}\right) \Gamma_{-k} \right\} \begin{bmatrix}
\cos m \alpha_i \\
\sin m \alpha_i
\end{bmatrix} b_\ell,
\]
(83)
where
\[
\Gamma_{-k} = \frac{r_i}{2R} \left[ J_k^2 \left( \frac{\beta_{mn} r_i}{R} \right) + (-1)^{2k-1} \left( \frac{2kR}{\beta_{mn} r_i} \right) J_k \left( \frac{\beta_{mn} r_i}{R} \right) J_{k-1} \left( \frac{\beta_{mn} r_i}{R} \right) \right] + J_k^2 \left( \frac{\beta_{mn} r_i}{R} \right)
\]
and
\[
\Gamma_{k} = \frac{r_i}{2R} \left[ J_k^2 \left( \frac{\beta_{mn} r_i}{R} \right) - \frac{2kR}{\beta_{mn} r_i} J_k \left( \frac{\beta_{mn} r_i}{R} \right) J_{k+1} \left( \frac{\beta_{mn} r_i}{R} \right) - J_k^2 \left( \frac{\beta_{mn} r_i}{R} \right) \right]
\]

The quantity \( \Gamma_{-k} \) is formed from \( \Gamma_k \) by use of the relation
\[
J_{-k}(z) = (-1)^k J_k(z)
\]
In addition,
\[
\Gamma_0 = \frac{r_i}{2R} \left[ J_0^2 \left( \frac{\beta_{0n} r_i}{R} \right) + J_0^2 \left( \frac{\beta_{0n} r_i}{R} \right) \right] = \frac{2}{\beta_{01} - \beta_{mn}} \left[ \beta_{01} J_0 \left( \frac{\beta_{mn} r_i}{R} \right) J_1 \left( \frac{\beta_{mn} r_i}{R} \right) - \beta_{mn} J_0 \left( \frac{\beta_{01} r_i}{R} \right) J_1 \left( \frac{\beta_{mn} r_i}{R} \right) \right]
\]

The final form of the first approximation of the total sink function is, therefore,
\[
S(\theta) = \sum_{i=1}^{k} S_i(\theta) = \sum_{i=1}^{k} C_i \left\{ a_{mn} \right\} b_{\ell} J_m \left( \frac{\beta_{mn} r_i}{R} \right) \left\{ \cos m\theta \right\} \sin \frac{\ell \pi z}{H}, \quad (84)
\]
where
\[
C_i = 4r_i/R
\]
As was stated previously, we are seeking the first eigenfunction and first eigenvalue, so in what follows it is understood that this is the case.

Knowing the first approximation, \( S(\theta) \), of the sink, we find, from eq. (33c), the first approximation of the eigenfunction:
\[
\phi(\theta) = \sum_{i=1}^{k} C_i \sum_{m,n,\ell} a_{mn} b_{\ell} J_m \left( \frac{\beta_{mn} r_i}{R} \right) \left\{ \cos m\theta \right\} \sin \frac{\ell \pi z}{H}, \quad (85)
\]
with the first approximation $A^{(1)}$ of the eigenvalue $A$ to be determined by the use of eq. (33b). Integrations of $\phi^{(i)}$ over the surfaces of the control rods require integrating eq. (77) over these surfaces. These integrations would be zero unless $k = 0$, as can be seen by considering first the integration with respect to $\psi$.

In order to get the interaction of these control rods, we let the $i^{th}$ + 1 control rod now become the $j^{th}$ control rod, with $i \neq j$ unless $i = 1$, that is, unless one rod is present only. Equation (77) for this integration then becomes effectively

$$J_m \left( \frac{\beta_{mn}}{R} \right) e^{im\theta} = J_m \left( \frac{\beta_{mn}}{R} \right) J_0 \left( \frac{\beta_{mn}}{R} \right) e^{im\alpha_i} \quad (86)$$

Substituting eq. (86) in eq. (85), we then form the integrals

$$\sum_{j=1}^{k} \phi_i = \sum_{i=1}^{k} C_i \sum_{m,n,\ell} a_{mn} b_{\ell} \sum_{j=1}^{k} \left\{ J_m \left( \frac{\beta_{mn}}{R} \right) \left[ \cos m\alpha_j \right] \right\}$$

$$= \sum_{i=1}^{k} \sum_{m,n,\ell} a_{mn} b_{\ell} \sum_{j=1}^{k} \left\{ J_m \left( \frac{\beta_{mn}}{R} \right) \left[ \cos m\alpha_j \right] \right\}$$

Substituting eq. (86) in eq. (85), we then form the integrals

$$\sum_{j=1}^{k} \phi_i = \sum_{i=1}^{k} C_i \sum_{m,n,\ell} a_{mn} b_{\ell} \sum_{j=1}^{k} \left\{ J_m \left( \frac{\beta_{mn}}{R} \right) \left[ \cos m\alpha_j \right] \right\}$$

$$x \left[ \int r_j J_0 \left( \frac{\beta_{mn}}{R} \right) \rho d\rho \sin \frac{\ell \pi h_j}{H} \int_0^{2\pi} d\psi + r_j J_0 \left( \frac{\beta_{mn}}{R} \right) \right]$$

$$\sum_{i=1}^{k} \sum_{m,n,\ell} a_{mn} b_{\ell} \sum_{j=1}^{k} \left\{ J_m \left( \frac{\beta_{mn}}{R} \right) \left[ \cos m\alpha_j \right] \right\}$$

$$= \sum_{i=1}^{k} \sum_{m,n,\ell} a_{mn} b_{\ell} \sum_{j=1}^{k} \left\{ J_m \left( \frac{\beta_{mn}}{R} \right) \left[ \cos m\alpha_j \right] \right\}$$

Carrying out the integrations by use of the relation

$$\int_0^r \rho J_0 \left( \frac{\beta_{mn}}{R} \right) \rho d\rho = \frac{R r}{\beta_{mn}} J_1 \left( \frac{\beta_{mn} r}{R} \right)$$

we have that the first approximation $A^{(1)}$, must satisfy the equation

$$2\pi \sum_{i=1}^{k} C_i \sum_{m,n,\ell} a_{mn} b_{\ell} \sum_{j=1}^{k} \left\{ r_j J_m \left( \frac{\beta_{mn}}{R} \right) \left[ \cos m\alpha_j \right] \right\}$$

$$+ \frac{H}{\ell \pi} J_0 \left( \frac{\beta_{mn}}{R} \right) \left( 1-\cos \frac{\ell \pi h_j}{H} \right) \left\{ \cos m\alpha_j \sin m\alpha_j \right\} = 0 \quad (88)$$
Equation (88) implicitly shows the complicated interaction of the control rods, for it must be remembered that \( \{ \hat{a}_{mn} \} \) are also functions of the control rod positions.

The second-order approximation, \( \Lambda^{(2)} \), of the eigenvalue \( \Lambda \) is obtained by use of eq. (33d). Using eq. (86) and the fact that the \( i^{th} + 1 \) rod is designated by \( j \), we form the integrals

\[
\int_{D-D_1} \sum_{i=1}^{k} S_i^{(k)} = \sum_{i=1}^{k} \left\{ \sum_{m,n,L} \left[ \frac{a_{mn}}{\hat{a}_{mn}} \right] \frac{b_L \left( \Lambda^{(2)} - \lambda_{mnL} \right)}{\Lambda^{(1)} - \lambda_{mnL}} \right. \\
\times \left[ \int_{0}^{R} \int_{0}^{H} \int_{0}^{2\pi} J_0 \left( \frac{\beta_{mn}}{R} \right) \cos^m \theta \sin^m \theta \right. \\
\left. \sin \frac{\ell \pi z}{R} - \int_{0}^{j} J_0 \left( \frac{\beta_{mn}}{R} \right) \cos^{\alpha_j} \theta \sin^{\alpha_j} \theta \right. \\
\left. \rho d \rho d z d \theta \right\} = 0.
\]

(89)

Unless \( m = 0 \), the first integration in eq. (89) is zero. Carrying out the integrations in eq. (89), we have that the second-order approximation, \( \Lambda^{(2)} \), must satisfy the equation

\[
\sum_{i=1}^{k} \left\{ \sum_{m=0,n,L} \left[ \frac{a_{mn}}{\hat{a}_{mn}} \right] \frac{b_L \left( \Lambda^{(2)} - \lambda_{mnL} \right)}{\Lambda^{(1)} - \lambda_{mnL}} \right. \\
\times \left[ \int_{0}^{R} \int_{0}^{H} \int_{0}^{2\pi} J_0 \left( \frac{\beta_{mn}}{R} \right) \cos^m \theta \sin^m \theta \right. \\
\left. \sin \frac{\ell \pi z}{R} - \int_{0}^{j} J_0 \left( \frac{\beta_{mn}}{R} \right) \cos^{\alpha_j} \theta \sin^{\alpha_j} \theta \right. \\
\left. \rho d \rho d z d \theta \right\} = 0.
\]

(90)

Knowing \( \Lambda^{(2)} \), we may find \( S^{(2)} \) by use of eq. (33e). Using eq. (33f) and (84) through (88), we may find \( \Lambda^{(3)} \). Using eq. (33g), we have the second-order approximation of \( \phi \):
\[
\phi^{(2)} = \sum_{i=1}^{k} C_i \sum_{m,n,l}^{\infty} \left\{ \frac{a_{mn}}{a_{mn}} b_{l} \left( \lambda^{2} - \lambda \right)_{mn} l \right\} J_{m} \left( \frac{\beta_{mn}}{R} \right) \left\{ \cos m \theta \right\} \sin \frac{l \pi z}{H}.
\]

(91)

Since this is an eigenvalue problem, the solution may be multiplied by an arbitrary constant. It is now quite clear how this procedure may be continued in this application of the theory to obtain eq. (33h) through (33k).

As previously mentioned in Sect. II, the theory developed may be used equally well when the homogeneous boundary conditions imposed may be of the form shown in eq. (11a). In this case, the roots $\beta_{mn}$ would have to be such that

\[
\frac{\partial}{\partial r} \left|_{\Sigma r} \right| J_{m} \left( \frac{\beta_{mn}}{R} \right) = 0
\]

and the integers $l$ would have to be such that

\[
\frac{\partial}{\partial z} \left|_{\Sigma z} \right| \left( \sin \frac{l \pi z}{H} \right) = 0
\]

where $\Sigma r$ and $\Sigma z$ are those portions of the surface which are normal to the components $\nu_r$ and $\nu_z$ of the surface normal $\nu$. The remaining boundary conditions in eq. (11a) may be satisfied by proceeding as already shown. Similar arguments may be applied to the other boundary-value problems mentioned in Sect. II.
VIII. THE SPECIAL CASE OF A FULLY INSERTED CENTRAL ROD AND A RESULTING IDENTITY

The derivation of the first eigenfunction of eq. (1) for the case of a central rod fully inserted in a finite, bare, homogeneous thermal reactor, subject to homogeneous Dirichlet boundary conditions on the extrapolated boundaries of the reactor and the effective boundaries of the control rod may be found in reference 18. The solution of this problem is

\[ \Theta(r,z) = A \left[ J_0 \left( \frac{\beta r}{R} \right) - \frac{J_0(\beta)}{Y_0(\beta)} Y_0 \left( \frac{\beta r}{R} \right) \right] \sin \frac{\pi z}{H} , \tag{92} \]

where \( Y_0 \left( \frac{\beta r}{R} \right) \) is the zero-order Bessel function of the second kind, and \( A \) is an arbitrary normalization constant.

Since this geometry admits of a separate solution, the first eigenvalue, \( \Lambda_1 \), is of the form

\[ \Lambda_1 = \left( \frac{\beta}{R} \right)^2 + \left( \frac{\pi}{H} \right)^2 , \tag{93} \]

where \( \left( \frac{\beta}{R} \right)^2 \) is the radial component and \( \left( \frac{\pi}{H} \right)^2 \) is the vertical component of the eigenvalue. For the case of the rod radius \( r_1 \ll R \), the reactor radius, \( \beta \), may be shown to be of the form

\[ \beta = \frac{\beta_{01}}{R} + \frac{3.75}{\beta_{01}R} \left( 0.116 + \frac{\ell_n R}{2.4 r_1} \right) , \tag{94} \]

where \( \beta_{01} \) is such that

\[ J_0 \left( \beta_{01} \right) = 0 \]

If in the complete solution of the reactor equation for many partially inserted rods, we let \( i = 1 \) only, and form the limit

\[ \lim_{h_1 \to H, d_1 \to 0} \phi , \]

we must get, because of the uniqueness of the solution, the same solution as in eq. (92). Carrying out the limiting process:

\[ \lim_{h_1 \to H, d_1 \to 0} \sum_{m,n,\ell} \left\{ \frac{a_{mn\ell}}{a_{mn}} \right\} b_{\ell} \prod_{s=1}^{\infty} \left( \Lambda_2 s \right)^{-\lambda_{mn\ell}} \prod_{s=0}^{\infty} \left( \Lambda_2 (s+1) \right)^{-\lambda_{mn\ell}} J_0 \left( \frac{\beta_{mn\ell} r}{R} \right) \left\{ \cos \theta \right\} \left\{ \sin m \theta \right\} \sin \frac{\ell \pi z}{H} \]
we find after interchanging the sum and limit operations and applying this limiting process to eq. (83) that

\[
\frac{4r_1}{R} \sum_n \prod_{s=1}^{\infty} \frac{\left( \Lambda (2s) - \lambda_{0n1} \right) J_1 \left( \frac{\beta_{0n} r_1}{R} \right)}{\prod_{s=0}^{\infty} \left( \Lambda (2s+1) - \lambda_{0n1} \right) \beta_{01} J_1 \left( \frac{\beta_{0n}}{R} \right)} J_0 \left( \frac{\beta_{0n} r}{R} \right) \sin \frac{\pi z}{H}
\]

\[
= A \left[ J_0 \left( \frac{\beta r}{R} \right) - \frac{J_0(\beta)}{Y_0(\beta)} \right] Y_0 \left( \frac{\beta r}{R} \right) \sin \frac{\pi z}{H} ,
\]

\[(95)\]

where

\[
\lambda_{0n1} = \left( \beta_{0n}/R \right)^2 + \left( \pi/H \right)^2 .
\]

Since A is arbitrary, we may set A = 4r_1/R and obtain from eq. (95) the identity

\[
\sum_n \prod_{s=1}^{\infty} \frac{\left( \Lambda (2s) - \lambda_{0n1} \right) J_1 \left( \frac{\beta_{0n} r_1}{R} \right)}{\prod_{s=0}^{\infty} \left( \Lambda (2s+1) - \lambda_{0n1} \right) \beta_{01} J_1 \left( \frac{\beta_{0n}}{R} \right)} J_0 \left( \frac{\beta_{0n} r}{R} \right)
\]

\[
= J_0 \left( \frac{\beta r}{R} \right) - \frac{J_0(\beta)}{Y_0(\beta)} \right] Y_0 \left( \frac{\beta r}{R} \right) \sin \frac{\pi z}{H} .
\]

\[(96)\]
IX. A METHOD FOR EVALUATING DEFINITE INTEGRALS

If we expand the right-hand side of eq. (96) in the Fourier-Bessel series:

\[
\sum_{n} A_n J_0 \left( \frac{\beta_{on} r}{R} \right) = J_0 \left( \frac{\beta r}{R} \right) - \frac{J_0(\beta)}{Y_0(\beta)} Y_0 \left( \frac{\beta r}{R} \right)
\]

we obtain by use of the formulas (15)

\[
\int_{r_1}^{R} r J_\nu^2 (kr) dr = \frac{r^2}{2} \left[ \left\{ J_\nu'(kr) \right\}^2 + \left( \frac{1 - \nu^2}{k^2 r^2} \right) \left\{ J_\nu (kr) \right\}^2 \right]_{r_1}^{R}
\]

and

\[
\int_{r_1}^{R} Y_0(k_1 r) J_0(kr \sin \phi) rdr = \left[ \frac{r}{\beta^2 - \beta^2_{on}} \left\{ k_1 J_0(kr \sin \phi) Y_1(k_1 r) - k \sin \phi Y_0(k_1 r) J_1(kr \sin \phi) \right\} \right]_{r_1}^{R}
\]

that

\[
A_n = \frac{R^2}{2} \left[ J_1^2 \left( \frac{\beta_{on}}{R} \right) + J_0 \left( \frac{\beta_{on}}{R} \right) \right] - \frac{r^2}{2} \left[ J_1^2 \left( \frac{\beta_{on} r}{R} \right) + J_0 \left( \frac{\beta_{on} r}{R} \right) \right]
\]

\[
- \frac{J_0(\beta)}{Y_0(\beta)} \left[ \frac{R^2}{\beta^2 - \beta^2_{on}} \left\{ \beta J_0(\beta_{on}) Y_1(\beta) - \beta_{on} Y_0(\beta) J_1(\beta_{on}) \right\} - \frac{r^2}{\beta^2 - \beta^2_{on}} \left\{ \beta J_0 \left( \frac{\beta_{on} r}{R} \right) Y_1 \left( \frac{\beta r}{R} \right) - J_0 \left( \frac{\beta_{on} r}{R} \right) J_1 \left( \frac{\beta_{on} r}{R} \right) \right\} \right]
\]

Equating the left-hand side of eq. (96) to the left-hand side of eq. (97), we find that the Fourier-Bessel components \( A_n \) satisfy the equation

\[
A_n = \frac{\prod_{s=1}^{\infty} \left( \Lambda^{(2s)} - \lambda_{on1} \right) J_1 \left( \frac{\beta_{on} r}{R} \right)}{\prod_{s=0}^{\infty} \left( \Lambda^{(2s+1)} - \lambda_{on1} \right) \beta_{on1} J_1^2 (\beta_{on})}.
\]
In this way, we have derived the value of the sum of two integrals:

\[
\int_{r_1}^{R} r J_0^2 \left( \frac{\beta_{on}}{R} \right) dr + \frac{J_0(\beta)}{Y_0(\beta)} \int_{r_1}^{R} r Y_0(\beta r) J_0(\beta_{on} r) \, dr
\]

\[
\prod_{S=1}^{\infty} \left( \frac{\Lambda^{(2S)} - \lambda_{on1}}{\lambda_{on1}} \right) J_1 \left( \frac{\beta_{on1} r}{R} \right) = \prod_{S=0}^{\infty} \left( \frac{\Lambda^{(2S+1)} - \lambda_{on1}}{\lambda_{on1}} \right) \beta_{on1} \cdot J_1^2(\beta_{on1})
\]

without carrying out the integrations themselves. This procedure may be helpful for evaluating definite integrals which do not admit of any analytic form. Note that the various orders of approximation, \( \Lambda^{(k)} \), depend on \( R \), since they were derived by integrating the eigenfunctions over \( \Gamma \) and \( D - D_1 \).
X. DETERMINATION OF BUCKLING AND FLUX FOR A RECTANGULAR REACTOR WITH PARTIALLY INSERTED CROSS-SHAPED CONTROL RODS

We now will find the first eigenfunction and first eigenvalue for a finite, rectangular, bare, homogeneous thermal reactor with partially inserted cross-shaped control rods. The physical theory here is the same as presented in Sect. VII. Figure 3 shows how the x-, y-, and z-axes were chosen for this geometry. Only two rods are shown in Fig. 3, but, as before, the theory to be developed holds for \(k\) rods. Figure 4 shows how the parameters for the rod positions were chosen, as seen in the plane \(z = 0\).

![Figure 3](image)

**Fig. 3**

Rectangular Parallelepiped Reactor with Partially Inserted Cross-shaped Rods.

![Figure 4](image)

**Fig. 4**

Control Rod Positions in the Plane \(z = 0\).

We wish to determine the sink function, \(S\), such that

\[
\frac{\partial^2 \phi}{\partial k^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} + \Lambda \phi = S \text{ in } \bar{D}_1
\]

\[= 0 \text{ in } \bar{D} - \bar{D}_1, \tag{102}\]

where \(\bar{D}\) denotes the extrapolated volume of the reactor and \(\bar{D}_1\) the effective volume of the cross-shaped control rods and such that the eigenfunction \(\phi\) is zero on \(\Sigma\), the reactor surface, and on the effective control rod surfaces \(\Sigma_1\).
We assume as a first approximation to the sink function $S_i$,

$$S_i^{(1)} = \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c} \text{ in } \overline{D_i}$$  \hspace{1cm} (103)

$$= 0 \text{ in } \overline{D-D_i} \hspace{1cm} (i = 1, 2, \cdots, k) \hspace{1cm} (104)$$

The unnormalized eigenfunctions $f_{mn\ell}$ are

$$f_{mn\ell} = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{\ell\pi z}{c} .$$

We form the eigenfunction expansion

$$S_i^{(1)} = \sum_{m,n,\ell}^\infty a_{mbnc\ell} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{\ell\pi z}{c} . \hspace{1cm} (105)$$

Multiply eq. (105) by

$$\sin \frac{m'\pi x}{a} \sin \frac{n'\pi y}{b} \sin \frac{\ell'\pi z}{c}$$

and integrate over $\overline{D}$. By use of the orthogonality properties,

$$\int_0^a \int_0^b \int_0^c \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{\ell\pi z}{c} dx dy dz = \frac{abc}{8} \begin{cases} m = m' \\ n = n' \\ \ell = \ell' \end{cases}$$

$$= 0 \begin{cases} m \neq m' \\ n \neq n' \\ \ell \neq \ell' \end{cases}$$

From eq. (103) and (104) we have

$$a_{mbnc\ell} = \frac{8}{abc} \left[ \left( \int_{x_1}^{x_i} \int_{y_1}^{y_i} \int_{z_1}^{z_i} \right) + \left( \int_{x_2}^{x_i} \int_{y_2}^{y_i} \int_{z_1}^{z_i} \right) + \left( \int_{x_3}^{x_i} \int_{y_1}^{y_i} \int_{z_2}^{z_i} \right) \right] \sin \frac{\pi x}{a} \sin \frac{m\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{n\pi y}{b} \sin \frac{\pi z}{c} \sin \frac{\ell\pi z}{c} dx dy dz .$$
By repeated use of the equation

\[
\int_{\alpha}^{\beta} \sin \frac{\pi \xi}{\gamma} \sin \frac{p\pi \xi}{\gamma} \, d\xi = \frac{\gamma}{2} A_p \bigg|_{\alpha}^{\beta}
\]

where

\[
A_p \bigg|_{\alpha}^{\beta} = \frac{\gamma}{p} \left[ \frac{\pi \xi}{2\gamma} - \frac{1}{4} \sin \frac{2\pi \xi}{\gamma} \right] \bigg|_{\alpha}^{\beta}, \quad p = 1
\]

we find that

\[
a^i_{mnk} c^i_{kl} = A_m \begin{pmatrix} x_i^1 & y_i^3 & z_i^1 \\ x_i^2 & y_i^4 & z_i^2 \\ x_i^3 & y_i^3 & z_i^3 \end{pmatrix} + A_n \begin{pmatrix} x_i^1 & y_i^4 & z_i^1 \\ x_i^2 & y_i^2 & z_i^2 \\ x_i^3 & y_i^3 & z_i^3 \end{pmatrix}
\]

The first approximation, \( \Lambda^{(1)} \), of the first eigenvalue \( \Lambda \) is found by use of eq. (33b) and (33c). We find, therefore, that \( \Lambda^{(1)} \) satisfies the equation

\[
\sum_{i=1}^{k} \sum_{m,n,l}^{\infty} \frac{a^i_{mnk} c^i_{kl}}{\Lambda^{(1)} - \lambda_{mnl}} \sum_{j=1}^{k} \sum_{p=1}^{12} P_{mnk}^{jp} = 0, \quad (106)
\]

where

\[
\lambda_{mnl} = \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{l^2}{c^2} \right)
\]

\[
P_{mnk}^{jp} = \int_{\sigma_j} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{l\pi z}{c} \, dx \, dy \, dz
\]

and \( \sigma_j \) is the \( p \)th part of the total surface \( \sigma_j \) of the \( j \)th cross-shaped control rod. Knowing \( \Lambda^{(1)} \), we have, from eq. (33c), the first approximation of the first eigenfunction:

\[
\phi^{(1)} = \sum_{i=1}^{k} \sum_{m,n,l}^{\infty} \frac{a^i_{mnk} c^i_{kl}}{\Lambda^{(1)} - \lambda_{mnl}} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{l\pi z}{c} \quad (107)
\]
The second approximation, \( \Lambda^{(2)} \), of \( \Lambda \) is obtained by use of Eq. (33d); \( \Lambda^{(2)} \) satisfies the equation

\[
\sum_{i=1}^{k} \sum_{m,n,l}^{\infty} \frac{a_m^i b_n^i c_n^i (\Lambda^{(2)} - \lambda_{mnl})}{\Lambda^{(i)} - \lambda_{mnl}} \left\{ Q_{mnl} - \sum_{j=1}^{k} \sum_{p=1}^{4} R_{jmn}^p \right\}, \quad (108)
\]

where

\[
Q_{mnl} = \int_0^a \int_0^b \int_0^c \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \sin \frac{l \pi z}{c} \, dx \, dy \, dz,
\]
\[
R_{jmn}^p = \int_{D_j^p} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \sin \frac{l \pi z}{c} \, dx \, dy \, dz,
\]

and \( D_j^p \) is the \( p^{th} \) of the total volume \( D_j \) of the \( j^{th} \) cross-shaped control rod. Processing in this manner, we may successively derive Eq. (33f) through (33k) for this geometry.
XI. NUMERICAL RESULTS FOR THE CYLINDRICAL REACTOR WITH TWO PARTIALLY INSERTED CONTROL RODS

Calculations were made with the formulas derived in Sect. VII and IX for the cases of one and two control rods inserted into the reactor in a variety of positions. The reactor and control rod parameters used were $R = 1$, $H = 2$, $r_1 = 0.02$, and $r_2$ equal 0.02, all in centimeters, where $R$ is the extrapolated reactor radius, $H$ is the extrapolated reactor height, and $r_1$ and $r_2$ are the effective radii of the control rods. The distances between the centers of the control rods and the reactor center in centimeters were designated by the parameters $d_1$ and $d_2$. The lengths of insertion in centimeters were designated by the parameters $h_1$ and $h_2$. The buckling or first eigenvalue calculated for these various configurations was in inverse square centimeters.

A complete description of how the equations for the cylindrical reactor with partially inserted control rods as derived in Sect. VII were coded for the IBM 704 is included in the appendix. This description includes convergence behavior of the equations in Sect. VII, the program listing the preparation of data, and operating procedures for the code.

The fluxes calculated by this code for a reactor with one and two partially inserted control rods in a variety of positions are plotted in Figs. 5 to 11. The calculated fluxes were multiplied by the arbitrary scale factor $10^4$. Before the flux calculation can be made, three different series must be evaluated a number of times. All of these series are, of course, partially summed and affect, in turn, the accuracy of the partially summed series for the calculation of the flux itself. The partial sums for the various approximations of the flux are designated as $\Phi_{m,n,x}^{(k)}(r,z,\theta)$ and the location of where the flux was calculated may be determined by use of Fig. 1. Since the various orders of approximations of the flux and buckling require the repeated use of the series in eq. (85) through (91), all the series used in the calculations were summed over an equal number of terms, so the partial sums will be designated by the common symbol $T_{m,n,x,l}$. The point $(r,z,\theta)$ is given in terms of the reactor dimensions and is designated as $(r_0R, z_1H, \theta)$. The distances between the centers of the control rods and the center of the reactor are given in terms of the reactor radius $R$ and are designated as $\delta_iR$, $i = 1, 2$.

Which order of approximation of the flux is calculated depends on the error $\epsilon_k$ stipulated. The resulting plots shown in Figs. 5 to 11 were made with $\epsilon_k = 0.01$.

In Fig. 6, calculations for the flux made by the author by a different but related approximate theory are also shown and are designated by $\Phi_{m,n}(r,z)$. For an explanation of the meaning of $\Phi_{m,n}(r,z)$, see reference 16.
Fig. 5 Radial Flux Plots for $Z = Z_1H$ for One Central Control Rod Inserted Effectively a Length 0.625

Fig. 6 Radial Flux Plots for $Z = Z_1H$ for One Central Control Rod Inserted Effectively a Length 1.00

Fig. 7 Radial Flux Plots for $Z = Z_1H$ for One Central Control Rod Inserted Effectively a Length 1.50

Fig. 8 Radial Flux Plots for $Z = Z_1H$ for One Central Control Rod Fully Inserted.
Effective Lengths of 0.960.

$$\phi = 0.4, \quad a = 0.01$$

Rod inserted at $d_1 = 0.4, \quad a_1 = 0$ and $d_2 = 0.4$.

FIG. 11 Angular Flux Plot for Two Control Rods with Axes at $\theta$.

FIG. 10 Radial Flux Plot for Two Control Rods with Axes at $\theta$.

FIG. 9 Radial Flux for One Control Rod Inserted Effectively at 0.960. $Z = Z^1 = Z^2 = 0.4, \quad a = 0.4, \quad d_1 = 0, \quad a_1 = 0.01$.
Table I shows how the various orders of approximation, $A^{(k)}$, for a half-way inserted control rod varied with the choice of $m$, $n$, and $\ell$ in the series used for these calculations.

Table I

Orders of Approximation of the Buckling, $A^{(k)}$, for the Central Control Rod Halfway Inserted vs Partial Sums Taken

| Values of the Indices for the Partial Sum $T_{m \times n \times \ell}$ | Orders of Approximation of $A^{(k)}$ |
|---|---|---|---|---|
| $m$ | $n$ | $\ell$ | $A^{(1)}$ | $A^{(2)}$ | $A^{(3)}$ | $A^{(4)}$ |
| 1 | 50 | 5 | 9 8349791 | 9 6532998 | 9 6543206 | 9 6543207* |

† The mathematically effective length was found to be 1.00 for the last partial sum taken. The author's previous value for the buckling of a half-way inserted control rod was 9.508 (see Ref 16). Here the boundary condition at the tip was neglected. The mathematically effective length was 0.975, which was slightly smaller than in the above calculation.

*The actual length was 1.15

In Fig. 8, calculations made by use of eq (92), which is the exact solution for this case for $A = \phi_{m \times n \times \ell}(0,3,1,2\pi)/\Theta(0,3,1)$ is also shown and is designated by $\Theta(r,z)$, which was plotted using the value $\beta = 2.8842250$ found by use of eq. (92).*

Tables II-IV show how the various orders of approximation $A^{(k)}$ for control rods in a variety of positions varied with the choice of $m$, $n$, and $\ell$.

Table II

Orders of Approximation of the Buckling, $A^{(k)}$, for the Central Control Rod Fully Inserted vs Partial Sums Taken

| Values of the Indices for the Partial Sum $T_{m \times n \times \ell}$ | Orders of Approximation of $A^{(k)}$ |
|---|---|---|---|---|
| $m$ | $n$ | $\ell$ | $A^{(1)}$ | $A^{(2)}$ | $A^{(3)}$ | $A^{(4)}$ |
| 1 | 50 | 1 | 10 786831 | 10 555445 | 10 555555 | 10 555555 |
| 1 | 150 | 1 | 10 786236 | 10 694038 | 10 694055 | 10 694055 |
| 1 | 300 | 1 | 10 786165 | 10 736356 | 10 736366 | 10 736366* |

*Buckling calculated using eq (93) and (94) is 10 874274. Buckling calculated with the IBM 704 using eq (92) and (93) is 10 786154 for a radius of $r_1 = 0.02$.

*Equation (92) was evaluated at $r = 0.02$, set equal to zero, and the value found by use of the IBM 704 to satisfy this equation was 2.8842250.
### Table III

Orders of Approximation of the Buckling, $\Lambda^{(k)}$, for One Halfway Inserted Control Rod at $d_1 = .4$ vs. Partial Sums Taken†

| Values of the Indices for the Partial Sum $T_{m \times n \times \ell}$ | Orders of Approximation of $\Lambda^{(k)}$ |
|---|---|---|---|---|
| $m$ | $n$ | $\ell$ | $\Lambda^{(1)}$ | $\Lambda^{(2)}$ | $\Lambda^{(3)}$ | $\Lambda^{(4)}$ |
| 15 | 40 | 2 | 9.3148143 | 8.7440069 | 8.7592590 | 8.7592585* |
| 20 | 50 | 4 | 9.1456788 | 8.7813159 | 8.7814810 | 8.7814809** |

† The mathematically effective length was found to be 0.960.

*The actual length was 1.25.

**The actual length was 1.13.

### Table IV

Orders of Approximation of the Buckling, $\Lambda^{(k)}$, for Two Halfway Inserted Control Rods at $d_1 = .4$, $d_2 = .4$, $\alpha_1 = 0$ and $\alpha_2 = 90^\circ$ vs. Partial Sums Taken†

| Values of the Indices for the Partial Sum $T_{m \times n \times \ell}$ | Orders of Approximation of $\Lambda^{(k)}$ |
|---|---|---|---|---|
| $m$ | $n$ | $\ell$ | $\Lambda^{(1)}$ | $\Lambda^{(2)}$ | $\Lambda^{(3)}$ | $\Lambda^{(4)}$ |
| 15 | 40 | 2 | 10.330864 | 9.6180513 | 9.6296294 | 9.6296357* |

† The mathematically effective lengths were found to be 0.960.

Table III and Table IV illustrate the shadow effect or interaction effect of two control rods.

*The actual lengths were 1.25.
Figure 12 shows how the fractional change in buckling varied as a function of the mathematically effective (see discussion) control rod position for a central control rod. In addition, similar curves calculated at KAPL by use of CURE code,\(^{(17)}\) by a method developed by Schindler,\(^{(21)}\) and by a method developed by Murray\(^{(22)}\) are also shown. These plots show the same S-shape, a property which is characteristic of such calculations. It should be noted that the KAPL calculation was made using slightly different reactor and control rod parameters. A calculation using exactly the KAPL parameter was also made and is shown in Fig. 12.

Figure 13 shows how the fractional change in buckling varies as a function of control rod position for a halfway inserted control rod moved away from the center a distance \(d_1\). The case of a fully inserted rod moved off center calculated by another method\(^{(20)}\) is also shown. It should be noted that different parameters were used, yet both curves display the same characteristic shape.

Figure 14 shows the control rod worth for the central rod and off-center rod for a typical migration area of 300 cm\(^2\).

Discussion of Numerical Results

Since the method developed is an analytic one, the cutting off of the various series leads to certain inaccuracies. In order, for example, to represent the initial sink function, \(S^{(4)}\), a great many terms would have to be taken since this function is a "step" function in three-space. Consequently, at the surfaces of discontinuity this function would have its poorest representation. In effect, this discrepancy in the partial-sum representation shrinks the subdomains which, in the case of control rods, shortens the length.

However, because of the iterative procedure developed for a fair amount of terms, the series solution will converge and be forced to satisfy the boundary conditions for somewhat smaller subdomains than initially stipulated in the given calculation. Thus, one is guaranteed in any case that the solution satisfy the partial differential equation and the boundary condition for a somewhat smaller domain, which will be designated as the mathematically effective subdomain.

In the case of control rods, one would have a mathematically effective insertion length corresponding to the length one wishes to consider. There is also some variation in diameter, of course, but this is very slight. One, then, in reactor calculations would perform a series of calculations with lengths in the vicinity of the mathematically effective lengths to be considered. This procedure would have to be adopted since to analyze mathematically this shrinkage or error introduced by partially summing is a formidable task.
Frac. 14  

Two Rods Length 0.960  

Rod from Center inserted a  

Fig. 13 Fractional Change in Buckling  

Frac. 12 Fractional Change in Buckling  

When Rods inserted  

Fig. 11 Rod from Center  

Distance from Center  

Fig. 9 Rod Worth vs. Effective Length of Control Rod.
Certain methods exist for speeding up convergence of series; these may or may not work in the series developed by this method. These possibilities will be considered in the code developed.

The various tables and graphs illustrate the convergence of these solutions and, in addition, illustrate how, by using fewer terms and longer control rods, one gets effectively the same results as using many terms for slightly shorter control rods.

Fig. 12 indicates that after the control rod is inserted considerably past the halfway point, the theory is not quite as good. This behavior would become more marked for off-center control rods. However, for many more terms this situation would be alleviated. But since the theory is primarily for partially inserted control rods and not particularly for almost fully inserted control rods, this sort of calculation with many terms in the series would not have to be made, since there is a great deal of work in the literature for off-center fully inserted control rods.
XII. THEOREMS II AND III. EXTENSION OF THE THEORY TO THE WAVE EQUATION WITH VARIABLE COEFFICIENTS

We now extend the theory to solve the wave equation with variable coefficients which have one or more independent variables. We treat in this section the three-dimensional case again, but, as before, it is understood to apply to considerations involving n-dimensional problems as well.

First, we consider the somewhat special case in which the coefficients are step functions over the domains of interest. This special case of step-function variable coefficients is of interest in solving reactor physics problems involving multi-region calculations.

In this section we consider three-dimensional rectangular geometries only, but the treatment applies equally well to other geometries. We restrict our arguments to self-adjoint partial differential equations of the form

$$a(x,y,z) \frac{\partial^2 \phi}{\partial x^2} + b(x,y,z) \frac{\partial^2 \phi}{\partial y^2} + c(x,y,z) \frac{\partial^2 \phi}{\partial z^2} + d(x,y,z) \Lambda \phi = 0$$

where the coefficients $a$, $b$, $c$, and $d$ are to have certain restrictive properties as stated in the two theorems to follow. We consider the homogeneous Dirichlet problem, but, as before, we may solve other types as well.

Theorem II: Let the coefficients $a$, $b$, $c$, and $d$ be constants in the open domain $D - D_1$. Then there exists a function $S(sink)$ which has the following properties:

(i) $a \frac{\partial^2 \phi}{\partial x^2} + b \frac{\partial^2 \phi}{\partial y^2} + c \frac{\partial^2 \phi}{\partial z^2} + d \Lambda \phi = S$ in $D_1$ (110)

(ii) $= 0$ in $D - D_1$ (111)

and is such that it makes $\phi$ satisfy the boundary condition IV (iii).

Here we are considering the simple case of a partial differential equation with constant coefficients. The coefficients are constant over the entire domain $D - D_1$; thus we do not expand these coefficients into their respective Fourier expansions.

The $f_n$ in the following equation satisfy eq. (4) and the boundary condition in eq. (5). Further, we form the eigenfunction expansion:

$$\phi = \sum_{n} e_n f_n$$ (111a)
and the eigenfunction expansion of the first approximation, \( S^{(1)} \), of the sink \( S \):

\[
S^{(1)} = \sum_{n} \alpha_{n} f_{n},
\]

(111b)

where, in all the above expansions, the Fourier coefficients are found by similar integrations as in eq. (19a). We now let the index \( n \) become the indices \( m, n, \ell \). Substituting these expansions into eq. (110) and interchanging summation and differentiation, we have by use of eq. (16) that

\[
e_{mn\ell} = \alpha_{mn\ell} \left( d_{mn\ell} - a_{mn\ell} m^2 - b_{mn\ell} n^2 - c_{mn\ell} \ell^2 \right)
\]

(112)

From eq. (112) one readily finds the first approximation to be

\[
\phi^{(1)} = \sum_{m, n, \ell} \frac{\alpha_{mn\ell} f_{mn\ell}}{d_{mn\ell} - a_{mn\ell} m^2 - b_{mn\ell} n^2 - c_{mn\ell} \ell^2}
\]

(113)

The first approximation \( \Lambda^{(1)} \) may also be found by use of eq. (23) again. As in eq. (29), \( \Lambda^{(2)} \) may be found and so the same procedure is carried out as before by use of eq. (31) through (33k).

Since the denominators of the coefficients of the expansion in eq. (113) would make the expansion converge even more strongly than those derived for the proof of Theorem I, the proof of this theorem follows in exactly the same way.

A more general case involving variable coefficients is considered in the next theorem.

**Theorem III.** Let the coefficients \( a, b, c, \) and \( d \) be of the form \( a(x, y, z) \), \( b(x, y, z) \), \( c(x, y, z) \), and \( d(x, y, z) \). Further let these coefficients admit of the Fourier expansion

\[
a(x, y, z) = \sum_{m, n, \ell} a_{mn\ell} f_{mn\ell}
\]

\[
b(x, y, z) = \sum_{m, n, \ell} b_{mn\ell} f_{mn\ell}
\]

\[
c(x, y, z) = \sum_{m, n, \ell} c_{mn\ell} f_{mn\ell}
\]

\[
d(x, y, z) = \sum_{m, n, \ell} d_{mn\ell} f_{mn\ell}
\]
in the domain $D-D_1$, where

$$f_{mnk} = f_m^{(1)}(x) f_n^{(2)}(y) f_\ell^{(3)}(z).$$  \hfill (114)

(The eigenfunction in eq. (114a) is, of course, the actual form of these eigenfunctions used throughout this section, but is displayed here to emphasize what is to follow.) Then there exists a function $S(sink)$ which has the following properties:

(i) $a(x,y,z) \frac{\partial^2 \phi}{\partial x^2} + b(x,y,z) \frac{\partial^2 \phi}{\partial y^2} + c(x,y,z) \frac{\partial^2 \phi}{\partial z^2} + d(x,y,z) \Lambda \phi = S$ in $\overline{D_1}$

$$= 0 \text{ in } \overline{D-D_1}$$  \hfill (114a)

$$= 0 \text{ in } \overline{D-D_1}$$  \hfill (115)

and is such that it makes $\phi$ satisfy the boundary condition IV (iii). We form the expansions in eq. (111a) and (111b) and substitute them and the expressions for the coefficients into eq. (114a). We now use for $\phi$ and $S^{(i)}$ the eigenfunctions

$$f_m^{(1)}(x) f_n^{(2)}(y) f_\ell^{(3)}(z)$$

and integrate over $\overline{D-D_1}$. We now directly integrate eq. (114a) after substituting into it the above expansions and after interchanging the various operations; we have by the orthogonality properties of

$$f_m^{(1)}(x) f_n^{(2)}(y) f_\ell^{(3)}(z)$$

that

$$e_{mn\ell} = \frac{a_{mn\ell} \int_{D_1} f_{mn\ell}}{d_{mn\ell} a_{mn\ell} m^2 - b_{mn\ell} n^2 - c_{mn\ell} \ell^2}$$  \hfill (116)

as before. Eq. (116) illustrates that in this case as well the nature of the coefficients in the expansion of $\phi^i$ and, consequently, succeeding derivations made will be essentially the same as those involved in Theorems I and II. Thus the proof of Theorem III is made exactly in the same manner as were the proofs of Theorems I and II.
XIII. AN APPLICATION OF THE METHOD TO A TIME-DEPENDENT PROBLEM AND THEOREM IV

We seek now the solution of the time-dependent wave equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}$$  \hspace{1cm} (117)

subject to the following boundary and initial conditions.

$$\phi(x,y,t)\big|_{\Gamma} = 0, \quad t>0$$  \hspace{1cm} (118)

$$\phi(x,y,0) = F(x,y),$$  \hspace{1cm} (119)

where $c$ is a constant and $\Gamma$ is the boundary surface of an arbitrary domain $D-D_1$. Assume the solution to be of the form

$$\phi(x,y,t) = \bar{\phi}(x,y) e^{-i\omega t}$$  \hspace{1cm} (120)

Substituting eq. (120) into eq. (117), it is seen that $\bar{\phi}$ must satisfy the equation

$$\frac{\partial^2 \bar{\phi}}{\partial x^2} + \frac{\partial^2 \bar{\phi}}{\partial y^2} + \Lambda \bar{\phi} = 0$$  \hspace{1cm} (121)

where

$$\Lambda = \omega^2/c^2$$

Equation (118) now becomes equivalently the condition

$$\bar{\phi}(x,y)\big|_{\Gamma} = 0$$  \hspace{1cm} (122)

Equation (122), then, is the same as boundary condition IV (iii), since $\Gamma$ is the arbitrarily shaped surface for the arbitrarily shaped domain $D-D_1$. We now find the sequence of eigenfunctions, $\{\phi_n\}$, and corresponding sequence of eigenvalues, $\{\Lambda_n\}$, by the methods already developed. The eigenfunctions $\bar{\phi}_n$ then have the representations

$$\bar{\phi}_n = \sum_m a_n^m f_m$$  \hspace{1cm} (123)

where the $f_m$ satisfy eq. (4) and (5) and the $a_n^m$ are the constants constructed by use of the theory developed so that each $\bar{\phi}_n$ satisfies the homogeneous boundary conditions on all other surfaces besides $\Gamma$. We now expand $\phi$: 
\[ \phi = \sum_{n} \phi_n e^{-i\Omega_n t} = \sum_{m,n} A_n a_m^* f_m e^{-i\Omega_n t}, \]

where

\[ \Omega_n = c\Lambda_n \frac{1}{2} \]

Before finishing the solution to this problem by making \( \phi \) satisfy eq. (119), we state and prove the following:

**Theorem IV.** The eigenfunctions of eq. (121) are orthogonal over \( \overline{D-D_1} \).

Denote the operator in eq. (121) by \( L \). Now let \( \Lambda_n \) and \( \Lambda_n' \) be the two eigenvalues corresponding to the two eigenfunctions \( \phi_n \) and \( \phi_n' \), respectively. From eq. (121) we have, therefore, that

\[ L\phi_n = -\Lambda_n \phi_n \]

and

\[ L\phi_n' = -\Lambda_n' \phi_n' \]

We now multiply eq. (126) by \( \phi_n' \), eq. (125) by \( \phi_n' \), and subtract eq. (126) from (125) to obtain

\[ \phi_n', L\phi_n - \phi_n L\phi_n' = (\Lambda_n' - \Lambda_n) \phi_n' \phi_n' \]

Since the operator \( L \) is self-adjoint, we have from Lagrange's formula:

\[ \phi_n', L\phi_n - \phi_n L\phi_n' = \nabla \cdot \left[ \phi_n' \nabla \phi_n' - \phi_n' \nabla \phi_n \right] \]

that

\[ (\Lambda_n' - \Lambda_n) \phi_n' \phi_n = \nabla \cdot \left[ \phi_n' \nabla \phi_n' - \phi_n' \nabla \phi_n \right] \]

We now integrate both sides of eq. (128) over \( \overline{D-D_1} \) and obtain after an application of Green's theorem that

\[ (\Lambda_n' - \Lambda_n) \int_{\overline{D-D_1}} \phi_n' \phi_n = \int_{\Gamma} (\phi_n' \phi_n' - \phi_n' \nabla \phi_n) \cdot \nabla \phi_n \]

But we have constructed the \( \phi_n' \) s so that

\[ \phi_n' \bigg|_{\Gamma} = 0 \]
and, therefore, we have that

$$(A_n - A_{n'}) \int_{D-D_1} \bar{\phi}_{n'} \bar{\phi}_n = 0$$

(130)

Since $n' \neq n$, we have

$$\int_{D-D_1} \bar{\phi}_{n'} \bar{\phi}_n = 0$$

(131)

Now, by the orthogonality property in eq. (131), we find from eq. (119) and (124) that, after interchanging summation and integration,

$$A_n = \sum_{m} a_{m}^{n} \int_{D-D_1} F(x,y) f_{m}$$

which thus completes the solution of this problem.
XIV. AN APPLICATION OF THE METHOD OF THE SOLUTION OF POISSON'S EQUATION

In a manner similar to the one in Sect. XIII we may employ the method developed to solve Poisson's equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \rho(x,y),$$  \hspace{1cm} (132)

in an arbitrarily shaped domain $\overline{D-D_1}$, with $\phi$ subject to the homogeneous Dirichlet condition:

$$\phi \bigg|_{\Gamma} = 0,$$

where $\Gamma$ is the surface of the domain $\overline{D-D_1}$.

We assume that $\rho(x,y)$ is a distributed source in the open domain $D-D_1$ and that it can be expanded into the series of eigenfunctions:

$$\rho(x,y) = \sum_{n=0}^{\infty} B_n \phi_n,$$  \hspace{1cm} (133)

where

$$\phi_n \bigg|_{\Gamma} = 0,$$

by an application of the method developed. Further, we make the expansion

$$\phi = \sum_{n} A_n \phi_n \equiv \sum_{m,n} A_{mn} \phi_{mn} \equiv \sum_{m,n,m',n'} A_{mn} a_{mn} f_{m'n'},$$  \hspace{1cm} (134)

for the solution. Substituting eq. (133) and (134) into eq. (132), we have by use of Theorem IV that

$$A_{mn} = -B_{mn}/(m^2+n^2).$$

Since

$$\frac{\partial^2 \phi_n}{\partial x^2} = -\sum_{m,n,m',n'} m^2 A_{mn} a_{mn} f_{m'n'},$$
then

\[
\frac{\partial^2 \phi_n}{\partial y^2} = - \sum_{m,n,m',n'}^\infty n^2 A_{mn} a_{mn}^{m'n'} f_{m'n'}
\]

Therefore, the solution to Poisson's equation, subject to homogeneous Dirichlet boundary conditions, is

\[
\phi = - \sum_{m,n,m',n'}^\infty \frac{B_{mn} a_{mn}^{m'n'}}{m^2 + n^2} f_{m'n'}
\]

(135)

XV. REMARKS ON VERY IRREGULAR DOMAINS

If the domains are extremely irregular for the problems considered above, it is still possible to apply the method developed. In these cases, one may approximate these irregular domains by many small regular subdomains which allow one to perform the various integrations necessary in carrying out the solution.
REFERENCES


ACKNOWLEDGEMENT

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In addition, the author wishes to express his thanks to Mr. James K. Butler for his helpful remarks and constructive criticisms.

Finally, I would like to thank the IBM Customer Engineers, Messrs. Thomas Hogan, Robert Murphy, Leroy McMahon, and Kenneth Madsen for their joint cooperation with the IBM Corporation in aiding me to carry out the various computations on the IBM 704.
I. Description of Code

The reactor control rod formulas derived in Section VII were coded by use of the Argonne National Laboratory version of FORTRAN II, an automatic coding system developed for use on the IBM 704. Since the nature of this calculation consisted primarily of summing series and evaluating formulas, it was felt that the FORTRAN coding technique would be rapid and effective for these calculations.

The complete code consists of ten decks. However, only three decks comprise the reactor calculation portion of the code, and the other six decks are auxiliary programs which would be used very infrequently at each installation; they do serve their purpose in preparing or correcting magnetic tapes, making preliminary calculations, interrupting the main reactor program, and restarting the main program. The main program is thus divided into three phases which will now be described along with the auxiliary programs.

In addition to this description and the FORTRAN listing of the code, a table defining the various FORTRAN counterparts of the mathematical symbols used is included just prior to the listing. The program listing included in the back of this Appendix is for calculations involving up to ten control rods inserted along the axis of the reactor at arbitrary angles, distances, and depths.

Phase 1. In phase 1, the Fourier–Bessel coefficients expressed by eq. (83), the expression in braces in eq. (88), i.e., the surface integral coefficients, and the expressions in brackets in eq. (90), i.e., the volume integral coefficients, are calculated and written on binary magnetic tape. The roots of the Bessel functions $\beta_{mn}$ are calculated by use of the formula*

$$\beta_{mn} = \beta - \frac{q-1}{8\beta} (1 + \frac{Q_1}{3(4\beta)^2} + \frac{2Q_2}{15(4\beta)^4} + \frac{Q_3}{105(4\beta)^6} + \cdots)$$

where

- $q = 4 \ m^2$
- $\beta = (m-0.5+2n) \pi/2$
- $Q_1 = 7q-31$
- $Q_2 = 83q^2-982q+3779$
- $Q_3 = 6949q^3-153855q^2+1585743q-6277237$

and are used as part of the arguments of the Bessel function subroutines throughout phase 1. The eigenvalues $\lambda_{mn}$ are also written on binary magnetic tape.

In addition, the following operations are performed:

1. The dimension statement sets aside storage for the subscripted variables appearing in the calculation of $a_{mn}$, floating point integers $q_{n}$, and for the subscripted variables necessary to calculate the Bessel functions using the N.Y.U. Bessel function subroutine. These latter subscripted variables appear as the various TABLE variables.

2. FORTRAN statements are used to define additional Bessel function subroutines for large argument and index of approximately equal magnitude.

3. The data describing the rod positions, initial flux points, increments in flux plots, number of rods, etc., are read in (see operating procedure for phase 1). All input concerning rod positions, number of terms, number of rods, date, and heading of table for Fourier-Bessel coefficients, surface integral coefficients, eigenvalues, volume integral coefficients, current values of indices and the current rod number are printed on line.

4. The coefficients $c_{mn}$ are set equal to one and written on binary tape. Since the first root of the first Bessel function is in error by a considerable amount, this root is set equal to the correct value in the program.

5. When phase 1 is finished, all input data necessary to start phase 2 is punched out.

Phase 2. The data from phase 1 is read in. The various coefficients calculated in phase 1 are used to sum the series in eq. (88) and (90) for the various orders of approximations of the first eigenvalue and eigenfunction. Once again pertinent input data regarding rod positions, terms, etc., is printed on line.

The following operations are performed:

1. For a some initial guess on $\Lambda$, namely, $\Lambda_0^{(1)}$, the series in eq. (88) is summed. Since the theory shows that this series is either positive or negative (and one can always start positively by making $\Lambda_0^{(1)}$ close to $\Lambda_{01}$), the program tests for a negative value. If this series is negative, the increment
$\Delta \Lambda$ is subtracted from $\Lambda^{(1)}$. This increment is then divided by an arbitrary number $S$ and the series in eq. (88) is resummed using $(\Delta \Lambda/S) + \Lambda^{(1)} = \Lambda^{(2)}$ until the sign of the series changes again. Thus one may increment to the accuracy $\Delta \Lambda/S J_4^{+1}$, where $J_4$ is the number of sign changes requested.

(2) Once $\Lambda^{(1)}$ is known, then $\Lambda^{(2)}$ is found from eq. (90) as the ratio of two series, as is shown more clearly in eq. (64).

(3) $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are compared to see if $|\Lambda^{(1)} - \Lambda^{(2)}| \leq \epsilon$, where $\epsilon$ is previously specified as input data.

(4) If $|\Lambda^{(1)} - \Lambda^{(2)}| > \epsilon$, then the iterative procedure is repeated.

(5) When phase 2 is completed, the necessary data to start phase 3 is punched out.

Phase 3. Phase 3 consists of primarily making flux plots in various regions of the reactor. Once phases 1 and 2 are completed, the necessary binary tapes for calculations in phase 3 for the case of interest are prepared and may be used repeatedly to make flux calculations. Thus it is possible to make flux traverses in any desired region of the reactor by simply changing some of the input data cards (such as initial values for starting flux plots or the increments used in these flux plots).

The following operations are performed:

(1) Pertinent data from phase 2 is read in.

(2) Pertinent data regarding rod position parameters, etc., is printed on line.

(3) The series of the form expressed in eq. (91) on a higher-order approximation is summed for any choice of parameters stipulated by the user.

(4) All final information is written on B.C.D. tape and is printed off-line. Intermediate print outs may also be had by proper sense switch setting.

(5) Phase 3 may be interrupted, in which case current data is then punched out to either be used for continuing the calculation or it may be altered to start a new flux plot.
Auxiliary Program No. 1. This program is used to calculate the buckling for fully inserted central rod by the Nordheim-Scalletar theory, i.e., by use of eq. (93) and (94) and by use of the exact equation as expressed in eq. (92). The flux is also calculated using eq. (92) and normalized according to a given constant.

Auxiliary Program No. 2. This program is used to initialize magnetic tape number four if for some reason phase 2 is to be restarted from its beginning any time after iteration number one for the buckling has been completed. This sort of situation may occur in the event that because iteration number one for the buckling was not calculated accurately enough, leading consequently to a larger inaccuracy in the determination of iteration number two, one may wish to restart phase 2 for a more accurate determination of the buckling.

Auxiliary Program No. 3. In the event that calculations involving more than one control rod are to be made, one uses this program after using phase 1, if phase 1 was interrupted after the various coefficients were calculated for k control rods, where 1 < k < 10. For a large number of control rods one would use the interrupt features available in phase 1 and then use this auxiliary program to reset the various tapes used in phase 1 in order to continue the phase 1 calculation for the coefficients of the remaining control rods. In addition to resetting the proper tapes, this program also prints on line the last records which contain the pertinent coefficients calculated by phase 1, so that one has a way of checking to be sure the tapes have been reset properly.

Auxiliary Program No. 4. (In three parts) Since the formula for calculating the roots $\hat{\beta}_{mn}$ of the Bessel Functions $J_m(\hat{\beta}_{mn})$ was inaccurate for indices of the order three and up, it was necessary to devise this program to find out precisely which of these roots were in error. These discrepancies were determined by calculating the roots from the formula and then substituting the appropriate root or zero into the appropriate Bessel function. Once this was done, one could readily see which roots went astray.

Once this was known, these roots were corrected by use of the Harvard tables for Bessel functions and these corrections were punched on cards. Using these cards and the rest of auxiliary program No. 4, one makes up a master binary tape having $m \times n = 50 \times 100$ roots. As checks on these roots, the root and the value of the corresponding Bessel function, i.e., the Bessel function which has this root as a zero, are written on B.C.D. tape for off-line printing.

Auxiliary Program No. 5. Once the master tape containing 5000 correct roots $\hat{\beta}_{mn}$ has been made, then this program allows one to make a B.C.D. and binary tape of $m \times n$ roots where $m \leq 50$ and $n \leq 100$. The binary tape is then used as tape number seven for calculations in phases 1 and 2.
II. Operating Procedures

Phase 1.

(1) Preparation of Data. The data \( R, H, z_{in}, \theta_{in}, \Delta A, \Delta r, \Delta z, \Delta \theta, \epsilon, S, \Lambda_{bo}, \Lambda_{in}, \Delta 'r, \Delta 'z, \Delta ' \theta, (r_{i}, i = 1,10), \)
\((d_{i}, i = 1,10), (a_{i}, i = 1,10), (M/R_{1})^2 \) and TEST are punched on the first twelve cards according to FORMAT (1H 1P5E14.7).

The data \( J_1, J_2, J_3, J_4, n_r, i_1, i_2 \) and \( J_k \) are punched on the thirteenth card according to FORMAT (8I4).

The total number of terms to be taken in summing the various series, namely, \( J_{15} = n_r \times m \times n \times \ell \), is to be punched on the fourteenth card according to FORMAT (1I6).

On the fifteenth card the information for the date, namely, \( J_{12}, J_{13}, \) and \( J_{14} \), will be punched according to FORMAT (3I3).

The sixteenth card allows up to seventy-one Hollerith characters to be read in so as to be used for problem identification. Such information as the originator and the originator's affiliation may be punched in according to FORMAT (72HO [71]).

(2) Magnetic Tapes. The magnetic tapes needed in phase 1 are tapes set to logical 2, 4, 8 and 7. Tape number 7 is prepared by use of auxiliary programs 4 and 5.


(4) Sense Switch Settings

<table>
<thead>
<tr>
<th>Sense Switch</th>
<th>Setting</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sense Switch #1</td>
<td>Up</td>
<td>No print out of table entries - Fourier-Bessel coefficients, rod number, etc.</td>
</tr>
<tr>
<td>Sense Switch #1</td>
<td>Down</td>
<td>Print out table entries and monitor progress of calculation</td>
</tr>
<tr>
<td>Sense Switch #2</td>
<td>Not used</td>
<td></td>
</tr>
<tr>
<td>Sense Switch #3</td>
<td>Not used</td>
<td></td>
</tr>
<tr>
<td>Sense Switch #4</td>
<td>Up</td>
<td>No print out of ( \Gamma_0 ) and ( \Gamma_k )</td>
</tr>
<tr>
<td>Sense Switch #4</td>
<td>Down</td>
<td>Print out ( \Gamma_0 ) and ( \Gamma_k )</td>
</tr>
</tbody>
</table>
Sense Switch #5  Up - No print out of $B_0^d$, $RT(1,1)$, $RT(1,2)$, 
            $\cdots$, $RT(1,5)$.
Down - Print out $B_0^d$, $RT(1,1)$, $RT(1,2)$, 
            $\cdots$, $RT(1,5)$.

Sense Switch #6  Up - No print out of Bessel functions 
            and argument.
Down - Print out the following information:
            $$\beta_{mn}d_i/R, \quad J_m\left(\frac{\beta_{mn}d_i}{R}\right),$$
            $$J_{m+1}\left(\beta_{mn}\right), J_{m+k}\left(\frac{\beta_{mn}d_i}{R}\right),$$
            $$J_{m-k}\left(\frac{\beta_{mn}d_i}{R}\right), J_{k}\left(\frac{\beta_{mn}r}{R}\right),$$
            $m, n, k, \text{and } i$.

(5) **Stops**

(a) **HPR 10** - Unacceptable argument for the G.E. Bessel 
            function subroutine. Probably one of the control rod 
            position parameters is identically zero.

(b) **N.Y.U. Bessel Function Error** - An error may occur 
            in the execution of the N.Y.U. Bessel function sub­ 
            routine due to a very small argument also. The 
            program will then print out this information. How­ 
            ever, there is a possibility that this is a machine 
            error. This machine error may be due to the core 
            malfunctioning or because of an error on one of the 
            tapes. The tape error probably occurred in prepara­ 
            tion of tape #7. An HPR 666666 will then occur.

(c) **HPR 555555** - End of phase 1 calculation.

(d) For other object program stops see FORTRAN II 
            operator's manual.

(6) **Interrupt and Restart Procedures.** If in phase 1 coefficients 
    are to be calculated for $n_r$ control rods starting at the $i_1^{th}$ 
    and ending at the $i_2^{th}$, one may accomplish this by punching 
    data card number 15 with this information. The program 
    will then interrupt itself and print out the pertinent in­ 
    formation as to how far the calculation has proceeded and 
    as to how many control rod coefficients are ultimately to 
    be calculated. By use of auxiliary program number 3,
one resets the tapes used in phase 1 so that the coefficients for the remaining rods may be calculated. At this point one replaces $i_1$ by $i_1 + 1$ and $i_2$ by $i_2 + 1$ in data card number 15, reads in the phase 1 deck and continues the calculation. This procedure is continued until the coefficients of the $n_{th}$ rod are calculated.

If one wishes not to initialize tape #4 in phase 1, one makes the data TEST on data card number 12 non-zero. If it is zero then tape #4 will be initialized, i.e., all coefficients $c_{mn\ell}$ will be set equal to one. This feature probably will not be used very much.

When phase 1 is finished, all pertinent data to start phase 2 is punched on line.

Note: Since this program is a FORTRAN II written program, the data cards are put in their proper order behind the object deck. These cards are put into the reader, readied, and loaded into the 704 in the ordinary manner.

Phase 2.

(1) **Preparation of Data.** The necessary data cards to start phase 2 are obtained from phase 1. However, the third and fourth words on card number 12 must be punched according as to how phase 2 is to be restarted or interrupted. In this regard, see the section on "Interrupt and Restart Procedures" for this phase. If phase 2 is to be started, a blank card is put behind the sixteenth data card.

(2) **Magnetic Tapes.** Mount tapes 2, 4, and 8 from phase 1. In addition, set two tapes to logical 5 and 6.

(3) **On-Line Punch.** Put in seventeen cards and ready punch.

(4) **Sense Switch Settings**

Sense Switch #1

Up - does not interrupt phase 2.

Down - interrupts phase 2 in the phase 2(a) portion, i.e., after the series is summed using the value $\Lambda(2k+1)$. The purpose here is to avoid unnecessary machine time if a poor initial guess for the odd-order approximation of the eigenvalue was made. Pertinent data is then punched out to restart phase 2.
Sense Switch #2 Not used
Sense Switch #3 Up - proceed from phase 2(a) to phase 2(b).
   Down - interrupt phase 2 after phase 2(a) is completed and punch out pertinent data.
Sense Switch #4 Up - does not print
   Down - print iteration number for $\Lambda^{(2k+1)}$
   i.e., print $2k+1$ and print the value of $\Lambda^{(2k+1)}$. The odd-order approximation is printed out.
Sense Switch #5 Up - does not print.
   Down - prints out $\Lambda^{(2k+1)}$ and value of corresponding series utilizing $\Lambda^{(2k+1)}$ (odd-order approximation)
   when program is in phase 2(a). Prints out $\Lambda^{(2k+2)}$ (even-order approximation) and the values of the two series whose ratio is $\Lambda^{(2k+2)}$. The iteration number for the odd-order approximation is also printed out.
Sense Switch #6 Up - does not write B.C.D. on tape #5.
   Down - writes in B.C.D. on tape #5, the partial sums of the various series summed and the current values of the indices.

(5) **Stops.** HPR 111118 designates end of phase 2(a), i.e., completion of calculation of $\Lambda^{(2k+1)}$ and $\Lambda^{(2k+2)}$.
   If $|\Lambda^{(2k+1)} - \Lambda^{(2k+2)}| < \epsilon$, then program will stop after
   $c_{mn} \to c_{mn} \frac{\Lambda^{(2k+2)} - \lambda_{mn}}{\Lambda^{(2k+1)} - \lambda_{mn}}$.
   This calculation is performed in phase 2(b). If sense switch #3 is down, then phase 2 will be interrupted after START is hit and pertinent data will be punched out to restart phase 2. If sense switch #3 is up, then phase 2 will continue until $|\Lambda^{(2k+1)} - \Lambda^{(2k+2)}| < \epsilon$. HPR 777778 designates end of phase 2 calculation.

(6) **Interrupt and Restart Procedures.** Phase 2 as previously mentioned requires a blank card behind the sixteenth data card. The third and fourth words on data card number 12 are used to tell the program where in phase 2 the calculation should be started. These words are designated as TEST 1 and TEST 2.
If TEST 1 ≤ 0 and TEST 2 > 0 then phase 2 will start with the calculation of $A^{(2k+1)}$ and proceed to the calculation of $A^{(2k+2)}$, where $k = 1$. An HPR 111118 will then occur and phase 2 may be interrupted at this point, if the error stipulated is still too great, by putting sense switch #3 down and hitting START.

Now phase 2 may be restarted, i.e., the calculation will proceed from the calculation of $A^{(2k+1)}$ and go on to the calculation of $A^{(2k+2)}$, where $k = 2$, until the difference in these approximations is within the error stipulated if TEST 1 > 0 and TEST 2 may be arbitrary. At the completion of phase 2, seventeen data cards are punched out as input for phase 3.

Note: Once again this program is loaded into the 704 as is done in phase 2.

Phase 3.

(1) Preparation of Data. All necessary data to start phase 3 is punched out by phase 2 except for the third word on card number 12. This word designates the actual radius of the reactor in centimeters and is left available to the problem requestor's liking. Initial values for the flux and increments on the flux plots may be changed before starting phase 3 or after phase 3 is interrupted depending once again on the originator's inclinations.

(2) Magnetic Tapes. Tapes 2 and 4 prepared by phase 2 are mounted and used in the phase 3 calculations. A blank tape set to logical 5 is also mounted and all input and output is written on tape number 5 for off-line printing.


(4) Sense Switch Settings

<table>
<thead>
<tr>
<th>Sense Switch</th>
<th>Setting</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>Not used</td>
<td></td>
</tr>
<tr>
<td>#2</td>
<td>Up - no print out</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Down - print out $r$, $z$, $\theta$, and flux</td>
<td></td>
</tr>
<tr>
<td>#3</td>
<td>Up - do not interrupt phase 3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Down - interrupt phase 3 punch out current data, put end of file on tape number 5, and rewind all tapes. Intermits after series for flux is summed</td>
<td></td>
</tr>
</tbody>
</table>
Sense Switch #4  Same as sense switch #3 except interrupts before series for flux is summed
Sense Switch #5  Not used
Sense Switch #6  Up - no print out
                Down - print out \((h_i, i = 1, 10)\) and \(\Lambda^{(2k+2)}\).

(5) **Stops.** HPR 222228 occurs after N.Y.U. Bessel function subroutine fails (see phase 1 stops). HPR 108 occurs if G.E. Bessel function subroutine fails.

Object program stops (see FORTRAN II operator's manual).

HPR 111118 - end of phase 3 calculation.

(6) **Interrupt and Restart Procedures.** If program is interrupted by sense switch number 3 or 4, the punch out may simply be used as data once again as before if the program is to be resumed. Once again this data may be altered to start the flux plot in a different region. One must, of course, keep tapes 2 and 4 for phase 3 calculations. These tapes may be used over again depending on what flux calculations are to be made.

Load and ready program and data cards as in the above programs.

**Auxiliary Program No. 1**

(1) **Preparation of Data.** The following information \(r_0, z_{in}, \Delta r, \Delta z, \Delta \alpha, t_{in}, R, H, \Delta r\) and \(C_2\) is punched on the first two data cards according to FORMAT (1P5E14.7). On the third card \(J_4\) is punched according to FORMAT (14).

(2) **Sense Switch Settings**

   Sense Switch #1  Up - no print
                   Down - print \(t_{in}\) and value of the transcendental function

\[
J_0\left(\frac{\alpha r_0}{R}\right) - \frac{J_0(\alpha)}{Y_0(\alpha)} Y_0\left(\frac{\alpha r_0}{R}\right)\]

(3) **Stops.** HPR 777778 - end of program.
Auxiliary Program No. 2

(1) Preparation of Data. The following information: $J_1$, $J_2$, $J_3$, $n_r$, and $i_3$, is punched on one card according to FORMAT (514). This program is used to initialize tape number 4 if, after $\Lambda^{(2k+2)}$ is calculated, the accuracy is not satisfactory, which means starting phase 2 from the beginning without preparing tapes 2 and 8 by use of phase 1. The total number of coefficients initialized is equal to $n_r \times J_1 \times J_2 \times J_3$.

(2) Magnetic Tapes. Mount blank tape and set to logical 4.

(3) Stops. HPR 111118 - end of program.

Auxiliary Program No. 3

(1) Preparation of Data. The following information: $J_1$, $J_2$, $J_3$, $n_r$, and $i_3$, is punched on the first card according to FORMAT (514). On the second card is punched the information, $J_{12}$, $J_{13}$, and $J_{14}$ according to FORMAT (313). This program positions tapes 2, 4, and 8 in phase 1 when coefficients ultimately are to be calculated for $n_r$ rods, but the calculation has been interrupted after the coefficients for $i_2$ rods have been made. Then $i_3$ is equal to $i_2 + 1$, and $\leq n_r$.

(2) Stops. HPR 444448 - end of program.

Auxiliary Program No. 4

Part 1.

(1) Preparation of Data. Punch on one card the information $J_1$, $J_2$, $J_3$, $J_4$, $n_r$, $i_1$, $i_2$ and $J_k$ according to FORMAT (814).

(2) Magnetic Tapes. Mount a blank tape and set to logical 5. On this tape will be written the values of $J_m (\beta_{mn})$, $\beta_{mn}$, $m$, and $n$, where $\beta_{mn}$ is calculated by the formulas mentioned in the description of phase 1. This information is written in B.C.D. and is printed off-line. From this output one can find which roots are unsatisfactory. An end of file is written by the program on tape 5. This tape is saved for use in part 2.

(3) Stops. HPR 333338 - end of program.
Part 2.

1) Preparation of Data. From the results on tape 5 in part 1, one finds, by use of the Harvard Tables on Bessel functions, which roots cannot be accurately calculated by the usual formulas. These corrected roots designated as CORTMN, the indices designated as MM and NN, and a tag IND used to designate end of corrections are all punched on cards (one at a time) according to FORMAT (1H1PE14.7, 314). The tag IND if not equal to zero means that no more corrections are to be made after that card and if equal to zero means that corrections are to be continued.

2) Magnetic Tapes. From part 1, tape number 5 is used as an input tape for examination of which roots are to be corrected. The corrected roots and appropriate indices are written on tape number 3, which must be mounted. A manual end of file must be put on tape number 3. Tape number 3 is in B.C.D. and may be printed off-line as a check to see if the proper roots have been corrected when compared with the output of tape 5.

3) Stops. No machine stops have been designated.

Part 3.

1) Preparation of Data. See (2) below.

2) Magnetic Tapes. Tape Number 3 from Part 2, containing the corrected roots \( \beta_{mn} \), is mounted and set to logical 3. This tape is now read in as an input tape. Mount two blank tapes and set to logical 1 and 5. The program will now write in binary on tape number 1 the roots \( \beta_{mn} \) and in B.C.D. the information \( J_m(\beta_{mn}) \), \( \beta_{mn} \), m and n, which can be printed off-line for final inspection as to how good the roots \( \beta_{mn} \) are. Manual end of files are put on tapes 1 and 5.

3) Stops. No stops are designated for this program.

Auxiliary Program No. 5

1) Preparation of Data. The following information: \( J_{11}, J_{22}, J_1 \) and \( J_2 \), is punched on one card according to FORMAT (1H4I4).
(2) **Magnetic Tapes.** Mount tape number 1 from Auxiliary Program No. 4, Part 2, and set to logical 1. Mount a blank tape and set to logical 7. Mount a blank tape and set to logical 6. The program will read the 5000 roots $\beta_{mn}$ from tape 1 into the memory. It will then make up a tape 7 with $m \times n$ roots, where $m \leq 50$ and $n \leq 100$ according to the requestor's demands for the partial sums to be calculated in the various series. As a check to see if tape 7 is properly written, the information $\beta_{mn}$, $m$, and $n$ is written on tape 6 in B.C.D. and may be printed off-line.

(3) **Stops.** HPR 777778 - end of program.

Procedure for the Determination of the Number of Terms to be Taken in Summing the Various Series. Since, as previously mentioned, the number of terms to be taken is a function of the geometry being considered, it is felt that it would be more efficient to run a few test cases. The procedure to be outlined to run test cases for one or two control rods will save a great deal of machine time when calculations are made for many control rods.

A particular calculation may be considered reliable if, with increasing numbers of terms taken in evaluating the various series, the buckling or first eigenvalue shows no substantial change and the flux along the control surfaces satisfies the homogeneous Dirichlet boundary conditions. Of particular importance is the satisfaction of the boundary condition across the control rod tips.

There are two reasons which make the satisfaction of the boundary condition across the control tip extremely important. The first is that the actual mathematical control rod length may be determined by this procedure. In order to determine this length, one may form flux plots along the effective radii of the control; the position where the flux begins to increase away from the zero boundary condition is the point where the actual mathematical control rod ends. Of course, because a Fourier-type series is involved here this point may fluctuate a little for various numbers of terms taken. However, after a certain number of terms, this point essentially does not change. As a result of this procedure, one may determine the least amount of terms to get high accuracy.

The second reason that makes this approach very important is that the buckling will automatically be determined for the exactly or very close to the exactly correct control rod length considered.

The various calculations made in this paper will serve as useful guides in these preliminary studies. Three cases that certainly should be tried first are the fully inserted central control rod using the Auxiliary
Program No. 1 as a checking feature, the central halfway inserted control rod, and the halfway off-centered control rod. In this way, a typical picture may be obtained for the determination of the proper choice of the indices involved. In particular, a helpful check point is that the buckling for a small \( r_0 \ll R \) control rod inserted halfway is equal to \( \sim 0.55 \) of the total change in buckling due to a fully inserted central control rod.

There is no provision to take care of underflow except by a built-in device. The limits that the indices may take are \( m \leq 50 \), \( n \leq 100 \) and \( l \leq 100 \).

To further aid the user of this code some general information will be given regarding limitations on the rod position parameters, general behavior of the mathematically effective length as a function of the number of terms used, and on estimates of the running times for phases 1, 2, and 3.

In regard to the range of validity of the position parameters on what in essence is the same thing the ranges of the arguments of the subroutines, it was found that the following bounds on these parameters yielded acceptable calculations, namely,

\[
10^{-4} \leq d_i \leq 0.65R \\
0.25H \leq h_i \leq 0.75H \\
0.01R \leq r_i \leq 0.10R \\
0 \leq \alpha_i \leq 2\pi
\]

It was also found that one had to specify a mathematically effective control rod length of 10 to 25% longer than the actual length one wished to consider. Some approximate, but indicative, figures are given as follows:

<table>
<thead>
<tr>
<th>Percent Increase of Mathematically Effective Length over Actual Length</th>
<th>Total Number of Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>One</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>15 - 10</td>
<td>250 - 500</td>
</tr>
<tr>
<td>30 - 20</td>
<td></td>
</tr>
</tbody>
</table>

There is some irregularity or change of rod radius also, but this is really a much smaller effect than the length of the rod being shortened because of the partial sum representation of the sink absorption distribution.
assumed initially in the theory. It was also found that the number of terms for the m and n indices was always much greater than the number of terms required for the 1 index.

Since tape 8 in phase 1 will have $n_r \times m \times n \times 1$ records on it, the maximum number of terms for $n_r = 10$ would then have to be about $m \times n \times 1 = 320$. Thus, for a ten-rod calculation one would pick something like $10 \times 16 \times 2$ terms, but chances are that the accuracy of the calculation would suffer somewhat. However, the total change in buckling would be great and might result in at least a fair order of approximation of the buckling and flux. The restriction is due to the smallness of the particular IBM 704 for which this code was written and not due to the theory. Thus, on larger machines more rods and more terms may be easily added by altering the code slightly. As it stands, however, it is not strongly recommended that ten-rod calculations be made. Four–six rod calculations would be more reasonable problems to consider.

Some idea of the running times for phases 1, 2, and 3 will now be given. For phase 1 the running time is about

$$n_r \times m \times n \times 1 \times (2 \times 10^{-4}) \text{ hr}.$$ 

For phase 2 the running time is about

$$n_r \times m \times n \times 1 \times J_4 \times J_{15} \times (4 \times 10^{-5}) \text{ hr}.$$ 

For phase 3 the running time per flux point is about

$$n_r \times m \times n \times 1 \times (4 \times 10^{-5}) \text{ hr}.$$ 

These running times are estimates, it must be remembered, and would tend to be slightly less if one- or two-rod calculations were made. They would tend to be a little greater if three- to six-rod calculations were made.

It was also found that $J_k$ gave good results when set equal to one, i.e., one term in the series arising from use of the addition formulas was usually enough for the geometries considered.

Certain techniques for improving the convergence of series* have been tried. These techniques, however, did not seem to improve the convergence of the series developed in the above theory and were thus not incorporated in the code.

---

<table>
<thead>
<tr>
<th>FORTRAN Symbol</th>
<th>Meaning or Mathematical Symbol Represented</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>U(M)</td>
<td>Numbers 0, 1, 2, ..., M-1</td>
<td>Floating point integers generated.</td>
</tr>
<tr>
<td>Qϕ(M)</td>
<td>q</td>
<td>Used to calculate roots of Bessel functions.</td>
</tr>
<tr>
<td>Q1(M)</td>
<td>7q - 31</td>
<td>&quot;     &quot;     &quot;</td>
</tr>
<tr>
<td>Q2(M)</td>
<td>83q^2 - 982q + 3779</td>
<td>&quot;     &quot;     &quot;</td>
</tr>
<tr>
<td>Q3(M)</td>
<td>6949q^3 - 153855q^2 + 1585743q - 6277237</td>
<td>&quot;     &quot;     &quot;</td>
</tr>
<tr>
<td>Q(J)</td>
<td>Numbers 1, 2, ..., J</td>
<td>Floating point integers generated.</td>
</tr>
<tr>
<td>Q(N)</td>
<td>&quot;     1, 2, ..., N</td>
<td></td>
</tr>
<tr>
<td>Q(L)</td>
<td>&quot;     1, 2, ..., L</td>
<td></td>
</tr>
<tr>
<td>RT(M, N)</td>
<td>β_{mn}</td>
<td>Roots of Bessel Functions.</td>
</tr>
<tr>
<td>Cϕ(I)</td>
<td>C_i' = \frac{4r_i}{R}; C_i = \frac{8\pi r_i}{R}</td>
<td>Control rod radii.</td>
</tr>
<tr>
<td>RR(I)</td>
<td>r_i</td>
<td>Control rod angles made with reference rod.</td>
</tr>
<tr>
<td>D(I)</td>
<td>d_i</td>
<td>Insertion length of control rods.</td>
</tr>
<tr>
<td>HR(I)</td>
<td>h_i</td>
<td>Reactor radius.</td>
</tr>
<tr>
<td>ALPH(I)</td>
<td>α_i</td>
<td></td>
</tr>
<tr>
<td>R</td>
<td>R</td>
<td></td>
</tr>
</tbody>
</table>

Table V

FORTRAN Symbols and Meaning in Code
<table>
<thead>
<tr>
<th>FORTRAN Symbol</th>
<th>Meaning or Mathematical Symbol Represented</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>H</td>
<td>Reactor height.</td>
</tr>
<tr>
<td>RIN</td>
<td>( r_{in} )</td>
<td>Initial point for radial plot.</td>
</tr>
<tr>
<td>ZIN</td>
<td>( z_{in} )</td>
<td>Initial point for vertical plot.</td>
</tr>
<tr>
<td>THETIN</td>
<td>( \theta_{in} )</td>
<td>Initial point for angular plot.</td>
</tr>
<tr>
<td>DELTAE</td>
<td>( \Delta \Lambda )</td>
<td>Used to increment ( \Lambda ) when finding root.</td>
</tr>
<tr>
<td>DELTAR</td>
<td>( \Delta r )</td>
<td>Increment for radial plot.</td>
</tr>
<tr>
<td>DELTAZ</td>
<td>( \Delta z )</td>
<td>Increment for vertical plot.</td>
</tr>
<tr>
<td>DELTAO</td>
<td>( \Delta \theta )</td>
<td>Increment for angular plot.</td>
</tr>
<tr>
<td>ERROR</td>
<td>( \epsilon )</td>
<td>Error stipulated between ( \Lambda^{(2k+1)} ) and ( \Lambda^{(2k+2)} ).</td>
</tr>
<tr>
<td>S</td>
<td>S</td>
<td>Variable used to divide ( \Delta \Lambda ).</td>
</tr>
<tr>
<td>EO</td>
<td>( \Lambda_0 )</td>
<td>First guess of eigenvalue ( \Lambda^{(1)} ). ( \Lambda_0 \gtrsim \lambda_{mn\ell} ).</td>
</tr>
<tr>
<td>EIN</td>
<td>( \Lambda_{in} )</td>
<td>First guess of eigenvalue ( \Lambda^{(2)} ). ( \Lambda^{(2)} \gtrsim \lambda_{mn\ell} ).</td>
</tr>
<tr>
<td>DELPR</td>
<td>( \Delta'r )</td>
<td>Used to initialize radial plot.</td>
</tr>
<tr>
<td>DELPZ</td>
<td>( \Delta'z )</td>
<td>Used to initialize vertical plot.</td>
</tr>
<tr>
<td>DELPO</td>
<td>( \Delta'\theta )</td>
<td>Used to initialize angular plot.</td>
</tr>
<tr>
<td>AREA</td>
<td>M</td>
<td>Migration area.</td>
</tr>
<tr>
<td>J1</td>
<td>( m )</td>
<td>Index ( m ) in series.</td>
</tr>
<tr>
<td>FORTRAN Symbol</td>
<td>Meaning or Mathematical Symbol Represented</td>
<td>Remarks</td>
</tr>
<tr>
<td>----------------</td>
<td>-------------------------------------------</td>
<td>---------</td>
</tr>
<tr>
<td>J2</td>
<td>n</td>
<td>Index n in series.</td>
</tr>
<tr>
<td>J3</td>
<td>l</td>
<td>Index l in series.</td>
</tr>
<tr>
<td>J4</td>
<td>$\Delta\Lambda / S^4$ and no. of sign changes.</td>
<td>Number of sign changes near roots.</td>
</tr>
<tr>
<td>NR</td>
<td>$n_r$</td>
<td>Total number of rods.</td>
</tr>
<tr>
<td>J15</td>
<td>$J_{15} = m \times n \times l$</td>
<td>Total number of coefficients $c_{mn\ell}$ to be calculated using tape 4.</td>
</tr>
<tr>
<td>J12</td>
<td>Mo.</td>
<td>Integers for date.</td>
</tr>
<tr>
<td>J13</td>
<td>Day</td>
<td>&quot; &quot; &quot;</td>
</tr>
<tr>
<td>J14</td>
<td>Yr.</td>
<td>&quot; &quot; &quot;</td>
</tr>
<tr>
<td>PI</td>
<td>$\pi$</td>
<td>$\pi$ set equal to 3.1415926 in program.</td>
</tr>
<tr>
<td>DELTIN</td>
<td>$\Delta \Lambda_{in}$</td>
<td>Used to initialize $\Delta \Lambda$ after iteration.</td>
</tr>
<tr>
<td>BE</td>
<td>$\beta$</td>
<td>Dummy variable to conserve storage.</td>
</tr>
<tr>
<td>BB</td>
<td>$\lambda_{011}$</td>
<td>$\lambda_{011} = \left(\frac{\beta_{01}}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2$.</td>
</tr>
<tr>
<td>JJ5</td>
<td>$J_{J5}$</td>
<td>Counter for number of iterations made.</td>
</tr>
<tr>
<td>ROOTMN</td>
<td>$\beta_{mn}$</td>
<td>Dummy variable for roots of Bessel functions used to conserve storage.</td>
</tr>
<tr>
<td>JBESSF(X, N)</td>
<td>$J_n (x)$</td>
<td>Bessel function of index n.G.E.</td>
</tr>
<tr>
<td>FORTRAN Symbol</td>
<td>Meaning or Mathematical Symbol Represented</td>
<td>Remarks</td>
</tr>
<tr>
<td>----------------</td>
<td>--------------------------------------------</td>
<td>---------</td>
</tr>
<tr>
<td>SINF(X)</td>
<td>( \sin(x) )</td>
<td>Sine function.</td>
</tr>
<tr>
<td>A</td>
<td>( a_{mn} )</td>
<td>Dummy variable used to conserve storage.</td>
</tr>
<tr>
<td>B</td>
<td>( b_l )</td>
<td>&quot; &quot; &quot; &quot;</td>
</tr>
<tr>
<td>QU</td>
<td>( Q_{nm}^l = J_m \left( \frac{\beta_{mn}}{R} \right) J_1 \left( \frac{\beta_{mn}}{R} \right) ) * ( \sin \frac{\ell \pi h_1}{H} + \frac{H}{\ell \pi} J_0 \left( \frac{\beta_{mn}}{R} \right) ) * ( \left( 1 - \cos \frac{\ell \pi h_j}{H} \right) \left( \cos \frac{m a_i}{H} \cos \frac{m a_j}{H} + \sin \frac{m a_i}{H} \sin \frac{m a_j}{H} \right) )</td>
<td>Dummy variable used to conserve storage--arises from surface integration.</td>
</tr>
<tr>
<td>P</td>
<td>( P_{mn}^l = R^2 H J_1 \left( \frac{\beta_{on}}{\ell} \right) \left( 1 - \cos \frac{\ell \pi}{\ell} \right) )</td>
<td>Dummy variable used to conserve storage--arises from volume integrations.</td>
</tr>
<tr>
<td>E</td>
<td>( \lambda_{mn} )</td>
<td>Dummy variable used to conserve storage.</td>
</tr>
<tr>
<td>C</td>
<td>( C_{mn}^l = \frac{\Lambda^{(2k+2)} - \lambda_{mn} \ell}{\Lambda^{(2k+1)} - \lambda_{mn} \ell} ) * ( \sum_{i=1}^{n_k} \sum_{j=1}^{n_{\text{mod}}} \frac{a_{mn}}{n_{i,\text{mod}}} J_0 \left( \frac{\beta_{mn}}{R} \right) \sum_{j=1}^{n_r} r_j Q_{nm,i,j} )</td>
<td>&quot; &quot; &quot; &quot;</td>
</tr>
<tr>
<td>SUMA</td>
<td>Partial sum for odd order approximation of eigenvalue.</td>
<td></td>
</tr>
<tr>
<td>El</td>
<td>( \Lambda^{(2k+1)} )</td>
<td>Dummy variable for odd order approximations of the eigenvalue.</td>
</tr>
<tr>
<td>SUMC</td>
<td>Partial sum for odd order approximation for eigenvalue.</td>
<td></td>
</tr>
<tr>
<td>FORTRAN Symbol</td>
<td>Meaning or Mathematical Symbol Represented</td>
<td>Remarks</td>
</tr>
<tr>
<td>----------------</td>
<td>---------------------------------------------</td>
<td>---------</td>
</tr>
<tr>
<td>SUME</td>
<td>$\sum_{i=1}^{n_r} \sum_{m,n,l} \frac{a_{mn} b_{n+l} c_{mn+l}}{d^{(i)}(1) - \lambda_{mn+l} m + \lambda_{mn+l} n} \left( p_{mn+l} + \sum_{j=1}^{n_r} s_{mn+j} \right)$</td>
<td>Partial sum for even order approximations for eigenvalue.</td>
</tr>
<tr>
<td>E2</td>
<td>$A(2k+2)$</td>
<td>Dummy variable for even order approximations of the eigenvalue.</td>
</tr>
<tr>
<td>DELKOK</td>
<td>$\Delta k/k$</td>
<td>Reactivity worth.</td>
</tr>
<tr>
<td>SUMH</td>
<td>$\sum_{i=1}^{n_r} \sum_{m,n,l} \frac{a_{mn} b_{n+l} c_{mn+l}}{d^{(i)}(1) - \lambda_{mn+l} m + \lambda_{mn+l} n} \left( \frac{b_{mn+l}}{R} \right)$</td>
<td>Partial sum for flux.</td>
</tr>
<tr>
<td>NOFLX</td>
<td>$n_{\text{flux}}$</td>
<td>Number of approximations made for eigenfunction.</td>
</tr>
<tr>
<td>TEST</td>
<td>Constant for altering path of calculation.</td>
<td>Variable used in phase 1 to set all coefficients on tape 4, equal to one if TEST = 0. If TEST \neq 0, initializing tape 4 will be omitted and calculation will proceed.</td>
</tr>
<tr>
<td>TABLE(M)</td>
<td>Reserves storage.</td>
<td>Used in conjunction with N.Y.U. Bessel function subroutine.</td>
</tr>
<tr>
<td>TABL2(M)</td>
<td>&quot; &quot;</td>
<td>&quot; &quot; &quot; &quot;</td>
</tr>
<tr>
<td>TABL3(M)</td>
<td>&quot; &quot;</td>
<td>&quot; &quot; &quot; &quot;</td>
</tr>
<tr>
<td>TABL4(M)</td>
<td>&quot; &quot;</td>
<td>&quot; &quot; &quot; &quot;</td>
</tr>
<tr>
<td>TABL5(M)</td>
<td>&quot; &quot;</td>
<td>&quot; &quot; &quot; &quot;</td>
</tr>
<tr>
<td>ALPF(X, Y)</td>
<td>Function statement.</td>
<td>Used to define a subroutine which in turn is used in a Bessel function subroutine.</td>
</tr>
<tr>
<td>FORTRAN Symbol</td>
<td>Meaning or Mathematical Symbol Represented</td>
<td>Remarks</td>
</tr>
<tr>
<td>---------------</td>
<td>---------------------------------------------</td>
<td>---------</td>
</tr>
<tr>
<td>BETF(X, Y)</td>
<td>Function statement.</td>
<td>Used to define a subroutine which in turn is used in a Bessel function subroutine.</td>
</tr>
<tr>
<td>JBES1(X, Y)</td>
<td>&quot; &quot;</td>
<td>Used to define a subroutine for the asymptotic form of $J_m(X)$ for large index $m$, where $m &lt; x$. See Morse and Feshbach, Vol. II.</td>
</tr>
<tr>
<td>JBES2(X, Y)</td>
<td>&quot; &quot;</td>
<td>Used to define a subroutine for the asymptotic form of $J_m(X)$ for large index $m$, where $m &gt; x$. See Morse and Feshbach, Vol. II.</td>
</tr>
<tr>
<td>GNU</td>
<td>Part of N.Y.U. Bessel Function subroutine.</td>
<td>Fractional part $v$ in the function $J_n \pm v(X)$ where $n$ is an integer.</td>
</tr>
<tr>
<td>LE</td>
<td>&quot; &quot;</td>
<td>Index $k$ in the series arising from using the additional formulas for Bessel function.</td>
</tr>
<tr>
<td>LEE</td>
<td>&quot; &quot;</td>
<td>Index used to calculate coefficients in phase 1. from $i_1$ to $n_r$ rods.</td>
</tr>
<tr>
<td>LEEE</td>
<td>&quot; &quot;</td>
<td>Index used to interrupt the calculation of coefficients in phase 1. when calculating from $i_1$ to $i_2$ rods where $i_2 \leq n_r$.</td>
</tr>
</tbody>
</table>
Table V (Cont'd.)

<table>
<thead>
<tr>
<th>FORTRAN Symbol</th>
<th>Meaning or Mathematical Symbol Represented</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>S8</td>
<td>$S_8 = J_1 \left( \frac{\beta_{01} r_i}{R} \right)$</td>
<td>Dummy variable for the value of G.E. Bessel function subroutine.</td>
</tr>
<tr>
<td>S9</td>
<td>$S_9 = J_0 \left( \frac{\beta_{01} r_i}{R} \right)$</td>
<td>&quot; &quot; &quot;</td>
</tr>
<tr>
<td>S16</td>
<td>$S_{16} = J_0 \left( \frac{\beta_{01} d_i}{R} \right)$</td>
<td>&quot; &quot; &quot;</td>
</tr>
<tr>
<td>ARG</td>
<td>$\beta_{mn} d_i$</td>
<td>Argument of Bessel function subroutines.</td>
</tr>
<tr>
<td>AS</td>
<td>Exit location for N.Y.U. Bessel function subroutine. If $AS = 0$ subroutine has been properly executed. If $AS \neq 0$, subroutine incorrectly executed.</td>
<td></td>
</tr>
<tr>
<td>S6</td>
<td>$S_6 = J_{m+1} \left( \beta_{mn} \right)$</td>
<td>Dummy variable for N.Y.U. Bessel function subroutine of argument $\beta_{mn}$.</td>
</tr>
<tr>
<td>S7</td>
<td>$S_7 = J_m \left( \beta_{mn} \right)$</td>
<td>Dummy variable for G.E. Bessel function subroutine of argument $\beta_{mn}$.</td>
</tr>
<tr>
<td>GAMK</td>
<td>$\Gamma_k = \sum_{k=0}^{\infty} \left( \frac{i}{\beta_{01} - \beta_{mn}} \right) \left( \beta_{01} s_{10} s_{13} - \beta_{mn} s_{12} s_{11} \right) s_{17} \left( -1 \right)^k s_{10} s_{12}$</td>
<td>Partial sum of series.</td>
</tr>
<tr>
<td>GAMO</td>
<td>$\Gamma_0 = \left( \frac{1}{\beta_{01} - \beta_{mn}} \right) \left( \beta_{01} s_7 s_8 - \beta_{mn} s_9 s_2 \right)$</td>
<td>First term in series.</td>
</tr>
</tbody>
</table>
Table V (Cont'd.)

<table>
<thead>
<tr>
<th>FORTRAN Symbol</th>
<th>Meaning or Mathematical Symbol Represented</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>BS</td>
<td></td>
<td>Similar to AS.</td>
</tr>
<tr>
<td>SS10</td>
<td>( S_{S10} = J_{k-1} \left( \frac{\beta mn^r_i}{R} \right) )</td>
<td>Dummy variable for N.Y.U Bessel function subroutine of argument ( \beta mn^r_i/R )</td>
</tr>
<tr>
<td>S10</td>
<td>( S_{10} = J_k \left( \frac{\beta mn^r_i}{R} \right) )</td>
<td>&quot; &quot; &quot;</td>
</tr>
<tr>
<td>S11</td>
<td>( S_{11} = J_{k+1} \left( \frac{\beta mn^r_i}{R} \right) )</td>
<td>&quot; &quot; &quot;</td>
</tr>
<tr>
<td>S17</td>
<td>( S_{17} = J_k \left( \frac{\beta d_i}{R} \right) )</td>
<td>Dummy variable for G.E. Bessel function subroutine of argument ( \beta d_i/R )</td>
</tr>
<tr>
<td>DS</td>
<td>new ( mk = m + k )</td>
<td>Same as AS.</td>
</tr>
<tr>
<td>NEWMK</td>
<td>( new ) ( mk = m + k )</td>
<td>Change of variable to allow index ( m + k ).</td>
</tr>
<tr>
<td>S14</td>
<td>( S_{14} = J_m^\pm k \left( \frac{\beta mn^d_i}{R} \right) )</td>
<td>Dummy variable for N.Y.U. Bessel function subroutine.</td>
</tr>
<tr>
<td>T</td>
<td>( t = m + k )</td>
<td>Variable used to float fixed point variable.</td>
</tr>
<tr>
<td>TT</td>
<td>( tt = m - k )</td>
<td>&quot; &quot; &quot;</td>
</tr>
<tr>
<td>DS</td>
<td>( new ) ( mk = m + k )</td>
<td>Same function as AS.</td>
</tr>
<tr>
<td>SS14</td>
<td>( S_{S14} = (-1)^{m-k} J_{m-k} \left( \frac{\beta mn^d_i}{R} \right) )</td>
<td>Dummy variable for Bessel function subroutines depending on argument and index.</td>
</tr>
</tbody>
</table>
Table V (Cont'd.)

<table>
<thead>
<tr>
<th>FORTRAN Symbol</th>
<th>Meaning or Mathematical Symbol Represented</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>LI</td>
<td>$l_1 = k - m$</td>
<td>Change of variable to allow index $k - m$.</td>
</tr>
<tr>
<td>L2</td>
<td>$l_2 = m - k$</td>
<td>Change of variable to allow index $m - k$.</td>
</tr>
<tr>
<td>XLOCF</td>
<td>Part of N.Y.U. Bessel function.</td>
<td>Location function subroutine.</td>
</tr>
<tr>
<td>FK</td>
<td>$f_k = k$</td>
<td>Float fixed point variable.</td>
</tr>
<tr>
<td>S5</td>
<td>$s_5 = J_m\left(\frac{\beta_{mn}d_1}{R}\right)$</td>
<td>Dummy variable for G.E. Bessel function subroutine.</td>
</tr>
<tr>
<td>ES</td>
<td></td>
<td>Same function as AS.</td>
</tr>
<tr>
<td>SS5</td>
<td>$s_{ss} = J_m\left(\frac{\beta_{mn}d_j}{R}\right)$</td>
<td>Dummy variable for Bessel function subroutines for $j$th control rod.</td>
</tr>
<tr>
<td>Z</td>
<td>$z_{mnk}^j = \frac{-R_j H}{I_{\beta_{mn}}} s_{ss} j_{l_{\beta_{mn}r_j}} \left[1 - \cos \frac{l_{\pi h_j}}{H} \right] \left(\cos m_1 \cos m_1 + \sin m_1 \sin m_1 \right)$</td>
<td>Term arising when interaction of $i$th and $j$th control rod is considered.</td>
</tr>
<tr>
<td>TEST1</td>
<td>Test constant.</td>
<td>If TEST1 &gt; 0 then the phase 2. calculation will begin iterations at $\lambda^{(2k+1)}$ where $k &gt; 1$.</td>
</tr>
<tr>
<td>TEST2</td>
<td>&quot;$&quot; &quot;&quot;</td>
<td>If TEST1 ≤ 0 and TEST 2 &gt; 0 then the phase 2. calculation will begin at $\lambda^{(2k+1)}$, where $k = 0$. If TEST2 ≤ 0, then the phase 2. calculation will begin with making up the coefficient $c_{mn} \ell$.</td>
</tr>
</tbody>
</table>
### Table V (Cont'd.)

<table>
<thead>
<tr>
<th>FORTRAN Symbol</th>
<th>Meaning or Mathematical Symbol Represented</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>R₁</td>
<td>Actual radius of reactor considered in centimeters.</td>
</tr>
<tr>
<td>THETAF(X,Y)</td>
<td>$\theta(x,y) = J_0(x) - \frac{J_0(y)}{Y_0(y)}Y_0(x)$</td>
<td>Subroutine defining the exact solution for one central rod fully inserted.</td>
</tr>
<tr>
<td>RO</td>
<td>r₀</td>
<td>Effective radius for fully inserted central rod.</td>
</tr>
<tr>
<td>DETAL</td>
<td>$\Delta a$</td>
<td>Increment for root of transcendental equation for fully inserted central rod.</td>
</tr>
<tr>
<td>ALPHIN</td>
<td>$\alpha_{in}$</td>
<td>Initial guess for root of transcendental equation for fully inserted central rod.</td>
</tr>
<tr>
<td>C2</td>
<td>$C_2 = C_1 \theta(\alpha_{in} \cdot 15 \text{r } \alpha_{in})$</td>
<td>Normalization constant used to normalize solution obtained by use of the transcendental equation for the fully inserted central rod so as to compare flux plots obtained by use of phases 1., 2., and 3.</td>
</tr>
<tr>
<td>FUNCT</td>
<td>$F = \theta(x,y)$</td>
<td>Value of transcendental function for the flux for the fully inserted central rod at various points r, and Z.</td>
</tr>
<tr>
<td>EIGVL1</td>
<td>$B^2 = \left(\frac{\beta}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2$</td>
<td>Buckling obtained by the use of the Nordheim-Scalletar theory for a fully inserted central rod.</td>
</tr>
<tr>
<td>EIGVL2</td>
<td>$B^2 = \left(\frac{\alpha}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2$</td>
<td>Buckling obtained by use of the transcendental equation for a fully inserted central rod.</td>
</tr>
<tr>
<td>FORTRAN Symbol</td>
<td>Meaning or Mathematical Symbol Represented</td>
<td>Remarks</td>
</tr>
<tr>
<td>----------------</td>
<td>------------------------------------------</td>
<td>---------</td>
</tr>
<tr>
<td>DELBE</td>
<td>$\Delta \beta$</td>
<td>Change in radial component of buckling using Nordheim-Scalletar Theory.</td>
</tr>
<tr>
<td>BETA</td>
<td>$\beta = \left( \frac{2.4048}{R} + \Delta \beta \right)$</td>
<td>Radial component of buckling using Nordheim-Scalletar Theory.</td>
</tr>
<tr>
<td>I3</td>
<td>$i_3$</td>
<td>Used in conjunction with $J_1$, $J_2$, and $J_3$ to initialize tape 4 in phase 1. It is also used to reset tapes 2, 4, and 8 in phase 1 if only part of the control rod coefficients have been calculated.</td>
</tr>
<tr>
<td>CORTMN</td>
<td>$\beta'_m^n$</td>
<td>Corrected roots of $J_m$ found by use of Harvard Tables for Bessel functions.</td>
</tr>
<tr>
<td>MM</td>
<td>$m'$</td>
<td>Index used to tell program which roots are to be corrected.</td>
</tr>
<tr>
<td>NN</td>
<td>$n'$</td>
<td>&quot; &quot; &quot; &quot;</td>
</tr>
<tr>
<td>IND</td>
<td>Index</td>
<td>Tag to signal program when to stop correcting roots.</td>
</tr>
<tr>
<td>S6</td>
<td>$S_6 = J_m (\beta'_m^n)$</td>
<td>$S_6$ has this slightly different meaning in Auxiliary Program No. 4 than in phase 1.</td>
</tr>
<tr>
<td>J11</td>
<td>$J_{11}$</td>
<td>Used in Auxiliary Program No. 5 to read in $m$ roots.</td>
</tr>
<tr>
<td>J22</td>
<td>$J_{12}$</td>
<td>Used in Auxiliary Program No. 5 to read in $n$ roots.</td>
</tr>
<tr>
<td>J1</td>
<td>$J_1$</td>
<td>Used in Auxiliary Program No. 5 to write $m$ roots.</td>
</tr>
<tr>
<td>FORTRAN Symbol</td>
<td>Meaning or Mathematical Symbol Represented</td>
<td>Remarks</td>
</tr>
<tr>
<td>----------------</td>
<td>--------------------------------------------</td>
<td>---------</td>
</tr>
<tr>
<td>J2</td>
<td>( J_2 )</td>
<td>Used in Auxiliary Program No. 5. to write ( n ) roots.</td>
</tr>
<tr>
<td>S13</td>
<td>( S_{13} = J_{k+1} \left( \frac{\beta_{01} r_i}{R} \right) )</td>
<td>Dummy Variable</td>
</tr>
<tr>
<td>S12</td>
<td>( S_{12} = J_k \left( \frac{\beta_{01} r_i}{R} \right) )</td>
<td>&quot; &quot;</td>
</tr>
<tr>
<td>SS12</td>
<td>( S_{S12} = J_{k-1} \left( \frac{\beta_{01} r_i}{R} \right) )</td>
<td>&quot; &quot;</td>
</tr>
<tr>
<td>S2</td>
<td>( S_2 = J_1 \left( \frac{\beta_{mn} r_i}{R} \right) )</td>
<td>&quot; &quot;</td>
</tr>
</tbody>
</table>
C PROGRAM FOR THE CALCULATION OF THE BUCKLING AND FLUX FOR A BARE CYLINDRICAL HOMOGENEOUS REACTOR WITH PARTIALLY INSERTED CONTROL RODS

PHASE 1-CALCULATION OF FOURIER-BESSEL COEFFICIENTS AND VOLUME INTEGRAL COEFFICIENTS

4/30/59

E.J.BETINIS

DIMENSION U(51), Q(50), Q(50), Q(50), Q(100), RT(100),
10(10), R(10), D(10), E(10), A(10), T(120), T(120), T(120), T(120), T(120),
20(1,100), RT(1,100),

ALPF(Y)=.5*LOGF((1.+SQRTF(1.-X/Y)*2))/(1.-SQRTF(1.-X/Y)*2))

BETF(Y)=ATANF(SQRTF((X/Y)**2-1.))

JBES1F(Y)=EXPF(Y*SQRTF(ALPF(Y)**2-ALPF(Y)))/SQRTF(2.*PI*Y**2)

JBES2F(Y)=SQRTF(2./(PI*Y**2)*(SINF(BETF(Y))/COSF(BETF(Y))-PI/4.))

READ 5.R.H.RIN.ZIN.THETIN.DELTAE.DELTAR.DELTAZ.DELTA0.

ERROR.S.EO.EIN.DELPR.DELPZ.DELPO.

TABLE(120), TABL2(120), TABL3(120), TABL4(120), TABL5(120)

PRINT 301

PRINT 302, J1, J2, J3, J4, NR, I1, I2, JK

PRINT 303

PRINT 370

FORMAT(72H0)

ERROR.S.EO.EIN.DELPR.DELPZ.DELPO.

PRINT 307

FORMAT(301)

PRINT 301

FORMAT(301)

FORMAT(301)

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RT(1,1)=2.4048256000
1) 51 9,10
9 PRINT 308*RT(1,1)*RT(1,2)*RT(1,3)*RT(1,4)*RT(1,5)
308 FORMAT(26H0THE FIRST FIVE ROOTS ARE IP5tl4.7)
10 BB=RT(1,1)/R)**2+(0(1)*PI/H)**2
11 PRINT 309,RT(1.1 ) .RT(1.2) .RT(1.3) .RT (1.4) .RT(1.5)
309 FORMAT(56H0THE GEOMETRIC BUCKLING FOR THE REACTOR WITHOUT RODS IS
12 IF(TEST) 3006.2.3006
13 J5=1
14 J16=1
C=1.0
305 WRITE TAPE 4,C
15 IF(J16-J15) 100.101.101
16 J16=J16+1
GO TO 305
17 END FILE 4
18 REWIND 4
19 PRINT 310
310 FORMAT(102H0 A(M.N) B(L) O(M.N.L)
2 E(M.N.L) Z(M.N.L) M+1 N L RDNO)
3006 DO 1119 I=1,NR
1200 S8=JBESSF(RT(1.1)*RR(I)/R.1)
13 S9 = JBESSF( RTd.1 )*RR ( I ) /R.0)
14 DO 1119 M=1,J1
15 Y=M
16 DO 1119 N=1,J2
17 READ TAPE 7.R00TMN
18 IF(M-1) 1111.1111.1113
19 IF(N-1) 1112.1112.1113
20 R00TMN=2.4048256000
21 S2=JBESSF(R00TMN*RR(I)/R.1)
22 ARG=D(I)*R00TMN
23 IF(ARG-50.) 2222,2223,2224
24 S6=JBESSF(R00TMN,Y)
25 GO TO 9100
26 GAM0=2./(RT(l.l)**2-ROOTMN**2)»(RT(1.1)•S7*S8-ROOTMN*S9*S2)
27 GO TO 9100
28 GAM0=RR(I)/(2.»R)*(S2**2+S7»»2)
29 IF(D(I)-.001) 3130.3130.7777
30 DO 4444 K=1,JK
31 BS=BESJF(R00TMN#RR(I)/R,GNU,K+1,LEE,LMM)
32 SS10=TABL2(K)
33 S10=TABL2(K+1)
34 S11=TABL2(K+2)
35 CS=BEsJF{RTd,1)*RR(I)/R,GNU,K+1,LEEE,LMMM)
36 SS12=TABL3(K)
37 S12=TABL3(K+1)
38 S13=TABL3(K+2)
39 S17=JBESSF(R00TMN,M)
40 IF(M+K-16) 1234.5678.5678
41 IF(R00TMN*D(I)/R-50.)7789.778 9,8701
42 DS=BESJF(R00TMN»D(I)/R,GNU,M-1+K,LFEE,LNMM)
43 NEWMK=M+K
44 S14=TABL4(NEWMK)
45 GO TO 9104
46 IF(ROOTMN«D(I)/R-5*T) 9102,9102,8802
47 IF(ROOTMN«D(I)/R-4.*T) 9103,9101.9101
48 1234 IF(RO0TMN*D(I)/R-50.)789. 789.9101

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E=(\text{ROOTMN} / R)^2+(0(L)*\pi/H)^2  
\text{GO TO 199}  

800 B = \frac{1}{2} \left( \text{PI} * \text{HR(I)} \right) / \left( 2 \text{PI} \times \text{HR(I)} / H \right)  
\text{GO TO 199}  

801 B = \frac{1}{2} \left( \text{PI} * \text{HR(I)} / H \right) / \left( \text{PI} \times \text{HR(I)} / H \right)  
\text{GO TO 199}  

199 \text{P} = R^2 \times H \times \text{JBESSF} \left( \text{ROOTMN} / R, \frac{\text{NR}}{2} \right) \times \text{SINF} \left( \text{Q(L)} \times \text{PI} \times \text{HR(I)} / H \right)  
\text{WRITE TAPE 2, A, B, P, E}  
\text{DO 119 J=1, NR}  

4113 \text{IF} \left( \text{ROOTMN} / R - 50. > 4113 \text{PRINT 55, A, B, E, Z, M, N, L, I} \right)  
\text{STOP 6666}  

5551 \text{IF} \left( \text{SENSE SWITCH 1} \right)  
\text{PRINT 301 PH}  
\text{STOP 55555}  

END(0,1,0,0,0)
PROGRAM FOR THE CALCULATION OF THE BUCKLING AND FLUX FOR A BARE CYLINDRICAL HOMOGENEOUS REACTOR WITH PARTIALLY INSERTED CONTROL RODS

PHASE 2 (PHASE 2A) AND PHASE 2B - CALCULATION OF THE VARIOUS ORDERS OF APPROXIMATION OF THE GEOMETRIC BUCKLING.

4/30/59

E.J. BETINIS

DIMENSION CO(10),RR(10),HR(10),ALPH(10),RT(11)

READ 5,RR,HR,ALPH,RT

ERROR=EO*EIN+DELPR+DELPZ+DELP0*(HR(I),I=1,10)+RR(I),I=1,10)+ALPH(I),I=1,10)*AREA+TEST1+TEST2

5 FORMAT(1H,1PE14.7)

READ 3,RR,HR,ALPH

FORMAT(8I4)

READ 304,RR

FORMAT(116)

READ 303,RR,HR,ALPH

FORMAT(3I3)

READ 370

FORMAT(72H0)

READ 556,RR

FORMAT(116)

PI = 3.1415926

DELTIN = DELTAE

RT(I) = 2.4048256

DO 14 K = 1, NR

C0(K) = 8.*RR(K)/R*PI

PRINT 306

FORMAT(IH1)

PRINT 307,RR,HR,ALPH

FORMAT(3I3)

READ OUTPUT TAPE 5,307,RR,HR,ALPH

FORMAT(6H DATE , IH/I2, IH/I2)

PRINT 370

PRINT 308

FORMAT(72H0)

PRINT 312

FORMAT(24H0R0D POSITION PARAMETERS)

PRINT 313,RR,HR,ALPH

FORMAT(10F10.3)

PRINT 6000

6000 FORMAT(24H0BEGINNING OF ITERATIONS)

IF(ISTEST1) 1146,1146,6002

1146 IF(ISTEST2) 1046,1046,115

115 SUMA = 0.0

DO 19 I = 1, NR

DO 19 M = 1, J1

DO 19 N = 1, J2

DC 19 L = 1, J3

READ TAPE 2, A, B, P, E

READ TAPE 4, C

DO 19 J = 1, NR

READ TAPE 8, QU, Z

IF(ONSE SWITCH 6) 99.19

99 WRITE OUTPUT TAPE 5, 9911

9911 FORMAT(5OSUMA)

WRITE OUTPUT TAPE 5, 9999, SUMA=SUMA+P*E/(EO-E)*IRR(J)*QU)

19 SUMA = SUMA + CO(I)*A*B*C/(EO-E)*IRR(J)*QU)

REWIND 8

REWIND 2

REWIND 4

IF(ONSE SWITCH 5) 221,22

221 PRINT 5, EO, SUMA

21 IF(ONSE SWITCH 1) 5022,5021

5021 IF(SUMA) 23,22,22

22 EO = EO + DELTAE

GO TO 115
23 IF(J5=J4) 27,26,26
26 E1=EO
GO TO 288
27 EO=EO-DELTAE
DELTAE=DELTAE/S
J5=J5+1
GO TO 22
288 JJ5=JJ5+1
IF(SENSE SWITCH 4) 2888,28
2888 PRINT 3888,JJ5,EO
3888 FORMAT(22H0F0R ITERATION NUMBER I3,24H THE GEOMETRIC BUCKLING=1PE1)
28 SUMC=0.0
DO 32 1=1,NR
DO 32 M=1,J1
DO 32 N=1,J2
DO 32 L=1,J3
READ TAPE 2,A,B,P,E
READ TAPE 4,C
IF(M-l) 1028,1028,1029
1028 SUMC=SUMC+CO(I)*A*B*C*E/(El-E)*P
1029 DO 32 J=1,NR
READ TAPE 8,QU,Z
IF(SENSE SWITCH 6) 8832,32
8832 WRITE OUTPUT TAPE 5,9922
9922 FORMAT(5H0SUMC)
WRITE OUTPUT TAPE 5,9999,SUME,M,N,L,J,I
32 SUME=0.0
DO 38 1=1,NR
DO 38 M=1,J1
DO 38 N=1,J2
DO 38 L=1,J3
READ TAPE 2,A,B,P,E
READ TAPE 4,C
IF(M-l) 1038,1038,1039
1038 SUME=SUME+CO(I)*A*B*C/(E1-E)*P
1039 DO 38 J=1,NR
READ TAPE 8,QU,Z
IF(SENSE SWITCH 6) 8838,38
8838 WRITE OUTPUT TAPE 5,9933
9933 FORMAT(5H0SUME)
WRITE OUTPUT TAPE 5,9999,SUME,M,N,L,J,I
38 SUME=SUME+CO(I)*A*B*C/(E1-E)*Z
9999 FORMAT(1H1PE14.7,5I4)
END FILE 5
REWIND 8
REWIND 4
REWIND 2
41 E2=SUMC/SUME
JJ5=JJ5+1
IF(SENSE SWITCH 5) 42,43
42 PRINT 5,E2,CO(I)*SUMC/SUME
43 IF(SENSE SWITCH 4) 2889,4443
2889 PRINT 3888,JJ5,EO
4443 IF(ABS(E2-E1)-ERROR) 48,48,1046
C
C PHASE 2(B)
C
1046 DO 47 I=1,NR
DO 47 M=1,J1
DO 47 N=1,J2
DO 47 L=1,J3
READ TAPE 4,C
READ TAPE 2,A,B,P*E
C=C*ARSF((E2-E)/(E1-E))
47 WRITE TAPE 6,C
END FILE 6
REWIND 2
REWIND 4
REWIND 6
J16=1
772 READ TAPE 6,C
WRITE TAPE 4,C
IF(J16=J16) 772,771,771
770 J16=J16+1
GO TO 772
PROGRAM FOR THE CALCULATION OF THE BUCKLING AND FLUX FOR A BARE CYLINDRICAL HOMOGENEOUS REACTOR WITH PARTIALLY INSERTED CONTROL RODS

PHASE 3—CALCULATION OF THE FLUX IN VARIOUS REGIONS OF THE REACTOR. THE REACTIVITY WORTH AND THE BUCKLING FOR THE REACTOR CONSIDERED.

4/30/59 E.J. BETINIS

DIMENSION U(51),Q(100),CO(10),RR(10),D(10),HR(I),ALPH(I),TABLE X(120)

ALPF(X,Y)=.5*L0GF((1.+SQRTF(1.-(X/Y)**2))/(1.-SORTF(1.-(X/Y)**2))
BETF(X,Y)=ATANF(SQRTF((X/Y)**2-1.))
JBES1F(X,Y)=EXPF(Y*(TANHF(ALPF(X,Y)))-ALPF(X,Y))/SORTF(2.*PI*Y*TANHF(ALPF(X,Y)))
JBES2F(X,Y)=SQRTF(2./(PI*Y*SINF(BETF(X,Y))/COSF(BETF(X,Y))))*COSF(1.)*SINF(BETF(X,Y))/COSF(BETF(X,Y))-Y*BETF(X,Y)-PI/4.)

READ 5,R,H,RIN,ZIN,THETIN,DELTAE,DELTAZ,DELTA0,

WRITE(IH 1P5E14.7)

READ 3,J1,J2,J3,J4,NR,I,12,JK

WRITE(26H BEGIN MONITORING PHASE 3.)

5 FORMAT(IH 1I2,IH/I 2,IH/I 2)

READ 308 FORMAT(26H0BEGIN MONITORING PHASE 3.)
READ TAPE 7,ROOTMN
DO 956 L=1,J3
READ TAPE 2,A,B,P,E
READ TAPE 4,C
956 IF(M-16) 9114,9113,9113
9113 IF(ROOTMN*RIN/R-50.) 9115,9115,8905
9115 A=5E-JF(ROOTMN*RIN/R,A=GNU,M=LE,LM)
IF(A) 1111,9006,1111
9006 S5=TABLE(M)
GO TO 9125
8905 IF(ROOTMN*RIN/R-4.*Y) 9213,9213,8906
8906 IF(ROOTMN*RIN/R-50.) 9114,9113,9113
9114 IF(ROOTMN*RIN/R-50.) 9124,9124,9124
9124 A=5E-JF(ROOTMN*RIN/R,A=GNU,M=LE,LM)
IF(A) 1111,9125,1111
9125 S5=TABLE(M)
9126 IF(M-16) 9114,9113,9113
9127 S5=TABLE(M)
GO TO 956
8905 IF(ROOTMN*RIN/R-.5*Y) 9213,9213,8906
8906 IF(ROOTMN*RIN/R-.5*Y) 9223,9134,9134
9213 SUMH=SUMH+CO(M)*A*B*JBES1F(RootMN*RIN/R,Y-1.)•
1(COSF(U(M)*ALPH(I))*COSF(U(M)*THETIN)+SINF(U(M)*ALPH(I))*SINF(U(M)*THETIN))•
2(U(M)*THETIN))•SINF(GL)*PI*ZIN/H)/(EO-E)
GO TO 956
9223 SUMH=SUMH+CO(M)*A*B*JBES2F(RootMN*RIN/R,Y-1.)•
1(COSF(U(M)*ALPH(I))*COSF(U(M)*THETIN)+SINF(U(M)*ALPH(I))*SINF(U(M)*THETIN))•
2(U(M)*THETIN))•SINF(GL)*PI*ZIN/H)/(EO-E)
GO TO 956
956 CONTINUE
REPRINT 4
REWIND 2
REWIND 7
4501 IF(SENSE SWITCH 2) 60,62
60 PRINT 61,RIN,ZIN,THETIN,SUMH
61 F0RMAT(3H R=F6.3,3H Z=F6.3,7H THETA=F6.3,6H FLUX=1PE14.6)
62 WRITE OUTPUT TAPE 5,61,RIN,ZIN,THETIN,SUMH
IF(SENSE SWITCH 4) 1000,4500
4500 RIN=RIN+DELTAR
IF(RIN-R1 55,55,64
64 RIN=RIN-DELPZ
ZIN=ZIN-DELTAZ
IF(ZIN-H) 55,66,66
66 ZIN=ZIN-DELPZ
THETIN=THETIN+DELTAO
IF(THETIN-6.2831852)55,68,68
68 THETIN=THETIN-DELP0
END FILE 5
REWIND 5
PRINT 320
320 FORMAT(25H END MONITORING PHASE 3.)
PRINT 306
STOP 77777
1000 PRINT 321
321 FORMAT(31H PHASE 3. HAS BEEN INTERRUPTED.)
END FILE 5
REWIND 5
PRINT 306
556 FORMAT(1P2E14.7,I3)
PUNCH 556,J1,J2,J3,J4,NR,RI,AREA,Rl
PUNCH 330304,J15
PUNCH 356,J12,J13,J14
PUNCH 356356,J13,J14
PUNCH 556356,EZ,EB,J15
556 FORMAT(1P2E14.7,I3)
STOP 22222
1111 PRINT 6666
6666 FORMAT(11H AN ERROR OCCURRED IN THE EXECUTION OF NYU BESSEL FUNCTION)
10N SUBROUTINE. THIS IS EITHER A MACHINE ERROR OR A DATA ERROR.
PRINT 306
STOP 1111
END (0.1,0,0.0)
AUXILIARY PROGRAM NO. 1.-FOR CALCULATING THE
BUCKLING OF A FULLY INSERTED CENTRAL ROD BY THE
NORDHEIM-SCALLETAR THEORY AND FROM THE EXACT
SOLUTION. THE FLUX FROM THE EXACT SOLUTION IS ALSO
CALCULATED.

4/30/59

E.J.BETINIS

THETAF(X,Y) = JBESSF(X*0)-JBESSF(Y*0)/YBESSF(Y,0)*YBESSF(X,0)

READ3. RO.ZIN.DELTAR.DELTAZ.DELTAL.ALPHIN.R,H,DELPR,C2

3 FORMAT (1PE14.7)

READ 4.R0

4 FORMAT (14)

RO=2.4048256
PI=3.1415926

PRINT 76.R0.R.H

76 FORMAT (26H 0FOR THE EFFECTIVE RADIUS F7.4,7H AND R=F6.3,3H H=F6.3)

DELBE=3.75/(ROOT*(.116+LOGF(R/(2.4*RO))))

BETA=ROOT+DELBE

EIGVL1=(BETA/R)**2+(PI/H)**2

J5=1

115 X=ALPHIN*RO

Y=ALPHIN

FUNCT=THETAF(X,Y)

IF (SENSE SWITCH 1) 555,556

555 PRINT 5,ALPHIN,FUNCT

5 FORMAT (7H ALPHA=1PE15.7,12H TRANSCENDENTAL FUNCTION=1PE14.7)

556 IF(FUNCT) 23,22,22

22 ALPHIN=ALPHIN+DELTAZ

GO TO 115

23 IF(J5-J4) 27,27,22

27 ALPHIN=ALPHIN-DELTAZ

DELTAZ=DELTAZ/2.

J5=J5+1

GO TO 22

26 EIGVL2=(ALPHIN/R)**2+(PI/H)**2

PRINT 66.BETA+EIGVL1+ALPHIN+EIGVL2

66 FORMAT (22H 0ROOT NORD-SCALL THEO=1PE14.7,12H AND GEO BK=1PE14.7,12H)

X=ALPHIN*15.*R0

Y=ALPHIN

F=THETAF(X,Y)

C1=C2/F

700 X=ALPHIN*RO

F=THETAF(X,Y)*SINF(PI*ZIN/H)

F=CI+F

PRINT 666.R0.ZIN+F

666 FORMAT(3H R=F6.3,3H Z=F6.2,10H FUNCTION=1PE14.61

RO=RO+DELTAZ

IF(RO-R) 700,700.64

64 RO=RO-DELPR

ZIN=ZIN+DELTAZ

IF(ZIN-H) 700,667.667

667 STOP 77777

END(1.0,0.1)
AUXILIARY PROGRAM NO. 2.-FOR INITIALIZING TAPE NO. 4 IF PHASE 2(A) IS TO BE REPEATED.

4/30/59

E.J.BETINIS

READ J1,J2,J3,NR,I3

1 FORMAT(5I4)

C=1.0

DO 2 J=1,J1
    DO 2 M=1,J1
    DO 2 N=1,J2
    DO 2 L=1,J3
    WRITE TAPE 4,C

END FILE 4

REWIND 4

STOP

AUXILIARY PROGRAM NO. 3.-FOR RESETTING TAPES 2,4, AND 8 WHEN NR RODS ARE CONSIDERED AND PHASE 1. IS CALCULATED FOR J1 TO NR RODS TAKEN 12 AT A TIME.

4/30/59

E.J.BETINIS

READ 1,J1,J2,J3,NR,I3

1 FORMAT(5I4)

READ 303,J12,J13,J14

303 FORMAT(3I3)

PRINT 302,J12,J13,J14

302 FORMAT(6H DATE 12*1H/12*1H/12)

DO 2 J=1,J1
    DO 2 M=1,J1
    DO 2 N=1,J2
    DO 2 L=1,J3
    II = I
    MM = M
    NN = N
    LL = L

READ TAPE 2,A,B,P,E

READ TAPE 4,C

DO 2 J=1,NR
    READ TAPE 8,QU,Z

PRINT 3

3 FORMAT(H0THE LAST RECORDS NOW ON TAPES 2 AND 8 ALONG WITH RESPECTIVE INDICES ARE GIVEN IN THE FOLLOWING TABLE.)

PRINT 310

310 FORMAT(102HO A(M,N) B(L) Q(M+N,L) P(M+N,L)

1 E(M,N,L) Z(M+N,L) M+1 N L RDNO)

PRINT 55,A+B+QU+Z+MN+NL+II

55 FORMAT(1H 1P6E14.7*14)

PRINT 4,C

4 FORMAT(31H0 THE LAST RECORD ON TAPE 4 WAS 1PE14.7)

STOP

END

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AUXILIARY PROGRAM NO. 4.-FOR GENERATING MXN
ROOTS RT(M,N) SUCH THAT \( J(RT(M,N)) = 0 \), WHERE
M=50, N=100, AND J IS THE BESSSEL FUNCTION OF
THE FIRST KIND, IN THREE PARTS.
4/30/59
E.J.BETINIS

PART 1. TAPE 5.

DIMENSION U(51), Q0(50), Q1(50), Q2(50), Q3(50), Q4(50), TABLE(120).

10
BETF(X,Y) = ATANF(SQRTF((X/Y)**2-1))
JBES2F(X,Y) = SQRTF(2./(PI*Y*SINF(BETF(X,Y)))/COSF(BETF(X,Y)))
LY = SINF(BETF(X,Y))/Y*BETF(X,Y)-PI/4.
READ 3, J1, J2, J3, J4, NR, I1, I2, JK

3 FORMAT(8I4)

GNU=0.0
LE=120
LM=XL0CF(TABLE(11))
Ul1=0.0
DO 4 K=1, J1
4 U(K+1) = U(K-1)
10 PI = 3.1415926
DO 6 M=1, J1
600(M) = 4.*U(M)**2
601(M) = 7.*Q0(M)**2-982.*Q0(M)+3779.
604(M) = 70197.*Q0(M)**4-2479316.*Q0(M)**3+48010494.*Q0(M)**2-
148.*Q0(M)+2092163573.
6 03(M) = 6949.*Q0(M)**3-153855.*Q0(M)**2+1585743.*Q0(M)-627723.
0(J) = Q(J)+1.
DO 7 I=1, J1
7 0(J) = Q(J-1)+1
DO 8 I=1, J1
8 0(IN) = 1.*Q0(IN)/C0SF(AUX4)*E**8
EM = FLOATF(M)
Y = EM
ROOTMN = BE-1000(M)-1./8.*BE)*(1.+Q1(M)/13.*4*BE**2)+2.*Q2(M)/11
15.*4*BE**4)+Q3(M)/15.*4*BE**6)+Q4(M)/12520.*8.*4*BE**8)
IF(M=15) 3677, 3676, 3667
3677 IF(RootMN-50.) 8801, 8802, 3688
3676 AS = BESJF(ROOTTMN, GNU, M, LE, LM)
3675 S6 = TABLE(M)
3667 IF(RootMN-50.) 8801, 8802, 3688
3688 IF(RootMN-4.*EM) 8802, 8802, 3698
3687 S6 = JBES2F(ROOTTMN-4.)
GO TO 8
3698 S6 = JBESFF(ROOTTMN+M-1)
8 WRITE OUTPUT TAPE 5, 3030+S6+ROOTTMN+M+N
3030 FORMAT(1H 1P2E14.7, 2I4)
END
REWIND 5
STOP33333
END(0,1,0,0,1)
C PART 2, TAPE 5.

11 READ 1000, CORTMN, MM, NN, IND
1000 FORMAT(1H 1P2E14.7, 3I4)
1001 READ INPUT TAPE 5, 3030+S6+ROOTTMN+M+N
3030 FORMAT(1H 1P2E14.7, 2I4)
IF(MN=M) 4008, 4003, 4003
4003 IF(NN=M) 4008, 4003, 4003
4008 WRITE OUTPUT TAPE 5, 31000+S6+ROOTTMN+M+N
GO TO 1001
4000 WRITE OUTPUT TAPE 5, 31000+CORTMN+M+N
IF(IND) 11, 1001, 11
C MANUAL END OF FILE TAPE 3
END(0,1,0,0,1)
C PART 3, TAPE 5.

DIMENSION TABLE(120)
BETF(X,Y) = ATANF(SQRTF((X/Y)**2-1.1))
JBES2F(X,Y) = SQRTF(2./PI*Y*SINF(BETF(X,Y))/COSF(BETF(X,Y)))
LY = SINF(BETF(X,Y))/Y*BETF(X,Y)-PI/4.

AUX4 000
AUX4 001
AUX4 002
AUX4 003
AUX4 004
AUX4 005
AUX4 006
AUX4 007
AUX4 008
AUX4 009
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AUX4 077
AUX4 078
AUX4 079
3.1415926
0
120
XXOF(TABLE(1))
2000 READ INPUT TAPE 3,3031,ROOTMN,M,N
Y=M
4008 WRITE TAPE 1,ROOTMN
IF(M=15) 3677,3667,3667
3677 IF(ROOTMN-GNU, 3678,3668,3698
3678 AS=BESJF(ROOTMN,GNU,GNU+M-LE+LM)
S6=TABLE(M)
GO TO 8
3668 IF(ROOTMN-50.) 3698,3698,3698
3698 AS=BESJF(ROOTMN,GNU,GNU+M-LE+LM)
S6=TABLE(M)
GO TO 8
3688 IF(ROOTMN-4.) 3801,3801,3801
3801 AS=BESJF(ROOTMN,GNU,GNU+M-LE+LM)
S6=TABLE(M)
GO TO 8
3667 IF(ROOTMN-50.) 3802,3802,3802
3802 AS=BESJF(ROOTMN,GNU,GNU+M-LE+LM)
S6=TABLE(M)
GO TO 8
3030 FORMAT(1H 1P2E14.7,2I4)
3031 FORMAT(1H 1PE14.7*214)
C MANUAL END OF FILE TAPES 1,5
C AUXILIARY PROGRAM NO. 5.--FOR PUTTING MXN
C ROOTS RT(M,N) ON TAPE 7,WHERE M AND N ARE VARIABLE,
C 4/30/59
C
DIMENSIONRT(50,100)
READ 1,J11,J22,J1,J2
1 FORMAT(1H 414)
DO 2 M=1,J1
DO 2 N=1,J2
2 READ TAPE 1,RT(M,N)
REWIND 1
DO 3 M=1,J1
DO 3 N=1,J2
WRITE OUTPUT TAPE 6,4,RT(M,N),M,N
3 WRITE TAPE 7,RT(M,N)
END FILE 6
REWIND 6
END FILE 7
REWIND 7
4 FORMAT(1H 1PE14.7*214)
STOP 77777